ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES:: RAJAMPET
(An Autonomous Institution)

DEPARTMENT OF MECHANICAL ENGINEERING

LECTURE NOTES

Operations Research

[20A37AT]

ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES RAJAMPET
(An Autonomous Institution)
Department of Mechanical Engineering

Title of the Course Operations Research
Category PEC
Course Code 20A37AT

Year IV B. Tech
Semester I Semester
Branch ME

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	0	0	3

Course Objectives:

- To enable the students to the nature and scope of various decision making situations within business contexts, understand and apply operations research techniques to industrial applications.
- To learn the fundamental techniques of Operations Research and to choose a suitable OR technique to solve problem.

Unit 1 **10**

Development – Definition– Characteristics and Phases – Types of operation and Research models– applications.
Linear Programming Problem Formulation – Graphical solution – Simplex method –Artificial variables techniques - Two–phase method, Big-M method – Duality Principle.

Learning Outcomes: At the end of the unit, the student will be able to:

- Formulate practical problems given in words into a mathematical model. (L6)
- Quantify OR models to solve optimization problems. (L5)
- Formulate linear programming problems and appreciate their limitations. (L6)

Unit 2 **10**

Transportation Problem: Formulation – Optimal solution, unbalanced transportation problem –Degeneracy.
Assignment Problem – Formulation – Optimal solution - Variants of Assignment Problem-Travelling Salesman problem

Learning Outcomes: At the end of the unit, the student will be able to:

- Model linear programming problems like the transportation. (L3)
- Solve the problems of transportation from origins to destinations with minimum time and cost. (L6)

Unit 3 **10**

Replacement Models: Introduction – Replacement of items that deteriorate with time – with change in money value - without change in money value – Replacement of items that fail completely, group replacement.
Theory Of Games: Introduction – Minimax - Maximin – Criterion and optimal strategy – Solution of games with saddle points – Rectangular games without saddle points – 2 X 2 games – m X 2, 2 X n & m x n games -Graphical method, Dominance principle.

Learning Outcomes: At the end of the unit, the student will be able to:

- Apply the concept of replacement model. (L3)
- Identify strategic situations and represent them as games. (L3)
- Solve simple games using various techniques. (L6)

Unit 4 **10**

Waiting Lines: Introduction – Single Channel – Poisson arrivals – exponential service times – with infinite queue length models.
Simulation: Definition – Types of simulation models – phases of simulation– applications of simulation – Queuing problems – Advantages and Disadvantages – Simulation Languages.

Learning Outcomes: At the end of the unit, the student will be able to:

- Understand and will apply the fundamentals of waiting lines in real life situations. (L3)
- Simulate queuing models. (L3)

Unit 5 **10**

Inventory: Introduction – Single item – Deterministic models – Purchase inventory models with one price break and multiple price breaks.

Dynamic Programming: Introduction – Bellman's Principle of optimality – Applications of dynamic programming- shortest path problem – linear programming problem.

Learning Outcomes: At the end of the unit, the student will be able to:

- Understand and will apply the fundamentals of inventory in real life situations. (L3)
- Have aware of applying Dynamic Programming technique to solve the complex problems by breaking them into a series of sub- problems. (L3)

Prescribed Text Books:

1. Operations Research, PS Gupta, DS Hira, S Chand Publications, 10th Edition, 2016, ISBN-13978-8121902816.
2. Operations Research, S.D. Sharma, Kedarnath and Ramnath Publications, 2012, ISBN-135551234001596.

Reference Books:

1. Introduction to Operations Research. Taha, PHI, 10 th edition, 2016, ISBN-13978-0134444017.
2. Operations Research. R. Panneerselvam, PHI Publ, 2nd edition, 2004, ISBN: 9788120319233.
3. Operations Research: Theory and Applications,Sharma J.K., 4th Edition, Laxmi Publications, 2009.

Course Outcomes:

A student will be able to		Blooms Level of Learning
1.	Solve the Linear Programming Problems using Graphical, Simplex and Artificial Variable Techniques	L3
2.	Solve the Transportation, Assignment and Travelling Salesmen problems	L3
3.	Solve the problems of replacement and Game Theory	L3
4.	Analyze the waiting lines in real life situations and Simulate the queuing models	L4
5.	Apply the inventory models related to market and Dynamic Programming technique for complex problems	L3

CO-PO-PSO Mapping:

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
20A37AT.1	3	2	1	2	-	-	-	1	-	-	-	-	1	2
20A37AT.2	3	2	1	2	-	-	-	1	-	-	-	-	1	2
20A37AT.3	3	2	1	2	-	-	-	1	-	-	-	-	1	2
20A37AT.4	3	3	2	2	-	-	-	1	-	-	-	-	1	1
20A37AT.5	3	2	1	2	-	-	-	1	-	-	-	-	1	1

Unit - I

Linear Programming Models

Linear programming deals with the optimization (Maximization or minimisation) of a function of variables known as "objective function", subject to a set of linear equations and/or inequalities known as 'constraints'.

The objective function may be either profit, production capacity, sales, efficiency, crop yeild which are needed to be maximized or time, cost, losses, breakdown etc which are needed to be minimized. The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability etc.

The linear programming model first conceived by "George B Dantzig" in 1947. Koopmans coined the term 'Linear Programming' in 1948. Dantzig also developed the most powerful mathematical tool known as "simplex method" to solve LP problems in 1949.

Requirements of Linear Programming Problem:

1. Objective function: There must be a well defined objective function present which is to be either maximized & minimized. ~~and which~~ It is ~~can~~ to be expressed as a linear function of "decision variables".

2. Decision variables: These are the problem variables like x_1 & x_2 and they represent the solution or the output decision from the given LP problem.

3. Constraints: These are the conditions matching the resources availability and resources requirements. These usually limit (or Restrict) the values of decision variables can take.

④ ~~we have~~ It also needed to explicitly state that the decision variable ~~should~~ Must be "non-negative values". This is called "non-negativity restriction" ($x_1, x_2 \geq 0$)

⑤ Problem formulation: The problem that we have written down in algebraic form represents the mathematical model of the given system and is called the problem formulation.

Linear Programming Problem (LPP) formulation:

Ex ①: A company manufactures two products A & B. Both the Products pass through two machines M_1 & M_2. The time required to process each unit of products A & B on each m/c and available capacity of each m/c (in hrs) is given below.

Products	M/c Processing time M_1	M_2
A	6	2
B	4	4
Available capacity (hrs)	3600	2000

The availability of materials is sufficient to product 500 units of Product A and 400 units of Product B. Each unit of product 'A' gives a profit of Rs. 25 and each unit of Product B gives a profit of Rs. 20. Construct LP model to determine the quantity of each product to be manufactured to maximize the profit.

Sol

Let x_1 - no of Product 'A' type units.

x_2 - no of units of Product 'B' type

Objective function: Maximize $Z = 25x_1 + 20x_2$

Subjected to the constraints

$$\left.\begin{aligned} 6x_1 + 4x_2 &\leq 3600 \\ 2x_1 + 4x_2 &\leq 2000 \end{aligned}\right\} \text{m/c hrs} \qquad \left.\begin{aligned} x_1 &\leq 500 \\ x_2 &\leq 400 \end{aligned}\right\} \text{material}$$

and $x_1, x_2 \geq 0$.

Q2: The XYZ company manufactures two different type of products A & B. Each product is to be processed in 3 different departments ie, casting, machining and final inspection. The available times (capacities) of the departments are limited to 45 hrs, 35 hrs and 30 hr per week respectively. Product A requires 10 hrs in casting dept. 12 hr in the machining dept. and 6 hr in inspection where as B require 7 hrs, 6 hrs and 8 hrs for the same. The profit contribution for an unit product A & B is Rs 50 & Rs 40 respectively. Formulate the mathematical model.

Sol

let

	C	M	R	Profit
A	10	12	6	Rs. 50
B	7	6	8	Rs. 40
	45	35	30	

let x_1 - No of units of Product 'A' type

x_2 - No of units of Product 'B' type

Objective function:

Maximise $Z = 50x_1 + 40x_2$

Subjected to the constraints

$$10x_1 + 7x_2 \leq 45$$

$$12x_1 + 6x_2 \leq 35$$

$$6x_1 + 8x_2 \leq 30$$

and $x_1, x_2 \geq 0$ (non-negativity constraint)

	R1	R2	Profit
A (x)	1	3	Rs 6
B (y)	1	2	Rs 5
Available	5	12	

Max $Z = 6x_1 + 5x_2$

$x + y \leq 5$

$3x + 2y \leq 12$

$x, y \geq 0$

1. Solution
2. Basic Solution
3. BFS ≥ 0
4. Degenerate solution 2 cost make

Ex ③: A food company is developing a low-calorie, high protein diet supplement called Hi-Pro. The specifications for Hi-Pro have been established by a panel of medical experts. These specifications along with calorie, protein & vitamin content of 3 basic foods are given in below table.

Nutritional Elements \ Basic foods	Units of Nutritional Element 1	2	3	Basic foods Hi-Pro specification
Calories	350	250	200	300
Proteins	250	300	150	200
Vitamin-A	100	150	75	100
Vitamin-C	75	125	150	100
Cost/serving (Rs)	1.50	2.0	1.20	

What quantities of foods 1, 2, & 3 should be used. Formulate this problem as an LP model to minimize cost of serving.

Sol: let x_1, x_2, x_3 - Quantities of foods 1, 2, 3.

Objective function:

$$\text{Minimize } Z = 1.5x_1 + 2.0x_2 + 1.2x_3$$

Subjected to the constraints:

$$350x_1 + 250x_2 + 200x_3 \geq 300$$
$$250x_1 + 300x_2 + 150x_3 \geq 200$$
$$100x_1 + 150x_2 + 75x_3 \geq 100$$
$$75x_1 + 125x_2 + 150x_3 \geq 100$$

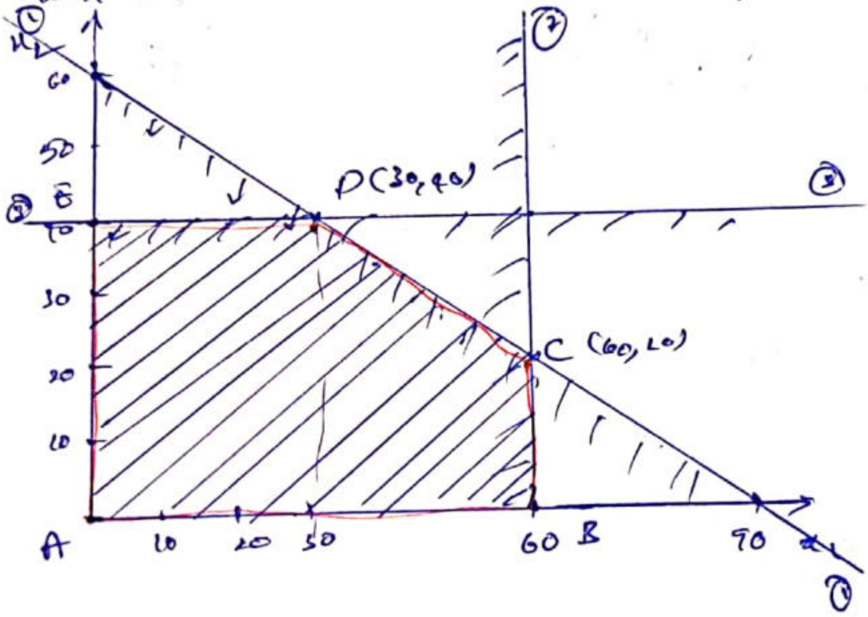
and $x_1, x_2, x_3 \geq 0$.

Graphical Method to Solve LPP:

① Solve the following LPP by using graphical Method

$$\text{Max. } Z = 15x_1 + 10x_2$$
$$4x_1 + 6x_2 \le 360$$
$$3x_1 \le 180$$
$$5x_2 \le 200$$

Sol ① Take x_1 - on horizontal axis and x_2 - on vertical axis. ② Plot each constraint on the graph by treating it as a linear equation and it is then the inequality conditions will be used to mark the area of feasible solutions



Constraints

$4x_1 + 6x_2 = 360$

① Put $x_1 = 0$; $x_2 = 60$
$x_2 = 0$; $x_1 = 90$

② $x_1 = 60$

③ $x_2 = 40$

At: A = (0,0) : Z = 0
B = (60,0) : Z = 900
C = (60,20) : Z = 1100 ✓
D = (30,40) : Z = 850
E = (0,40) : Z = 400

At $x_1 = 60$, $x_2 = 20$ $\boxed{\text{Max } Z = 1{,}100}$.

Point:
→ Highlight only the area in I^{st} quadrant
→ RHS constraint should be +ve.

Case ①: Maximize $Z = 5x_1 + 4x_2$

Subject to
$$x_1 - 2x_2 \le 1$$
$$x_1 + 2x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

Sol

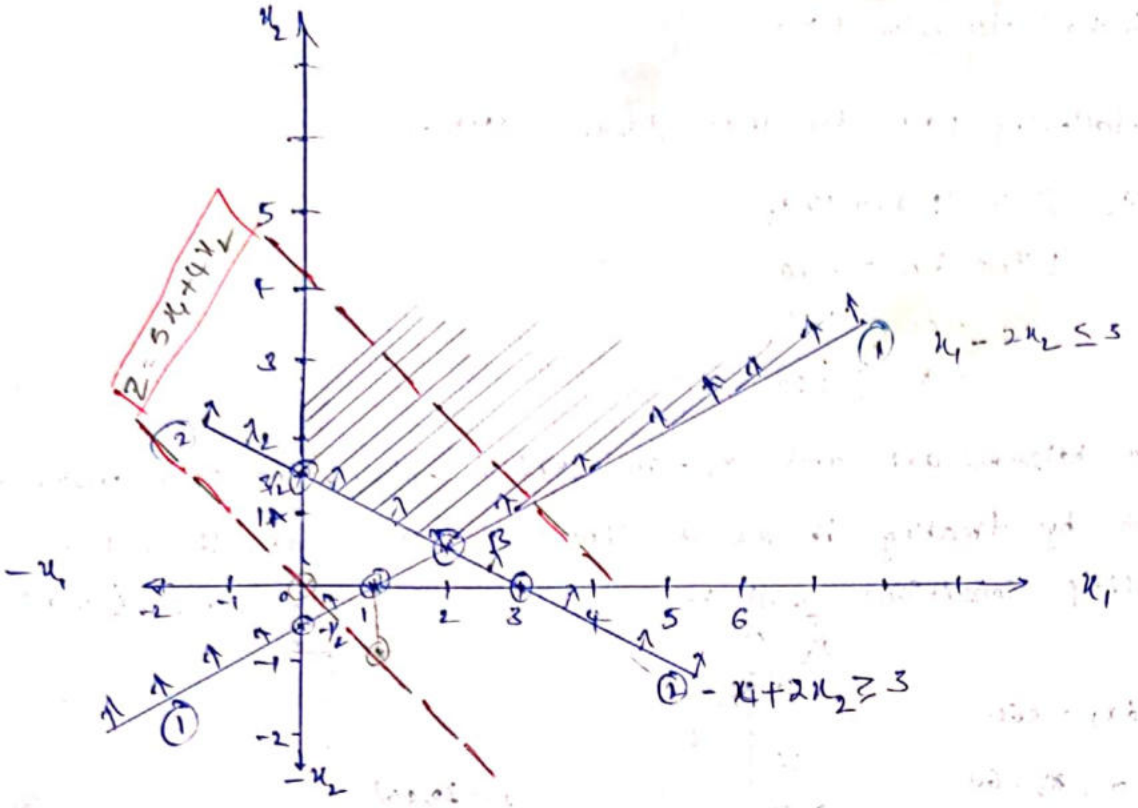
Constraints:

① $x_1 - 2x_2 = 1$

Put $x_1 = 0$; $x_2 = -1/2$
$x_2 = 0$; $x_1 = 1$

② $x_1 + 2x_2 = 3$

$x_1 = 0$; $x_2 = 3/2$
$x_2 = 0$; $x_1 = 3$



When $Z=0$; $5x_1+4x_2=0$, which is shown by dotted line passing through origin 'O'. As Z increased from zero, this dotted line move to right, parallel to itself. Since we are interested in maximizing Z, we increase the value of Z till the dotted line passes through the farthest corner of the shaded convex region from origin; the maximum value of 'Z' can not be found as it occurs at infinity only. Therefore, the problem has an "unbounded solution".

Case (1): If the maximum value of Z is un-able to find, as it is extended upto infinity in the shaded feasible zone then the solution to the problem has "Unbounded solution".

Case (2): Maximize $Z = 3x+2y$

Subject to $-2x+3y \le 9$ — Conversion compulsory

$3x-2y \le -20$ ⟶ $-3x+2y \ge 20$

and $x, y \ge 0$

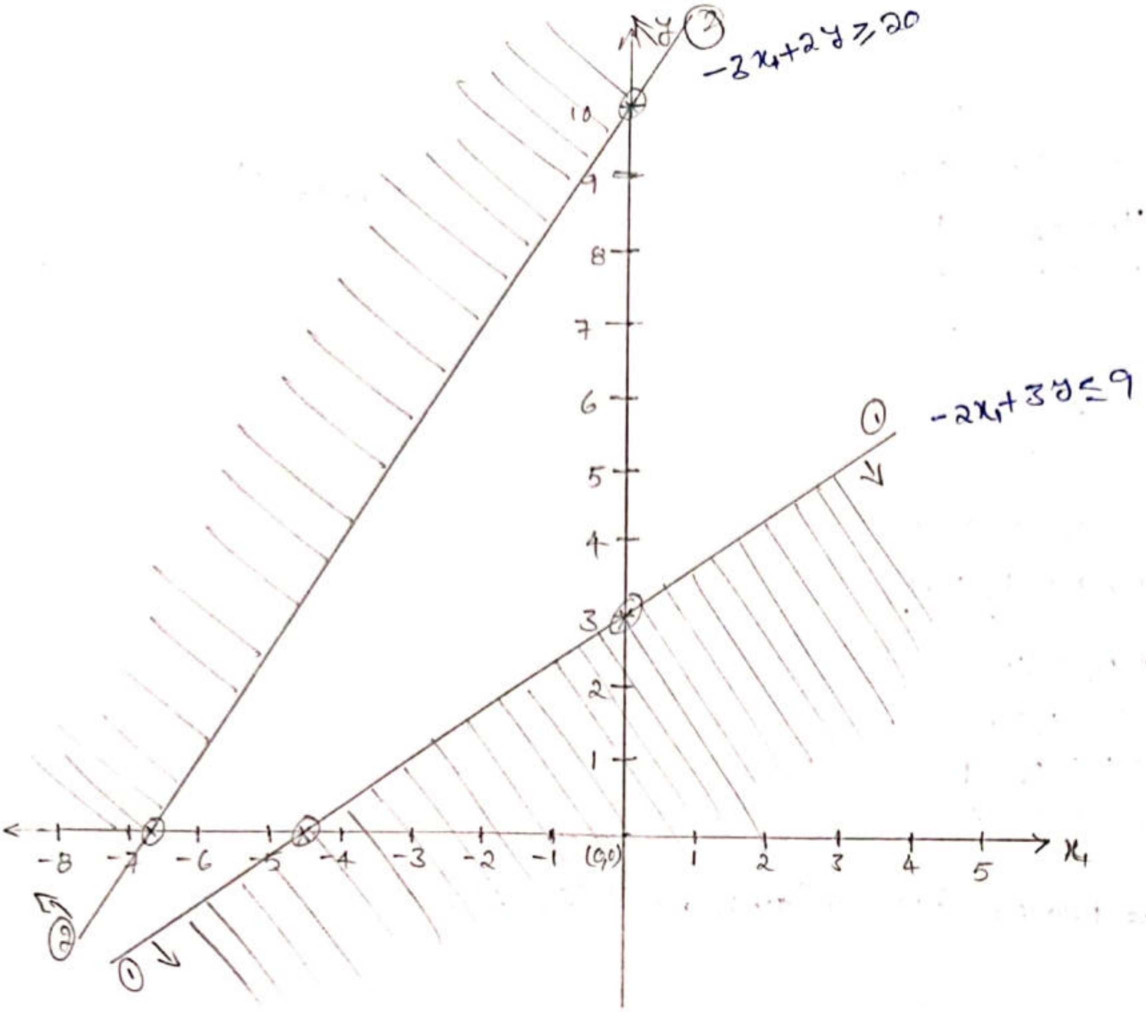
$x=0$, $y=10$
$y=0$, $x=\frac{-20}{3}$

s/f

Constraints:

(1) $-2x+3y=9$
$x=0$; $y=3$ $(0,3)$
$y=0$; $x=-9/2$ $(-4.5,0)$

(2) $3x-2y=-20$
$x=0$; $y=10$ $(0,10)$
$y=0$; $x=-20/3$ $(-6.67,0)$



Since there is no common zone and each constraint leads to a diversified zone, there is 'no feasible solution' to the problem (or) The solution is said to be "infeasible".

(2) Solve the following LPP using graphical method.

Max $Z = 3x_1 + 2x_2$

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Sol Constraints:

(1) $x_1 + x_2 = 4$

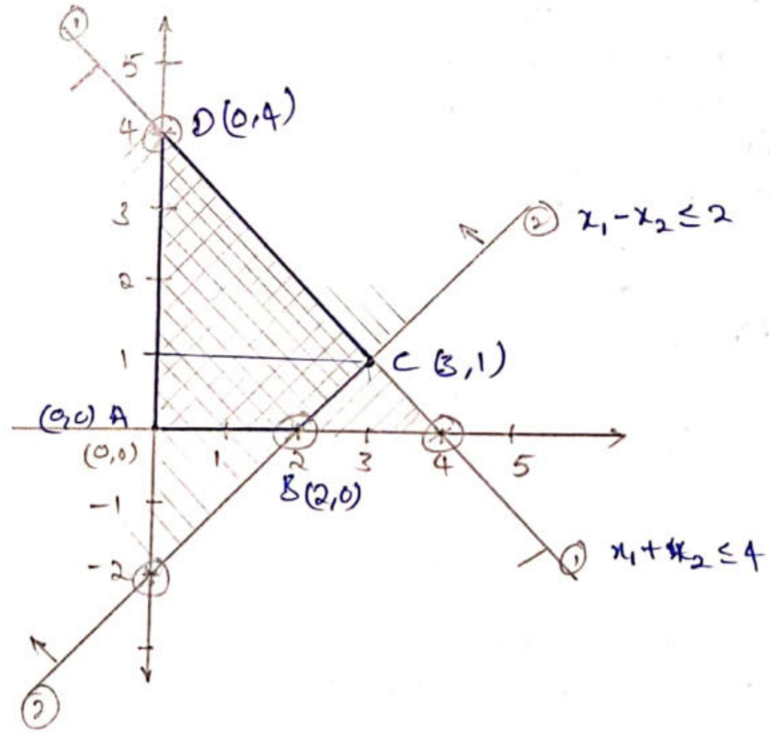
$x_1 = 0$; $x_2 = 4$ (0,4)

$x_2 = 0$; $x_1 = 4$ (4,0)

(2) $x_1 - x_2 = 2$

$x_1 = 0$; $x_2 = -2$ (0,-2)

$x_2 = 0$; $x_1 = 2$ (2,0)



AT! $A = (0,0) \Rightarrow Z = 0$; $B = (2,0) \Rightarrow Z = 6$

$\boxed{C = (3,1)} \Rightarrow \boxed{Z = 11}$; $D = (0,4) \Rightarrow Z = 8$

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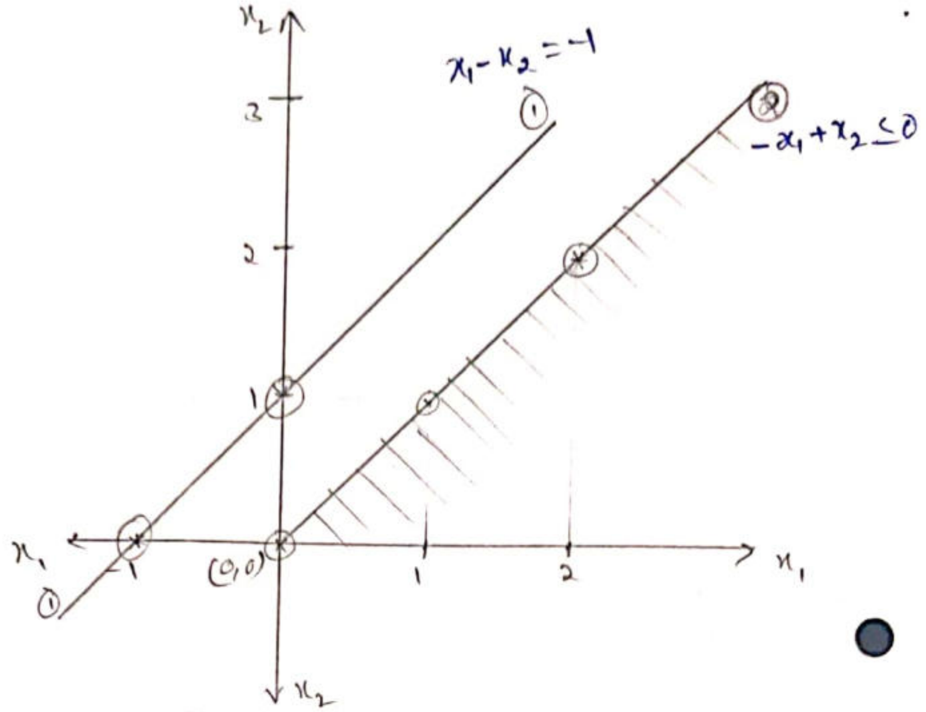
(3) Solve the following LPP using graphical Method

$$\text{Max } Z = 3x_1 + 4x_2$$

$$x_1 - x_2 = -1$$

$$-x_1 + x_2 \leq 0$$

$$x_1, x_2 \geq 0$$



Sol: Constraints

(1) $x_1 - x_2 = -1$

$x_1 = 0$; $x_2 = 1$ $(0,1)$

$x_2 = 0$; $x_1 = -1$ $(-1,0)$

(2) $-x_1 + x_2 = 0$

$x_1 = 0$; $x_2 = 0$

$x_1 = 1$; $x_2 = 1$

$x_1 = 2$; $x_2 = 2$

The problem has an "unbounded solution".

(4) $\text{Max } Z = x_1 + \frac{x_2}{2}$

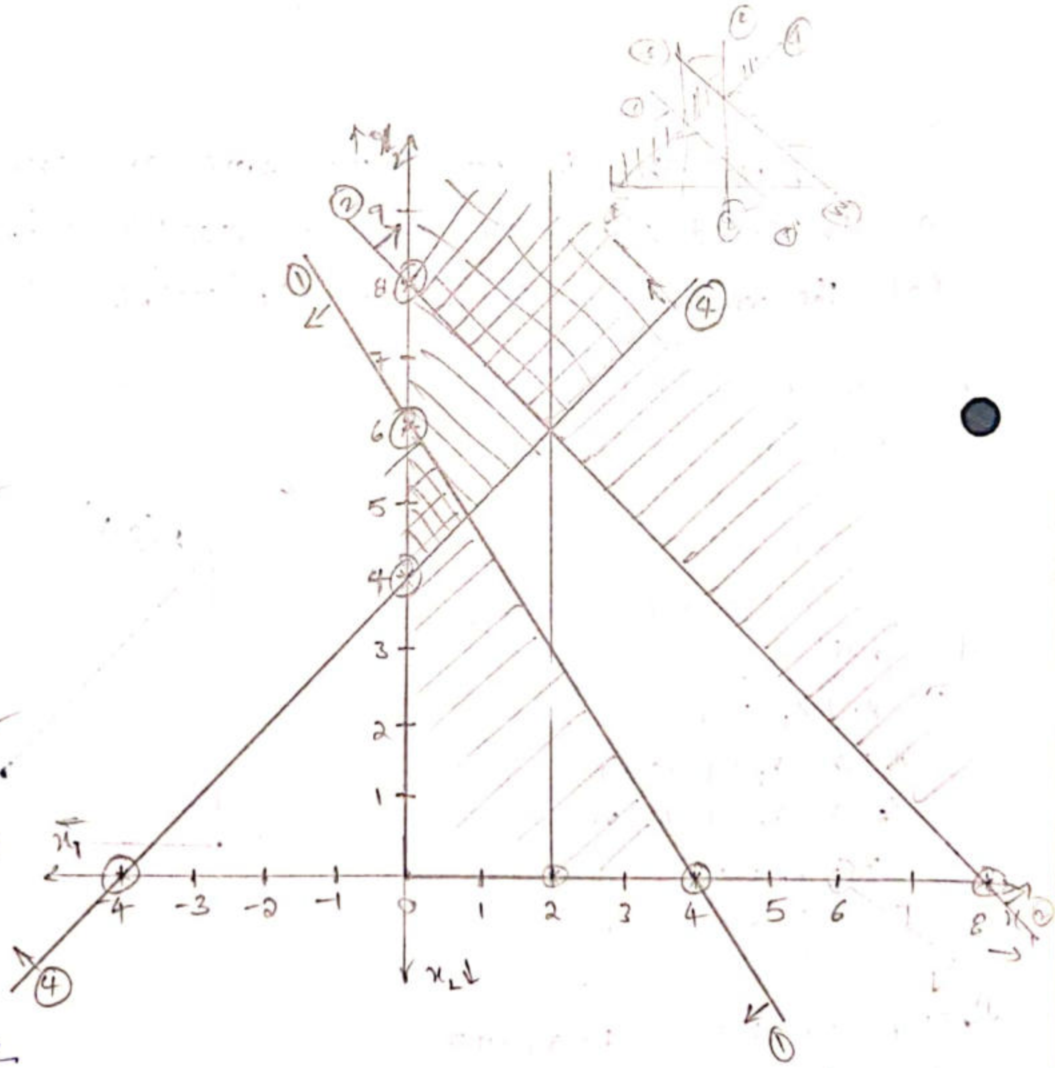
$3x_1 + 2x_2 \leq 12$

$5x_1 = 10$

$x_1 + x_2 \geq 8$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$



Sol

(1) $3x_1 + 2x_2 = 12$

$x_1 = 0$; $x_2 = 6$ $(0,6)$

$x_2 = 0$; $x_1 = 4$ $(4,0)$

(2) $x_1 = 2$

(3) $x_1 + x_2 = 8$

$x_1 = 0$; $x_2 = 8$ $(0,8)$

$x_2 = 0$; $x_1 = 8$ $(8,0)$

(4) $-x_1 + x_2 = 4$

$x_1 = 0$; $x_2 = 4$ $(0,4)$

$x_2 = 0$; $x_1 = -4$ $(-4,0)$

No common zone

Hence solution is infeasible

Simplex Method :

Problem ① : Solve the following LPP using simplex method.

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to $\quad x_1 + x_2 \le 4$

$\quad x_1 - x_2 \le 2 \quad$ where $x_1, x_2 \ge 0$.

Sol^n:

CANON structure :

① objective function should be 'Maximization'

② All constraints should consist of '$\le$' inequality.

③ Right Hand side (RHS) of constraints should be 'Positive' +ve

The given problem satisfies the CANON Structure.

Constraints : ① $x_1 + x_2 + s_1 = 4$

② $x_1 - x_2 + s_2 = 2$

where s_1, s_2 are the slack variables. If $x_1 = x_2 = 0$ then $s_1 = 4$ and $s_2 = 2$. The new objective function will be

New objective function is $\quad \text{Max } Z^* = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Simplex table :

(C_j) Coeff. obj. fn $\rightarrow$ 3 2 0 0

Entering variable & Key column)

Basic Variable	Coeff. of BV in obj. fun C_B	Solution	x_1	x_2	s_1	s_2	Min Ratio (θ) Replacement Ratio (RR)
s_1	0	4	1	1	1	0	$4/1 = 4$ FR $= 1/1 = 1$
s_2	0	2	1 Key element	-1	0	1	$2/1 = 2$ ← Key row
Net Evaluation Row $Z_j - C_j$			$\boxed{-3}$	-2 Key column	0	0	Key Element = 1
s_1	0	2	0	2 K.E	1	-1	$2/2 = 1$ ✓
x_1	3	2	1	-1	0	1	$2/-1 = -2$ FR $= -1/2$
Net Evaluation Row (NER)			0	$\boxed{-5}$	0	3	K.E = 2
x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
		NER	0	0	5/2	1/2	

Leaving variable & Key row

Leaving row

$2 - (2 \times -\frac{1}{2}) = 3$; $\; -1 - (2 \times -\frac{1}{2}) = 0$; $\; -1 + 1/2 - 0$

$1 - (-2 \times 1/2)$

Net evaluation Row : $NER = Z_j - C_j$

where C_j = Coefficient of variable in objective function.

$Z_j = \sum C_B a_{ij}$ where C_B = Coeff. of Basic variable

a_{ij} = variable coeff. in constraints

NER (x_1): $Z_j = 0 \times 1 + 0 \times 1 = 0$

$C_j = 3$ $Z_j - C_j = -3$

Note: Select highest -ve (negative) value of NER as "key column" & least positive value of minimum ratio (MR) as "key row".

New table/solution:-

① Key row Elements = $\frac{\text{Number in key row in Previous table}}{\text{Key Element}}$

$\frac{2}{1}, \frac{1}{1}, \frac{4}{4}$

② Non key row elements

= Old number $-$ {Corresponding number in key row $\times$ Fixed Ratio}

fixed Ratio = $\frac{\text{Element in key column}}{\text{key Element}}$

∴ All the Net evaluation Row values are non-negative the obtained Solution is a feasible solution and is optimal at $\boxed{x_1 = 3 ; x_2 = 1}$

Simplex Procedure:

Slack variable – Addition of a variable in LHS of a constraint to remove the inequality.

Surplus Variable – Substraction of a variable in LHS of a constraint to remove inequality in constraint and make it equality constraint.

① Observe whether all the RHS constraints are non-negative, If not, it can be changed into positive value

② Convert the inequality constraints to equality constraints by introducing slack & surplus variables. of non-negativity. The coefficients of slack & surplus variables are always taken as 'Zero' in obj. function. Formulate new objective function with slack & surplus variables

③ Construct the simplex table. (First table)

Optimality Test:

If all the value of the Net Evaluation Row (NER) are greater than & equal to 'zero' (non negative), then the Solution is an Optimal Solution. (All the -ve elements has to eliminate & Revise the Solution)

Revising the table:

① Identify key column:

It is the column which is having the most negative value of the Net Evaluation Row (NER). Variable in that key column is the incoming variable in the next revised table.

② Identify key row:

$$\text{Replacement Ratio (RR) \& Minimum Ratio} = \frac{\text{Solution column number}}{\text{key column number}}$$

Key row is the row which contains min. Positive replacement ratio. The variable in that key row is the "Outgoing / leaving variable".

③ Key Element:

Point of intersection of key row & key column

④ Replacement of key row:

$$\therefore \text{New key row elements} = \frac{\text{Key row Element}}{\text{Key Element}}$$

⑤ Replacement of non-key row:

$$\therefore \text{New Element} = \text{Old Element in Prev. table} - \left[\text{Corresponding number in key row} \times \text{Fixed Ratio}\right]$$

$$\text{Fixed Ratio} = \frac{\text{Number in key column}}{\text{Key Element}}$$

Problem ②: Use simplex method to maximise $Z = 3x_1 + 5x_2$, the constraints are.

$$3x_1 + 2x_2 \leq 18$$
$$x_1 \leq 4$$
$$x_2 \leq 6$$

where $x_1, x_2 \geq 0$.

sol

The Problem satisfies the con CANON structure.

Constraints:

$$3x_1 + 2x_2 + s_1 = 18 \quad \text{—①}$$
$$x_1 + s_2 = 4 \quad \text{—②}$$
$$x_2 + s_3 = 6 \quad \text{—③}$$

if $x_1 = x_2 = 0$ then $s_1 = 18$; $s_2 = 4$; $s_3 = 6$

∴ New objective function is

$$\text{Max } Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$$

$C_j \rightarrow$ Entering variable key column

Basic variable	Coeff of BV in obj fn	Solution	x_1 (3)	x_2 (5)	s_1 (0)	s_2 (0)	s_3 (0)	Replacement Ratio (RR)
s_1	0	18	3	2	1	0	0	$18/2 = 9$ $\boxed{FR_1 = 2/1 = 2}$
s_2	0	4	1	0	0	1	0	$4/0 = \infty$ $\boxed{FR_2 = 0/1 = 0}$
Outgoing variable ← s_3	0	6	0	1	0	0	1	$6/1 = 6$ ✓ key Row
		NER	-3	$\boxed{-5}$	0	0	0	Key Element = 1
Outgoing variable ← s_1	0	6	$\boxed{3}$	0	1	0	-2	RR $6/3 = 2$ ✓ key row
s_2	0	4	1	0	0	1	0	$4/1 = 4$ $\boxed{FR_2 = 1/3}$
x_2	5	6	0	1	0	0	1	$6/0 = \infty$ $\boxed{FR_3 = 0/3 = 0}$
		NER	$\boxed{-3}$	0	0	0	5	K.E = 3
x_1	3	2	1	0	$1/3$	0	$-2/3$	
s_2	0	2	0	0	$-1/3$	1	$+2/3$	
x_2	5	6	0	1	0	0	1	
		NER	0	0	1	0	3	

$18 - (6 \times 2) = 6$

$4 - 6 \times \frac{1}{3} = 2$

∴ All the net evaluation Row elements are non-negative. The obtained solution is optimal at $x_1 = 2$; $x_2 = 6$

obj. Max $Z = 3x_1 + 5x_2$

$= 3(2) + 5(6) = 36$

∴ $\boxed{\text{Max. } Z = 36}$

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Problem ③: Solve the following LPP using simplex method.

$$\text{Max } Z = 4x_1 + 3x_2 + 6x_3$$

Subjected to

$$2x_1 + 3x_2 + 2x_3 \le 440$$
$$4x_1 + 3x_3 \le 470 \qquad \text{where } x_1, x_2, x_3 \ge 0$$
$$2x_1 + 5x_2 \le 430$$

Sol

The problem is in 'CANON' structure.

Constraints:

$$2x_1 + 3x_2 + 2x_3 + S_1 = 440$$
$$4x_1 + 3x_3 + S_2 = 470$$
$$2x_1 + 5x_2 + S_3 = 430$$

If $x_1 = x_2 = x_3 = 0$ then
$S_1 = 440$
$S_2 = 470$
$S_3 = 430$

The new objective function is

$$\text{Max } Z = 4x_1 + 3x_2 + 6x_3 + 0S_1 + 0S_2 + 0S_3$$

Simplex table

key column / Incoming variable: x_3

B.V	CBV	Sol	4 x_1	3 x_2	6 x_3	0 S_1	0 S_2	0 S_3	RR
S_1	0	440	2	3	2	1	0	0	$\frac{440}{2} = 220$ $FR_1 = 2/3$
S_2 (leaving variable)	0	470	4	0	3	0	1	0	$\frac{470}{3} = 156.66$ ✓ key row
S_3	0	430	2	5	0	0	0	1	$\frac{430}{0} = \infty$ $FR_3 = \frac{0}{3} = 0$
		NER	−4	−3	$\boxed{-6}$	0	0	0	K.E = 3
S_1 (leaving variable)	0	380/3	−2/3	3	0	1	−2/3	0	$\frac{380}{9} = 43.33$ ✓ key row
x_3	6	$\frac{470}{3}$	4/3	0	1	0	1/3	0	$\frac{470}{3}/0 = \infty$ $FR_2 = 0$
S_3	0	430	2	5	0	0	0	1	$\frac{430}{5} = 86$ $FR_3 = 5/3$
		NER	4	$\boxed{-3}$	0	0	−2	0	K.E = 3
x_2	3	$\frac{380}{9}$	−2/9	1	0	1/3	−2/9	0	
x_3	6	470/3	4/3	0	1	0	1/3	0	
S_3	0	970/9	28/9	0	0	−5/3	10/9	1	
		NER	10/3	0	0	1	4/3	0	

All the values of NER are non-negative.

Optimal solution:

$$x_1 = 0$$
$$x_2 = \frac{380}{9} = 43.33$$
$$x_3 = \frac{470}{3} = 156.66$$
$$\text{Max } Z = 3 \times \frac{380}{9} + 6 \times \frac{470}{3} = 1066.66$$
$$= 1066.66$$

Unbounded Solution :

Ex: ①

$$\text{Max } Z = 2x_1 + x_2$$

Subject to $-x_1 + x_2 \le 1$
$x_1 - 2x_2 \le 2$ & $x_1 = x_2 \ge 0$

Sol

Constraints: CANON Structure - yes.

$-x_1 + x_2 + s_1 = 1$
$x_1 - 2x_2 + s_2 = 2$

if $\Rightarrow x_1 = x_2 = 0$
$s_1 = 1$
$s_2 = 2$

Obj function:

$$\text{Max } Z = 2x_1 + x_2 + 0s_1 + 0s_2$$

B.V	CBV	Solution	$c_j \rightarrow$ 2 x_1 ↓	1 x_2 ↓ enter	0 s_1	0 s_2	RR	
s_1	0	1	-1	1	1	0	$1/-1 = -1$	$FR_1 = -1/1 = -1$
leave ← s_2	0	2	1	-2	0	1	$2/1 = 2$ ✓	
		NER	$\boxed{-2}$	-1	0	0	K.E = 1	
s_1	0	3	0	-1	1	1	-3	$1-2(-1)$
x_1	2	2	1	-2	0	1	-1	no key row; $-1-(1\times -1)$
		NER	0	$\boxed{-5}$	0	2		$1-(-2\times 1)$; $0-(-1)$

∴ The Replacement Ratio values ~~#~~ are negative. So the solution is ~~unbounded~~ "Unbounded".

Check by Graphical Method :

① $-x_1 + x_2 = 1$; $x_1 = 0$; $x_2 = 1$
$x_2 = 0$; $x_1 = -1$

② $x_1 - 2x_2 = 2$; $x_1 = 0$ $x_2 = -1$
$x_2 = 0$ $x_1 = 2$

Minimization in simplex method:

Ex(1)

$$\text{Min } Z = x_1 - 3x_2 + 3x_3$$

Subject to $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \ge -12$ & $x_1, x_2, x_3 \ge 0$

$-4x_1 + 3x_2 + 8x_3 \le 10$

4/3

Sol:

The problem not satisfying CANON structure. so convert it.

$\text{Min } Z = -\text{Max}(-Z)$

Constraints: $3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 - 4x_2 + s_2 = 12$ $(-2x_1 - 4x_2 \le 12)$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

If $x_1 = x_2 = x_3 = 0$ then $s_1 = 7$; $s_2 = 12$; $s_3 = 10$

Objective function: $\text{Max}(-Z) = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$

B.V	CBV	Solution	x_1 (−1)	x_2 (3)	x_3 (−3)	s_1 (0)	s_2 (0)	s_3 (0)	RR	
s_1	0	7	3	−1	2	1	0	0	$\frac{7}{-1} = -7$	$FR_1 = -1/3$
s_2	0	12	−2	−4	0	0	1	0	$12/-4 = -3$	$FR_2 = -4/3$
s_3 (leaving)	0	10	−4	3	8	0	0	1	$10/3 = 3.33$ ✓ key row	
		NER	1	$\boxed{-3}$	3	0	0	0	Key Element = 3	
s_1	0	31/3	5/3	0	14/3	1	0	1/3	31/5 ✓ k.R	
s_2	0	76/3	−22/3	0	32/3	0	1	4/3	$-\frac{76}{22}$	$FR_2 = \frac{-22}{5}$
x_2	3	10/3	−4/3	1	8/3	0	0	1/3	−10/4	$FR_3 = \frac{-4}{5}$
		NER	$\boxed{-3}$	0	11	0	0	1	K.E = 5/3	
x_1	−1	31/5	1	0	14/5	3/5	0	1/5		
s_2	0	$\frac{1062}{15}$	0	0	$\frac{468}{15}$	$\frac{22}{5}$	1	$\frac{82}{15}$		
x_2	3	$\frac{174}{15} = \frac{58}{5}$	0	1	$\frac{96}{15}$	4/5	0	$\frac{9}{15}$		
		NER	0	0	$\frac{201}{15}$	9/5	0	$\frac{24}{15}$		

$7 - 10 \times \frac{-1}{3} = \frac{21+10}{3}$; $12 + 10 \times \frac{4}{3} = 76/3$

$\frac{76}{3} + \left[\frac{31}{3} \times \frac{22}{5}\right] = \frac{5 \times 76 + 31 \times 22}{15} = \frac{1062}{15}$

$\frac{32}{3} - \left(\frac{14}{3} \times \frac{-22}{5}\right) = \frac{5 \times 32 + 14 \times 22}{15} = \frac{468}{15}$

$\frac{4}{3} - \left[\frac{1}{3} \times \frac{-22}{5}\right] = \frac{5 \times 4 + 22}{15}$

$\frac{10}{3} + \frac{31}{3} \times \frac{4}{5} = \frac{50 + 124}{15} = \frac{174}{15}$

$\frac{8}{3} - \left[\frac{14}{3} \times \frac{-4}{5}\right] = \frac{5 \times 8 + 14 \times 4}{15}$

$\frac{1}{3} - \left[\frac{1}{3} \times \frac{-4}{5}\right] \Rightarrow \frac{5+4}{15}$

$\frac{14}{5} + \frac{3 \times 96}{15} - 3 = \frac{288 - 42 - 45}{15} = \frac{201}{15}$

$-\frac{1}{5} + \frac{27}{15} \Rightarrow \frac{-3+27}{15}$; $-\frac{3}{5} + \frac{12}{5}$

$x_1 = \frac{31}{5}$; $x_2 = \frac{174}{15}$; $x_3 = 0$

$\text{Min } Z = -\frac{31}{5} - 3\left(\frac{174}{15}\right) + 3(0)$

$= \frac{31 - 174}{5} = \frac{-143}{5}$

Big-M Method (or) Penalty Method:

Min Max
$\leq$ +ve
$\leq$ +ve } Simplex
$\geq$ +ve

Cases of RHS +ve
$\leq$, $\leq$, $=$ | $\geq$, $\leq$, $=$ | $\geq$ +ve, $\leq$ +ve (Big M)
$\leq$ -ve, $\leq$ +ve

① Max. $z = 3x_1 - x_2$

Subject to $2x_1 + x_2 \leq 2$

$x_1 + 3x_2 \geq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0.$

✓ 2/8

Sol: Constraints:

① → $2x_1 + x_2 + S_1 = 2$

② → $x_1 + 3x_2 - S_2 + \boxed{A_1} = 3$

③ → $x_2 + S_3 = 4$

if $x_1 = x_2 = 0$

$S_1 = 2$

$A_1 = 3$

$S_3 = 4$

S_1, S_3 = are slack variable

S_2 = Surplus variable

A_1 = Artificial variable.

New obj function:

$$\text{Max. } Z = 3x_1 - x_2 + 0S_1 + 0S_2 + 0S_3 - mA_1$$

$-m \rightarrow$ penalty.

Table:

Entering variable

B.V	C.BV (C_B)	Solution	x_1	x_2	S_1	S_2	S_3	A_1	RR sol/key
$C_j \rightarrow$			3	-1	0	0	0	-m	
S_1	0	2	2	1	1	0	0	0	$2/1 = 2$ $FR_1 = 1/3$
A_1 (Leaving variable)	-m	3	1	3	0	-1	0	1	$3/3 = 1$ ✓
S_3	0	4	0	1	0	0	1	0	$4/1 = 4$ $FR_3 = 1/3$
		NER	$-m-3$	$\boxed{-3m+1}$	0	m	0	0	K.E = 3
S_1	0	1	5/3	0	1	1/3	0	~~1/3~~	3/5 ✓
x_2	-1	$3/3 = 1$	1/3	1	0	-1/3	0	~~1/3~~	3 $FR_2 = 1/5$
S_3	0	3	-1/3	0	0	1/3	1	~~-1/3~~ Do not consider	-9 $FR_3 = -1/5$
		NER	$\boxed{-10/3}$	0	0	1/3	0		K.E = 5/3
x_1	3	3/5	1	0	3/5	1/5	0		
x_2	-1	4/5	0	1	-1/5	-2/5	0		
S_3	0	16/5	0	0	+1/5	2/5	1		
		NER	0	0	2	1	0		

$\frac{9}{5} + \frac{1}{5} - 0$

$\frac{3}{5} + \frac{2}{5} = \frac{1}{5}$

$3 + \frac{1}{5}$

$\frac{16}{5}$

$\frac{1}{3}(1 + \frac{1}{5})$

Optimal Solution: $x_1 = 3/5$; $x_2 = 4/5$; $S_3 = 16/5$

$$\text{Max } z = 3\left(\frac{3}{5}\right) - \left(\frac{4}{5}\right) = \underline{1}$$

(2) $\text{Min } Z = 12x_1 + 20x_2$

Subject to $6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

where $x_1, x_2 \geq 0$

Rough

Sol

Constraints:

$$6x_1 + 8x_2 + (-S_1) + A_1 = 100$$

$$7x_1 + 12x_2 - S_2 + A_2 = 120$$

$$x_1, x_2, S_1, S_2 \geq 0;$$

If $x_1 = x_2 = S_1 = S_2 = 0$

$\boxed{A_1 = 100 \quad A_2 = 120}$

New obj fn:

$$\text{Max}(-Z) = -\text{Min}(Z) = -12x_1 - 20x_2$$

$$\text{Max}(-Z) = -12x_1 - 20x_2 - S_1 0 - S_2 0 + mA_1 - mA_2$$

$C_j \rightarrow$ -12, -20, 0, 0, -m, -m

B.V	C.B.V C_B	Solution	x_1 ↓	x_2 ↓ entering	S_1	S_2	A_1	A_2	RR	FR
A_1	-m	100	6	8	-1	0	1	0	$\frac{100}{8} = 12.5$	$FR_1 = \frac{8}{12} = 2/3$
leaving variable ← A_2	-m	120	7	12	0	-1	0	1	$\frac{120}{12} = 10$ ✓	
		NER	$-13m+12$	$\boxed{-20(m-1)}$	m	m	0	0	K.E = 12	
← A_1	-m	20	4/3	0	-1	2/3	1	$\cancel{-2/3}$	$\frac{20}{4/3} = 15$ ✓	
x_2	-20	$\frac{120}{12} = 10$	7/12	1	0	-1/12	0	$\cancel{1/12}$	$\frac{10}{7/12} = \frac{120}{7} = 17.14$	$FR_2 = \frac{7}{12} \times \frac{3}{4} = \frac{7}{16}$
		NER	$\boxed{\frac{-4m+31}{3}}$	0	m	$\frac{-2m+5}{3}$	0		K.E = 4/3	
x_1	-12	15	1	0	-3/4	1/2				
x_2	-20	5/4	0	1	7/16	$-9/24 \rightarrow -3/8$				
		NER	0	0	1/4	3/2				

All the Net Evaluation Row values are 'non-negative' the solution is an optimal solution.

$$x_1 = 15$$

$$x_2 = 5/4$$

$$\text{Max}(-Z) = -12(15) - 20\left(\frac{5}{4}\right) = -205$$

&

$$\text{Min } Z = -\text{Max}(-Z) = 205$$

$-12 + \frac{15}{2}$ $\quad 9 - \frac{35}{4}$ $\quad \frac{36-35}{4} \Rightarrow 1/4$

$-\frac{1}{12} - \left[\frac{2}{3} \times \frac{7}{16}\right] \Rightarrow \frac{-2-7}{24} \rightarrow -9/24 \Rightarrow -3/8$

$\frac{16 \times 10 - 20 \times 7}{16} \Rightarrow \frac{20}{16} = \frac{5}{4}$

Problem (3) : Max. $Z = 2x_1 + 3x_2 + 4x_3$

Subject to $3x_1 + x_2 + 4x_3 \le 600$

$2x_1 + 4x_2 + 2x_3 \ge 480$

$2x_1 + 3x_2 + 3x_3 = 540$

where, $x_1, x_2, x_3 \ge 0$.

Sol: Constraints:

$$3x_1 + x_2 + 4x_3 + S_1 = 600$$
$$2x_1 + 4x_2 + 2x_3 - S_2 + A_1 = 480$$
$$2x_1 + 3x_2 + 3x_3 + A_2 = 540$$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \ge 0$;

If $x_1 = x_2 = x_3 = 0 = S_2 = 0$

$S_1 = 600$

$A_1 = 480$

$A_2 = 540$

$600 - [480 \times \frac{1}{4}] = 480$

$3 - (2 \times \frac{1}{4}) = \frac{5}{2}$

$4 - (2 \times \frac{1}{4}) = \frac{7}{2}$

$0 - (-1 \times \frac{1}{4}) = 1/4$

$540 - [480 \times \frac{3}{4}]$

$\Rightarrow 540 - 360$

$\Rightarrow$

$2 - (2 \times \frac{3}{4})$

$\Rightarrow \frac{4-3}{2} (1/2)$

New obj fn:

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 + 0S_1 - 0S_2 - mA_1 - mA_2$$

			2	3	4	0	0	-m	-m	→ sol/K.C	
B.V	CB.V (C_B)	solution	x_1	x_2 ↓	x_3 ↓	S_1	S_2	A_1	A_2	R.R	F.R
S_1	0	600	3	1	4	1	0	0	0	600/1 = 600	$FR_1 = 1/4$
← A_1	-m	480	2	4	2	0	-1	1	0	480/4 = 120 ✓	
A_2	-m	540	2	3	3	0	0	0	1	540/3 = 180	$FR_3 = 3/4$
	NER		$-4m-2$	$\boxed{-7m-3}$	$-5m-4$	0	m	0	0	K.E = 4	
S_1	0	480	5/2	0	7/2	1	1/4	-1/4	0	960/7 = 137.14	$FR_1 = 7/3$
x_2	3	120	1/2	1	1/2	0	-1/4	1/4	0	240	$FR_2 = 1/3$
← A_2	-m	180	1/2	0	3/2	0	+3/4	-3/4	1	120 ✓	
	NER		$\frac{-m-1}{2}$	0	$\boxed{\frac{-3m-5}{2}}$	0	$\frac{-3m-3}{4}$		0		
S_1	0	60	4/3	0	0	1	-3/2				
x_2	3	60	1/3	1	0	0	-1/2				
x_3	4	120	1/3	0	1	0	1/2		~~2/3~~		
	NER		1/3	0	0	0	1/2				

∵ NER values are non-negative.

Optimal solution: ∴

$x_1 = 0$

$x_2 = 60$

$x_3 = 120$

Max $Z = 2(0) + 3(60) + 4(120)$

$Z \Rightarrow 180 + 480$

$\boxed{Z \Rightarrow 660}$

$1 + \frac{4}{3} - 2$

$\Rightarrow \frac{3+4-6}{3}$

$-\frac{3}{2} + 2$

$\Rightarrow \frac{4-3}{2}$

$120 - [180 \times \frac{1}{3}]$

$\frac{1}{2} - [\frac{1}{2} \times \frac{1}{3}]$

$-1/4 - [\frac{3}{4} \times \frac{1}{2}] \frac{1}{4}$

$\Rightarrow -\frac{1}{4} - \frac{1}{2}$

$480 - [180 \times \frac{7}{3}] = 60$

$\frac{5}{2} \Rightarrow [\frac{1}{2} \times \frac{7}{3}]$

$\Rightarrow \frac{15-7}{6} \Rightarrow \frac{8}{6}$

$- [\frac{3}{4} \times \frac{7}{3}]$

$\Rightarrow \frac{1-7}{4}$

$3 - (2 \times \frac{3}{4})$

$-\frac{3}{4} - \frac{3m}{4} \Rightarrow \frac{-3m-3}{4}$

$\frac{3}{2} - \frac{m}{2} - 2$

$\Rightarrow \frac{3-m}{2} - 2$

$\Rightarrow \frac{-m-1}{2}$

$\frac{3}{2} - \frac{3m}{2} - 4$

$\Rightarrow 3 - 3m - 8$

Un Bounded Solution:

Problem (4): Max. $Z = 3x_1 + 2x_2$

Subject to $x_1 - x_2 \geq 1$

$x_1 + x_2 \geq 3$

where, $x_1, x_2 \geq 0$

S.F constraints:

$x_1 - x_2 - S_1 + A_1 = 1$

$x_1 + x_2 - S_2 + A_2 = 3$

BFS: $x_1 = x_2 = S_1 = S_2 = 0$

$A_1 = 1$

$A_2 = 3$

New obj. fun: Max $Z^* = 3x_1 + 2x_2 + 0S_1 + 0S_2 - mA_1 - mA_2$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

B.V	C_{BV}	Solution	x_1	x_2	S_1	S_2	A_1	A_2	R.R $= \frac{Sol}{KE}$	F.R $= \frac{KCE}{KE}$
		$Z_j \rightarrow$	3	2	0	0	$-m$	$-m$		
← A_1	$-m$	1	1	-1	-1	0	1	0	$1/1 = 1$ ✓	
A_2	$-m$	3	1	1	0	-1	0	1	$3/1 = 3$	$FR = 1/1 = 1$
		NER	$\boxed{-2m-3}$	-2	m	m	0	0	K.E = 1	
x_1	3	$1/1 = 1$	1	-1	-1	0		0	$1/-1 = -1$	$FR = -1/2$
← A_2	$-m$	2	0	2	1	-1		1	$2/2 = 1$ ✓	
		NER	0	$\boxed{-2m-5}$	$-m-3$	m		0	K.E = 2	
x_1	3	2	1	0	$-1/2$	$-1/2$			$2/-1/2 = -4$	
x_2	2	$2/2 = 1$	0	1	$1/2$	$-1/2$			$1/-1/2 = -2$	
		NER	0	0	$-1/2$	$-5/2$				

Since all the replacement ratio elements are negative and it is not possible to reach optimal solution, Hence the above LPP has an "unbounded" Solution

Two Phase Simplex Method :

Construct an auxiliary LPP leading to final simplex table containing a Basic feasible solution to the original problem.

Phase I :

① Convert the LPP into Maximization form and ensure that all constraint terms are non-negative. If some of them are negative, make them non-negative by multiplying both sides '−1'.

② Add artificial variables A_i to the LHS of constraints having equations "= & ≥" to complete the identity matrix.

③ Express the given LPP in standard form.

④ Obtain an initial basic feasible solution.

⑤ Auxiliary LPP :

Assign a cost of '−1' to each artificial variable and a cost of '0' (zero) to all other variables (instead of their original coefficients i.e, costs) in the objective function.

The new auxiliary obj function is

$$\text{Max } Z^* = 0x_1 + 0x_2 \dots + 0S_1 + 0S_2 \dots - A_1 - A_2 \dots - A_n$$

where,

$x_1, x_2 \dots$ = Decision variables

$S_1, S_2 \dots$ = Slack / surplus variables

$A_1, A_2 \dots$ = Artificial variables.

⑥ Formulate the simplex table for the new auxiliary obj function subjected to given constraints.

⑦ Solve the auxiliary LPP by simplex method until either of the following three possibilities arise

ⓐ

(a) Max $z^* \cancel{>} 0$; and At least one artificial variable appear in the optimum basis at a +ve level. In this case there is no feasible solution exist, stop the procedure.

(b) Max $z^* = 0$; and at least one decision variable appears in the optimum basis at zero level. (OR)

(c) Max $z^* = 0$, and no artificial variable appears in the optimum basis.

→ In case (b) & (c) arise, proceed to phase - II

Phase - II :

(1) Use the optimum basic feasible solution of phase - I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a cost of '0' to every artificial variable in the basis at zero level.

(2) Delete the artificial variables column from the table which is eliminated from the basis in phase - I.

(3) Apply simplex method to the modified simplex table obtained at the end of phase - II till an optimum basic feasible solution is obtained. (or) till there is an indication of 'unbounded solution'.

Remarks:

(1)

Problem ①: Use Two phase method to solve the following LPP

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3$$

Subject to
$$2x_1 + x_2 - 6x_3 = 20$$
$$6x_1 + 5x_2 + 10x_3 \le 76$$
$$8x_1 - 3x_2 + 6x_3 \le 50$$

2/8

Sol:

Constraints:

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

If $x_1 = x_2 = x_3 = 0$
$A_1 = 20$
$S_1 = 76$
$S_2 = 50$

Phase - I :

The new auxiliary objective function is

$$\text{Max } Z = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1$$

Basic Variable	Coeff. of Basic Var	Solution	x_1 (entering)	x_2	x_3	S_1	S_2	A_1	Replacement Ratio (RR)	Fixed Ratio
			0	0	0	0	0	-1		
A_1	-1	20	2	1	-6	0	0	1	$\frac{20}{2} = 10$	$FR_1 = \frac{2}{8} = \frac{1}{4}$
S_1	0	76	6	5	10	1	0	0	$\frac{76}{6} = 9.33$	$FR_2 = \frac{6}{8} = \frac{3}{4}$
leaving S_2	0	50	8	-3	6	0	1	0	$\frac{50}{8} = \boxed{6.2}$	
		NER	$\boxed{-2}$	-1	6	0	0	0	K.E = 8	
A_1	-1	$\frac{30}{4} = \frac{15}{2}$	0	7/4	-15/2	0	-1/4	1	$\frac{15}{2} \times \frac{4}{7} = \boxed{30/7}$ 4.25	
S_1	0	$\frac{154}{4} = \frac{77}{2}$	0	29/4	22/4	1	-3/4	0	$\frac{77}{2} \times \frac{4}{29} = \frac{154}{29}$ 5.31	$FR_2 = 29/7$
x_1	0	$\frac{50}{8} = \frac{25}{4}$	1	-3/8	3/4	0	1/8	0	$\frac{50}{8} \times \frac{8}{-3} = -\frac{50}{3}$	$FR_3 = -\frac{3}{8} \times \frac{4}{7} = -\frac{3}{14}$
		NER	0	$\boxed{-7/4}$	15/2	0	1/4	0	K.E = 7/4	
x_2	0	30/7	0	1	-30/7	0	-1/7			
S_1	0	$\frac{104}{14} = \frac{52}{7}$	0	0	256/7	1	2/7			
x_1	0	55/7	1	0	-6/7	0	1/14			
		NER	0	0	0	0	0			

All the NER values are non-negative & no artificial variable is available, then proceed to phase - II with the Phase - I output as input (Starting table) and solve the objective function with actual coefficients.

Rough:

$76 - (50 \times \frac{3}{4}) \Rightarrow \frac{4 \times 76 - 150}{4} = \frac{154}{4}$

$20 - (50 \times \frac{1}{4}) = \frac{30}{4}$

$\frac{77}{2} - (\frac{15}{2} \times \frac{29}{7})$

Phase - II :

New objective function is : (no need to consider artificial variables)

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3 + 0S_1 + 0S_2 .$$

$+\frac{30\times 4}{7} - \frac{30}{7} - 3$ $\Rightarrow \frac{+89}{7} - \frac{5}{1}$ $\Rightarrow 69/7$

$\frac{4}{7} + \frac{5}{14}$ $\Rightarrow \frac{13}{14}$

		$z_j \rightarrow$	5	-4	3	0	0
B.V	C.B.V C_B	Solution	x_1	x_2	x_3	S_1	S_2
x_2	-4	30/7	0	1	-30/7	0	-1/7
S_1	0	52/7	0	0	256/7	1	2/7
x_1	5	55/7	1	0	-6/7	0	1/14
		NER	0	0	69/7	0	13/14

$\because$ All the NER values are non-negative.

The solution is Optimal $\Rightarrow x_1 = \frac{55}{7}$; $x_2 = \frac{30}{7}$; $x_3 = 0$

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3$$

$$= 5\left(\frac{55}{7}\right) - \left(4 \times \frac{30}{7}\right) + 3(0)$$

$\therefore$ $\boxed{\text{Max } Z = 22.14}$

② Solve the following LPP using Two-phase Simplex method.

$$\text{Max } Z = x_1 - 2x_2 - 3x_3$$

subjected to
$$-2x_1 + x_2 + 3x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 1$$
$$x_1, x_2, x_3 \geq 0.$$

Sol

Constraints:

$$-2x_1 + x_2 + 3x_3 + A_1 = 2$$
$$2x_1 + 3x_2 + 4x_3 + A_2 = 1$$

$x_1 = x_2 = x_3 = 0$
$A_1 = 2$
$A_2 = 1$

$2 - \left(\frac{4 \times 3}{4}\right)$, $\frac{3 \cdot 3}{4}$, $-2 - \frac{6}{4}$, $1 - \frac{9}{4}$, $-5/4$

The new auxiliary objective function is

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0x_3 - 1A_1 - 1A_2$$

BV	CBV	$C_j \to$ Solution	0 x_1	0 x_2	0 x_3	−1 A_1	−1 A_2	RR	FR
A_1	−1	2	−2	1	3	1	0	2/3	PR = 3/4
A_2	−1	1	2	3	4	0	1	1/4 ✓	
		NER	0	−4	$\boxed{-7}$	0	0	K.E = 4	
A_1	−1	5/4	−14/4	−5/4	0	1			
x_3	0	1/4	1/2	3/4	1	0	~~1/4~~		
		NER	7/2	5/4	0	0			

→ All the NER values are +ve, but there is an artificial variable present in the problem basis. Hence this problem has no feasible solution.

(3) Use Two-phase simplex method to solve.

$$\text{Max. } Z = 5x_1 + 3x_2$$

subject to $2x_1 + x_2 \le 1$

$x_1 + 4x_2 \ge 6$

where, $x_1, x_2 \ge 0$

Sol

Constraints:

$$2x_1 + x_2 + S_1 = 1$$
$$x_1 + 4x_2 - S_2 + A_1 = 6$$

BFS: $x_1 = x_2 = S_2 = 0$

$S_1 = 1$

$A_1 = 6$

Phase-I: The new auxilliary objective function is

$$\text{Max. } Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1$$

			0	0	0	0	−1	Min Ratio = Sol/KCE	F.R = KCE/KE
B.V	C_{BV}	Sol.	x_1	x_2 ↓	S_1	S_2	A_1		
← S_1	0	1	2	1	1	0	0	1/1 = 1 ✓	
A_1	−1	6	1	4	0	−1	1	6/4 = 1.5	FR = 4/1
		NER	−1	−4	0	1	0	K.E = 1	
x_2	0	1	2	1	1	0	0		
A_1	−1	2	−7	0	−4	−1	1		
		NER	7	0	4	1	0		

All the NER values are non-negative but still there is an artificial variable in the basis. Hence the given LPP has an infeasible solution.

Duality Method:

(1) Convert into duality.

$$\text{Max } Z = x_1 - x_2 + 3x_3$$

Subject to
$$x_1 + x_2 + x_3 \le 10 \mid y_1$$
$$2x_1 + x_2 - x_3 \le 2 \mid y_2$$
$$2x_1 - 2x_2 - 3x_3 \le 6 \mid y_3$$
$$x_1, x_2, x_3 \ge 0$$

Sol

$$\boxed{\begin{array}{c} \text{Max} \Rightarrow \text{Min.} \\ \le \Rightarrow \ge \end{array}}$$

Constraints $y_1, y_2 --$

$$\text{Min } Z = 10y_1 + 2y_2 + 6y_3$$

Subject to
$$y_1 + 2y_2 + 2y_3 \ge 1$$
$$y_1 - y_2 - 2y_3 \ge -1$$
$$y_1 - y_2 - 3y_3 \ge 3$$
$$y_1, y_2, y_3 \ge 0.$$

Primal	Dual
No. of Constraints →	no. of variables in obj. fun.
obj. fun. Min →	obj. fun. Max
All constraints must be min → $\ge$ (or) Max → $\le$ →	Max → $\le$ (or) Min → $\ge$
RHS constraints →	Coefficients of variables in obj. function.
Coefficients 'one' variable in all constraints. →	coefficients all variables in one constraint

(2) $$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

$$3x_1 + 5x_2 + 4x_3 \ge 7 \mid y_1$$
$$6x_1 + x_2 + 3x_3 \ge 4 \mid y_2$$
$$7x_1 - 2x_2 - x_3 \le 10 \mid y_3 \quad (-1)$$
$$\checkmark\ -7x_1 + 2x_2 + x_3 \ge -10$$
$$x_1 - 2x_2 + 5x_3 \ge 3 \mid y_4$$
$$4x_1 + 7x_2 - 2x_3 \ge 2 \mid y_5$$
$$x_1, x_2, x_3 \ge 0$$

Constraint (3): $-7x_1 + 2x_2 + x_3 \ge -10$

Sol: $$\text{Max } Z = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$$
$$5y_1 + y_2 + 2y_3 + (-2)y_4 + 7y_5 \le -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4$$
$$y_1, y_2, y_3, y_4, y_4 \ge 0.$$

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③ Min $Z = x_1 + 2x_2$

$$2x_1 + 4x_2 \leq 160 \quad — ①$$
$$x_1 - x_2 = 30 \quad — ②$$
$$x_1 \geq 10 \quad — ③$$

Sol obj. fun. Max $Z^* = -160y_1 + 30y_2' - 30y_2'' + 10y_3$

Constraint ② is having '=' conveys it has two opportunities '≤' or '≥'. It can be reached equality condition from origin (0,0) which will be suitable to use '≤' (or) It can reach optimal solution from equality constraint which will be suitable to use '≥'.

$$x_1 - x_2 \leq 30 \rightarrow (-1)$$
$$x_1 - x_2 \geq 30$$

All the constraints are

$-2x_1 - 4x_2 \geq -160$	y_1	y_1
$-x_1 + x_2 \geq -30$	y_2''	y
$x_1 - x_2 \geq 30$	y_2'	
$x_1 \geq 10$	y_3	

obj. fun:

$$\text{Max } Z^* = -160y_1 - 30y_2'' + 30y_2' + 10y_3$$

Constraints:

$$-2y_1 - y_2'' + y_2' + y_3 \leq 1$$
$$-4y_1 + y_2'' - y_2' \leq 2$$

$y_1, y_3 \geq 0$ & y_2 - is unrestricted.

Let $y_2 = y_2'' - y_2'$

Max $Z^* = -160y_1 - 30y_2 + 10y_3$

$$-2y_1 - y_2 + y_3 \leq 1$$
$$-4y_1 + y_2 \leq 2$$

$y_1, y_3 \geq 0$ & y_2 - is unrestricted

(4) Max $z = x_1 - 2x_2 + 3x_3$

$$2x_1 + x_2 + 3x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 1$$

Sol

Constraints: are re written as.

$$\left.\begin{array}{l} 2x_1 + x_2 + 3x_3 \le 2 \\ (-1) \to 2x_1 + x_2 + 3x_3 \ge 2 \end{array}\right\} y_1$$
$$\left.\begin{array}{l} 2x_1 + 3x_2 + 4x_3 \le 1 \\ (-1) \quad 2x_1 + 3x_2 + 4x_3 \ge 1 \end{array}\right\} y_2$$

Convert

$$\begin{array}{l|l} 2x_1 + x_2 + 3x_3 \le 2 & y_1' \\ -2x_1 - x_2 - 3x_3 \le -2 & y_1'' \\ 2x_1 + 3x_2 + 4x_3 \le 1 & y_2' \\ -2x_1 - 3x_2 - 4x_3 \le -1 & y_2'' \end{array}$$

New obj. fun.:

$$\text{Min } z^* = 2y_1' - 2y_1'' + y_2' - y_2''$$

Constraints:

$$2y_1' - 2y_1'' + 2y_2' - 2y_2'' \ge 1$$
$$y_1' - y_1'' + 3y_2' - 3y_2'' \ge -2$$

Unit - II

Transportation Problem

Transportation problem is a special class of linear programming problem in which the objective is to transport a single commodity from different sources to different destinations at minimum total transportation cost (TTC).

Transportation Model

- Balanced Model (Demand = Supply)
- Un Balanced Model (Demand ≠ Supply)

Mathematical Model formulation:

Sources \ Destinations	D_1	D_2	D_3	--	D_n	Supply/Capacity
S_1	c_{11}, x_{11}	c_{12}, x_{12}	c_{13}, x_{13}		c_{1n}, x_{1n}	a_1
S_2	c_{21}, x_{21}	c_{22}, x_{22}	c_{23}, x_{23}		c_{2n}, x_{2n}	a_2
S_3	c_{31}, x_{31}	c_{32}, x_{32}	c_{33}, x_{33}		c_{3n}, x_{3n}	a_3
⋮						⋮
S_m	c_{m1}, x_{m1}	c_{m2}, x_{m2}	c_{m3}, x_{m3}		c_{mn}, x_{mn}	a_m
Demand:	b_1	b_2	b_3	-	b_n	

Sources/Supply - a

Demand/Destination - b

The main objective of transportation problem is to minimize the transportation cost.

Terms used in model:

Sources/Origin $\Rightarrow S_1, S_2, S_3, \ldots S_m$

Destination $\Rightarrow D_1, D_2, D_3 \ldots D_n$

Transportation Cost $\Rightarrow C_{ij}$ (Transportation cost of 1 unit from i^{th} source to j^{th} destination)

Supply/capacity $\Rightarrow a_1, a_2, \ldots a_m$

Demand/Requirement $\rightarrow b_1, b_2, b_3, \ldots b_n$

No. of items/commodities per each transit $\Rightarrow x_{ij}$ (b/w i^{th} source to j^{th} Destination)

$\therefore$ Total transportation Cost $= x_{11}c_{11} + x_{12}c_{12} + x_{13}c_{13} + \cdots$
$+ x_{21}c_{21} + x_{22}c_{22} + x_{23}c_{23} + \cdots$
$+ x_{31}c_{31} + x_{32}c_{32} + x_{33}c_{33} + \cdots$
$+ x_{m1}c_{m1} + x_{m2}c_{m2} + x_{m3}c_{m3} + \cdots$
$+ x_{mn}c_{mn}$

$$\therefore TTC = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

Objective function = Minimization of TTC

Obj. function $= \text{Min}(x_{11}c_{11} + x_{12}c_{12} + \cdots + x_{mn}c_{mn})$

Constraints:

① Total no. of product allocated in a row must be equals to its correspond supply.

$$x_{11} + x_{12} + x_{13} + \cdots + x_{1n} = a_1$$
$$x_{21} + x_{22} + x_{23} + \cdots + x_{2n} = a_2$$
$$\vdots$$
$$x_{m1} + x_{m2} + x_{m3} + \cdots + x_{mn} = a_m$$

② Total no of products allocated in a column must be equal to its corresponding Demand/Requirement.

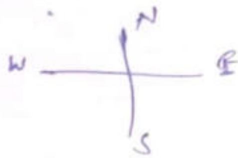
$$x_{11} + x_{21} + x_{31} + \cdots + x_{m1} = b_1$$
$$x_{12} + x_{22} + x_{32} + \cdots + x_{m2} = b_2$$
$$\vdots$$
$$x_{1n} + x_{2n} + x_{3n} + \cdots + x_{mn} = b_n$$

where $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, \ldots x_{mn} \geq 0$ and integers

Transportation Algorithm:

① Balancing the given transportation problem (made Supply = Demand)
② Obtain initial basic feasible solution (IBFS). There are three methods to obtaining it.
   (a) North-West corner rule (NWC)
   (b) Least Cost Method (LCM) & Inspection method.
   (c) Vogel's Approximation Method. (VAM).
③ Testing the Optimality of initial basic feasible solution.
   (i) Stepping stone method
   (ii) Modified Distribution Method (MODI) or u-v method.
④ If the solution is not Optimal, Revise the basic feasible solution
⑤ Repeat the steps ~~2~~ 3 and 4 until obtaining the optimal solution.

North West corner Rule:



① According to this rule, the first allocation is made to the cell occupying the north-west corner is, first cell (1,1).
② If the origin/source capacity is exhausted first, then move down to the first column for allocation
③ If the destination requirement (demand) is (exhausted) completed/fulfilled, then move right ~~+~~ in that row for next allocation.

Likewise move to last cell, until all remaining requirements (demands) and capacities are satisfied.

Problem ①: Solve the following Transportation Problem & find IBFS ?

	D_1	D_2	D_3	D_4	Supply
S_1	2	3	11	7	6
S_2	1	0	6	1	1
S_3	5	8	15	9	10
Demand:	7	5	3	2	

Solution 1

Mathematical Model:

	D_1	D_2	D_3	D_4	
S_1	(2) 6	(3) x	(11) x	(7) x	6 ✓
S_2	(1) 1	(0) x	(6) x	(1) x	1 ✓
S_3	(5) x	(8) 5	(15) 3	(9) 2	10 ✓
	7 ✓	5 ✓	3 ✓	2 ✓	

Step ①: Balancing Supply: $6+1+10=17$

Demand: $7+5+3+2=17$ } Supply = Demand

The given problem is balanced problem.

Step ②: obtain Initial Basic feasible solution.

Total Transportation Cost (TTC) $= 6\times2+1\times1+5\times8+3\times15+2\times9$

$= 116/-$

Degeneracy Test: No. of Allocated Cells $= m+n-1$

where m - no. of rows

n - no. of columns.

No. of allocated cells $= 5$

$m+n-1 = 3+4-1 = 6$ } $\boxed{5 \neq 6}$

The ~~given problem~~ obtain solution is 'degenerate' solution.

Problem ②: Solve the following transportation problem using NWC & find its initial basic feasible solution

					Supply
(12) 40	(4) 15	(9) x	(5) x	(9) x	~~55~~ 15 ✓
(8) x	(1) 5	(6) 40	(6) x	(7) x	~~45~~ 40 ✓
(1) x	(12) x	(4) 10	(7) 20	(7) x	30 ✓
(10) x	(15) x	(8) x	(9) 10	(1) 40	~~50~~ 40 ✓
Demand: ~~40~~ ✓	~~20~~ 5 ✓	~~50~~ 10 ✓	~~30~~ 10	40 ✓	

$TTC = 40\times12+15\times4+5\times1+40\times6+10\times4+20\times7+10\times9+40\times1$

$= 1095/-$

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Least Cost Method (LCM)/Inspection Method:

This method consists in allocating as much as possible in the lowest cost cell/cells & then further allocation is done in the cells with next lowest cost and so on. In case of tie among the cost, select the cell where allocation of more number of units can be made.

Problem ①: Determine the initial basic feasible solution to the following transportation problem using least cost method?

Solution:

6 (2)	x (3)	x (11)	x (7)	6 ✓
x (1)	1 (0)	x (6)	x (1)	1 ✓
1 (5)	4 (8)	3 (15)	2 (9)	~~10~~ ~~9~~ ~~5~~ 3
~~7~~ 1 ✓	~~5~~ 4 ✓	3 ✓	2 ✓	

Balancing: Supply = 6+1+10 = 17

Demand = 7+5+3+2 = 17

$\boxed{\text{Supply = Demand}}$

∴ Balanced problem.

Total Transportation Cost = 6×2 + 1×0 + 5×1 + 4×8 + 3×15 + 2×9

= 112 /-

Degeneracy Test: no. of allocated cells = m+n−1

6 = 3+4−1 = 6

∴ The solution has non-degeneracy.

Problem ②:

					Supply
10 (12)	x (4)	15 (9)	30 (5)	x (9)	55 25 10
x (8)	20 (1)	25 (6)	x (6)	x (7)	45 25 ✓
30 (1)	x (12)	x (4)	x (7)	x (7)	30 ✓
x (10)	x (10)	10 (6)	x (9)	40 (1)	50 10 ✓
Demand: 40 10	20 ✓	50 35 15 ✓	30 ✓	40 ✓	

Balancing!

Supply = 55+45+30+50 = 180

Demand = 40+20+50+30+40 = 180

$\boxed{\text{Supply = Demand}}$

∴ Balanced problem.

Total Transportation Cost = 10×12 + 15×9 + 30×5 + 20×1 + 25×6 + 30×1 + 10×6 + 40×1

= 120 + 135 + 150 + 20 + 150 + 30 + 60 + 40

= 705 /-

Degeneracy: no of allocated cells = m+n−1

8 = 4+5−1 = 8

∴ The condition is satisfied i.e., non-degeneracy solution.

Vogel's Approximation Method (VAM) / Penalty method :

The Vogel's approximation method to obtain initial basic feasible solution it uses penalty which means the difference of least two element i.e. unit transportation costs in a row (row penalty) & column (column penalty).

The VAM consists of the following steps.

1. Calculate row penalty and column penalty which represent the difference between the two cheapest routes from the origin to the destination.
2. Identify the highest difference.
3. First allocation is made to the cheapest cell in that highest difference (penalty) column & row.
4. Remove the column & row & both which is exhausted.
5. Repeat the steps from 1 to 4 until all allocations are completed.

Problem ①:

					Supply	Penalty					
	1 (2)	5 (3)	x (11)	x (7)	6	1	1	5	-	-	-
	x (1)	x (0)	x (6)	1 (1)	1	1	-	-	-	-	-
	6 (5)	x (8)	3 (15)	1 (9)	10	3	3	4	4	4	5
Demand:	~~7~~ 6	5	3	~~2~~ 1							
	1	3	5	6							
	3	5	4	2							
	3	-	4	2							
	5	-	15	9							
	5	-	-	9							
	5	-	-	9							

Total transportation cost = 1×2 + 5×3 + 1×1 + 6×5 + 3×15 + 1×9

∴ TTC = 102 /-

Modified Distribution Method (MODI) & U-V method for determining the optimal solution of a transportation Problem:
[Non - degeneracy]

This method consists of the following steps; after determining the initial basic feasible solution.

Step ①: ⓐ Determine the variables U, V of the occupied cells

is, $$\boxed{C_{ij} = U_i + V_j}$$ for 'occupied cells' only & 'Allocated cells' & 'filled up cells'

where

C_{ij} = cost of transportation of one unit from 'i' to 'j'

U_i = Row number 'i' cost variable

V_j = Column number 'j' cost variable

Note: Initially assume $U_1 = 0$ (or any U_i).

ⓑ Determine Opportunity cost of empty cells using the following equation.

$$\therefore \boxed{\text{Opportunity cost of empty cells } (i, j) = C_{ij} - (U_i + V_j)}$$

ⓒ **Inferences:**

(i) If the opportunity costs of all the empty cells are positive & zero (Non-negative), then the solution is optimal.

(ii) otherwise (negative opportunity cost is present), revise the solution.

* The negative opportunity cost means if we add one unit to this cell, the cost of transportation decreases equal to the opportunity cost.

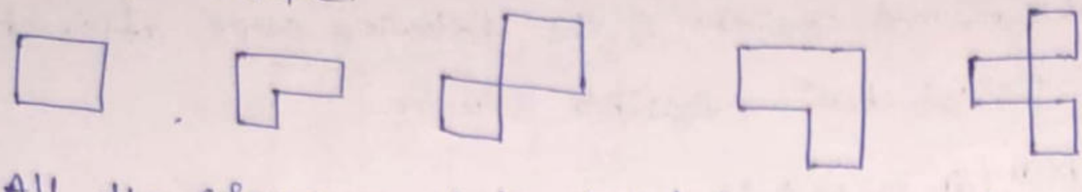
Step ②: Revising the solution:

ⓐ Identify the cell which is having Maximum negative opportunity cost.

ⓑ Form the closed loop with the cell having maximum negative opportunity cost value.

Conditions to form closed loop:

(i) closed loop shape need not be Rectangular it may be any complex shape.



(ii) All the corners must be placed in allocated cells only, but not in any of the empty cells other than maximum negative opportunity cost cell (starting point of loop & ending point of loop. should be same i.e max negative value cell).

(iii) No line of a closed loop should be drawn digonally from one cell to another.

(c) Starting with the empty cell alternatively put +, − signs for the remaining occupied cells of the closed loop corners.

(d) Find the minimum value of allocation in '− sign' cells of the closed loop. Add that minimum value wherever "+ sign" and substract from quantity wherever "− sign" present.

Step ③: Repeat the step ① & ② until obtaining of an Optimal solution. (All opportunity costs should be non-negative).

Problem ①: A bfs for general transportation problem must have exactly m+n−1 (no of rows + no of col. − 1) positive allocations in transportation table. If the no of allocations are less than the required no of allocations (m+n−1) then the solution is called degenerate solution.

Degeneracy Test! To find out the optimal solution the previous solution must be non-degeneracy solution.

According to Degeneracy test the condition for non-degeneracy is given below.

$$\boxed{\text{Number of Allocated Cells} = m+n-1}$$

where m − no of rows
n − no of columns

Note! Otherwise we need to convert degeneracy into non-degeneracy

If the Condition is satisfied then the solution is non-degeneracy

Problem①: Considering the same problem which solved initial basic feasible solution using Vogel's Approximation method.

Sol

IBFS:

	$v_1=2$	$v_2=3$	$v_3=12$	$v_4=6$	
$u_1=0$	(2) 1 −	(3) 5	(11) +	(7)	6
$u_2=-5$	(1)	(0)	(6)	(1) 1	1
$u_3=3$	(5) 6 +	(8)	(15) 3 −	(9) 1	10
	7	5	3	2	

Degeneracy Test:

No. of Allocated cells $= m+n-1$

$6 = 3+4-1$

$\boxed{6=6}$

Non-degeneracy solution

MODI Method: (u-v Test):

u-v test for Allocated cells only. & Assume $u_1=0$

$$C_{ij} = u_i + v_j$$

$(1,1) \Rightarrow c_{11} = u_1+v_1$

$2 = 0+v_1 \Rightarrow \boxed{v_1=2}$

$(1,2) \Rightarrow c_{12} = u_1+v_2 \Rightarrow 3 = 0+v_2 \quad \boxed{v_2=3}$

$(2,4) \Rightarrow c_{24} = u_2+v_4 \Rightarrow 1 = u_2+v_4 \quad 1 = u_2+6 \quad \boxed{u_2=-5}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 5 = u_3+2 \quad \boxed{u_3=3}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 15 = 3+v_3 \quad \boxed{v_3=12}$

$(3,4) \Rightarrow c_{34} = u_3+v_4 \Rightarrow 9 = 3+v_4 \quad \boxed{v_4=6}$

Opportunity Cost (for empty cells $= c_{ij} - (u_i+v_j)$

$(1,3)$: $c_{13} - (u_1+v_3) \Rightarrow 11 - (0+12) = -1$ ← To Select for Allocation

$(1,4)$: $c_{14} - (u_1+v_4) \Rightarrow 7 - (0+6) = 1$

$(2,1)$: $c_{21} - (u_2+v_1) \Rightarrow 1 - (-5+2) = 4$

$(2,2)$: $c_{22} - (u_2+v_2) \Rightarrow 0 - (-5+3) = 2$

$(2,3)$: $c_{23} - (u_2+v_3) \Rightarrow 6 - (-5+12) = -1$

$(3,2)$: $c_{32} - (u_3+v_2) \Rightarrow 8 - (3+3) = 2$

Revised Solution:

	$v_1 = 1$	$v_2 = 3$	$v_3 = 11$	$v_4 = 5$	
$u_1 = 0$	(2)	(3) 5	(11) 1	(7)	6
$u_2 = -4$	(1)	(0)	(6) +	(1) − 1	1
$u_3 = 4$	(5) 7	(8)	(15) 2 −	(9) +1	10
	7	5	3	2	

Total Transportation cost:
$= 5\times3 + 1\times11 + 1\times1 + 7\times5 + 2\times15 + 1\times9$
$= 15 + 11 + 1 + 35 + 30 + 9$
$= \boxed{101}$/-

Degeneracy test! no of Allocated cells = m+n-1

$6 = 3+4-1 = 6$ $\boxed{6=6}$

∴ The solution has non-degeneracy condition.

U-V method! (for allocated cells only) $C_{ij} = u_i + v_j$

$(1,2) \Rightarrow C_{12} = u_1 + v_2 \Rightarrow 3 = 0 + v_2$ $\boxed{v_2 = 3}$

$(1,3) \Rightarrow C_{13} = u_1 + v_3 \Rightarrow 11 = 0 + v_3$ $\boxed{v_3 = 11}$

$(2,4) \Rightarrow C_{24} = u_2 + v_4 \Rightarrow 1 = u_2 + v_4$ $\boxed{u_2 = -4}$

$(3,1) \Rightarrow C_{31} = u_3 + v_1 \Rightarrow 5 = u_3 + v_1$ $\boxed{v_1 = 1}$

$(3,3) = C_{33} = u_3 + v_3 \Rightarrow 15 = u_3 + v_3$ $\boxed{u_3 = 4}$ $(v_3 = 11)$

$(3,4) \Rightarrow C_{34} = u_3 + v_4 \Rightarrow 9 = 4 + v_4$ $\boxed{v_4 = 5}$

Opportunity cost: for empty cells $= C_{ij} - (u_i + v_j)$

$(1,1) \Rightarrow C_{11} - (u_1 + v_1) \Rightarrow 2 - (0+1) = 1$

$(1,4) \Rightarrow C_{14} - (u_1 + v_4) \Rightarrow 7 - (0+5) = 2$

$(2,1) \Rightarrow C_{21} - (u_2 + v_1) \Rightarrow 1 - (-4+1) = 4$

$(2,2) \Rightarrow C_{22} - (u_2 + v_2) \Rightarrow 0 - (-4+3) = 1$

$(2,3) \Rightarrow C_{23} - (u_2 + v_3) \Rightarrow 6 - (-4+11) = -1$ ← To select for allocation

$(3,2) \Rightarrow C_{32} - (u_3 + v_2) \Rightarrow 8 - (4+3) = 1$

→ closed loop selected & minimum -ve sign value is '1'

Revised Solution:

	$v_1 = 1$	$v_2 = 3$	$v_3 = 11$	$v_4 = 5$	
$u_1 = 0$	(2)	(3) 5	(11) 1	(7)	6
$u_2 = -5$	(1)	(0)	(6) 1	(1)	1
$u_3 = 4$	(5) 7	(8)	(15) 1	(9) 2	10
	7	5	3	2	

Total transportation Cost $= 5\times3 + 1\times11 + 1\times6 + 7\times5 + 1\times15 + 9\times2$
$= 15 + 11 + 6 + 35 + 15 + 18$
$= \boxed{100}$/-

Degeneracy Test: no of Allocated cells = m+n-1

$6 = 3+4-1 = 6$ $\boxed{6=6}$

∴ There is no degeneracy

U-V-test: (for allocated cells only) $C_{ij} = u_i + v_j$

(1,2) ⇒ $C_{12} = u_1 + v_2 \Rightarrow 3 = 0 + v_2$ $\boxed{v_2 = 3}$

(1,3) ⇒ $C_{13} = u_1 + v_3 \Rightarrow 4 = 0 + v_3$ $\boxed{v_3 = 4}$

(2,3) ⇒ $C_{23} = u_2 + v_3 \Rightarrow 6 = u_2 + 11$ $\boxed{u_2 = -5}$

(3,1) ⇒ $C_{31} = u_3 + v_1 \Rightarrow 5 = u_3 + v_1$ $\boxed{v_1 = 1}$

(3,3) ⇒ $C_{33} = u_3 + v_3 \Rightarrow 15 = u_3 + 11$ $\boxed{u_3 = 4}$

(3,4) ⇒ $C_{34} = u_3 + v_4 \Rightarrow 9 = 4 + v_4$ $\boxed{v_4 = 5}$

Opportunity cost for empty cells = $C_{ij} - (u_i + v_j)$

(1,1) ⇒ $C_{11} - (u_1 + v_1) \Rightarrow 2 - (0+1) = 1$

(1,4) ⇒ $C_{14} - (u_1 + v_4) \Rightarrow 7 - (0+5) = 2$

(2,1) ⇒ $C_{21} - (u_2 + v_1) \Rightarrow 1 - (-5+1) = 5$

(2,2) ⇒ $C_{22} - (u_2 + v_2) \Rightarrow 0 - (-5+3) = 2$

(2,4) ⇒ $C_{24} - (u_2 + v_4) \Rightarrow 1 - (-5+5) = 1$

(3,2) ⇒ $C_{32} - (u_3 + v_2) \Rightarrow 8 - (4+3) = 1$

∴ There is no negative opportunity cell i.e, the solution 100% is the optimal solution.

Problem ②: Use Vogels' Approximation method for finding initial basic feasible solution. Then determine optimal solution using MODI method,

	Mumbai	Banglore	Delhi	Chennai	Supply
Kolkata	6	5	8	8	30
Ranchi	5	11	9	7	40
Ahmedabad	8	9	7	13	50
Demand:	35	28	32	25	

7

(3) Ans

		Mumbai D_1 ($v_1=6$)	Banglore D_2 ($v_2=5$)	Delhi D_3 ($v_3=2$)	Chennai D_4 ($v_4=8$)	Supply	Penalty
S_1	Kolkata $u_1=0$	x (6)	28 (5)	x (8)	2 (8)	~~30~~ 2	1 2 0 8 (8) — —
S_2	Ranchi $u_2=-1$	35 − (5)	x (11)	x (9)	+ 5 (7)	~~40~~ 5	2 (2) 2 7 7 7 —
S_3	Ahmedabad $u_3=5$	x + (8)	x (9)	32 (7)	− (18) (13)	~~50~~ 18	1 1 (4) (13) — —
	Demand:	35 ✓	28 ✓	32 ✓	25 ✓	120 / 120	

D_1	D_2	D_3	D_4
1	(4)	1	1
1	—	1	1
—	—	1	1
—	—	—	1
—	—	—	1
—	—	—	7
—	—	—	—

Balancing: Supply = 30+40+50 = 120

Demand: 35+28+32+25 = 120

Supply = Demand.

∴ The given problem is balanced model.

IBFS: initial basic feasible solution is -

Total Transportation Cost $= 28\times5 + 2\times8 + 35\times5 + 5\times7 + 32\times7 + 18\times13$

$= 140 + 16 + 175 + 35 + 224 + 234$

Optimality:

TTC = 824/-

Degeneracy test: no of allocated cells = $m+n-1$

$6 = 3+4-1 = 6$ $(6=6)$

∴ There is no degeneracy.

u-v-Test: (for Allocated Cells only) $c_{ij} = u_i + v_j$

$(1,2) \Rightarrow c_{12} = u_1+v_2 \Rightarrow 5 = 0+v_2$ $\boxed{v_2=5}$

$(1,4) \Rightarrow c_{14} = u_1+v_4 \Rightarrow 8 = 0+v_4$ $\boxed{v_4=8}$

$(2,1) \Rightarrow c_{21} = u_2+v_1 \Rightarrow 5 = u_2+v_1$ $\boxed{v_1=6}$

$(2,4) \Rightarrow c_{24} = u_2+v_4 \Rightarrow 7 = u_2+8$ $\boxed{u_2=-1}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 7 = u_3+v_3$ $\boxed{v_3=2}$

$(3,4) \Rightarrow c_{34} = u_3+v_4 \Rightarrow 13 = u_3+8$ $\boxed{u_3=5}$

(Next we go for Calculation of opportunity cost of each empty cell for allocation of units to get optimal solution.)

Opportunity Cost: for empty cells $= c_{ij} - (u_i + v_j)$

$(1,1) = c_{11} - (u_1+v_1) = 6-(0+6) = 0$

$(1,3) = c_{13} - (u_1+v_3) = 8-(0+2) = 6$

$(2,2) = c_{22} - (u_2+v_2) = 11-(-1+5) = 7$

$(2,3) = c_{23} - (u_2+v_3) = 9-(-1+2) = 8$

$(3,1) = c_{31} - (u_3+v_1) = 8-(5+6) = \boxed{-3}$ ← Select (3,1) cell for allocation

$(3,2) = c_{32} - (u_3+v_2) = 9-(5+5) = -1$

⇒ (3,1) cell is selected and closed loop formed. from the closed loop 18 is the least -ve sign value. Add 18 to +ve sign cell and substract 18 to -ve sign cell quantity.

Revised Solution:

	$v_1 =$	$v_2 = 5$	$v_3 = 5$	$v_4 = 8$	
$u_1 = 0$	(6)	(5) 28	(8)	(8) 2	20
$u_2 = -1$	(5) 17	(11)	(9)	(7) 23	40
$u_3 = 2$	(8) 18	(9)	(7) 32	(13)	50
	35	28	32	25	

Total Transportation Cost:

$TTC = 28\times5 + 2\times8 + 17\times5 + 23\times7 + 18\times8 + 32\times7$
$= 140 + 16 + 85 + 161 + 144 + 224$
$= 770/-$

Degeneracy test: no of filled up cells $= m+n-1$

$6 = 3+4-1 = 6$ $(6=6)$ no degeneracy

u-v Test: (for filled up cells only) $c_{ij} = u_i + v_j$

$(1,2) \Rightarrow c_{12} = u_1 + v_2 \Rightarrow 5 = 0 + v_2 \Rightarrow \boxed{v_2 = 5}$

$(1,4) \Rightarrow c_{14} = u_1 + v_4 \Rightarrow 8 = 0 + v_4 \quad \boxed{v_4 = 8}$

$(2,1) \Rightarrow c_{21} = u_2 + v_1 \Rightarrow 5 = u_2 + v_1 \quad \boxed{v_1 = 6}$

$(2,4) \Rightarrow c_{24} = u_2 + v_4 \Rightarrow 7 = u_2 + 8 \quad \boxed{u_2 = -1}$

$(3,1) \Rightarrow c_{31} = u_3 + v_1 \Rightarrow 8 = u_3 + 6 \quad \boxed{u_3 = 2}$

$(3,3) \Rightarrow c_{33} = u_3 + v_3 \Rightarrow 7 = 2 + v_3 \quad \boxed{v_3 = 5}$

Opportunity Cost: for empty cells $= c_{ij} - (u_i + v_j)$

$(1,1) = c_{11} - (u_1+v_1) = 6-(0+6) = 0$

$(1,3) = c_{13} - (u_1+v_3) = 8-(0+5) = 3$

$(2,2) = c_{22} - (u_2+v_2) = 11-(-1+5) = 7$

$(2,3) = c_{23} - (u_2+v_3) = 9-(-1+5) = 5$

$(3,2) = c_{32} - (u_3+v_2) = 9-(2+5) = 2$

$(3,4) = c_{34} - (u_3+v_4) = 13-(2+8) = 3$

no negative cost.

∴ The 770/- is the optimal cost.

Degeneracy in Transportation Problem:

If number of allocated cells is not equal to sum of number of rows and number of columns -1, then there exist 'degeneracy'. i.e., No of Allocated cells $\neq m+n-1$

Conversion of degeneracy into non-degeneracy:

(i) Select the requisite number of vacant cells with least unit transportation cost (in case of tie choose arbitrarily) so that these cells plus existing no of allocated cells is equal to $m+n-1$.

(ii) These $m+n-1$ cells are in independent positions i.e, "no closed loop can be formed among them." If a loop is formed the cell/cells with next lower cost is selected, so that no loop is formed among them.

Problem ①: consider same problem ① solved by NWC rule for which the solution has degeneracy

	$v_1=2$	$v_2=1$	$v_3=8$	$v_4=2$	
$u_1=0$	(2) 6	(3)	(11)	(7)	6
$u_2=-1$	(1) $-$1	(0) ε_1 $+$	(6)	(1)	1
$u_3=7$	(5) $+$	(8) $-$5	(15) 3	(9) 2	10
	7	5	3	2	

TTC = 116/-

Degeneracy: no of Allocated cells $= m+n-1$

$5 = 3+4-1 = 6 \qquad 5 \neq 6$

∴ There exist degeneracy

U-V Test: for Allocated cells only $c_{ij} = u_i + v_j$

$(1,1) \Rightarrow c_{11} = u_1 + v_1 \Rightarrow 2 = 0 + v_1 \quad \boxed{v_1 = 2}$

$(2,1) \Rightarrow c_{21} = u_2 + v_1 \Rightarrow 1 = u_2 + 2 \quad \boxed{u_2 = -1}$

$(3,2) \Rightarrow c_{32} = u_3 + v_2 \Rightarrow 8 = u_3 + v_2$

$(3,3) \Rightarrow c_{33} = u_3 + v_3 \Rightarrow 15 = u_3 + v_3$

$(3,4) \Rightarrow c_{34} = u_3 + v_4 \Rightarrow 9 = u_3 + v_4$

Resolve degeneracy by selecting a cell with least cost of transportation & denote it with $\mathcal{E}_1$ (means no unit is allocated) then in u-v test consider that least cost cell.

$(2,2) \Rightarrow C_{22} = u_2 + V_2 \Rightarrow 0 = -1 + V_2 \; \boxed{V_2 = 1}$

$(3,2) \Rightarrow C_{32} = u_3 + V_2 \Rightarrow 8 = u_3 + 1 \; \boxed{u_3 = 7}$

$(3,3) \Rightarrow C_{33} = u_3 + V_3 \Rightarrow 15 = 7 + V_3 \; \boxed{V_3 = 8}$

$(3,4) \Rightarrow C_{34} = u_3 + V_4 \Rightarrow 9 = 7 + V_4 \; \boxed{V_4 = 2}$

$\therefore$ no of allocated cell including $\mathcal{E}_1 = m+n-1$

$5+1 = 3+4-1 = 6$

$\boxed{6 = 6}$

no - degeneracy

Opportunity cost: for empty cells $= C_{ij} - (u_i + v_j)$ ($\because$ exclude '$\mathcal{E}_1$' cells)

$(1,2) = C_{12} - (u_1 + V_2) = 3 - (0+1) = 2$

$(1,3) = C_{13} - (u_1 + V_3) = 11 - (0+8) = 3$

$(1,4) = C_{14} - (u_1 + V_4) = 7 - (0+2) = 5$

$(2,3) = C_{23} - (u_2 + V_3) = 6 - (-1+8) = -1$

$(2,4) = C_{24} - (u_2 + V_4) = 1 - (-1+2) = 0$

$\boxed{(3,1)} = C_{31} - (u_3 + V_1) = 5 - (7+2) = -4 \rightarrow$ Select for Allocation

* Formation of closed loop by joining cells (3,1), (2,1), (2,2) (3,2).

Revised Solution:

	$v_1 = 2$	$v_2 = 5$	$v_3 = 12$	$v_4 = 6$	
$u_1 = 0$	(2) 6 −	(3) +	(4)	(7)	6
$u_2 = -5$	(1)	(0) 1	(6)	(1)	1
$u_3 = 3$	(5) 1 +	(8) − 4	(15) 3	(9) 2	10
	7	5	3	2	

TTC: $6 \times 2 + 1 \times 0 + 1 \times 5 + \ldots 8 + 3 \times 15 + 2 \times 9$

TTC $= 12 + 5 + 32 + 45 + 18$

$= 112/-$

Total transportation cost $= 112/-$

Degeneracy: no of Allocated cells $= m+n-1$

$6 = 3+4-1 = 6$ $(6 = 6)$

$\therefore$ There is no degeneracy.

u-v Test: for allocated cells. $C_{ij} = u_i + v_j$

$(1,1) \Rightarrow C_{11} = u_1 + V_1 \Rightarrow 2 = 0 + V_1 \; \boxed{V_1 = 2}$

$(2,2) \Rightarrow C_{22} = u_2 + V_2 \Rightarrow 0 = u_2 + V_2 \; \boxed{u_2 = -5}$

$(3,1) \Rightarrow C_{31} = u_3 + V_1 \Rightarrow 5 = u_3 + 2 \; \boxed{u_3 = 3}$

$(3,2) \Rightarrow C_{32} = u_3 + V_2 \Rightarrow 8 = 3 + V_2 \; \boxed{V_2 = 5}$

$(3,3) \Rightarrow C_{33} = u_3 + V_3 \Rightarrow 15 = 3 + V_3 \; \boxed{V_3 = 12}$

$(3,4) \Rightarrow C_{34} = u_3 + V_4 \Rightarrow 9 = 3 + V_4 \; \boxed{V_4 = 6}$

Opportunity Cost: for empty cells $= c_{ij} - (u_i + v_j)$

$\boxed{(1,2)} \Rightarrow c_{12} - (u_1 + v_2) \Rightarrow 3 - (0+5) = -2$ ← Select for allocation

$(1,3) \Rightarrow c_{13} - (u_1 + v_3) \Rightarrow 11 - (0+12) = -1$

$(1,4) \Rightarrow c_{14} - (u_1 + v_4) \Rightarrow 7 - (0+6) = 1$

$(2,1) \Rightarrow c_{21} - (u_2 + v_1) \Rightarrow 1 - (-5+2) = 4$

$(2,3) \Rightarrow c_{23} - (u_2 + v_3) \Rightarrow 6 - (-5+12) = -1$

$(2,4) \Rightarrow c_{24} - (u_2 + v_4) \Rightarrow 1 - (-5+6) = 0$

Revised Solution:

	$v_1 = 2$	$v_2 = 3$	$v_3 = 12$	$v_4 = 6$	
$u_1 = 0$	(2) 2 −	(3) 4 +	(11)	(7)	6
$u_2 = -3$	(1)	(0) 1 −	(6) +	(1)	1
$u_3 = 3$	(5) 5 +	(8)	(15) 3 −	(9) 2	10
	7	5	3	2	

Total transportation cost:

$= 2\times2 + 4\times3 + 1\times0 + 5\times5 + 3\times15 + 2\times9$

$= 4 + 12 + 25 + 45 + 18 = 104$

$\boxed{TTC = 104/-}$

Degeneracy: no of allocated cells $= m+n-1$

$6 = 3+4-1 = 6$

∴ no degeneracy.

U-V Test: for Allocated cells $\boxed{c_{pq} = u_p + v_q}$

$(1,1) \Rightarrow c_{11} = u_1 + v_1 \Rightarrow 2 = 0 + v_1$ $\boxed{v_1 = 2}$

$(1,2) \Rightarrow c_{12} = u_1 + v_2 \Rightarrow 3 = 0 + v_2$ $\boxed{v_2 = 3}$

$(2,2) \Rightarrow c_{22} = u_2 + v_2 \Rightarrow 0 = u_2 + 3$ $\boxed{u_2 = -3}$

$(3,1) \Rightarrow c_{31} = u_3 + v_1 \Rightarrow 5 = u_3 + 2$ $\boxed{u_3 = 3}$

$(3,3) \Rightarrow c_{33} = u_3 + v_3 \Rightarrow 15 = 3 + v_3$ $\boxed{v_3 = 12}$

$(3,4) \Rightarrow c_{34} = u_3 + v_4 \Rightarrow 9 = 3 + v_4$ $\boxed{v_4 = 6}$

Opportunity Cost for empty cells: $= c_{ij} - (u_i + v_j)$

$(1,3) \Rightarrow c_{13} - (u_1 + v_3) \Rightarrow 11 - (0+11) = 0$

$(1,4) \Rightarrow c_{14} - (u_1 + v_4) \Rightarrow 7 - (0+6) = 1$

$(2,1) \Rightarrow c_{21} - (u_2 + v_1) \Rightarrow 1 - (-3+2) = 2$

$\boxed{(2,3)} \Rightarrow c_{23} - (u_2 + v_3) \Rightarrow 6 - (-3+12) = -3$ ←

$(2,4) \Rightarrow c_{24} - (u_2 + v_4) \Rightarrow 1 - (-3+6) = -2$

$(3,2) \Rightarrow c_{32} - (u_3 + v_2) \Rightarrow 8 - (3+3) = 2$

→ Select (2,3) cell for allocation.

→ form the closed loop connecting (2,3).

Revised Solution:

	$v_1=2$	$v_2=3$	$v_3=12$	$v_4=6$	
$u_1=0$	1 (2)	5 (3)	(11)	(7)	6
$u_2=-6$	(1)	(0)	1 (6)	(1)	1
$u_3=3$	6 (5)	(8)	2 (15)	2 (9)	10
	7	5	3	2	

Total transportation cost:

$= 1\times2+5\times3+1\times6+6\times5+2\times15+2\times9$

$= 2+15+6+30+30+18$

$\boxed{TTC = 101/-}$

Degeneracy: no of Allocated cells = m+n-1

$6 = 3+4-1 = 6$ ∴ $\boxed{\text{no degeneracy}}$

Optimality of U-V Test for allocated cells

$(1,1) \Rightarrow c_{11} = u_1+v_1 \Rightarrow 2 = 0+v_1$ $\boxed{v_1 = 2}$

$(1,2) \Rightarrow c_{12} = u_1+v_2 \Rightarrow 3 = 0+v_2$ $\boxed{v_2 = 3}$

$(2,3) \Rightarrow c_{23} = u_2+v_3 \Rightarrow 6 = u_2+v_3$ $\boxed{u_2 = -6}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 5 = u_3+2$ $\boxed{u_3 = 3}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 15 = 3+v_3$ $\boxed{v_3 = 12}$

$(3,4) \Rightarrow c_{34} \Rightarrow u_3+v_4 \Rightarrow 9 = 3+v_4$ $\boxed{v_4 = 6}$

Opportunity cost for empty cells:

$\boxed{(1,3)} \Rightarrow c_{13}-(u_1+v_3) \Rightarrow 11-(0+12) = -1$

$(1,4) \Rightarrow c_{14}-(u_1+v_4) \Rightarrow 7-(0+6) = 1$

$(2,1) \Rightarrow c_{21}-(u_2+v_1) \Rightarrow 1-(-6+2) \Rightarrow 5$

$(2,2) \Rightarrow c_{22}-(u_2+v_2) \Rightarrow 0-(-6+3) \Rightarrow 3$

$(2,4) \Rightarrow c_{24}-(u_2+v_4) \Rightarrow 1-(-6+6) = 1$

$(3,2) \Rightarrow c_{32}-(u_3+v_2) \Rightarrow 8-(3+3) = 2$

Revised Solution:

	$v_1=1$	$v_2=3$	$v_3=11$	$v_4=5$	
$u_1=0$	* (2)	5 (3)	1 (11)	(7)	6
$u_2=-5$	(1)	(0)	1 (6)	(1)	1
$u_3=4$	7 (5)	(8)	1 (15)	2 (9)	10
	7	5	3	2	

Total transportation cost: =

$TTC = 5\times3+1\times11+1\times6+7\times5+1\times15+2\times9$

$= 15+11+6+35+15+18$

$\boxed{TTC = 100/-}$

Degeneracy: no of Allocated cells = m+n-1

$6 = 3+4-1 = 6$ $\boxed{\text{no degeneracy}}$

Optimality: U-V test! (for allocated Cells)

$(1,2) \Rightarrow c_{12} = u_1+v_2 \Rightarrow 3 = 0+v_2$ $\boxed{v_2 = 3}$

$(1,3) \Rightarrow c_{13} = u_1+v_3 \Rightarrow 11 = 0+v_3$ $\boxed{v_3 = 11}$

$(2,3) \Rightarrow c_{23} = u_2+v_3 \Rightarrow 6 = u_2+11$ $\boxed{u_2 = -5}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 5 = u_3+v_1$ $\boxed{v_1 = 1}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 15 = u_3+11$ $\boxed{u_3 = 4}$

$(3,4) \Rightarrow c_{34} = u_3+v_4 \Rightarrow 9 = 4+v_4$ $\boxed{v_4 = 5}$

Opportunity cost for empty cells:

$= c_{ij} - (u_i+v_j)$

$(1,1) \Rightarrow c_{11}-(u_1+v_1) \Rightarrow 2-(0+1) = 1$

$(1,4) \Rightarrow c_{14}-(u_1+v_4) \Rightarrow 7-(0+5) = 2$

$(2,1) \Rightarrow c_{21}-(u_2+v_1) \Rightarrow 1-(-5+1) = 5$

$(2,2) \Rightarrow c_{22}-(u_2+v_2) \Rightarrow 0-(-5+3) = 2$

$(2,4) \Rightarrow c_{24}-(u_2+v_4) \Rightarrow 1-(-5+5) = 1$

$(3,2) \Rightarrow c_{32}-(u_3+v_2) \Rightarrow 8-(4+3) = 1$

∴ There is no negative opportunity Cost means the obtained Solution 100/- is the optimal solution.

Problem ①: A company has 3 manufacturing plants with capacities of 60, 70, 80 units to meet the demands of 3 warehouses with requirements of 50, 80, 80 units. Using the following per unit cost of transportation, find the optimum plan?

Solution:

		$v_1=-3$ D_1	$v_2=1$ D_2	$v_3=3$ D_3	Supply	Penalty
$u_1=0$	S_1	× (8)	× (7)	60 (3)	60	4 4 3 3 –
$u_2=6$	S_2	50 (3)	× (8)	20 (9)	~~70~~ 20	5 1 9 – –
$u_3=2$	S_3	× (11)	80 (3)	E_1 × (5)	80	2 2 – – –
	Demand	50	80	~~80~~ 60		
		5	4	2		
		–	4	2		
		–	–	4		
		–	–	3		
		–	–	–		

Degeneracy Test:

No. of allocated cells = 4

$m+n-1 = 3+3-1 = 5$

∴ Degeneracy existed.

Step ①: Balancing the problem:

Supply $= 60+70+80 = 210$
Demand $= 50+80+80 = 210$

$\boxed{\text{Supply} = \text{Demand}}$ The problem is 'Balanced model'.

Step ②: Initial basic feasible solution:

∴ Total transportation cost $= 60\times3 + 50\times3 + 20\times9 + 80\times3$

$= 180 + 150 + 180 + 240 = 750/-$

Resolving the degeneracy:

→ The cheapest cell is (3,3) and is selected from which no closed loop is formed. ∴ No. of allocated cells $= 4+1 \Rightarrow 5$ equal to "$m+n-1$". ∴ $\boxed{\text{no degeneracy}}$

Optimality: u-v test: for allocated cells only

$c_{ij} = u_i + v_j$

$(1,3) \Rightarrow c_{13} = u_1+v_3 \Rightarrow 3 = 0+v_3 \Rightarrow \boxed{v_3=3}$

$(2,1) \Rightarrow c_{21} = u_2+v_1 \Rightarrow 3 = u_2+v_1 \Rightarrow \boxed{v_1=-3}$

$(2,3) \Rightarrow c_{23} = u_2+v_3 \Rightarrow 9 = u_2+3 \boxed{u_2=6}$

$(3,2) \Rightarrow c_{32} = u_3+v_2 \Rightarrow 3 = u_3+v_2 \boxed{v_2=1}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 5 = u_3+3 \boxed{u_3=2}$

Opportunity cost for empty cell: $= C_{ij} - (u_i + v_j)$

$(1,1) \Rightarrow C_{11} - (u_1 + v_1) \Rightarrow 8 - (0 - 3) = 11$

$(1,2) \Rightarrow C_{12} - (u_1 + v_2) \Rightarrow 7 - (0 + 1) \Rightarrow 6$

$(2,2) \Rightarrow C_{22} - (u_2 + v_2) \Rightarrow 8 - (6 + 1) \Rightarrow 1$

$(3,1) \Rightarrow C_{31} - (u_3 + v_1) \Rightarrow 11 - (2 - 3) \Rightarrow 12$

$\cancel{(3,3) \Rightarrow C_{33} - (u_3 + v_3) \Rightarrow 5}$

∴ Hence there is no negative opportunity cost the obtained solution 750/- is the optimal value

Problem (3): A company has 4 warehouses and 6 stores. The warehouses altogether have a supply of 22 units of a given commodity, divided among them as follows

Warehouse:	1	2	3	4
Supply:	5	6	2	9

The six stores altogether need 22 units of commodity, the individual requirements of al stores 1, 2, 3, 4, 5 & 6 are 4, 4, 6, 2, 4, 2 units respectively. The cost of shipping one unit of commodity from warehouse to store in rupees is given below. Find the optimal transportation cost.

Sol

	$v_1=6$	$v_2=-4$	$v_3=9$	$v_4=-2$	$v_5=-2$	$v_6=-2$	Supply	Penalty							
$u_1=0$	9 x	12 x	9 (5)	6 x	9 x	10 x	5	3	3	0	0	0	0	9	-
$u_2=7$	7 x	3 (4)	7 x	7 x	E_1 5 x	5 (2)	~~6~~ 4	2	2	2	4	-	-	-	-
$u_3=0$	8 (1)	6 x	9 (1)	11 x	3 x	11 x	~~2~~ 1	2	2	2	1	3	3	9	9
$u_4=0$	6 (3)	8 x	11 x	2 (2)	2 (4)	10 x	~~9~~ ~~7~~ 3	0	0	4	2	5	-	-	-
Demand:	~~4~~ 1	4	~~6~~ 1	2	4	2									
	0	2	2	4	1	5									
	0	2	2	4	1	-									
	0	2	2	-	1	-									
	0	2	2	-	-	-									
	0	-	0	-	-	-									
	3	-	0	-	-	-									
	-	-	0	-	-	-									
	-	-	9	-	-	-									

Balancing: Supply = 5+6+2+9 = 22

Demand: 4+4+6+2+4+2 = 22

Supply = Demand [Balanced]

IBFS: Total transportation Cost = $5\times9+4\times3+2\times5+1\times6+1\times9+3\times6+2\times2+4\times2$

$= 45+12+10+6+9+18+4+8$

TTC = 112/-

Degeneracy: No. of allocated cells = m+n-1

$8 \neq 9$ There exist degeneracy

To resolve degeneracy one empty cell is selected as allocated cell by providing E_1 dummy no of allocations. (2,5) is selected as allocated cell with E_1 allocations.

Optimality / U-V test:

$c_{ij} = u_i + v_j$

$(1,3) \Rightarrow c_{13} = u_1+v_3 \Rightarrow 9 = 0+v_3$ $\boxed{v_3 = 9}$

$(2,2) \Rightarrow c_{22} = u_2+v_2 \Rightarrow 3 = u_2+v_2$ $\boxed{v_2 = -4}$

$(2,5) \Rightarrow c_{25} = u_2+v_5 \Rightarrow 5 = u_2+v_5$ $\boxed{u_2 = 7}$

$(2,6) \Rightarrow c_{26} = u_2+v_6 \Rightarrow 5 = u_2+v_6$ $\boxed{v_6 = -2}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 6 = u_3+v_1$ $\boxed{v_1 = 6}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 9 = u_3+9$ $\boxed{u_3 = 0}$

$(4,1) \Rightarrow c_{41} = u_4+v_1 \Rightarrow 6 = u_4+6$ $\boxed{u_4 = 0}$

$(4,4) \Rightarrow c_{44} = u_4+v_4 \Rightarrow 2 = \cancel{4}+v_4$ $\boxed{v_4 = -2}$

$(4,5) \Rightarrow c_{45} = u_4+v_5 \Rightarrow 2 = 4+v_5$ $\boxed{v_5 = -2}$

opportunity cost : of empty cells

$= c_{ij} - (u_i + v_j)$

$(1,1) \Rightarrow c_{11}-(u_1+v_1) \Rightarrow 9-(0+6) = 3$

$(1,2) \Rightarrow c_{12}-(u_1+v_2) \Rightarrow 12-(0-4) = 16$

$(1,4) \Rightarrow c_{14}-(u_1+v_4) \Rightarrow 6-(0-2) = 8$

$(1,5) \Rightarrow c_{15}-(u_1+v_5) \Rightarrow 9-(0-2) = 11$

$(1,6) \Rightarrow c_{16}-(u_1+v_6) \Rightarrow 10-(0-2) = 12$

$(2,1) \Rightarrow c_{21}-(u_2+v_1) \Rightarrow 7-(7+6) = -6$

$\boxed{(2,3)} \Rightarrow c_{23}-(u_2+v_3) \Rightarrow 7-(7+9) = \boxed{-9}$

$(2,4) \Rightarrow c_{24}-(u_2+v_4) \Rightarrow 7-(7-2) = 2$

$\cancel{(2,5)}$

$(3,2) \Rightarrow c_{32}-(u_3+v_2) \Rightarrow 5-(0-4) = 9$

$(3,4) \Rightarrow c_{34}-(u_3+v_4) \Rightarrow 11-(0-2) = 13$

$(3,5) \Rightarrow c_{35}-(u_3+v_5) \Rightarrow 3-(0-2) = 5$

$(3,6) \Rightarrow c_{36}-(u_3+v_6) \Rightarrow 11-(0-2) = 13$

$(4,2) \Rightarrow c_{42}-(u_4+v_2) \Rightarrow 8-(0-4) = 12$

$(4,3) \Rightarrow c_{43}-(u_4+v_3) \Rightarrow 11-(0+9) = 2$

$(4,6) \Rightarrow c_{46}-(u_4+v_6) \Rightarrow 10-(0-2) = 12$

Revised solution:

(9)	(12)	(9) 5	(6)	(9)	(10)	5
(7)	(3) 4	(7) ε_1	(7)	(5)	(5) 2	6
(6) $1+\varepsilon_1$	(5)	(9) $1-\varepsilon_1$	(11)	(3)	(11)	2
(6) $3-\varepsilon_1$	(8)	(11)	(12) 2	(2) $4+\varepsilon_1$	(10)	9
4	4	6	2	4	2	

where ε_1 - dummy (0).

degeneracy: no of allocations = m+n-1

9 = 3+4-1 = 9

∴ no degeneracy.

	$v_1=6$	$v_2=5$	$v_3=9$	$v_4=2$	$v_5=2$	$v_6=7$	
$u_1=0$	(9)	(12)	(9) 5	(6)	(9)	(10)	5
$u_2=-2$	(7)	(3) 4	(7) ε_1	(7)	(5)	(5) 2	6
$u_3=0$	(6) 1	(5)	(9) 1	(11)	(3)	(11)	2
$u_4=0$	(6) 3	(8)	(11)	(2) 2	(2) 4	(10)	9
	4	4	6	2	4	2	

UV Test | $c_{ij} = u_i + v_j$

$(1,3) \Rightarrow c_{13} = u_1+v_3 \Rightarrow 9 = 0+v_3$ $\boxed{v_3 = 9}$

$(2,2) \Rightarrow c_{22} = u_2+v_2 \Rightarrow 3 = u_2+v_2$ $\boxed{v_2 = 5}$

$(2,3) \Rightarrow c_{23} = u_2+v_3 \Rightarrow 7 = u_2+9$ $\boxed{u_2 = -2}$

$(2,6) \Rightarrow c_{26} = u_2+v_6 \Rightarrow 5 = u_2+v_6$ $\boxed{v_6 = 7}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 6 = u_3+v_1$ $\boxed{v_1 = 6}$

$(3,3) \Rightarrow c_{33} = u_3+v_3 \Rightarrow 9 = u_3+9$ $\boxed{u_3 = 0}$

$(4,1) \Rightarrow c_{41} = u_4+v_1 \Rightarrow 6 = u_4+6$ $\boxed{u_4 = 0}$

$(4,4) \Rightarrow c_{44} = u_4+v_4 \Rightarrow 2 = 0+v_4$ $\boxed{v_4 = 2}$

$(4,5) \Rightarrow c_{45} = u_4+v_5 \Rightarrow 2 = 0+v_5$ $\boxed{v_5 = 2}$

TTC = $5\times9 + 4\times3 + 1\times6 + 1\times9 + 2\times5 + 3\times6 + 2\times2 + 4\times2$

$\Rightarrow 45+12+6+9+10+18+4+8$

$\Rightarrow 112$ /-

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Opportunity cost: Empty cells

$\Rightarrow c_{ij} - (u_i + v_j)$

$(1,1) \Rightarrow c_{11} - (u_1+v_1) \Rightarrow 9-(0+6) = 3$

$(1,2) \Rightarrow c_{12} - (u_1+v_2) \Rightarrow 12-(0+5) = 7$

$(1,4) \Rightarrow c_{14} - (u_1+v_4) \Rightarrow 6-(0+2) = 4$

$(1,5) \Rightarrow c_{15} - (u_1+v_5) \Rightarrow 9-(0+2) = 7$

$(1,6) \Rightarrow c_{16} - (u_1+v_6) \Rightarrow 10-(0+7) = 3$

$(2,1) \Rightarrow c_{21} - (u_2+v_1) \Rightarrow 7-(-2+6) = 3$

$(2,4) \Rightarrow c_{24} - (u_2+v_4) \Rightarrow 7-(-2+2) = 7$

$(2,5) \Rightarrow c_{25} - (u_2+v_5) \Rightarrow 5-(-2+2) = 5$

$(3,2) \Rightarrow c_{32} - (u_3+v_2) \Rightarrow 5-(0+5) = 0$

$(3,4) \Rightarrow c_{34} - (u_3+v_4) \Rightarrow 11-(0+2) = 9$

$(3,5) \Rightarrow c_{35} - (u_3+v_5) \Rightarrow 3-(0+2) = 1$

$(3,6) \Rightarrow c_{36} - (u_3+v_6) \Rightarrow 11-(0+7) = 4$

$(4,2) \Rightarrow c_{42} - (u_4+v_2) \Rightarrow 8-(0+5) = 3$

$(4,3) \Rightarrow c_{43} - (u_4+v_3) \Rightarrow 11-(0+9) = 2$

$(4,6) \Rightarrow c_{46} - (u_4+v_6) \Rightarrow 10-(0+7) = 3$

$\therefore$ No negative opportunity cost means 112/- is the optimal transportation cost

Un Balanced Transportation Problem:

The total capacities/supply are not equal to the total demand i.e, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, such problems are "Un balanced transportation Problems". As the feasible solution exists only for balanced problem it is necessary that the total capacities be made equal to the total demand.

Conversion:

1. If supply is more than demand then add dummy destination to take up the excess capacity and costs of shipping to this destination are set equal to zero.
2. Incase the demand is greater than supply then add dummy origin/source to fill the balance requirement and shipping costs are set equal to zero.

Problem 1: A product is produced by 4 factories A, B, C, & D. The unit production cost in them are Rs 2, Rs. 3, Rs. 1 and Rs 5 respectively. Their production capacities are: factory A - 50 units B - 70 units, C - 30 units and D - 50 units. These factories supply the products to four stores demands of which are 25, 35, 105, & 20 units respectively. Unit transportation cost in rupees from each factory to each store is given in below table.

Stores

Factories	1	2	3	4
A	2	4	6	11
B	10	8	7	5
C	13	3	9	12
D	4	6	8	3

Determine the extent of deliveries from each of the factories to each of the stores so that the total production and transportation cost is minimum.

Sol: First we re construct table with unit cost of both transport as well as production.

	1	2	3	4	
A	2+2	4+2	6+2	11+2	50
B	10+3	8+3	7+3	5+3	70
C	13+1	3+1	9+1	12+1	30
D	4+5	6+5	8+5	3+5	50
	25	35	105	20	

Transportation matrix | Dummy column/destination

		$v_1=4$	$v_2=6$	$v_3=8$	$v_4=3$	$v_5=-3$		
		1	2	3	4	d	Supply	Penalty
$u_1=0$	A	25 (4)	5 (6)	20 (8)	x (13)	x (0)	~~50~~ ~~25~~ 20	4 2 2 2 5 8 8 8
$u_2=2$	B	x (13)	x (11)	70 (10)	x (8)	x (0)	70	8 2 2 2 2 10 10 -
$u_3=-4$	C	x (14)	30 (4)	x (10)	x (13)	x (0)	30	4 6 - - - - - -
$u_4=3$	D	x (9)	x (11)	15 (13)	20 (8)	15 (0)	~~50~~ ~~35~~ 15	8 1 1 3 5 13 - -
	Demand	25	~~35~~ 5	~~105~~ ~~90~~ 20	20	15		
		5	2	2	0	0		
		5	2	2	0	-		
		5	5	2	0	-		
		-	5	2	0	-		
		-	-	2	0	-		
		-	-	2	-	-		
		-	-	2	-	-		
		-	-	8	-	-		

Balancing:

Supply = 50+70+30+50 = 200

Demand = 25+35+105+20 = 185

Demand ≠ Supply

$\boxed{\text{Un balanced Problem}}$

Degeneracy:

no of Allocated cells = m+n-1

8 = 4+5-1 = 8

8 = 8

$\boxed{\text{No Degeneracy}}$

IBFS: Total transportation cost = $25\times4 + 5\times6 + 20\times8 + 70\times10 + 30\times4 + 15\times13 + 20\times8 + 15\times0$

$= 100 + 30 + 160 + 700 + 120 + 195 + 160 + 0$

$= 1465/-$

Optimality: u-v-Test:

$c_{ij} = u_i + v_j$ (for allocated cells only)

$(1,1) \Rightarrow c_{11} = u_1 + v_1 \Rightarrow 4 = 0 + v_1$ $\boxed{v_1 = 4}$

$(1,2) \Rightarrow c_{12} = u_1 + v_2 \Rightarrow 6 = 0 + v_2$ $\boxed{v_2 = 6}$

$(1,3) \Rightarrow c_{13} = u_1 + v_3 \Rightarrow 8 = 0 + v_3$ $\boxed{v_3 = 8}$

$(2,3) \Rightarrow c_{23} = u_2 + v_3 \Rightarrow 10 = u_2 + 8$ $\boxed{u_2 = 2}$

$(3,2) \Rightarrow c_{32} = (u_3 + v_2) \Rightarrow 4 = u_3 + 8$ $\boxed{u_3 = -4}$

$(4,3) \Rightarrow c_{43} = u_4 + v_3 \Rightarrow 13 = u_4 + 8$ $\boxed{u_4 = 5}$

$(4,4) \Rightarrow c_{44} = u_4 + v_4 \Rightarrow 8 = 5 + v_4$ $\boxed{v_4 = 3}$

$(4,5) \Rightarrow c_{45} = u_4 + v_5 \Rightarrow 0 = 3 + v_5$ $\boxed{v_5 = -3}$

Opportunity cost: for empty cells

$= C_{ij} - (u_i + v_j)$

$(1,4) \Rightarrow C_{14} - (u_1+v_4) \Rightarrow 13-(0+3) = 10$

$(1,5) \Rightarrow C_{15} - (u_1+v_5) \Rightarrow 0-(0-3) = 3$

$(2,1) \Rightarrow C_{21} - (u_2+v_1) \Rightarrow 13-(2+4) = 7$

$(2,2) \Rightarrow C_{22} - (u_2+v_2) \Rightarrow 11-(2+6) = 3$

$(2,4) \Rightarrow C_{24} - (u_2+v_4) \Rightarrow 8-(2+3) = 3$

$(2,5) \Rightarrow C_{25} - (u_2+v_5) \Rightarrow 0-(2-3) = 1$

$(3,1) \Rightarrow C_{31} - (u_3+v_1) \Rightarrow 14-(-4+4) = 14$

$(3,5) \Rightarrow C_{35} - (u_3+v_2) \Rightarrow 10-(-4+8) = 6$

$(3,4) \Rightarrow C_{34} - (u_3+v_4) \Rightarrow 13-(-4+3) = 14$

$(3,5) \Rightarrow C_{35} - (u_3+v_5) \Rightarrow 0-(-4-3) = 7$

$(4,1) \Rightarrow C_{41} - (u_4+v_1) \Rightarrow 9-(3+4) = 2$

$(4,2) \Rightarrow C_{42} - (u_4+v_2) \Rightarrow 11-(3+6) = 2$

∴ There is no negative opportunity cost means the above solution i.e, Rs 1465/- is the optimal solution.

Problem (2): Consider the following unbalanced transportation problem;

From \ To	1	2	3	Supply
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
demand:	75	20	50	

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3, and 2 for destinations 1, 2, and 3 respectively. Find optimal solution.

Sol

Balancing: Supply $= 10+80+15 = 105$

Demand $= 75+20+50 = 145$ } $145-105 = 40$ units.

Supply ≠ Demand → Un balanced model

To resolve the un balance in supply & demand add dummy row.

	1	2	3	
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
d	5	3	2	40
	75	20	50	

In dummy row instead of 'zero' consider the penalty of not supply

① BFS:

		$v_1=3$ (1)	$v_2=1$ (2)	$v_3=3$ (3)	Supply	Penalty
$u_1=0$	1	(5) x	(1) 10	(7) x	10	4 – – – – –
$u_2=3$	2	(6) 60	(4) 10	(6) 10	80/20	2 2 2 2 2 4
from $u_3=0$	3	(3) 15	(2) x	(5) x	15	1 1 1 + – –
$u_4=-1$	4	(5) x	(3) x	(2) 40	40	1 1 – – – –
Demand		75/60	20/10	50/10	145/145	

2	1	3
2	1	3
3	2	1
6	4	6
–	4	6
–	4	–

Total transportation cost:

$= 10\times1 + 60\times6 + 10\times4 + 10\times6 + 15\times3 + 40\times2$

$= 10 + 360 + 40 + 60 + 45 + 80$

$= 595/-$

Degeneracy: no. of allocated cells $= m+n-1$

$6 = 4+3-1 = 6$. $\boxed{\text{no degeneracy}}$

Optimality: U-V Test:

$(1,2) \Rightarrow c_{12} = u_1+v_2 \Rightarrow 1 = 0+v_2$ $\boxed{v_2=1}$

$(2,1) \Rightarrow c_{21} = u_2+v_1 \Rightarrow 6 = u_2+v_1$ $\boxed{v_1=3}$

$(2,2) \Rightarrow c_{22} = u_2+v_2 \Rightarrow 4 = u_2+1$ $\boxed{u_2=3}$

$(2,3) \Rightarrow c_{23} = u_2+v_3 \Rightarrow 6 = 3+v_3$ $\boxed{v_3=3}$

$(3,1) \Rightarrow c_{31} = u_3+v_1 \Rightarrow 3 = u_3+3$ $\boxed{u_3=0}$

$(4,3) \Rightarrow c_{43} = u_4+v_3 \Rightarrow 2 = u_4+3$ $\boxed{u_4=-1}$

Opportunity cost: empty cells $= c_{ij} - (u_i+v_j)$

$(1,1) \Rightarrow c_{11}-(u_1+v_1) \Rightarrow 5-(0+3) = 2$

$(1,3) \Rightarrow c_{13}-(u_1+v_3) \Rightarrow 7-(0+3) = 4$

$(3,2) \Rightarrow c_{32}-(u_3+v_2) \Rightarrow 2-(0+1) = 1$

$(3,3) \Rightarrow c_{33}-(u_3+v_3) \Rightarrow 5-(0+3) = 2$

$(4,1) \Rightarrow c_{41}-(u_4+v_1) \Rightarrow 5-(-1+3) = 3$

$(4,2) \Rightarrow c_{42}-(u_4+v_2) \Rightarrow 3-(-1+1) = 3$

∴ There is no negative opportunity cost, hence above solution 595/- is the optimal solution.

Maximization Transportation Problem:

The transportation problem may involve 'maximization of profit rather than minimization of cost.

(a) Convert the maximization problem into minimization by substracting all profits from the highest profit in the matrix. The problem become minimization now it can solve in general way.

(b) It may be solved as a maximization problem itself. However while finding the IBFS, allocations are to be made in highest profit cells, rather than lowest cost cells. Also solution will be optimal when all cell evaluations are non-positive (≤ 0).

Ex ①: Profit maximization problem.

Factory	1	2	3	4	5	D	
1	2	2	6	10	5	0	140
2	10	8	9	4	7	0	190
3	5	6	4	3	8	0	115
	74	94	69	39	119		

Balancing:
Supply = 140+190+115 = 445
Demand = 74+94+69+39+119
= 395
445 − 395 = 50

Convert into minimization problem.

8	8	4	0	5	10	140
0	2	1	6	3	10	190
5	4	6	7	2	10	115
74	94	69	39	119	50	

Problem ① A Company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs are different from factory to factory and the sales prices from agency to agency. The shipping cost per unit product from each factory to each agency is known. Given the following data, find the production and distribution schedules most profitable to the company.

factory (i)	Production Cost / unit (Rs)	Max. capacity (No. of units)
1	18	140
2	20	190
3	16	115

factory (i)	1	2	2	6	10	5
	2	10	8	9	4	7
	3	5	6	4	3	8
Agency (j)		1	2	3	4	5
Demand		74	94	69	39	119
sales price (Rs)		35	37	36	39	34

} shipping cost

Sol:

Preparation of transportation matrix:

35-18-2 = 15	37-18-2 = 17	36-18-6 = 12	39-18-10 = 11	34-18-5 = 11
35-20-10 = 5	37-20-8 = 9	36-20-9 = 7	39-20-4 = 15	34-20-7 = 7
35-16-5 = 14	37-16-6 = 15	36-16-4 = 16	39-16-3 = 20	34-16-8 = 10

$\Rightarrow$

Profit table:

					Supply
15	17	12	11	11	140
5	9	7	15	7	190
14	15	16	20	10	115
Demand 74	94	69	39	119	

Balancing: Supply = 140+190+115 = 445

Demand: 74+94+69+39+119 = 395

Supply ≠ Demand ⇒ $\boxed{\text{Un balanced}}$

445 − 395 = 50 units to balance

∴ Supply is more than demand. Create ~~...~~ dummy column of demand 50 with zero profit lost.

Agencies

	1	2	3	4	5	dm	
f_1	15	17	12	11	11	0	140
f_2	5	9	7	15	7	0	190
f_3	14	15	16	20	10	0	115
	74	94	69	39	119	50	

The profit table is maximization problem convert it into loss i.e, minimization problem by substracting every profit ~~value~~ with maximum profit. The "minimization problem" is

					dm	Supply	Penalty							
46 (5)	94 (3)	x (8)	x (9)	x (9)	x (20)	~~140~~ ~~46~~	2	2	2	4	[illegible]	—	—	—
x (15)	x (11)	x (13)	39 (5)	101 (13)	50 (20)	~~190~~ ~~151~~ ~~101~~	6	2	2	2	2	7	7	7
28 (6)	x (5)	69 (4)	x (0)	18 (10)	x (20)	~~115~~ ~~46~~ 18	4	1	1	4	4	10	—	—
Demand: ~~74~~ 28	94	69	39	~~119~~ 101	50									
1	2	4	5	1	0									
1	2	4	—	1	0									
1	2	—	—	1	0									
1	—	—	—	1	0									
9	—	—	—	1	0									
—	—	—	—	3	0									
—	—	—	—	3	20									
—	—	—	—	13	—									

~~TTC: = 46×5+94×3+39×5+101×13+50×20+28×6+69×4+18×10~~

~~=~~

IBFS:

Since the objective is to maximize the profit. The maximum profit gained is ~~calc~~ the Sum of allocated units multiplied by the respective allocated cell profit cost.

$$\text{Total profit} = [46\times15 + 94\times17 + 39\times15 + 101\times7 + 50\times0 + 28\times14 + 69\times16 + 18\times10]$$

$$= 690 + 1{,}598 + 585 + 707 + 1000 + 0 + 392 + 1{,}104 + 180$$

$$= \cancel{5{,}292/-}$$

$$= 5256/-$$

Degeneracy: no of filled up cells = m+n-1

$$8 = 3+6-1 = 8 \quad \text{No degeneracy.}$$

$$8 = 8$$

Go for optimality. After two iterations the optimal solution will be obtained. The total profit is = 5292/-

Problem (2): A company has 4 mfg. plants and 5 warehouses. Each plant manufactures the same product, which is sold at different prices in each warehouse area. The cost of manufacturing and raw materials are different in each plant due to various factors. The capacities of the plants are also different. The relevant data given below.

Item	Plant 1	Plant 2	Plant 3	Plant 4
Manufacturing cost/unit	12	10	8	8
Raw material cost/unit	8	7	7	5
Capacity per unit time	100	200	120	80

Warehouse	Transportation cost/unit 1	2	3	4	Sale Price	Demand Per unit(Rs)
A	4	7	4	3	30	80
B	8	9	7	8	32	120
C	2	7	6	10	28	150
D	10	7	5	8	34	70
E	2	5	8	9	30	90

(a) Formulate the transportation problem in order to maximise profit.

(b) Find the initial basic feasible solution using VAM.?

(c) Test optimality and find the optimal solution?

Sol:

	A	B	C	D	E
P_1	30-12-8-4 = 6	32-12-8-8 = 4	28-12-8-2 = 6	34-12-8-10 = 4	30-12-8-2 = 8
P_2	30-10-7-7 = 6	32-10-7-9 = 6	28-10-7-7 = 4	34-10-7-7 = 10	30-10-7-5 = 8
P_3	30-8-7-4 = 11	32-8-7-7 = 10	28-8-7-6 = 7	34-8-7-5 = 14	30-8-7-8 = 7
P_4	30-8-5-3 = 14	32-8-5-8 = 11	28-8-5-10 = 5	34-8-5-8 = 13	30-8-5-9 = 8

Profit matrix

	A	B	C	D	E	Supply
P_1	6	4	6	4	8	100
P_2	6	6	4	10	8	200
P_3	11	10	7	14	7	120
P_4	14	11	5	13	8	80
P_d	0	0	0	0	0	10
	80	120	150	70	90	~~500~~ 510

Balancing:

Supply = 100 + 200 + 120 + 80 = 500

Demand = 80 + 120 + 150 + 70 + 90
= 510

510 − 500 = 10 units

[Add dummy row.]

Conversion of maximization into minimization problem.

+720
−900

		v_1 = A	v_2 = B	v_3 = C	v_4 = D	v_5 = E	Supply	VAM: Penalty
$u_1 = 0$	P_1	x (8)	x (10)	10 + (8)	x (10)	− 90 (6)	~~100~~ 10	2 2 2 2 2 8 8 8
$u_2 =$	P_2	x (8)	70 (8)	130 − (10)	x (4)	x + (6)	~~200~~ 130	2 2 2 2 2 10 10 —
$u_3 =$	P_3	x (3)	50 − (4)	x (7)	+ 70 (10)	x (7)	~~120~~ 50	3 4 3 — — — — —
$u_4 =$	P_4	80 (0)	x + (3)	x (9)	E − (1)	x (6)	80	1 — — — — — — —
$u_5 =$	P_d	x (14)	x (14)	10 (14)	x (14)	x (14)	10	0 0 0 0 0 14 — —
	Demand:	80	~~120~~ 70	~~150~~ 140	70	90		

A	B	C	D	E
3	1	1	1	0
—	4	1	4	0
—	4	1	—	0
—	2	2	—	0
—	2	2	—	—
—	—	2	—	—
—	—	2	—	—
—	—	2	—	—

Degeneracy:

No. of allocated cells = m+n−1

8 = 5+5−1 = 9

∴ There exist degeneracy.

(Select one empty cell) 2

I.B.F.S:

Total Profit = 10×6 + 90×8 + 70×6 + 130×4 + 50×10 + 70×14 + 80×14 + 10×0
= 60 + 720 + 420 + 520 + 500 + 980 + 1120 + 0

Total Profit = 3,112/-

Optimality: (U-V Test)

$C_{ij} = u_i + v_j$

$(1,3) \Rightarrow C_{13} = (u_1+v_3) \rightarrow 8 = 0+v_3$ $\boxed{v_3 = 8}$

$(1,5) \Rightarrow C_{15} = (u_1+v_5) \Rightarrow 6 = 0+v_5$ $\boxed{v_5 = 6}$

$(2,2) \Rightarrow C_{22} = (u_2+v_2) \Rightarrow 8 = u_2+v_2$ $\boxed{v_2 = 6}$

$(2,3) \Rightarrow C_{23} = (u_2+v_3) \Rightarrow 10 = u_2+8$ $\boxed{u_2 = 2}$

$(3,2) \Rightarrow C_{32} = (u_3+v_2) \Rightarrow 4 = u_3+6$ $\boxed{u_3 = -2}$

$(3,4) \Rightarrow C_{34} = (u_3+v_4) \Rightarrow 0 = -2+v_4$ $\boxed{v_4 = 2}$

$(4,1) \Rightarrow C_{41} = (u_4+v_1) \Rightarrow 0 = u_4+v_1$ $\boxed{v_1 = 1}$

$(4,4) \Rightarrow C_{44} = (u_4+v_4) \Rightarrow 1 = u_4+2$ $\boxed{u_4 = -1}$

$(5,3) \Rightarrow C_{53} = (u_5+v_3) \Rightarrow 14 = u_5+8$ $\boxed{u_5 = 6}$

Opportunity cost (Empty cells):

$(1,1) \Rightarrow C_{11}-(u_1+v_1) \rightarrow 8-(0+1) = 7$

$(1,2) \Rightarrow C_{12}-(u_1+v_2) \Rightarrow 10-(0+6) = 4$

$(1,4) \Rightarrow C_{14}-(u_1+v_4) \Rightarrow 10-(0+2) = 8$

$(2,1) \Rightarrow C_{21}-(u_2+v_1) \Rightarrow 8-(2+1) = 5$

$(2,4) \Rightarrow C_{24}-(u_2+v_4) \Rightarrow 4-(2+2) = 0$

$\boxed{(2,5)} \Rightarrow C_{25}-(u_2+v_5) \Rightarrow 6-(2+6) = \boxed{-2}$

$(3,1) \Rightarrow C_{31}-(u_3+v_1) \Rightarrow 3-(-2+1) = 4$

$(3,3) \Rightarrow C_{33}-(u_3+v_3) \Rightarrow 7-(-2+8) = 1$

$(3,5) \Rightarrow C_{35}-(u_3+v_5) \Rightarrow 7-(-2+6) = 3$

$\boxed{(4,2)} \Rightarrow C_{42}-(u_4+v_2) \Rightarrow 3-(-1+6) = -2$

$(4,3) \Rightarrow C_{43}-(u_4+v_3) \Rightarrow 9-(-1+8) = 2$

~~(4,4) => C44 - (u4+v4) => 6~~

$(4,5) \Rightarrow C_{45}-(u_4+v_5) \Rightarrow 6-(-1+6) = 1$

$(5,1) \Rightarrow C_{51}-(u_5+v_1) \Rightarrow 14-(6+1) = 7$

$(5,2) \Rightarrow C_{52}-(u_5+v_2) \Rightarrow 14-(6+6) = 2$

$(5,4) \Rightarrow C_{54}-(u_5+v_4) \Rightarrow 14-(6+2) = 6$

$(5,5) \Rightarrow C_{55}-(u_5+v_5) \Rightarrow 14-(6+6) = 2$

∴ (2,5) cell is selected for allocation

① Revised Table:-

	$v_1=1$	$v_2=6$	$v_3=8$	$v_4=2$	$v_5=4$	
$u_1=0$	(8)	(10)	100 (8)	(10)	(6)	100
$u_2=2$	(8)	70 (8)	40 (10)	(4)	90 (6)	200
$u_3=-2$	(3)	− 50 (4)	(7)	+ 70 (0)	(7)	120
$u_4=-1$	80 (0)	+ (3)	(9)	ε_1 − (1)	(6)	80
$u_5=6$	(14)	(14)	10 (14)	(14)	(14)	10
	80	120	150	70	90	

Degeneracy: no of allocated cells = m+n+1

9 = 5+5−1 = 9

no - degeneracy.

Optimality:- Iteration ①: U-V Test

$(1,3) \Rightarrow C_{13} = u_1+v_3 \Rightarrow 8 = 0+v_3$ $\boxed{v_3 = 8}$

$(2,2) \Rightarrow C_{22} = u_2+v_2 \Rightarrow 8 = u_2+v_2$ $\boxed{v_2 = 6}$

$(2,3) \Rightarrow C_{23} = u_2+v_3 \Rightarrow 10 = u_2+8$ $\boxed{u_2 = 2}$

$(2,5) \Rightarrow C_{25} = u_2+v_5 \Rightarrow 6 = 2+v_5$ $\boxed{v_5 = 4}$

$(3,2) \Rightarrow C_{32} = u_3+v_2 \Rightarrow 4 = u_3+6$ $\boxed{u_3 = -2}$

$(3,4) \Rightarrow C_{34} = u_3+v_4 \Rightarrow 0 = -2+v_4$ $\boxed{v_4 = 2}$

$(4,1) \Rightarrow C_{41} = u_4+v_1 \Rightarrow 0 = u_4+v_1$ $\boxed{v_1 = 1}$

$(4,4) \Rightarrow C_{44} = u_4+v_4 \Rightarrow 1 = u_4+2$ $\boxed{u_4 = -1}$

$(5,3) \Rightarrow C_{53} = u_5+v_3 \Rightarrow 14 = u_5+8$ $\boxed{u_5 = 6}$

Opportunity cost: Empty cells

$(1,1) \Rightarrow C_{11}-(u_1+v_1) \Rightarrow 8-(0+1) = 7$

$(1,2) \Rightarrow C_{12}-(u_1+v_2) \Rightarrow 10-(0+6) = 4$

$(1,4) \Rightarrow C_{14}-(u_1+v_4) \Rightarrow 10-(0+2) = 8$

$(1,5) \Rightarrow C_{15}-(u_1+v_5) \Rightarrow 6-(0+4) = 2$

$(2,1) \Rightarrow C_{21}-(u_2+v_1) \Rightarrow 8-(2+1) = 5$

$(2,4) \Rightarrow C_{24}-(u_2+v_4) \Rightarrow 4-(2+2) = 0$

$(3,1) \Rightarrow C_{31}-(u_3+v_1) \Rightarrow 3-(-2+1) = 4$

$(3,3) \Rightarrow C_{33}-(u_3+v_3) \Rightarrow 7-(-2+8) = 1$

$(3,5) \Rightarrow C_{35}-(u_3+v_5) \Rightarrow 7-(-2+4) = 5$

$\boxed{(4,2)} \Rightarrow C_{42}-(u_4+v_2) \Rightarrow 3-(-1+6) = \boxed{-2}$

$(4,3) \Rightarrow C_{43}-(u_4+v_3) \Rightarrow 9-(-1+8) = 2$

$(4,5) \Rightarrow C_{45}-(u_4+v_5) \Rightarrow 6-(-1+4) = 3$

$(5,1) \Rightarrow C_{51}-(u_5+v_1) \Rightarrow 14-(6+1) = 7$

$(5,2) \Rightarrow C_{52}-(u_5+v_2) \Rightarrow 14-(6+6) = 2$

$(5,4) \Rightarrow C_{54}-(u_5+v_4) \Rightarrow 14-(6+2) = 6$

$(5,5) \Rightarrow C_{55}-(u_5+v_5) \Rightarrow 14-(6+4) = 4$

17

4500/-

② Revised table!

	$v_1=3$	$v_2=6$	$v_3=8$	$v_4=8$	$v_5=4$	
$u_1=0$	(8)	(10)	(8) 100	(10)	(6)	100
$u_2=2$	(8)	(8) 70 −	(10) 40 +	(4) +	(6) 90	200
$u_3=-2$	(3)	(4) 50 +	(7)	(0) 70 −	(7)	120
$u_4=-3$	(0) 80	ε_1 (3)	(9)	(1)	(6)	80
$u_5=6$	(14)	(14)	(14) 10	(14)	(14)	10
	80	120	150	70	90	

Degeneracy: no of allocated cells = 9

m+n−1 = 9

no degeneracy.

Iteration ②: Optimality (u-v Test)

(1,3) ⇒ $c_{13}=(u_1+v_3)$ ⇒ $8=0+v_3$ $\boxed{v_3=8}$

(2,2) ⇒ $c_{22}=u_2+v_2$ ⇒ $8=u_2+v_2$ $\boxed{v_2=6}$

(2,3) ⇒ $c_{23}=u_2+v_3$ ⇒ $10=u_2+8$ $\boxed{u_2=2}$

(2,5) ⇒ $c_{25}=u_2+v_5$ ⇒ $6=2+v_5$ $\boxed{v_5=4}$

(3,2) ⇒ $c_{32}=u_3+v_2$ ⇒ $4=u_3+6$ $\boxed{u_3=-2}$

(3,4) ⇒ $c_{34}=u_3+v_4$ ⇒ $6=-2+v_4$ $\boxed{v_4=8}$

(4,1) ⇒ $c_{41}=u_4+v_1$ ⇒ $0=u_4+v_1$ $\boxed{v_1=3}$

(4,2) ⇒ $c_{42}=u_4+v_2$ ⇒ $3=u_4+6$ $\boxed{u_4=-3}$

(5,3) ⇒ $c_{53}=u_5+v_3$ ⇒ $14=u_5+8$ $\boxed{u_5=6}$

Opportunity cost: Empty cells

(1,1) ⇒ $c_{11}-(u_1+v_1)$ ⇒ $8-(0+3)=5$

(1,2) ⇒ $c_{12}-(u_1+v_2)$ ⇒ $10-(0+6)=4$

(1,4) ⇒ $c_{14}-(u_1+v_4)$ ⇒ $10-(0+8)=2$

(1,5) ⇒ $c_{15}-(u_1+v_5)$ ⇒ $6-(0+4)=2$

(2,1) ⇒ $c_{21}-(u_2+v_1)$ ⇒ $8-(2+3)=3$

$\boxed{(2,4)}$ ⇒ $c_{24}-(u_2+v_4)$ ⇒ $4-(2+8)=\boxed{-6}$

(3,1) ⇒ $c_{31}-(u_3+v_1)$ ⇒ $3-(-2+3)=2$

(3,3) ⇒ $c_{33}-(u_3+v_3)$ ⇒ $7-(-2+8)=1$

(3,5) ⇒ $c_{35}-(u_3+v_5)$ ⇒ $7-(-2+4)=5$

(4,3) ⇒ $c_{43}-(u_4+v_3)$ ⇒ $9-(-3+8)=4$

(4,4) ⇒ $c_{44}-(u_4+v_4)$ ⇒ $1-(-3+8)=-4$

(4,5) ⇒ $c_{45}-(u_4+v_5)$ ⇒ $6-(-3+4)=5$

(5,1) ⇒ $c_{51}-(u_5+v_1)$ ⇒ $14-(6+3)=5$

(5,2) ⇒ $c_{52}-(u_5+v_2)$ ⇒ $14-(6+6)=2$

(5,4) ⇒ $c_{54}-(u_5+v_4)$ ⇒ $14-(6+8)=0$

(5,5) ⇒ $c_{55}-(u_5+v_5)$ ⇒ $14-(6+4)=4$

③ Revised table!

	$v_1=3$	$v_2=6$	$v_3=8$	$v_4=2$	$v_5=4$	
$u_1=0$	(8)	(10)	100 (8)	(10)	(6)	100
$u_2=2$	(8)	(8)	(10) 40	(4) 70	(6) 90	200
$u_3=-2$	(3)	(4) 120	(7)	ε_2 (0)	(7) ~~90~~	120
$u_4=-3$	(0) 80	ε_1 (3)	(9)	(1)	(6)	80
$u_5=6$	(14)	(14)	(14) 10	(14)	(14)	10
	80	120	150	70	90	

Total profit: $100\times 8 + 40\times 4 + 70\times 10 + 90\times 8$
$120\times 10 + 80\times 14 + 10\times 0$
⇒ 4500/-

Degeneracy:

no of allocated cells = 8

m+n−1 = 5+5−1 = 9

There exist degeneracy

• convert into non-degeneracy

Optimality:

(1,3) ⇒ $c_{13}=u_1+v_3$ ⇒ $8=0+v_3$ $\boxed{v_3=8}$

(2,3) ⇒ $c_{23}=u_2+v_3$ ⇒ $10=u_2+8$ $\boxed{u_2=2}$

(2,4) ⇒ $c_{24}=u_2+v_4$ ⇒ $4=2+v_4$ $\boxed{v_4=2}$

(2,5) ⇒ $c_{25}=u_2+v_5$ ⇒ $6=2+v_5$ $\boxed{v_5=4}$

(3,2) ⇒ $c_{32}=u_3+v_2$ ⇒ $4=u_3+v_2$ $\boxed{v_2=6}$

(3,4) ⇒ $c_{34}=u_3+v_4$ ⇒ $0=u_3+2$ $\boxed{u_3=-2}$

(4,1) ⇒ $c_{41}=u_4+v_1$ ⇒ $0=u_4+v_1$ $\boxed{v_1=3}$

(4,2) ⇒ $c_{42}=u_4+v_2$ ⇒ $3=u_4+v_2$ $\boxed{u_4=-3}$

(5,3) ⇒ $c_{53}=u_5+v_3$ ⇒ $14=u_5+8$ $\boxed{u_5=6}$

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Opportunity cost: (empty cells). $c_{ij} - (u_i + v_j)$

$(1,1) \Rightarrow 8 - (0+3) = 5$

$(1,2) \Rightarrow 10 - (0+6) = 4$

$(1,4) \Rightarrow 10 - (0+2) = 8$

$(1,5) \Rightarrow 6 - (0+4) = 2$

$(2,1) \Rightarrow 8 - (2+3) = 3$

$(2,2) \Rightarrow 8 - (2+6) = 0$

$(3,1) \Rightarrow 3 - (-2+3) = 2$

$(3,3) \Rightarrow 7 - (-2+8) = 1$

$(3,5) \Rightarrow 7 - (-2+4) \Rightarrow 5$

$(4,3) \Rightarrow 9 - (-3+8) \Rightarrow 4$

$(4,4) \Rightarrow 1 - (-3+2) \Rightarrow 2$

$(4,5) \Rightarrow 6 - (-3+4) \Rightarrow 5$

$(5,1) \Rightarrow 14 - (6+3) \Rightarrow 5$

$(5,2) \Rightarrow 14 - (6+6) \Rightarrow 12$

$(5,4) \Rightarrow 14 - (6+2) \Rightarrow 6$

$(5,5) \Rightarrow 14 - (6+4) \Rightarrow 10$

All the opportunity costs are non-negative the above obtained solution is an optimal solution.

Poor perform / Inaccurate → Economical → Replacement

Replacement: Deteriorate (A. m/c, A unit, Individual, Group) — Obsolescence (due to new)

Replacement means fixing newer one instead of failured devices. This may be taken individually (or) Grouply.

Model - I: Replacement policy for items whose maintenance cost increase with time and money value is constant.

Problem ① The cost of the machine is Rs. 6100 and its scrap value is only Rs 100. The maintenance costs are found from experience.

Year :	1	2	3	4	5	6	7	8
Maintenance Cost (Rs) :	100	250	400	600	900	1250	1600	2000.

When would machine be replaced

Sol Given that

Machine initial cost is Rs. 6100

Scrap value & Resale value (RV) is Rs. 100

Cummulative maintenance cost (CMC) = $\sum_{i=1}^{8} MC_i$ (MC - Maintenance cost)

Year	Resale value (RV) (Rs)	Net cost (NC) = Initial Cost − RV (Rs)	Maintenance cost (MC) (Rs)	Cum. Maint Cost (CMC) (Rs)	Total cost = NC + CMC (Rs)	Avg Cost = TC/year (Rs)
1	100	6100 − 100 = 6000	100	100	6100	6100
2	100	6000	250	350	6350	3175
3	100	6000	400	750	6750	2250
4	100	6000	600	1350	7350	1837.5
5	100	6000	900	2250	8250	1650
6	100	6000	1250	3500	9500	1583.3
7	100	6000	1600	5100	11,100	1585.5
8	100	6000	2000	7100	13,100	1637.5

∴ Machine must be replaced ~~After~~ at the end of 6 years.

Problem (2): The maintenance cost and resale value per year of a machine whose purchase price is Rs 7000/- are given below.

Year	Resale value	Maintenance cost.
1	4000	900
2	2000	1000
3	1200	1600
4	600	2100
5	500	2800
6	400	3700
7	400	4700
8	400	5900

Sol

Given data.

Purchase price - Rs 7000

Year	Resale value (Rs)	Net Cost = IC - RV (Rs)	Maintenance Cost (MC) (Rs)	Cum Maint Cost (CMC) (Rs)	Total Cost = NC+CMC (Rs)	Avg Cost = $\frac{TC}{Year}$ (Rs)
1	4000	7000-4000 = 3000	900	900	3900	3900
2	2000	5000	1200	2100	7100	3550
3	1200	5800	1600	3700	9500	3166.67
4	600	6400	2100	5800	12200	3050
5	500	6500	2800	8600	15100	3020
6	400	6600	3700	12300	18,900	3150
7	400	6600	4700	17000	23,600	3371.4
8	400	6600	5900	22900	29,500	3687.5

It is better to replace the machine at the end of ~~after~~ 5 years.

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M.C → Running costs

1. (a) M/c A costs Rs 9000. Annual operating costs are Rs 200 for the first year, and then increases by Rs 2000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly (Annual) cost of owning and operating the machine? Assume that the machine has "no resale value" when replaced and that future costs are not discounted.

(b) M/c B costs Rs 10,000. Annual operating costs are Rs 400 for the first year and then increases by Rs 800 every year. You have now a machine type A which is one year old. Should you replace it with B, and if so when?

(c) Suppose you are just ready to replace machine A with another machine of the same type, when you hear that m/c B will become available in a year. What would you do?

Sol: (a) M/c A has no resale value when replaced.

Year	Resale value	Net Cost (Pc - Rv)	Maintenance Cost (MC)	Cum. Maint Cost	Total Cost = NC+CMC	Avg Cost = TC/year
1	0	9000	200	200	9200	9200
2	0	9000	2200	2400	11400	5700
3	0	9000	4200	6600	15600	$\boxed{5200}$
4	0	9000	6200	12800	21800	5450
5	0	9000	8200	21,000	30000	6000

We find that m/c A should be replaced at the end "3 years" and the avg yearly cost of owning and operating the m/c at this time of replacement is "Rs 5200/-

(b) M/C B :

Year	RV	NC = PC-RV	Mantenance cost (MC)	CMC	TC	Avg cost = TC/year
1	0	10,000	400	400	10,400	10,400
2	0	10,000	1200	1600	11,600	5800
3	0	10,000	2000	3600	13,600	4533.3
4	0	10,000	2800	6400	16,400	4100
5	0	10,000	3600	10,000	20,000	$\boxed{4000}$ ←
6	0	10,000	4,400	14,400	24,400	4066.67

4000-4100
= -100
-100 < 0 ...

M/c B should be replaced at the end of "5 years."

The lowest avg cost of m/c B is "Rs 4000" where as for m/c A is "Rs 5200", m/c A should be replaced by m/c B.

when m/c A should be replaced? M/c A should be replaced when the cost for next year of running this m/c becomes more than the avg yearly costs for m/c B.

Total cost of m/c A in the 1st year = Rs 9200
running cost of m/c A in the 2 year = Rs 11400 - 9200 = Rs 2,200
" A in the 3 year = Rs 4200 ✓
" A in the 4 year = Rs 6200.

As the running cost of m/c A in third year (4200) is more than the avg yearly cost for m/c B (Rs 4000); m/c A should be replaced at the end of two years i.e, one year after it is one year old. (One year hence)

(c) As seen from part (b), m/c A should be replaced one year hence and m/c B will also be available. at that time. Therefore m/c A should be replaced by m/c B after one year from now.

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(5) A company has a m/c whose cost is Rs 30,000. Its maintenance & resale costs at the end of different years are as follows.

Year	1	2	3	4	5	6
Maintenance cost (Rs)	4500	4,700	5000	5,500	6,500	7500
Resale value (Rs)	27000	25,300	24,000	21,000	18000	13,000

(a) What is the economic life of the m/c and what is the min. average cost?

(b) The company has obtained a contract to supply goods produced by the m/c for 5 years from now. After 5 years, the company does not intend to use the m/c. If the machine at present, is one year old, what replacement policy should the company adopt if it intends to replace the m/c not more than once?

sol

(a)

Year	Resale value (Rv)	Net cost IC − Rv	Maintenance cost	CMC	Total cost	Avg cost = TC/yr
1	27000	3000	4500	4500	7500	7 500
2	25300	4700	4700	9200	13900	6950
3	24,000	6000	5000	14,200	20,200	$\boxed{6733}$
4	21,000	7000	5,500	19,700	28,700	7175
5	18000	12,000	6,500	26,200	$\boxed{38,200}$	7640
6	13000	17000	7500	33700	50,700	8450

The economic life of the m/c is 3 years. ~~Since~~ the min avg cost is Rs 6,733.34

(b) First calculate yearly cost (and also cumulative cost) of keeping this one year old m/c in year 1, 2, .. 5 hence of its life.

Year	Maintenance Cost (Rs)	Depreciation (Rs) in Resale value	Total cost Cost	Cum. total cost
1	4700	27000-25300 = 1700	6400	6,400
2	5000	25300-24000 = 1300	6,300	12700
3	5500	3000	8500	21,200
4	6,500	3000	9500	30,700
5	7,500	5000	12500	43,200.

Alternative policies:

① Keep old m/c for zero year and the new one for full 5 years.

Total cost = Rs (0 + 38,200) = Rs. 38,200

② Keep old m/c for 1 year and the new one for full 4 years

Total cost = Rs (6400 + 28700) = Rs 35100 ↓

③ Keep old m/c for 2 years and the new one for 3 years

Total cost = Rs 12700 + 20200 = Rs $\boxed{32900}$ ↓ ✓

④ Keep old m/c for 3 years and the new one for 2 years

Total Cost = Rs 21200 + 13900 = Rs 35100 ↑

⑤ Keep old m/c for 4 years and the new one for one year

Total cost = Rs 30700 + 7500 = Rs 38200 ↑

⑥ Keep old m/c for all 5 years and do not buy new m/c

Total cost = Rs (43,200 + 0) = Rs 43200. ↑

∴ The company should keep the current machine running for two more years and then buy new m/c and use it for remaining 3 years.

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Model-② : Replacement of items whose maintenance costs increase with time and value of money also changes with time.

Let 'r' - rate of interest & inflation per year.

Present value of present worth of money to be spent a few years hence.

∴ Present value of a rupee spent 'n' years hence $= (1+r)^{-n} = d^n$

∴ $d = (1+r)^{-1} = \frac{1}{1+i}$

	Present value	1 year	future value 2 year
$r = 10\%$ →	100	$= 100 + 100 \times 10\%$ ⇒ 110/-	$110 + 110 \times 10\%$ → 121/-
	100	$= 100(1+r)$	$110(1+r)(r)$ $100(1+r)^2$
	$\frac{100}{(1+r)^2}$	= 100	100

discount $d_n = \frac{1}{(1+r)^n}$

1 year 1 y 100 10% 100

$\frac{100}{1+r} = 0.9091$ 100

$\frac{100}{(1+r)^2}$

1 year 2 year 100 100+100×10% 110+110×10 110 121 $100(1+r)$ $110(1+r)$ $100(1+r)^2$

"d - discount rate" is always less than unity.

Problem ①: The yearly cost of two m/cs A & B, when money value is neglected is given below. Find their cost patterns if money value is 10% per year and hence find which machine is more economical.

Year :	1	2	3	Total Cost
M/c A (Rs) :	1800	1200	1400	= 4,400
M/c B (Rs) :	2800	200	1400	= 4,400

Sol

The total expenditure for each m/c A is Rs 4400

" m/c B is Rs 4400

These two m/cs A & B are equally good if money has no value change over time.

Given that value of money is 10% per year i.e, $r = 10\%$.

∴ Discount rate (d) $= \frac{1}{1+r} = \frac{1}{1.1} = 0.9091$.

The discounted cost patterns for m/c A & B are.

Year :	1	2	3	Total Cost (Rs)
m/c A :	1800	$1200 \times 0.9091 = 1090.90$	$1400 \times 0.9091^2 = 1157.04$	4047.94
m/c B :	2800	$200 \times 0.9091 = 181.82$	$1400 \times 0.9091^2 = 1157.04$	4,138.86

As the total cost of m/c A is less. M/c A is more economical.

-710 ...

Problem ② The cost of a new m/c is Rs. 5000. The maintenance cost during the n^{th} year is given by M_n = Rs. $500(n-1)$, where $n = 1, 2, 3 \ldots$. If the discount rate per year is 0.05 after how many years will it be economical to replace the m/c by a new one?

Sol

Discount rate of money is 0.05 per year, the present worth of the money to be spent after a year is

$$d = \frac{1}{1+0.05} = \boxed{0.9523}$$

Year	Maintenance cost (C_m) Rs	Discount factor (d^{i-1})	Discounted maintenance cost ($R_i d^{i-1}$)	Cum. total Discounted cost $C + \sum_{i=1}^{n} R_i d^{i-1}$	Dividing factor $\sum_{i=1}^{n} d^{i-1}$	Weighted Avg Cost $\frac{C+\sum_{i=1}^{n} R_i d^{i-1}}{\sum_{i=1}^{n} d^{i-1}}$
1	0	1.000	0	5000	1.000	5000
2	500	0.9523	476	5476	1.9523	2,805
3	1000	0.9070	907	6,383	2.8593	2,232
4	1500	0.8638	1,296	7679	3.7231	2063
5	2000	0.8227	1,645	9324	4.5458	$\boxed{2051}$
6	2,500	0.7835	1,959	11,283	5.3293	2,117

It is economical to replace the m/c at the end of 5^{th} year.

$$\text{Weighted Avg Cost} = \frac{C + \sum_{i=1}^{n} R_i d^{i+1}}{\sum_{i=1}^{n} d^{i-1}}$$

Optimal replacement policy for "Model-②"

Present value of a rupee spent n years hence $= (1+r)^{-n} = d^n$

where, $d = \frac{1}{1+r}$ is the discount rate.

Let 'n' - no of years after which the m/c is replaced

c - Purchase price of m/c

$R_1, R_2 .. R_n$ - Running costs of period 1, 2, .. n years.

Assume the scrap value is zero, and all that payments (cash outflow) are made at the beginning of each year, the present worth of expenditure in 'n' years is

$$P_n = C + R_1 + dR_2 + d^2R_3 + \cdots + d^{n-1}R_n$$

P_n - Money required now to pay all future costs of Procurement & operating the machine assuming that it is to be replaced after 'n' years

P_n increases as n increases. which means that the present worth, if the machine is replaced after n+1 years is greater than if it is replaced after n years.

Let us assume that manufacturer invests the amount P_n by borrowing money at the interest rate 'r'. and repays it off in fixed annual payments, each of value 'x', throughout the life of the machine.

The present worth of ~~the~~ fixed annual payments

$$x + dx + d^2x + \cdots + d^{n-1}x = \frac{1-d^n}{1-d}x$$

Since this, is equal to the sum P_n borrowed.

$$P_n = \frac{1-d^n}{1-d}x$$

$$\therefore x = \frac{1-d}{1-d^n}P_n$$

Sinc d - less than unity $(1-d)$ is +ve. The period at which the m/c replace (n), which minimises the function $f_n = \frac{P_n}{1-d^n}$

f_n will be minimum if $\boxed{\Delta f_{n-1} < 0 < \Delta f_n}$

$\Delta f_{n-1} = f_n - f_{n-1}$

$$\Delta f_n = f_{n+1} - f_n$$

$$= \frac{P_{n+1}}{1-d^{n+1}} - \frac{P_n}{1-d^n}$$

$$= \frac{(1-d^n) P_{n+1} - (1-d^{n+1}) P_n}{(1-d^{n+1})(1-d^n)}$$

$$= \frac{1}{(1-d^{n+1})(1-d^n)} \left[(P_{n+1} - P_n) + d^{n+1} P_n - d^n P_{n+1} \right]$$

we know $P_{n+1} = (c + R_1 + R_2 d + \dots + d^{n-1} R_n) + d^n R_{n+1} = P_n + d^n R_{n+1}$ and substitute 'P_{n+1}' in 'Δf_n'.

$$\therefore \Delta f_n = \frac{1}{(1-d^{n+1})(1-d^n)} \left[d^n R_{n+1} + d^{n+1} P_n - d^n (P_n + d^n R_{n+1}) \right]$$

$$= \frac{1}{(1-d^{n+1})(1-d^n)} \left[R_{n+1} d^n (1-d^n) - d^n P_n (1-d) \right]$$

$$= \frac{d^n (1-d)}{(1-d^{n+1})(1-d^n)} \left[\frac{1-d^n}{1-d} R_{n+1} - P_n \right]$$

$$= \text{a Positive constant} \left[\frac{1-d^n}{1-d} R_{n+1} - P_n \right]$$

f_n has always the same sign as the quantity in brackets.

From above condition for optimal 'n'. if

$$\frac{1-d^{n-1}}{1-d} R_n - P_{n-1} < 0 \leq \frac{1-d^n}{1-d} R_{n+1} - P_n$$

$$\frac{1-d^n}{1-d} R_{n+1} - P_n \geq 0$$

(or)

$$R_{n+1} > P_n \frac{1-d}{1-d^n}$$

(or)

$$R_{n+1} > P_n \Big/ \frac{1-d^n}{1-d}.$$

4

(or) $R_{n+1} > \dfrac{C + R_1 + dR_2 + d^2R_3 + \dots + d^{n-1}R_n}{1 + d + d^2 + \dots + d^{n-1}}$

(weighted Avg cost of all costs upto 'n+1' (including n+1))

(or) $R_{n+1} > \dfrac{C + \sum_{i=1}^{n} R_i\, d^{i-1}}{\sum_{i=1}^{n} d^{i-1}}$

∴ Next periods cost > weighted average of previous costs, Since

$$R_n < \frac{C + R_1 + dR_2 + d^2R_3 + \dots + d^{n-2}R_{n-1}}{1 + d + d^2 + \dots + d^{n-2}}$$

(or) $R_n < \dfrac{C + \sum_{i=1}^{n} R_i\, d^{i-2}}{\sum_{i=1}^{n} d^{i-2}}$

Conclusions:

(a) The machine should be replaced if the next period's cost is greater than the weighted average of previous costs.

(b) The machine should not be replaced if the next periods cost is less than the weighted avg. of previous costs.

Case① : $R = 0$; $d = 1$ then $R_{n+1} > \dfrac{C + R_1 + R_2 + \dots + R_n}{1 + 1 + 1 - + \; n \text{ time}}$

$$R_{n+1} > \frac{P_n}{n}$$

Problem ③: A Scooter costs Rs. 6,000 when new. The running cost and salvage value (sale price) at the end of year is given in below. If the interest rate is 10% per year and the running costs are assumed to have occured at mid year, find when the scooter should be replaced.

Year :	1	2	3	4	5	6	7
Running cost (Rs) :	1,200	1,400	1,600	1,800	2,000	2,400	3,000
Salvage value (Rs) :	4,000	2,666	2,000	1,500	1,000	600	600

Sol. $d = \dfrac{1}{1+i} = \dfrac{1}{1+0.1} = 0.9091$ ∴ rate of interest (i) = 10%.

The given running cost is fd middle of the year and we can discount them to the start of year by multiplying by $d^{1/2} = \sqrt{0.9091} = 0.95346$.

$$
\begin{array}{r} 7144.2 \\ -3,636.4 \\ \hline 3,507.8 \end{array}
$$

$6000 + 1,144.2 - 3,636.4$

Year of Service (r)	Salvage value (RV)	Running Cost (mid-year) R_r'	Running Cost (start of year) $R_r = d^{1/2} R_r'$	Discount rate d^{r-1} (Previous year)	Discount rate d^r (next year)	Discounted running cost ($R_r d^{r-1}$)	Discounted salvage value (RV $\cdot d^r$)	Cost $C + \sum_{r=1}^{n} R_r d^{r-1}$	Total cost $C + \sum_{r=1}^{n} R_r d^{r-1} - R_v d^r$	Dividing factor $\sum_{r=1}^{n} d^r$	$\frac{C + \sum_{r=1}^{n} R_r d^{r-1}}{\sum_{r=1}^{n} d^{r-1}}$
1	4000	1,200	1,144.2	1.0000	0.9091	1,144.2	3,636.4	7144.2	3507.8	0.9091	3,859.0
2	2,666	1,400	1,334.8	0.9091	0.8264	1,211.6	2,203.2	8,355.8	6,152.6	1.7355	3,544.2
3	2,000	1600	1525.6	0.8264	0.7513	1,260.8	1,502.6	9,616.6	8,114.0	2.4868	3,262.6
4	1,500	1,800	1,716.2	0.7513	0.6830	1,289.2	1024.5	10,905.8	9,881.3	3.1698	3,117.2
5	1,000	2,000	1,907.0	0.6830	0.6207	1,302.4	620.9	12,208.2	11,587.3	3.7907	3,056.6
6	600	2,400	2,288.4	0.6209	0.5645	1,420.8	338.7	13,627.0	13,270.3	4.3552	$\boxed{3,051.6}$
7	600	3,000	2,860.4	0.5645	0.5132	1,614.7	307.9	16,243.7	14,935.8	4.8684	3,067.9

The scooter should be replaced after 6 years.

GROUP REPLACEMENT

Group replacement:

1. The following failure rates have been observed for a certain types of bulbs.

End of week :	1	2	3	4	5	6	7	8
Probability of failure to date :	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1
	5%	13%	25%					

The cost of replacing an individual bulb is Rs 2.25 /- The discussion is made to replace all the bulbs simultaneously at fixed intervals & also to replace individual bulbs as they fail in service. If the cost of group replacement is 60 Paise/bulbs and the total no of bulbs are 1000. what is the best interval b/n group replacement?

Sol

Let "P_i" be the probability of failure during i^{th} week

$P_1 \rightarrow$ Probability of failure in 1^{st} week $= 0.05$

$P_2 \rightarrow$ " 2^{nd} week $= 0.08$

$P_3 \rightarrow$ " 3^{rd} week $= 0.12$

$P_4 \rightarrow$ " 4^{th} week $= 0.18$

$P_5 \rightarrow$ " 5^{th} week $= 0.25$

$P_6 \rightarrow$ " 6^{th} week $= 0.2$

$P_7 \rightarrow$ " 7^{th} week $= 0.08$

$P_8 \rightarrow$ " 8^{th} week $= 0.04$

1.00

Let N_i - be no of replacements made at the end of the i^{th} week.

$N_1 = N_0 \times \overset{P_1}{0.05} = N_0 P_1 = 1000 \times 0.05 = 50$ bulbs $(N_0 = 1000)$

$N_2 = N_0 P_2 + N_1 P_1 = 1000 \times 0.08 + 50 \times 0.05 = 82.5 \cong 83$ bulbs.

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 1000 \times 0.12 + 50 \times 0.08 + 82.5 \times 0.05 = 128.125 \cong 129$ bulbs

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.18 + 50 \times 0.12 + 83 \times 0.08 + 128.125 \times 0.05 = 199$ bulbs

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.25 + 50 \times 0.18 + 83 \times 0.12 + 128.125 \times 0.08 + 199 \times 0.05 = 289.15$
$\cong 290$ bulbs

$N_6 = N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$

$= 1000 \times 0.2 + 50 \times 0.25 + 83 \times 0.18 + 128.125 \times 0.12 + 199 \times 0.08 + 289.15 \times 0.05$

$= 273.17 \cong 274$ bulbs..

$N_7 = N_0 P_7 + N_1 P_6 + N_2 P_5 + N_3 P_4 + N_4 P_3 + N_5 P_2 + N_6 P_1$

$= 1000 \times 0.08 + 50 \times 0.2 + 83 \times 0.25 + 128.125 \times 0.18 + 199 \times 0.12 +$

$290 \times 0.8 + 274 \times 0.05 =$

$= 194.76 \cong 195$ bulbs

$N_8 = N_0 P_8 + N_1 P_7 + N_2 P_6 + N_3 P_5 + N_4 P_4 + N_5 P_3 + N_6 P_2 + N_7 P_1$

$= 195.32 \cong 196$ bulbs

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End of week	Total Cost (Individual + Group replacements cost)	Avg Cost/week
1	$50 \times 2.25 + 1000 \times 0.6 = 712.5$	712.5
2	$(50+83)2.25 + 1000 \times 0.6 = 899.25$	$\frac{899.25}{2} = 449.625$
3	$(50+83+129)2.25 + 1000 \times 0.6 = 1189.5$	$\frac{1189.5}{3} = 396.5$ ✓
4	$(50+83+129+199)2.25 + 600 = 1637.5$	$\frac{1637.5}{4} = 409.875$
5	$(50+83+129+199+290)2.25+600$	458.4
6	$(50+83+129+199+290+274) \times 2.25+600$	484.75
7	$(50+83+129+199+290+274+195)2.25+600$	478.17
8	$(50+83+129+199+290+274+195+196)2.25 + 1000 \times 0.6$	

Average life of bulbs (Expected) $= \sum_{i=1}^{8} i \, p_i$

$= (1 \times P_1 + 2 \times P_2 + 3 \times P_3 + 4 \times P_4 + 5 \times P_5 + 6P_6 + 7P_7 + 8P_8)$

$= 4.62$ weeks

~~Also~~ Avg no of failures per week $= \frac{1000}{4.62} = 216.4 \cong 217$ bulbs/week

Individual cost/week $= 217 \times 2.25 =$ Rs 488.25

So, the group replacement policy is preferred since

$\boxed{\text{Rs } 396.5 < \text{Rs } 488.25}$

(2) The following mortality rates have been observed for a certain type of light bulbs in an installation with 1,000 bulbs.

End of week :	1	2	3	4	5	6
Probability of failure to date :	0.09	0.25	0.49	0.85	0.97	1.00

There are a large number of such bulbs which are to be kept in working order. If a bulb fails in service, it costs Rs 3 to replace but if all the bulbs which are replaced in the same operation, it can be done for only Rs. 0.70 a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing ~~replacing~~ burnt out bulbs as they fail.

(a) what is the best interval b/n group replacements?

(b) Also establish if the policy, as determined by you, is superior to the policy of replacing bulbs as and when they fail, there being nothing like 'group replacement'.

(c) At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to adopted policy? (Assume that all bulbs failing during a week might fail at any time of the week and that group replacements are made only at the end of a week)

Sol

let P_i - Probability that a new bulb fails in i^{th} week.

$P_1 = 0.09$

$P_2 = 0.25 - 0.09 = 0.16$

$P_3 = 0.49 - 0.25 = 0.24$

$P_4 = 0.85 - 0.49 = 0.36$

$P_5 = 0.97 - 0.85 = 0.12$

$P_6 = 1.00 - 0.97 = 0.03$

$\overline{1.00}$

N_i - number bulbs failed during i^{th} week.

$N_0 = 1000$ bulbs.

$N_1 = N_0 \times P_1 = 1000 \times 0.09 = 90$ bulbs

$N_2 = N_0 \times P_2 + N_1 \times P_1 = 1000 \times 0.16 + 90 \times 0.09 = 168$

$N_3 = N_0 \times P_3 + N_1 P_2 + N_2 P_1 = 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09 = 269$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1 = 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269 \times 0.09$
$= 432$

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$
$= 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09 = 274$

$N_6 = N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$
$= 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24 + 432 \times 0.16 + 274 \times 0.09$
$= 260.$

(a) Determination of optimal group replacement interval.

End of week	Total cost of replacement (Individual + Group)	Avg. cost/week
1	$90 \times 3 + 1000 \times 0.70 = 970$	970
2	$(90+168)3 + 700 = 1{,}474$	737.0 ←
3	$(90+168+269)3 + 700 = 2{,}281$	760.33

∴ It is optimal to have a group replacement after every 'two weeks'.

(b) Individual replacement policy :

Avg life of bulbs : $\sum_{i=1}^{6} i\,p_i$

$= 1 \times 0.09 + 2 \times 0.16 + 3 \times 0.24 + 4 \times 0.36 + 5 \times 0.12 + 6 \times 0.03$

$= 3.35$.

Avg no of failures per week $= \dfrac{1000}{3.35} = 299$. bulbs

Cost of individual replacement of bulbs per week = Rs 8×299 = $\boxed{\text{Rs } 897}$.

∴ Since the cost of group replacements per week is Rs 737 and that of individual replacements is Rs 897 per week, it is advisable to adopt the policy of group replacements.

(c) Let Rs 'x' be the group replacement price per bulb

$$\text{Rs. } 897 < \frac{1000x + 3(90+168)}{2}$$

$$\boxed{x > \text{Rs } 1.02}$$

Therefore, when the group replacement price per bulb exceeds Rs 1.02, the policy of strictly individual replacements becomes more economical.

Introduction to Simulation:

Simulation is the representative model for the real situation. Simulation technique is used in situations where it is not possible to construct mathematical tools like linear programming. Some major applications of the simulation are.

* Job shop scheduling
* Queueing problems
* Demand forecasting
* Inventory problems
* Capital budgeting problems
* Financial planning
* Replacement problems.

Advantages:

1. Many important managerial decisions problems cannot be solved by mathematical techniques. Simulation offers the solution by allowing experimentation with a model of the system without interfering with the real system.

2. Through simulation, management can predict the difficulties & bottlenecks which may come up due to the introduction of new machines, equipment or process. Costly trial and error methods ~~of~~ new concepts on costly equipment can be eliminated.

3. Simulation has the advantage of being relatively free from mathematics. Thus it can be easily understood by the operating personnel and non-technical managers.

4. Simulation models are comparatively flexible and can be modified due to the changes in the environment of real situation.

5. Simulation models are easier that mathematical models.

6. Simulation has advantageously been used for training the staff and workers. It is always advantageous to train the people on simulated models before engaging them on real system.

Limitations:

1. Optimal results cannot be guaranteed by simulation.
2. In many situations, it is not possible to quantify the variables which affect the behaviour of the system.
3. In a number of situations, simulation is comparitively costlier and time consuming.
4. It is very difficult to construct simulation models if the number of variables are large and their interrelationship is complex.

Queueing Problems:

① Two persons X & Y work on a two station assembly line. The distributions of activity times at their stations are given below.

Time (Seconds)	Time frequency for 'X'	for Y
10	4	2
20	7	3
30	10	6
40	15	8
50	35	12
60	18	9
70	8	7
80	3	3

ⓐ Simulate operation of the line for eight items.

ⓑ Assuming 'Y' must wait until 'X' completes the first item before starting the work, will he have to wait to process any of the other seven items? what is the average waiting time of items for 'Y'? use the following random numbers.

For X : 83, 70, 06, 12, 59, 46, 54, 04

for Y : 51, 99, 84, 81, 15, 36, 12, 54

ⓒ Determine the inventory items b/n the stations.

ⓓ What is the average production rate.

Sol:

(a) For 'X' person :

Activity time (seconds)	Time frequency for 'X'	Cumulative frequency	Range	Random numbers fitted
10	4	4	0 – 03	
20	7	11	04 – 10	06 (3), 04 (8)
30	10	21	11 – 20	12(4)
40	15	36	21 – 35	
50	35	71	36 – 70	70(2), 59(5), 46(6), 54(7)
60	18	89	71 – 88	83(1)
70	8	97	89 – 96	
80	3	100	97 – 99	

Thus the activity times of eight items for 'X' are 60, 50, 20, 30, 50, 50, 50 and 20.

For 'Y' person :

Activity time (seconds)	Time frequency for 'Y'	Cumulative frequency	Range	Random numbers fitted.
10	2	2 4	0 – 04	
20	3	5 10	05 – 09	
30	6	11 22	10 – 21	15(5), 12(7)
40	8	19 38	22 – 37	36(6),
50	12	31 62	38 – 61	51(1), 54 (8)
60	9	40 80	62 – 79	
70	7	47 94	80 – 93	84(3), 81 (4)
80	3	50 100	94 – 99	99(2),

Thus the activity times of Eight items for 'Y' are 50, 80, 70, 70, 30, 40, 30, and 50.

Item No.	Person 'X' Time in	Time out	Person 'Y' Time in	Time out	Waiting time of person 'Y'	Waiting time of Items.
1	0	60	60	110	60	–
2	60	110	110	190	–	–
3	110	130	190	260	–	60
4	130	160	260	330	–	100
5	160	210	330	360	–	120
6	210	260	360	400	–	100
7	260	310	400	430	–	90
8	310	330	430	480	–	100

Thus the person 'Y' will not have to wait for remaining seven items.

$$\text{Avg. waiting time of items} = \frac{0+0+60+100+120+100+90+100}{8}$$

$$= \frac{570}{8}$$

(c) In all there are 6 items waiting b/n two stations.

(d) The total time taken to process 8 items = 480 seconds.

$$\therefore \text{Avg. Production rate} = \frac{8}{480} = \frac{8}{8} = 1 \text{ item/minute.}$$

$(\because 480 \text{ sec} = 8 \text{ mins})$

(2) A dentist schedules all her patients for 30 minutes appointments. Some of the patients take more (or) less than 30 mins. depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and the time needed to complete the work. Simulate the dentist clinic for "four hours" and determine the avg. waiting time for patients as well as the idleness of the doctor.

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Category	Time required (minutes)	Probability of category.
filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
check-up	15	0.20

Assume that all the patients show up at the clinic at exactly their scheduled arrival times, starting at 8. Am. Use the following random numbers for handling the above problem. 40, 82, 11, 34, 25, 66, 17 & 79.

Sol:

Category	Time (minutes)	Probability	Cum. Probability	Random number interval (Range)	Random number fitted.
Filling	45	0.40	0.40	0 - 39	11(3), 34(4), 25(5), 17(7)
crown	60	0.15	0.55	40 - 54	40(1),
cleaning	15	0.15	0.70	55 - 69	66(6),
extraction	45	0.10	0.80	70 - 79	79(8).
Check-up	15	0.20	1.00	80 - 99	82(2),

Thus the times taken by the dentist to treat eight patients are 60, 15, 45, 45, 45, 15, 45, and 45

Simulation of the dentist clinic starting at 8 AM.

Patient No.	Arrival time	Dentist treatment starts	Dentist treatment Ends	waiting time of the patient (min)	Idle time for the dentist. (mins.)
1	8:00	8:00	9:00	-	-
2	8:30	9:00	9:15	30	-
3	9:00	9:15	10:00	15	-
4	9:30	10:00	10:45	30	-
5	10:00	10:45	11:30	45	-
6	10:30	11:30	11:45	60	-
7	11:00	11:45	12:30	45	-
8	11:30	12:30	01:15(PM)	60	-

Avg. waiting time for the patients $= \frac{0+30+15+30+45+60+45+60}{8}$

$= 35.625$ minutes.

Avg idleness of the dentist = NIL

(3) At a small store of readymade garments, there is one clerk at the counter who is to ~~create~~ check the bills, recieve the payments and place the packed garments into fancy bags etc. The customers arrival at the check counter is a random phenomenon and the time b/n the arrivals varies from 1 min. to 5 mins., the frequency distribution for which is given in the table. The service time (taken by counter clerk) varies from 1 min. to 3 mins. The manager of the store feels that the counter clerk is not sufficiently loaded with work and wants to assign to him some additional work. But before taking the decision he likes to know precisely by what % of time the counter clerk is idle.?

Frequency distribution of Inter-Arrival time.			
Time between Arrivals (mins)	Frequency	Cumulative Frequency	Random numbers Range
1	35	35	0 - 34
2	25	60	35 - 59
3	20	80	60 - 79
4	12	92	80 - 91
5	08	100	92 - 99

Frequency distribution of Service times.			
Service Time (Minutes)	Frequency	Cumulative Frequency	Random number Range.
1.0	20	20	0 - 19
1.5	35	55	20 - 54
2.0	25	80	55 - 79
2.5	15	95	80 - 94
3.0	05	100	95 - 99

Use the following random number for

Arrivals: 48, 51, 06, 22, 79, 56, 06, 91, 51, 13, 65, 59, 51, 50, 13, 94, 57, 26, 78, 33

Service: 22, 62, 25, 31, 23, 07, 93, 44, 12, 26, 93, 01, 17, 49, 58, 98, 61, 41, 13, 59

Arrivals	Arrival Random Number	Inter-Arrival time	Service Random Number	Service time	Cumulative Actual Arrival time	Service Starts	Service Ends	Counter Clerk Idle time	Customers waiting time
1	48	2	22	1.5	2	2.0	3.5	2	–
2	51	2	62	2.0	4	4.0	6.0	0.5	–
3	06	1	25	1.5	5	6.0	7.5	–	1.0
4	22	1	31	1.5	6	7.5	9.0	–	1.5
5	79	3	23	1.5	9	9.0	10.5	–	–
6	56	2	07	1.0	11	11	12.0	0.5	–
7	06	1	93	2.5	12	12.0	14.5	–	–
8	91	4	44	1.5	16	16.0	17.5	1.5	–
9	51	2	12	1.0	18	18.0	19.0	0.5	–
10	13	1	26	1.5	19	19.0	20.5	–	–
11	65	3	93	2.5	22	22	24.5	1.5	–
12	59	2	01	1.0	24	24.5	25.5	–	0.5
13	51	2	17	1.0	26	26.0	27.0	0.5	–
14	50	2	49	1.5	28	28.0	29.5	1.0	–
15	13	1	58	2.0	29	29.5	31.5	–	0.5
16	94	5	98	3.0	34	34.0	37.0	2.5	–
17	57	2	61	2.0	36	37.0	39.0	–	1.0
18	26	1	41	1.5	37	39.0	40.5	–	2.0
19	78	3	13	1.0	40	40.5	41.5	–	0.5
20	33	1	59	2.0	41	41.5	43.5	–	0.5
								10.5	7.5

∴ % of idleness of counter clerk = $\frac{10.5}{41} \times 100 = \underline{25.6}$ %

Manager allots some additional works to the clerk.

④ A firm has a single service station with the following arrival and service time probability distribution. (Nov, 2017)

Inter Arrival time (mins)	Probability	Service time (mins)	Probability
10	0.10	5	0.08
15	0.25	10	0.14
20	0.30	15	0.18
25	0.25	20	0.24
30	0.1	25	0.22
–	–	30	0.14.

The customer's arrival at the service station is random phenomenon, and the time b/n arrivals varies from 10 to 30 minutes. The service time varies from 5 to 30 minutes. The queueing process begins at 10.A.M and proceeds for nearly 8 hours. The queue discipline is first come first served. Simulate this queue for 10 arrivals. The following random numbers ~~are~~ can be used.

for arrival times : 20, 73, 30, 99, 66, 83, 32, 75, 04, 15.

for service times : 26, 43, 98, 87, 58, 90, 84, 60, 08, 50.

Sol Arrival times.

Inter Arrival time (Minutes)	Probability in (%)	Cum. Probability	Range	Random number fitted.
10	10	10	0 – 9	04(9)
15	25	35	10 – 34	20(1), 30(3), 32(7), 15(10)
20	30	65	35 – 64	
25	25	90	65 – 89	73(2), 66(5), 83(6), 75(8)
30	10	100	90 – 99	99(4)

Thus the times taken by the ten arrivals (inter arrival times) are : 15, 25, 15, 30, 25, 25, 15, 25, 10, and 15.

Service times:

Service time (mins)	Probability in %	Cum. Probability	Range	Random number fitted
5	8	8	0 - 7	
10	14	22	8 - 21	09(9)
15	18	40	22 - 39	26(1).
20	24	64	40 - 63	43(2), 58(5), 60(8), 50(10)
25	22	86	64 - 85	84 (7)
30	14	100	86 - 99	98(3), 87(4), 90(6)

Thus the ten service times required for 10 arrivals are: 15, 20, 30, 30, 20, 30, 25, 20, 10, and 20.

The simulation of service channel for 10 arrivals.

Arrival number	Inter-arrival time (mins)	Service time required (mins)	Actual arrival time (mins)	Service time (mins) Starting	Ending
1	15	15	10:15	10:15	10:30
2	25	20	10:40	10:40	11:00
3	15	30	10:55	11:00	11:30
4	30	30	11:25	11:30	12:00
5	25	20	11:50	12:00	12:20
6	25	30	12:15	12:20	12:50
7	15	25	12:30	12:50	01:15
8	25	20	12:55	01:15	01:35
9	10	10	01:05	01:35	01:45
10	15	20	01:20	01:45	02:05

Inventory Problems:

(May, 2018)

① A company trading in motor vehicles spares whishes to determine the level of stock it should carry for the items in its range. Demand is not certain and there is a lead time for stock replenishment for one item X, the following information is obtained.

Demand (units/day) :	3	4	5	6	7
Probability :	0.1	0.2	0.3	0.3	0.1

Carrying cost /unit/day = 20 paise.
Ordering cost /order : Rs 5
Lead time for replenishment = 3 days.

Stock in hand at the beginning of the simulation exercise was 20 units. You are required to carryout a simulation run over a period of 10 days with the objective of evaluating the following inventory rule.

" order 15 units when present inventory + any outstanding order falls below 15 units."

The sequence of random numbers used is 0, 9, 1, 1, 5, 1, 8, 6, 3, 5, 7, 1, 2, 9 using the first number for day one. Your calculation should include the total cost of operating the inventory rule for 10 days.

Sol: Demand in each of the 10 days : Table ①

Demand	Probability Distribution of Demand		Range	Random numbers fitted.
	(P(d)) Probability	Cummulative Probability		
3	0.1	0.1×10 = 1	0	0(1)
4	0.2	0.3×10 = 3	1 — 2	1(3), 1(4), 1(6), 1(12), 2(13)
5	0.3	0.6×10 = 6	3 — 5	5(5), 3(9), 5(10)
6	0.3	0.9×10 = 9	6 — 8	8(7), 6(8), 7(11)
7	0.1	1.0×10 = 10	9	9(2), 9(14)

Thus the demand for item 'X' on the ten days is 3, 7, 4, 4, 5, 4, 6, 6, 5, and 5 units respectively. The inventory carrying costs and ordering costs are computed in the below table.

Table ②: Simulation of demand, order delivery and inventory costs.

Day	Demand (units)	Units ordered	Lead time	Units Recieved	Inventory	Inventory carrying cost (Rs)	Ordering cost (Rs)
0	-				20	4.00	
1	3				17	3.40	
2	7	15	3		10	2.00	5.00
3	4				6	1.20	
4	4				2	0.40	
5	5	15	3	15	12	2.40	5.00
6	4				8	1.60	
7	6				2	0.40	
8	6	15	3	15	11	2.20	5.00
9	5				6	1.20	
10	5				1	0.20	
					Total	19.00	15.00

∴ The total cost of operating inventory for 10 days

$$= Rs.\ (19+15) = \boxed{Rs.\ 34}$$

Inventory Models

'Inventory' is nothing but stock of goods being held for future use or sale. Inventory is Essential for smooth running of system or organisation. Inventory can be categorised into two types:

① Direct Inventories

- Raw material inventory
- Work-in-Process inventory
- finished goods inventory

② Indirect inventories

- Pipeline inventory
- Buffer inventory
- Decoupling inventory.
- Lot size inventory
- Seasonal inventory
- Anticipation inventory.

The main problems of inventory management are to determine ① when to place an order (Replinish the inventory)? and ② How much to order? Inventory control helps in smoothing out irregularities in supply, minimizing the production cost and allowing organisations to cope up with perishable materials.

Costs involved in Inventory :

① **Ordering Cost :** This is a cost associated with ordering of inventory. It is denoted with C_o (Rs/order).

② **Setup Cost :** The cost associated with setting up of the machinery before starting the production, and is independent of ordering quantity for production.

③ Carrying (or) Holding Cost: (It is denoted as 'C_c'/unit/period)

The cost associated with storage (stocking or holding) like rent, interest on the money locked up, insurance of the stored equipment, production, taxes, depreciation of equipment, and special maintenance etc. (A.C. Room)

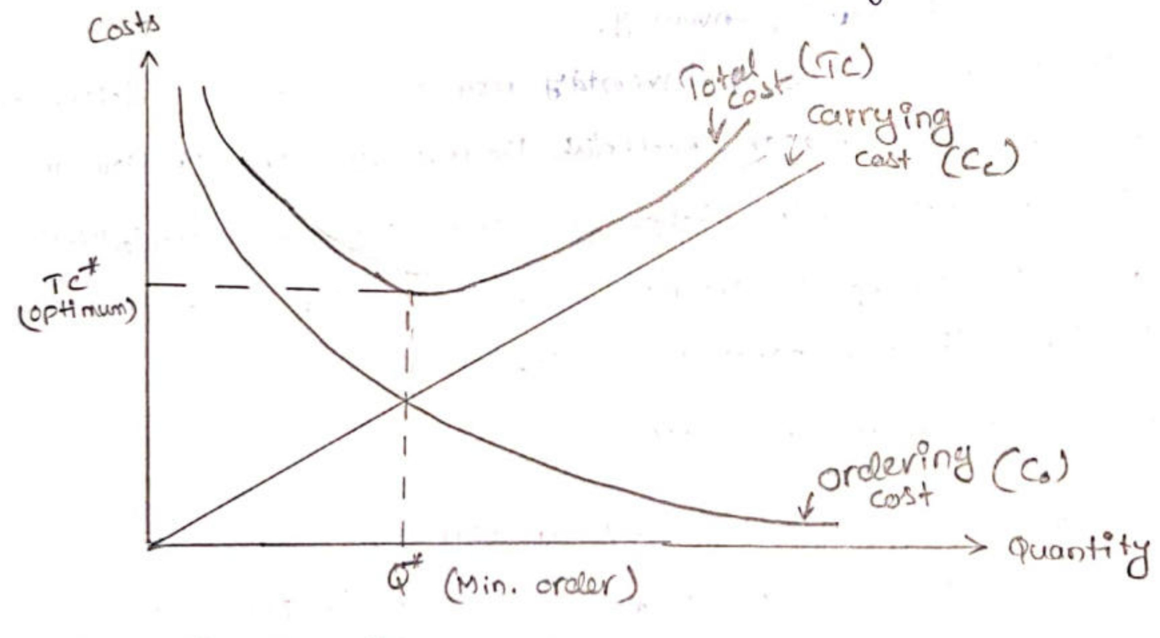
④ Purchase Cost: This is cost of purchasing (or) producing an item. It is very important when discounts are allowed.

⑤ Shortage Cost (or) Stock out Cost:

- Loss of profit
- Loss of opportunity
- Cost of addl capacity
- Rescheduling
- Under Utilization
- increased freight
- Loss of customer goodwill

The penalty cost for running out of stock. It includes the loss of potential profit through sales of item and loss of goodwill, in terms of permanent loss of customers and its associated lost profit in future sales.

Total cost of inventory for time t = Ordering cost + Purchase cost (or) set up cost + Carrying cost + Shortage cost.



Economic Order Quantity (EOQ):

It is the quantity ordered per order so that the total cost of the inventory is minimum. (TC^*).

Model - ①:

Purchasing model without shortages:

- Single item
- Continuous demand
- No shortages
- Instantaneous Replenishment

Let R = Annual demand

C = Unit cost of item

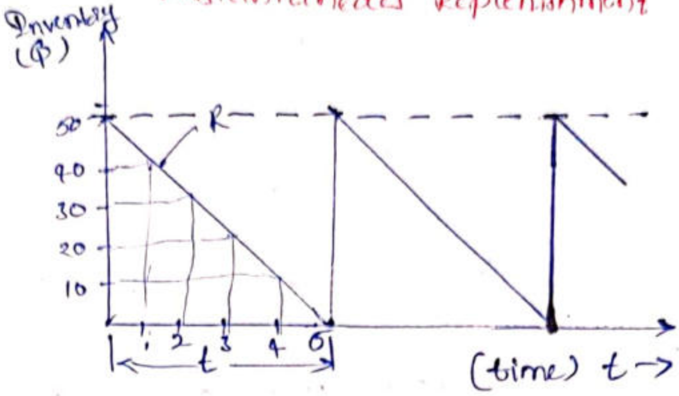
C_o = Ordering cost per order.

C_c = Carrying cost per unit per year.

$= i.C$ (some problems)

i = rate of interest (%)

4a



From graph $Q = Rt$

Area of $\Delta^{le} = \frac{1}{2} Q t$

$= \frac{1}{2} Rt \cdot t = \frac{1}{2} R t^2$ ✓

$\therefore$ Slope $= R = \frac{Q}{t}$

(or) Demand

$\boxed{Q = Rt}$

Slope: $R = \frac{Q}{t}$

$\boxed{Q = Rt}$

$\therefore$ Carrying cost $= C_c$ (or) C_1

holding stock $\frac{Qt}{2}$

$Qt\ C_c$

Cost of holding inventory during time, $t = \frac{1}{2} R t^2 C_c$

Ordering cost $= C_o$ (or) C_3

Total cost during time 't' $= \frac{1}{2} C_c R t^2 + C_o +$

Avg total cost / unit time $(C_t) = \left[\frac{1}{2} C_c R t^2 + C_o\right] \frac{1}{t}$

$$= \frac{1}{2} C_c R t + \frac{C_o}{t}$$

C_t will be minimum if $\frac{d(C_t)}{dt} = 0$

$$\frac{d}{dt}\left[\frac{1}{2} C_c R t + \frac{C_o}{t}\right] = 0$$

$$\frac{1}{2} C_c R + (-1)\frac{C_o}{t^2} = 0$$

$$\frac{C_c R t^2 - 2C_o}{2t^2} = 0$$

$$C_c R t^2 = 2C_o$$

$$\therefore \boxed{t = \sqrt{\frac{2C_o}{C_c R}}}$$

We know that $Q = Rt$

$$Q = R\sqrt{\frac{2C_0}{C_c R}}$$

$$Q = \sqrt{\frac{2C_0 R^2}{C_c R}}$$

$$Q = \sqrt{\frac{2RC_0}{C_c}} = EOQ$$

which is known as ~~se~~ economic lot size (or) Economic order quantity (EOQ).

The resulting min. ~~of~~ avg cost per unit time.

$\frac{Q}{2} C_c + \frac{R}{Q} C_0$

$\therefore$ Total cost $C_t = \frac{1}{2} C_c R t + \frac{C_0}{t}$

$$= \frac{1}{2} C_c R \sqrt{\frac{2C_0}{C_c R}} + C_0 \sqrt{\frac{R C_c}{2C_0}}$$

$$= \frac{1}{2}\sqrt{\frac{2C_c^2 R^2 C_0}{\cancel{C_c} \cancel{R}}} + \sqrt{\frac{R C_0^{\cancel{2}} C_c}{2\cancel{C_0}}}$$

$$= \sqrt{\frac{\cancel{2}}{\cancel{4}_2} \frac{R C_0 C_c}{1}} + \sqrt{\frac{R C_0 C_c}{2}}$$

$$= 2\sqrt{\frac{R C_0 C_c}{2}}$$

$$= \sqrt{\cancel{4}2 \frac{R C_0 C_c}{\cancel{2}}}$$

$$\therefore \boxed{C_t = \sqrt{2RC_0C_c}}$$

Total Annual Cost $= \frac{Q}{2} C_c + \frac{R}{Q} C_0 + RC$.

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Problem ①: A stockist has to supply 12,000 units of products/year to his customer. The demand is fixed and known, the shortage cost is assumed to be infinite. The inventory holding cost is Rs 0.20 /unit/month and the ordering cost is Rs 350/order. Determine. the optimum lot size

4 b

ⓐ The optimum lot size,
ⓑ Optimum scheduling period.
ⓒ Minimum total variable cost.

Sol

$R = 12000$ units/yr $= \frac{12000}{12} = 1000$ units/month

$C_c =$ Rs 0.20 /unit/month

$C_0 =$ Rs 350

(i) Optimum lot size (q) $EOQ = \sqrt{\frac{2 C_0 R}{C_c}}$

$EOQ = \sqrt{\frac{2 \times 350 \times 1000}{0.2}} = 1870$ units/order

(ii) Optimum scheduling period $(t) = \sqrt{\frac{2 C_0}{R C_c}}$

$t = \sqrt{\frac{2 \times 350}{1000 \times 0.2}} = 1.87$ months.

(iii) Min. total variable cost $C_t = \sqrt{2 R C_0 C_c}$

$= \sqrt{2 \times 350 \times (0.2 \times 12) \times (1000 \times 12)} =$ Rs 4490 /year

$C_t =$ Rs. 4490 /year

Problem ②: A particular item has a demand of 9000 units/year. The cost of one procurement is Rs 100 and the holding cost/unit is Rs. 2.40 /year. The ~~replacement~~ replinishment is instantaneous and no shortages are allowed. Determine

ⓐ Economic lot size
ⓑ The no of orders/year
ⓒ The time b/n orders.
ⓓ The total cost/year if the cost of one unit is Rs. 1

Sol Given that $R = 9000$ units / year

C_o = Rs 100 / order

C_c = Rs 2.40 /unit/year.

(a) Economic lot size (EOQ) $= \sqrt{\frac{2RC_o}{C_c}}$

$= \sqrt{\frac{2 \times 9000 \times 100}{2.40}} = 866$ units / order

(b) No of orders per year. $= \frac{1}{t} = \sqrt{\frac{RC_c}{2C_o}}$

$= \sqrt{\frac{9000 \times 2.40}{2 \times 100}} = 10.4$ orders / year

(c) Time b/n orders $= t = \frac{1}{10.4} = 0.09615$ years. (or) 1.15 months

(d) $C_t = \sqrt{2RC_oC_c} = \sqrt{2 \times 9000 \times 100 \times 2.40} = 2078.5$

Total cost /year $(C) = C_t + 9000 \times 1$

$= 2078.5 + 9000 =$ Rs 11078.5 / year.

Problem (8): The following table give the annual demand and unit price of 4 items.

Item	A	B	C	D
Annual Demand (units)	800	400	392	13,800
Unit Price (Rs)	0.02	1.00	8.00	0.20

The ordering cost is Rs 5 /order and holding cost is 10% of unit price. Determine

(a) The EOQ in Units

(b) Total variable cost

(c) Complete EOQ in Rs.

(d) EOQ in years of supply.

(e) No. of orders / year.

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Sol:

Item A:

$R = 800$ units/year

$C_o = Rs\ 5$ /order

$C_c = Rs\ \frac{10}{100} \times 0.02 = 0.002$

① $EOQ = \sqrt{\frac{2RC_o}{C_c}} = \sqrt{\frac{2 \times 800 \times 5}{0.02}} = 2000$ units

② $C_t = \sqrt{2C_oC_cR}$

$= \sqrt{2 \times 5 \times 0.002 \times 800} = 4$

③ EOQ in Rs $= 2000 \times 0.02 = Rs\ 40$

④ EOQ in years $= \frac{2000}{800} = 2.5$ years (t)

⑤ No of orders/year $= \frac{800}{2000} = 0.4$ orders. $\left(\frac{1}{t}\right) = N$

Item B:

$R = 400$ units/year

$C_o = Rs\ 5$ /order

$C_c = Rs\ \frac{1.00}{100} = Rs\ 0.1$

① $EOQ = \sqrt{\frac{2RC_o}{C_c}} = \sqrt{\frac{2 \times 400 \times 5}{0.1}} = 200$ units/order

② ~~EOQ~~ $C_t = \sqrt{2RC_oC_c} = \sqrt{2 \times 400 \times 5 \times 0.1} = Rs\ 20$ /year

③ EOQ in Rs $= 200 \times 1.0 = Rs\ 200$

④ EOQ in years $= \frac{200}{400} = 0.5$ years

⑤ No of orders/year $= \frac{400}{200} = 2$ orders/year

Item C:

$R = 392$ Units/year

$C_o = Rs\ 5$ /order

$C_c = 0.8$ /year.

① $EOQ = \sqrt{\frac{2 \times 392 \times 5}{0.8}} = 70$ units/order

② $C_t = \sqrt{2 \times 392 \times 5 \times 0.8} = Rs\ 56$ /year

③ EOQ in Rs $= 70 \times 8 = Rs\ 560$ /year

④ EOQ in years $= \frac{70}{392} = 0.18$ year.

⑤ No of orders/year $= \frac{392}{70} \sim 5.6$ order/year

Item D | $D = 13800$ units/year

$C_o = 5$; $C_c = 0.02$

① $EOQ = \sqrt{\dfrac{2 \times 13800 \times 5}{0.02}} = 2626.8$ units/order

② $C_t = \sqrt{2 \times 13800 \times 5 \times 0.02} = 52.54$ /year

③ EOQ in Rs = $\cancel{52.54 \times 0.2} = \cancel{Rs\ 10.51}$ /- $= 2626.8 \times 0.2 = 525.36$

④ EOQ in years = $\cancel{\dfrac{52.54}{13,800}} \cancel{0.0038}$ years $= \dfrac{2626.8}{13,800} = 0.19$ years

⑤ No of order per year $= \cancel{\dfrac{13,800}{52.54}} = \dfrac{1}{0.19} = 5.253$ /year.

④ ABC manufacturing company purchases 9,000 parts of a m/c for its annual requirement, ordering one month's usage at a time. Each part costs Rs 20. The ordering cost per order is Rs 15, and the carrying charges are 15% of the avg inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What ~~advice~~ advice would you offer and how much would it save the company per year?

Sol

$D = 9000$ parts/year

$Q = \dfrac{9000}{12} = 750$ parts.

$C = Rs.\ 20$ part

$C_o = Rs.\ 15$ /order

$C_c = Rs.\ 20 \times \dfrac{15}{100} = 3$ /part/year.

Total Annual Variable Cost $= \dfrac{Q}{2} C_c + \dfrac{D}{Q} C_o$

$= \left[\dfrac{750}{2} \times 3 + \dfrac{9000}{750} \times 15\right] = Rs\ 1,305/-$

$EOQ\ Q^* = \sqrt{\dfrac{2DC_o}{C_c}} = \sqrt{\dfrac{2 \times 9000 \times 15}{3}} = 300$ units.

Total annual variable cost $= \sqrt{2DC_oC_c} = \sqrt{2 \times 9000 \times 15 \times 3} = Rs\ 900.$

Hence if the company purchases 300 units each time and places 30 orders in the year, the net saving of company will be Rs $(1305 - 900) =$ Rs 405/- a year

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Model-② Purchasing model with shortages:

[Demand rate is uniform, Replinishment rate infinite, shortages Allowed]

In actual practice shortages may take place and hence shortage cost need to be considered. One advantage of allowing shortages is to increase the cycle time, and hence spreading the ordering cost over a long period, thereby reducing the total ordering cost over the planning period. Another advantage is decreased net stock in inventory, resulting in reduced inventory carrying cost.

Let R = demand rate

C_o = ordering cost

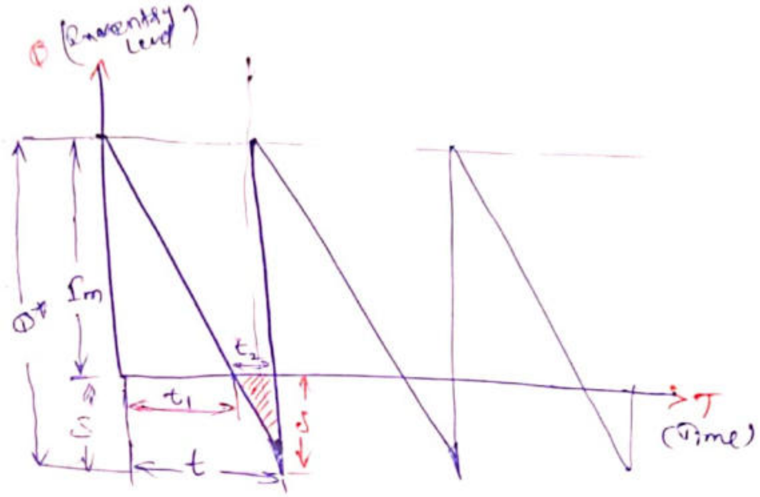
C_c = holding cost

C_s = shortage cost

Q = lot size/order

t = time b/n two orders

I_m = max inventory at beginning.



Formulae:

① Economic lot size $\boxed{EOQ = \sqrt{\dfrac{2RC_o}{C_c}}\sqrt{\dfrac{C_s+C_c}{C_s}}}$

$$\frac{t}{Q} = \frac{t_1}{I_m} = \frac{t_2}{S}$$

② Schedule period $t = \sqrt{\dfrac{2C_o}{RC_c}}\sqrt{\dfrac{C_s+C_c}{C_s}}$

③ Total variable cost (or) Avg cost per unit time $= C$

$$C = \sqrt{2C_oC_cR}\sqrt{\frac{C_s}{C_s+C_c}}$$

Optimum value of I_m, $I_{mo} = \sqrt{\dfrac{2RC_o}{C_c}}\sqrt{\dfrac{C_s}{C_s+C_c}}$

$$S = Q - I_m$$

Problem ①: The demand of an item is uniform at a rate of ~~25 units/month~~ 25 units/month. The fixed cost is ~~Rs 1.5~~ Rs. 15 each time a production run is made. The production cost is Rs 1 /item and inventory holding cost is Rs 0.30 /item/month. If the shortage cost is Rs 1.5 /item/month determine how often to make a production run and of what size it would be?

Sol

$R = 25$ units/month

$C_o =$ Rs. 15 /item/month

$C_c =$ Rs. 0.30 /item/month

$C_s =$ Rs. 1.50 /item/month

$$EOQ = q^* = \sqrt{\frac{2RC_o}{C_c}}\sqrt{\frac{C_s+C_c}{C_s}}$$

$$= \sqrt{\frac{2\times25\times15}{0.3}}\sqrt{\frac{1.5+0.3}{1.5}} = 54.7 \approx 55 \text{ units.}$$

$$t = \frac{Q^*}{R} = \frac{54}{25} = 2.16 \text{ months}$$

Problem ②: The demand for an item is 18000 units/year. The holding cost is Rs 1.20 /unit time and the cost of shortage is Rs 5. The ~~production~~ ordering cost is Rs 400 /-. Assuming that replacement rate is instantaneous, then determine the optimum order quantity?

Sol

$R = 18000$ units/year

$C_c =$ Rs 1.20.

$C_o =$ Rs 400 /-

$C_s =$ Rs 5 /-

$N_o =$ No of order

$N_o = \frac{R}{Q^o} = 4.67$ orders/year

$$EOQ = \sqrt{\frac{2RC_o}{C_c}}\sqrt{\frac{C_s+C_c}{C_s}}$$

$Q^* = 3857$ Units

$$t = \frac{Q^*}{R} = \frac{3857}{18,000} = 0.214 \text{ years}$$

Problem ③: A dealer supplies you the following information with regard to a product dealt-in by him:

Annual demand : 10,000 units

Ordering cost : Rs 10 per order

Inventory holding cost : 20% of value of inventory per year

Price : Rs 20 per unit.

The dealer considering the possibility of allowing some back-order (stock-out) to occur. He has estimated that the annual cost of back-ordering will be 25% of the value of inventory.

ⓐ What should be the optimum number of units of the product he should buy in one lot?

ⓑ What quantity of the product should be allowed to be back-ordered if any?

ⓒ What would be the max. quantity of inventory at any time of the year?

ⓓ Would you recommend to allow back-ordering? If so, what would be the annual cost saving by adopting the policy of back-ordering?

Sol Given $R = 10,000$ units/year

$C_o =$ Rs 10/order

$C_c =$ Rs $20 \times \frac{20}{100} =$ Rs. 4 /unit/year

$C_s =$ Rs $20 \times \frac{25}{100} =$ Rs. 5 /unit/year

ⓐ Optimum lot size $EOQ = \sqrt{\frac{2RC_o}{C_c}} \sqrt{\frac{C_s + C_c}{C_s}}$

$$Q^* = \sqrt{\frac{2 \times 10,000 \times 10}{4}} \sqrt{\frac{5+4}{5}} = 300 \text{ units/order}$$

ⓑ Quantity of back order $S = Q^* - I_m$

$$I_m = \sqrt{\frac{2RC_o}{C_c}} \sqrt{\frac{C_s}{C_s+C_c}}$$

$$= \sqrt{\frac{2 \times 10000 \times 10}{4}} \sqrt{\frac{5}{5+4}} = 167 \text{ units}$$

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back-orders stock $= S = Q^* - I_m$

$= 300 - 167 = 133$ units

(c) $I_m = 167$ units

(d) Annual Cost without shortages (back-orders) $= \sqrt{2RC_oC_c}$

$$C_t = \sqrt{2 \times 10{,}000 \times 10 \times 4} = Rs.\ 894$$

Annual cost allowing back-order $= \sqrt{2RC_oC_c}\sqrt{\frac{C_s}{C_s+C_c}}$

$$C_t = \sqrt{2 \times 10{,}000 \times 10 \times 4}\sqrt{\frac{5}{5+4}} = Rs\ 667.$$

There is a saving of Rs. 227 in annual cost if back-orders is allowed.

50

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Model ③ – "Manufacturing model" with "no shortages":

K

let R = demand rate

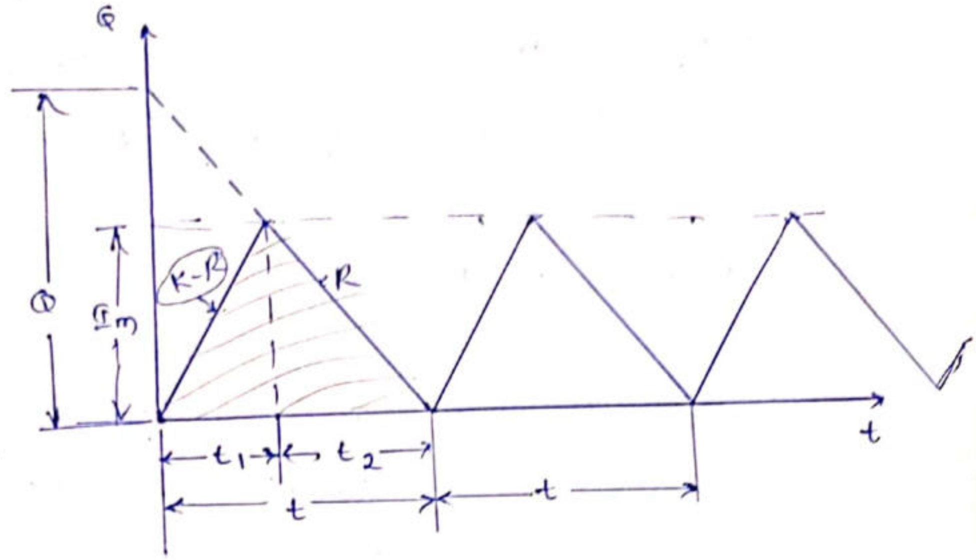
K = Production rate

C_c = holding cost

C_o = ordering cost (or) Cost of a setup a Production run.

Q = no of items produced per run = Rt

t = interval b/n runs.



$$\boxed{t = t_1 + t_2}$$

t_1 – time during which stock building up at constant rate of $K-R$ units per unit time.

t_2 – time during which there is no production (or supply of replinishment) and inventory decreasing at constant demand rate R per unit time.

I_m – max. Inventory available at the end of t_1

Formulae:

Optimum lot size $\boxed{EOQ = \sqrt{\frac{2RC_o}{C_c}}\sqrt{\frac{K}{K-R}}}$

Total minimum Production inventory cost $\boxed{C_t = \sqrt{2RC_oC_c}\sqrt{\frac{K-R}{K}}}$

Optimum length of each lot size production run

$$\therefore \boxed{t = \sqrt{\frac{2C_o}{RC_c}}\sqrt{\frac{K}{K-R}}}$$

$Q = Rt$
$1/t = \frac{R}{Q}$

Optimum no. of Production run/year.

$$\therefore \boxed{n_o = \frac{R}{Q} = \sqrt{\frac{C_cR}{2C_o}}\sqrt{\frac{K-R}{K}}}$$

$$\therefore \boxed{I_{m_o} = \frac{K-R}{K} Q_o} \quad (or) \quad \boxed{I_m = \sqrt{\frac{2RC_o}{C_c}}\sqrt{\frac{K-R}{K}}}$$

Problem ①: A contractor has to supply 10,000 bearings/day to an automobile manufacturer. He finds that when he starts production run, he can produce 25000 bearings/day. The holding cost of a bearing in stock is Rs 0.02 /year. Setup cost of a production is Rs. 18. How frequently should production run be made?

Sol

$R = 10,000$ units/day

$K = 25,000$ Units/day

$C_0 = Rs.\ 18$ /order (or) /setup

$C_c = Rs.\ 0.02$ /year

$= \frac{0.02}{365} = 0.000055$ /unit/day

$$EOQ = \sqrt{\frac{2RC_0}{C_c}}\sqrt{\frac{K}{K-R}}$$

$$= \sqrt{\frac{2\times 10,000\times 18}{0.000055}}\sqrt{\frac{25,000}{25,000-10,000}} = 1,04,447 \text{ units.}$$

$\therefore$ Time b/n orders $= \frac{Q^*}{R} = \frac{104,447}{10,000} = 10.4$ days

$\therefore$ Time of manufactures $= \frac{Q^*}{K} = \frac{104447}{25000} = 4$ days (Approx)

The production cycle starts at an interval of 10.4 days and production continues for 4 days. On each cycle a batch of 104,447 bearings are produced.

Problem ②: An item is produced at the rate of 50 /day. The demand occurs at the rate of 25 items/day. If the setup cost is Rs 100 /run and the holding cost is Rs ~~0.001~~ 0.01 per unit of item per day. Find the economic lot size for one run assuming the shortages are not permitted. Also find the time of the cycle and the min cost for one run.

$R = 25$ items/day

$K = 50$ items/day

$C_0 = $ Rs 100 /run

$C_c = $ Rs 0.01 /unit/day

$$EOQ = \sqrt{\frac{2RC_0}{C_c}} \times \sqrt{\frac{K}{K-R}}$$

$$Q^* = \sqrt{\frac{2\times100\times25}{0.01}} \times \sqrt{\frac{50}{50-25}} = 1000 \text{ units}$$

$$t = \frac{Q^*}{R} = \frac{1000}{25} = 40 \text{ days}$$

$$C = \sqrt{2C_0C_cR}\sqrt{\frac{K-R}{K}}$$

$$= \sqrt{2\times0.01\times100\times25}\sqrt{\frac{50-25}{50}} = 5$$

Min Cost per run $= 5\times 40 = $ Rs 200 $(c\times t)$

Problem (8): A company has a demand of 12000 units/year for an item and it can produce 2,000 such items per month. The cost of one setup is Rs 400 and the holding cost/unit/month is Rs 0.15. Find the optimum lot size and the total cost per year, assuming the cost of 1 unit is Rs. 4. Also find the maximum inventory, manufacturing time and total time.

Sol Given that $R = 12,000$ units/year

$K = 2,000 \times 12 = 24,000$ units/year

$C_0 = $ Rs 400 /setup

$C_c = $ Rs $0.15 \times 12 = $ Rs 1.80 /unit/year

Optimum lot size, $EOQ = \sqrt{\frac{2RC_0}{C_c}}\sqrt{\frac{K}{K-R}}$

$$EOQ = \sqrt{\frac{2\times400\times12000}{1.8}}\sqrt{\frac{24000}{24000-12000}} = 3266 \text{ units}$$

Max. Inventory $I_m = \frac{K-R}{K} \cdot Q^*$

$$= \frac{24000-12000}{24000} \times 3266$$

$I_m = 1632$ Units

Manufacturing time $= \frac{Q^*}{K}$

$$= \frac{3266}{24000} = 0.136 \text{ years}$$

Total time, $t = \frac{Q^*}{R} = \frac{3266}{12,000} = 0.272$ Years.

Total Cost $= C_t + R \times$ Unit Cost

$$C = \sqrt{2RC_oC_c}\sqrt{\frac{K-R}{K}} + R \times \text{Unit cost}$$

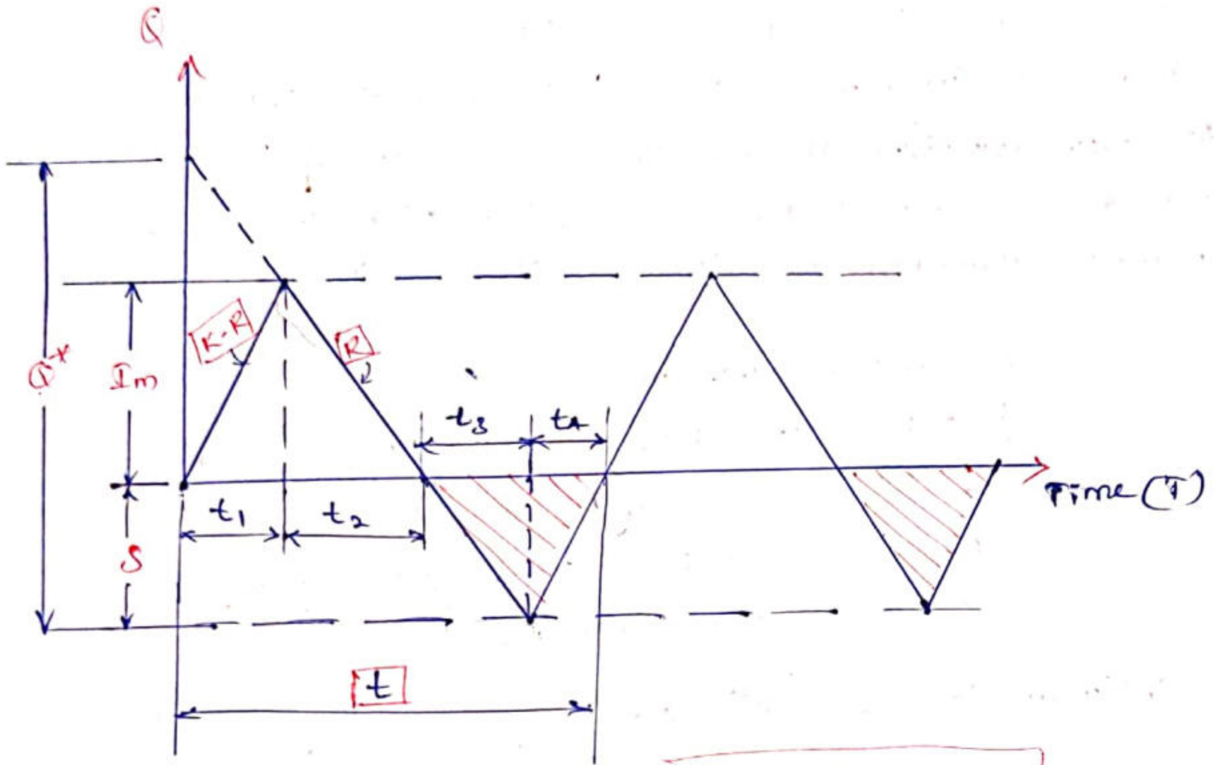
$$C = \sqrt{2 \times 12000 \times 400 \times 1.8}\sqrt{\frac{24,000-12,000}{24,000}} + 12000 \times 4$$

$$= \cancel{2,940} \; 2,940 + 48000$$

$C = 50,940$ /year.

Model - (4): Production model with shortages

Total cost $\Rightarrow C$



$$t = t_1 + t_2 + t_3 + t_4$$

R - Demand rate

K - Production rate

t - time interval to replinish the inventory

Optimum Lot size $\boxed{EOQ = \sqrt{\frac{2RC_0}{C_c}}\sqrt{\frac{C_s+C_c}{C_s}}\sqrt{\frac{K}{K-R}}}$

Time intervals b/n Production Runs $\boxed{t = \sqrt{\frac{2C_0}{RC_c}}\sqrt{\frac{K}{K-R}}\sqrt{\frac{C_s+C_c}{C_s}}}$

Annual variable cost $\boxed{C_t = \sqrt{2RC_0C_c}\sqrt{\frac{K-R}{K}}\sqrt{\frac{C_s}{C_s+C_c}}}$

Optimum order quantity (or) max Inventory $= I_m$

(or) (or)

$$\boxed{I_m = \sqrt{\frac{2RC_0}{C_c}}\sqrt{\frac{C_s}{C_s+C_c}}\sqrt{\frac{K-R}{K}}}$$

Shortage stock ~~S~~ = ~~$Q^* \frac{K-R}{K} \frac{C_c}{C_c+C_s}$~~ X

$$\boxed{S = Q^* \frac{K}{K-R} \frac{C_c}{C_c+C_s}}$$

Problem ① : [Same data of previous problem ② of model ③]

A company has a demand of 12000 units/year for an item and it can produce 2000 such items per month. The cost of one setup is Rs 400, and the holding cost/unit/month is Rs 0.15. Find the optimum lot size and Total cost per year. ~~also other results~~. Assume the cost of 1 unit is Rs. 4 and shortage cost of one unit is Rs. 20 per year. Also find the maximum inventory, manufacturing time and total time?

Sol

$R = 12{,}000$ units/year

$K = 2000 \times 12 = 24{,}000$ units/year

$C_0 = $ Rs. 400 /setup

$C_c = $ Rs $0.15 \times 12 = $ Rs 1.8 per unit/year

$C_s = $ Rs 20 per unit/year

Optimum lot size $Q^* = EOQ = \sqrt{\frac{2RC_0}{C_c}} \sqrt{\frac{K}{K-R}} \sqrt{\frac{C_s+C_c}{C_s}}$

$$= \sqrt{\frac{2 \times 12000 \times 400}{1.8}} \sqrt{\frac{24000}{24000-12000}} \sqrt{\frac{20+1.8}{20}}$$

$Q^* = 3{,}410$ units

Total cost $C = \sqrt{2RC_0C_c} \sqrt{\frac{K-R}{K}} \sqrt{\frac{C_s}{C_s+C_c}} + R \times \text{Unit Cost}$

$$= \text{Rs.} \sqrt{2 \times 12000 \times 400 \times 1.8} \sqrt{\frac{24000-12000}{24000}} \sqrt{\frac{20}{20+1.8}} + 12000 \times 4$$

$C = $ Rs. $2{,}185 + 48000 = $ Rs $50{,}185$ /year.

Max. inventory $I_m = \sqrt{\frac{2RC_0}{C_c}} \sqrt{\frac{C_s+C_c}{C_s}} \sqrt{\frac{K-R}{K}}$

$$= 3410 \times \frac{20}{20+1.8} \times \frac{24000-12000}{24000} = 1564.2 \text{ units}$$

$$= \sqrt{\frac{2 \times 12000 \times 400}{1.8}} \sqrt{\frac{20+1.8}{20}}$$

Maximum Inventory $Q_m = \sqrt{\frac{2RC_0}{C_c}} \sqrt{\frac{C_s}{C_s+C_c}} \sqrt{\frac{K-R}{K}}$

$= \sqrt{\frac{2 \times 12000 \times 400}{1.80}} \sqrt{\frac{20}{20+1.8}} \sqrt{\frac{24000-12000}{24000}}$

$= 1.564$ (or) $1,564.$ units /run

$2309.40 \times 0.958 \times 0.7071$

Manufacturing time interval $= t_1 + t_4$

$= \frac{Q^*}{K} = \frac{3410}{24,000} = 0.1421$ years (or) 1.705 months

Total time interval $t = t_1 + t_2 + t_3 + t_4$

$= \frac{Q^*}{R} = \frac{3410}{12000} = 0.2842$ years (or) 3.4 months

Problem ②: The demand for an item in a company is 48000 units/yr and the company can produce the item at a rate of 5000 units/month. The cost of one setup is Rs. 600. and the holding cost /unit/month is Rs 15 paise. The shortage cost of 1 units is Rs 15 /month. Determine the optimum manufacturing quantity and the no of shortages. Also determine the manufacturing time and the time b/n set ups.

Sol

$R = 48,000$ units / year $= \frac{48000}{12} = 4000$ units /month

$K = 5000$ units /month

$C_0 =$ Rs. 600 /setup

$C_c =$ Rs. 0.15 /unit/ month

$C_s =$ Rs 15 /unit /month

Optimum mfg Quantity $Q^* = EOQ = \sqrt{\frac{2RC_0}{C_c}} \sqrt{\frac{K}{K-R}} \sqrt{\frac{C_s+C_c}{C_s}}$

$= \sqrt{\frac{2 \times 4000 \times 600}{0.15}} \sqrt{\frac{5000}{5000-4000}} \sqrt{\frac{15+0.15}{15}}$

$Q^* = 12,712.2$ units.

$\simeq 12712.$ units

Shortage stock $s = Q^{*} - I_m$

$$I_m = \sqrt{\frac{2RC_o}{C_c}} \sqrt{\frac{C_s}{C_s+C_c}} \sqrt{\frac{K-R}{K}}$$

$$= \sqrt{\frac{2\times 4000\times 600}{0.15}} \sqrt{\frac{15}{15+0.15}} \sqrt{\frac{5000-4000}{5000}}$$

$$= 5656.85 \times 0.9949 \times 0.4472$$

$$= 2516.8 \text{ units.}$$

$$S = 12712 - 2516 = 10196 \text{ units}$$

Manufacturing time $= \frac{Q^{*}}{K}$

$$= \frac{12712}{5000} = 2.542 \text{ years}$$

Total time b/n setups $= \frac{Q^{*}}{R}$

$$= \frac{12712}{4000} = 3.18 \text{ years.}$$

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Inventory models with price breaks:

In the previous models, it was assumed that the unit cost is fixed. But coming to actual practices, the unit cost may vary with no. of items purchased. This variation in the cost of item/unit are called as price breaks.

Ex:

Upto 100 → 10/- — $0 \leq Q \leq 100$

101 to 500 → 9/- — $101 \leq q \leq 500$

more than 500 → 8.5/- — $Q > 500$

One Price break Inventory models:

(1) An automobile manufacturer purchases 2,400 castings over a period of 360 days. This requirement is fixed and known. These castings are subject to quantity discounts. Ordering cost is Rs. 70,000 /order and storage cost per day is 0.12 % of the unit cost. Determine the optimal purchase quantity if the supplier has offered the following unit prices for the castings.

Sol

Unit Price: Rs 1000 — $Q < 1000$

Rs 950 — $Q \geq 1000$.

Sol

EOQ for the unit price of casting is Rs 950 $= \sqrt{\frac{2RC_0}{C_c}}$

$$= \sqrt{\frac{2 \times \frac{2400}{360} \times 70{,}000}{\frac{0.12}{100} \times 950}} = 905 \text{ units}$$

∴ The manufacturer should place an order for 905 units. This is not acceptable solution, because for the purchase of ≥ 1000 products the discount is offered.

To avail the unit cost of Rs. 950, the manufacturers must place an order of atleast 1000 units.

Total cost /day for order quantity of 1000 units.

$$= \frac{\text{Avg. Cost}}{\text{Unit time}} + RC_i$$

$$= \left(\frac{1}{2} C_c R t + \frac{C_0}{t}\right) + R C_i$$

$$= \left(\frac{1}{2} C_c Q + C_0 \times \frac{R}{Q}\right) + R C_i$$

$Q = Rt$
$R = \frac{Q}{t}$
$t = \frac{Q}{R}$

$$= \left(\frac{1}{2}\, 0.12 \times 950 \times 1000 + \frac{70{,}000 \times \frac{2400}{360}}{1000}\right) + \frac{2400}{360} \times 1000$$

$C_{100} =$ Rs $7{,}370.00$

EOQ of the unit price of castings is Rs 1000.

$$EOQ_{1000} = \sqrt{\frac{2RC_0}{C_c}} = \sqrt{\frac{2 \times \frac{2400}{360} \times 70000}{0.12 \times 1000}} = 882 \text{ units.}$$

Total cost /day for ordering quantity of 882 units.

$$C_{882} = \sqrt{2RC_0C_c} + RC_i = \left(\frac{1}{2} C_c Q + C_0 \frac{R}{Q}\right) + RC_i$$

$$= \cancel{\sqrt{2 \times \frac{2400}{360} \times 70{,}000 \times 0.12 \times 1000}} + \left[\frac{2400}{360} \times 1000 + \frac{882}{2} \times 0.12 \times 1000 + 70000 \times \frac{2400}{360} \times \frac{1}{882}\right]$$

$=$ Rs 7725/-

The total cost curve is as follows.

Multiple price breaks :

① Find the optimal ordering quantity for a product for which the price breaks are as follows.

Quantity	Unit Cost (Rs)
$0 < Q < 500$	10
$500 \leq Q < 750$	9.25
$750 \leq Q$	8.75

The monthly demand for the product is 200 units, storage cost is 2 % of unit cost & the ordering cost is Rs 100 per order.

Sol

EOQ for unit price 8.75/ $= \sqrt{\frac{2DC_0}{C_c}} = \sqrt{\frac{2\times200\times100}{\frac{2}{100}\times8.75}} = 478$ units (Q) (Infeasible)

EOQ for unit price 9.25/- $= \sqrt{\frac{2\times200\times100}{\frac{2}{100}\times9.25}} = 465$ Units (Q) (Infeasible)

*** EOQ for unit price 10/ $= \sqrt{\frac{2\times200\times100}{\frac{2}{100}\times10}} = 447$ units (Q^*) (feasible)

Total cost per month for ordering quantity of 447 units (Q^*).

$$= \sqrt{2DC_0C_c} + DC$$

$$= \sqrt{2\times200\times100\times\frac{2}{100}\times10} + (200\times10) = \underline{2089.45}\ /-$$

Total cost per month for ordering quantity of 500 units (Q)

$$= \frac{Q}{2}C_c + \frac{D}{Q}C_0 + DC$$

$$= \frac{500}{2}\times\frac{2}{100}\times9.25 + \frac{200}{500}\times100 + 200\times9.25$$

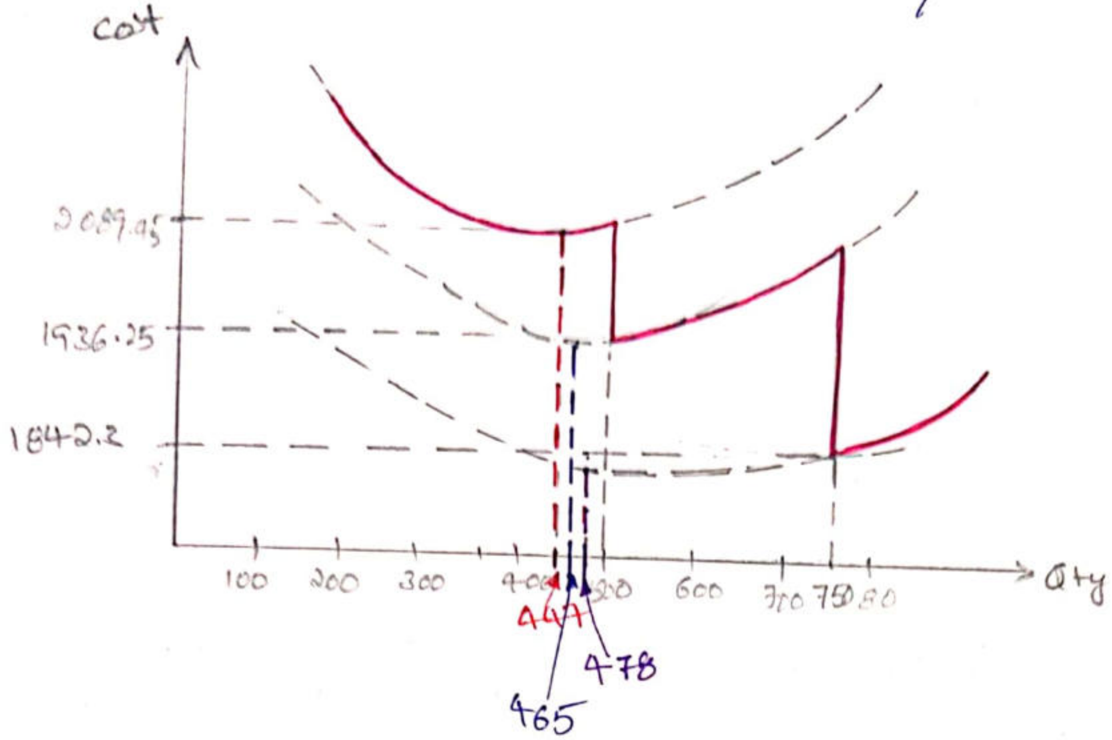
$$= \underline{1936.25}\ /-$$

Total cost per month for ordering Quantity of 750 units (Q)

$$= \frac{Q}{2}C_c + \frac{D}{Q}C_0 + DC$$

$$= \frac{750}{2}\times\frac{2}{100}\times8.75 + \frac{200}{750}\times100 + 200\times8.75 = \underline{1842.30}/-$$

∴ The optimal ordering Quantity is 750 units/order.



② The demand for a Product is 2400 units over 360 days. The storage cost is 0.06 % of the unit cost of the Product & Ordering cost is Rs 35,000. Find the optimal ordering quantity if the price breaks are as follows.

Quantity Range	Product Cost (Rs)
$0 < Q < 1000$	1,000
$1000 \le Q < 4000$	925
$4000 \le Q$	850

Sol!

$$EOQ_{C=850} = \sqrt{\frac{2DC_0}{C_c}} = \sqrt{\frac{2 \times \frac{2400}{360} \times 35000}{\frac{0.06}{100} \times 850}} = 956.57 \text{ units}$$
(Not feasible)

$$EOQ_{C=925} = \sqrt{\frac{2 \times \frac{2400}{360} \times 35000}{\frac{0.06}{100} \times 925}} = 916.97 \text{ units (not feasible)}$$

$$EOQ_{C=1,000} = \sqrt{\frac{2 \times \frac{2400}{360} \times 35000}{\frac{0.06}{100} \times 1000}} = 881.91 \text{ units (feasible)}$$

$$\text{Total Cost}_{Q=4000} = \frac{D}{Q} C_o + \frac{Q}{2} C_c + DC$$

$$= \frac{\frac{2400}{360}}{4000} 35000 + \frac{4000}{2} \times \frac{0.06}{100} \times 850 + \frac{2400}{360} \times 850$$

$$= Rs\ 6745/-$$

$$\text{Total Cost}_{Q=1000} = \frac{D}{Q} C_o + \frac{Q}{2} C_c + DC$$

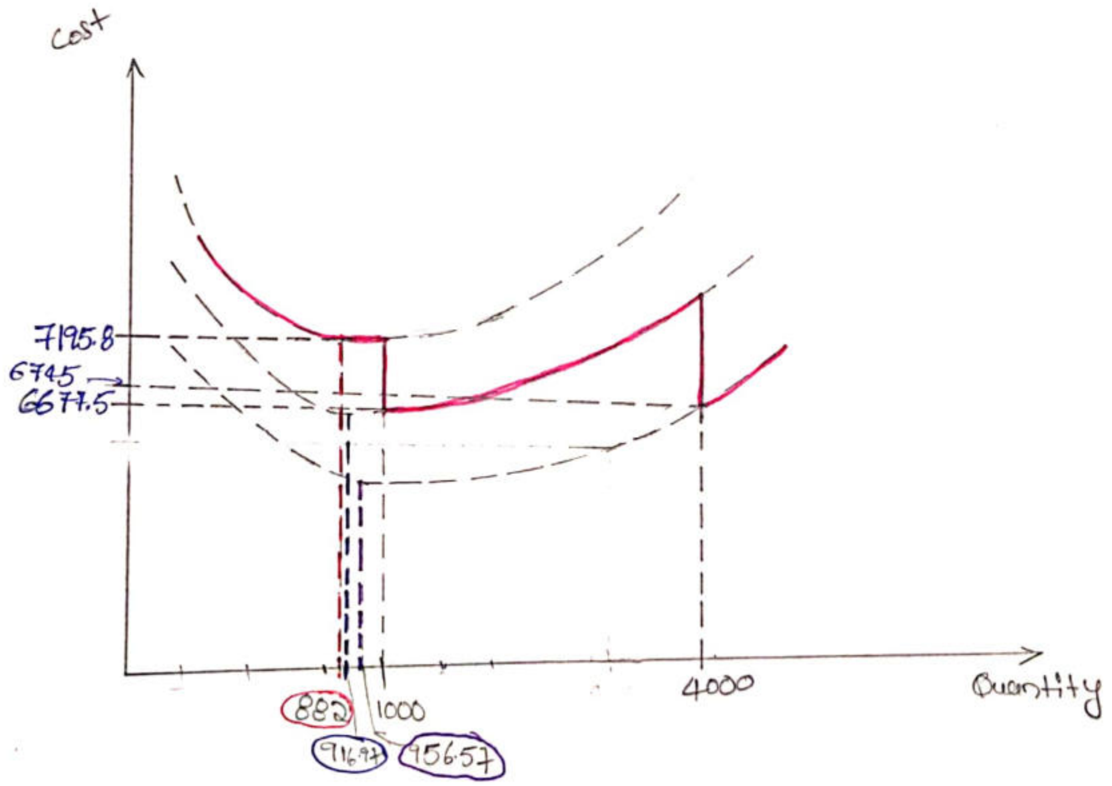
$$= \frac{\frac{2400}{360} \times 35000}{1000} + \frac{1000}{2} \times \frac{0.06}{100} \times 925 + \frac{2400}{360} \times 925$$

$$= Rs\ 6677.5/-$$

$$\text{Total Cost}_{Q^*=882} = \sqrt{2DC_oC_c} + DC$$

$$= \sqrt{2 \times \frac{2400}{360} \times 35000 \times \frac{0.06}{100} \times 1000} + \frac{2400}{360} \times 1000$$

$$= Rs\ 7195.8/-$$



(8) Find the optimal order quantity for a product when the annual demand for the product is 1500 units, the storage cost per unit per year is 10% of the unit cost and ordering cost per order is Rs 180. The unit costs are given below.

Quantity	Unit Cost (Rs)
$0 \leq Q_1 \leq 500$	25.00
$500 \leq Q_2 < 1500$	24.80
$1500 \leq Q_3 < 3000$	24.60
$3000 \leq Q_4$	24.40

Sol:

$D = 1500$ units/year

$C_c = 10\%$ of unit cost

$C_o = $ Rs $180/-$

$$EOQ_{c=24.4} = \sqrt{\frac{2 \times 1500 \times 180}{\frac{10}{100} \times 24.4}} = 271.6 \text{ units}$$

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Unit - V

DYNAMIC PROGRAMMING

Dynamic Programming is a mathematical technique dealing with the optimization of multi-stage decision process, where the decisions are taken at distinct stages.

"Richard Bellman" developed this technique in early 1950 and coined the term "Dynamic programming." This is also represented with some other terms called stochastic programming & recursive optimization. This technique converts large problems of n variables into n-sub problems (stages) each in one variable. Max.(or) Min. $[r(d)+F[T(s,d)]]$

"Bellmans's Principle of Optimality"

Problem ①:

A firm has divided its marketing area into three zones. The amount of sales depends upon the no. of salesmen in each zone. The firm has been collecting the data regarding sales and salesmen in each area over a no. of years. The information is summarised in the following table, for the next year firm has only 9 salesmen and the problem is to allocate these salesmen to three different zones so that the total sales are maximum.

No. of Salesmen	Zone 1	Zone 2	Zone 3
0	30	35	42
1	45	45	54
2	60	52	60
3	70	64	70
4	79	72	82
5	90	82	95
6	98	93	102
7	105	98	110
8	100	100	110
9	90	100	110

sol

Three zones represents three stages and the no. of salesmen represent the state variables.

stage ①:

No. of salesmen :	0	1	2	3	4	5	6	7	8	9
Profit (R_1) :	30	45	60	70	79	90	98	105	100	90

stage ②) Consider the first two stages is, zone ① & zone ②. Nine salesmen can be devided into among two zone in 10 different ways.

Zone 1 x_1 →		0	1	2	3	4	5	6	7	8	9
Zone 2 x_2 ↓	$f(x_2)$ ↓ / $f(x_1)$ →	30	45	60	70	79	90	98	105	100	90
0	35	65*	80*	95*	105*	114	125*	133	140	135	125
1	45	75	90	105*	115*	124	135*	143*	150	145	
2	52	82	97	112	122	131	142	150	157		
3	64	94	109	124	134	143*	154*	162			
4	72	102	117	132	142	151	162				
5	82	112	127	142	152	161					
6	93	123	138	153	163*						
7	98	128	143	158							
8	100	130	145								
9	100	130									

Stage ③: Now consider the distribution of 9 salesmen in three zones ②, ① and ③.

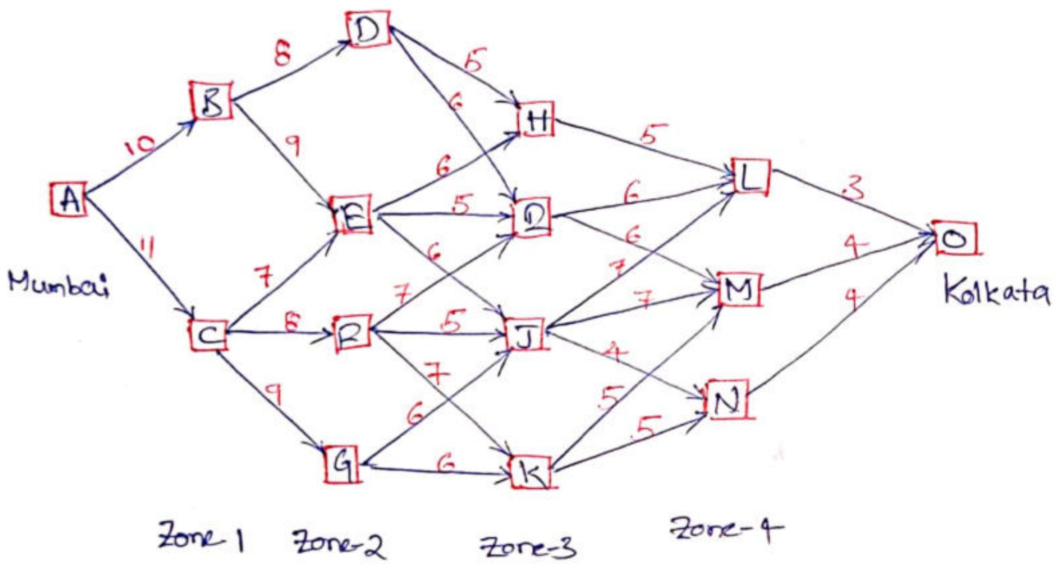
No. of salesmen :	0	1	2	3	4	5	6	7	8	9
Salesmen in zone ② & zone ① :	0+0	0+1	0+2	0+3 1+2	1+3	0+5	1+5	1+6 3+4	3+5	6+3
Total Profit $f_2(x_2)+f_1(x_1)$:	65	80	95	105	115	125	135	143	154	163
No. of salesmen in zone ③ :	9	8	7	6	5	4	3	2	1	0
Profit $f_3(x_3)$:	110	110	110	102	95	82	70	60	54	42
Total Profit $f_3(x_3)+f_2(x_2)+f_1(x_1)$:	175	190	205	207	$\boxed{210^*}$	207	205	203	208	205

∴ The maximum profit for 9 salesmen is Rs. 210 if 5 salesmen are allotted to zone ③, and one salesmen alloted to zone ② and 3 salesmen to zone ①.

Shortest path problem: (Stage - coach Problem)

The objective of this problem is to find the shortest distance and the corresponding path from a given source node to a given destination node in a given distance network.

Problem ①: A salesman is planning a business tour from Mumbai to Kolkata in the course of which he proposes to cover one city from each of the companys different marketing zones enroute. As he has limited time at his disposal, he has to compute his tour in the shortest possible time. The network is shown in the figure, and it shows the no. of days' time involved for covering any of the various intermediate cities (time includes travel as well as walking time). Determine optimum tour plan.



Sol Starting from A (Mumbai), the cities of various marketing zones may considered as distinct stages.

Stage-1 : B (or) C

Stage-2 : D, E, F & G

Stage-3 ; H, I, J, & K

Stage-4 ; L, M, or N

Stage-5 : Best route to 'O'.

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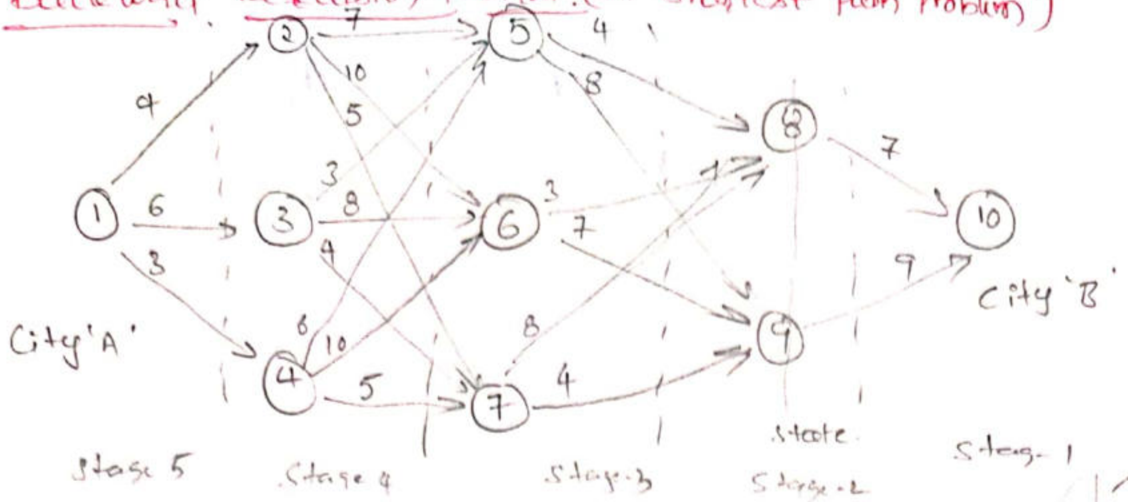
Backward Recursion Method: (In Shortest Path Problem)



N=1 One more stage to go

$f_1(s, x_1) = D_{sx_1}$

$f_1^*(s) = \text{Minimize } f_1(s, x_1)$

State s	Decision x_1	Distance function $f_1(s)$	x_1^*
8	10	7	10
9	10	9	10

Best Decision

x_1^* - Corresponds to x_1

N=2: Two more stages to go

$f_2(s, x_2) = D_{sx_2} + f_1^*(x_2)$

$f_2^*(s) = \text{Min } f_2(s, x_2)$

x_2^* = Corresponding to x_2

s	x_2: 8	9	$f_2^*(s)$ Min	x_2^*
5	4+7=11	8+9=17	11	8
6	3+7=10	7+9=16	10	8
7	8+7=15	4+9=13	13	9

N=3: Three more stages to go.

$f_3(s, x_3) = D_{s,x_3} + f_2^*(x_3)$

$f_3^*(s) = \text{Min } f_3(s, x_3)$

x_3^* = corr. x_3

s	x_3: 5	6	7	$f_3^*(s)$	x_3^*
2	7+11=18	10+10=20	5+13=18	18	5, 7
3	3+11=14	8+10=18	4+13=17	14	5
4	6+11=17	10+10=20	5+13=18	17	5

$n = 4$: Four stages to go

$$f_4(s, x_4) = C_{s,x_4} + f_3^*(x_4)$$

s	x_4 = 2	3	4	$f_4^*(s)$	x_4^*
1	4+18 = 22	6+14 = 20	3+17 = 20	20	3, 4

Path :

1 − 3 − 5 − 8 − 10 = "20"

Non-Linear Programming Problems:

① Determine the values of u_1, u_2, u_3 so as to maximize $(u_1 u_2 u_3)$ subject to $u_1 + u_2 + u_3 = 10$; and $u_1, u_2, u_3 \geq 0$.

Sol: At stage ① assume $x_1 = u_1 \quad = x_2 - u_2$

stage ② " $x_2 = u_2 + u_1 \quad = x_3 - u_3$

stage ③ " $x_3 = u_3 + u_2 + u_1$

$f_1(x_1) = u_1 = x_2 - u_2$

$f_2(x_2) = u_1 \cdot u_2 = (x_2 - u_2) \cdot u_2$

$= x_2 u_2 - u_2^2$

Differentiating w.r.t 'u_2' & equating to 'zero'.

$$\frac{\partial f_2(x_2)}{\partial u_2} = 0$$

$$\frac{\partial}{\partial u_2}(x_2 u_2 - u_2^2) = 0$$

$$x_2 - 2u_2 = 0$$

$$u_2 = \frac{x_2}{2}$$

$f_2(x_2) = u_1 \cdot u_2 = (x_2 - u_2) \cdot \frac{x_2}{2}$

$f_2(x_2) = (x_2 - \frac{x_2}{2}) \cdot \frac{x_2}{2} = \frac{x_2^2}{4}$

$\therefore f_3(x_3) = u_1 \cdot u_2 \cdot u_3$

$= \frac{x_2^2}{4} \cdot u_3 = \frac{(x_3 - u_3)^2 \cdot u_3}{4}$

Differentiating w.r.to 'u_3' & equating to zero

$$\frac{\partial f_3(x_3)}{\partial u_3} = 0$$

$$\frac{\partial}{\partial u_3}\left(\frac{u_3 (x_3 - u_3)^2}{4}\right) = 0$$

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$$\frac{\partial}{\partial u_3}\left(\frac{u_3}{4}\cdot(x_3^2+u_3^2-2x_3u_3)\right)=0$$

$$\frac{\partial L}{\partial u_3}\left(\frac{1}{4}(x_3^2u_3+u_3^3-2x_3u_3^2)\right)=0$$

$$\frac{\partial}{\partial u_3}(x_3^2u_3+u_3^3-2x_3u_3^2)=0$$

$$x_3^2+3u_3^2-4x_3u_3=0$$

$$x_3^2-x_3u_3-3x_3u_3+3u_3^2=0$$

$$x_3(x_3-u_3)-3u_3(x_3-u_3)=0$$

$$(x_3-u_3)(x_3-3u_3)=0$$

$x_3=u_3$

not feasible

$x_3=3u_3$

$u_3=x_3/3$

$x_3=u_1+u_2+u_3=10$

$u_3=10/3$

$$x_2=x_3-u_3=10-\frac{10}{3}=20/3$$

$$u_2=\frac{x_2}{2}=\frac{20}{3}\times\frac{1}{2}=\frac{10}{3}$$

$$u_1=x_2-u_2=\frac{20}{3}-\frac{10}{3}=10/3$$

Max. $Z=u_1\times u_2\times u_3$

$$=\frac{10}{3}\times\frac{10}{3}\times\frac{10}{3}$$

$$Z^*=\frac{1000}{27}$$

(2) Minimize $Z = y_1^2 + y_2^2 + y_3^2$

subject to $y_1 + y_2 + y_3 \geq 15$,

$y_1, y_2, y_3 \geq 0$.

Sol: Let the state variable s_1, s_2, s_3 can be defined as.

at stage ① $s_1 = y_1 = s_2 - y_2$

stage ② $s_2 = y_2 + y_1 = s_3 - y_3$

stage ③ $s_3 = y_3 + y_2 + y_1 \geq 15$

Return function:

$f_1(s_1) = y_1^2$

$f_2(s_2) = y_2^2 + y_1^2$

$f_3(s_3) = y_3^2 + y_2^2 + y_1^2$

Stage ②:

$$f_2(s_2) = y_2^2 + y_1^2 = y_2^2 + (s_2 - y_2)^2$$

$$= 2y_2^2 + s_2^2 - 2s_2 y_2$$

Differentiate $f_2(s_2)$ w.r.t. 'y_2' & square to 'zero' to get min. y_2

$$\frac{\partial f_2(s_2)}{\partial y_2} = 4y_2 - 2s_2 = 0$$

$$y_2 = \frac{s_2}{2} \text{ (or) } s_2 = 2y_2$$

Differentiate $f_2(s_2)$ further w.r.t. 'y_2' to get 2nd order derivative

$$\frac{\partial^2 f_2(s_2)}{\partial y_2^2} = 4 \quad (+ve) \therefore f_2(s_2) \text{ first order derivative is minimum.}$$

$$\therefore f_2(s_2) = \cancel{\tfrac{s_2^2}{4}} \left(\frac{s_2}{2}\right)^2 + \left(s_2 - \frac{s_2}{2}\right)^2 = 2\left(\frac{s_2}{2}\right)^2$$

$$f_2(s_2) = \frac{s_2^2}{2}$$

Stage ③:

$$f_3(s_3) = y_3^2 + \underline{y_2^2 + y_1^2}$$

$$= y_3^2 + f_2(s_2)$$

$$= y_3^2 + \frac{s_2^2}{2} = y_3^2 + \frac{(s_3 - y_3)^2}{2}$$

$$f_3(s_3) = \frac{3}{2}y_3^2 + \frac{s_3^2}{2} - s_3 y_3$$

Differentiate $f_3(s_3)$ w.r.t 'y_3' & square to '0'

$$\frac{\partial f_3(s_3)}{\partial y_3} = 3y_3 - s_3 = 0$$

$$y_3 = \frac{s_3}{3}$$

(I.W.K.T $s_3 = y_3 + y_2 + y_1 \geq 15$, say $s_3 = 15$)

$$y_3 = \frac{15}{3} = 5$$

Substitute in $s_2 = s_3 - y_3$

$$= 15 - 5 = 10$$

$$y_2 = \frac{s_2}{2} = \frac{10}{2} = 5$$

$$s_1 = s_2 - y_2$$

$$s_1 = 10 - 5 = 5$$

$$y_1 = 5 \quad \& \quad y_2 = 5, \quad y_3 = 5$$

Min. of $Z = 5^2 + 5^2 + 5^2$

$\therefore$ Minimum value of Z is '75'.

(3) Determine the values of y_1, y_2, y_3 to the function.

Min. $Z = y_1^2 + y_2^2 + y_3^2$

Subject to $y_1 + y_2 + y_3 = 10$

$y_1, y_2, y_3 \geq 0$.

Sol:

Let $s_1 = y_1 \quad = s_2 - y_2$

$s_2 = y_2 + y_1 \quad = s_3 - y_3$

$s_3 = y_3 + y_2 + y_1 = 10$

functions:

$f_1(s_1) = \text{Min. } y_1^2 \rightarrow$ Stage (1)

$f_2(s_2) = \text{Min}(y_2^2 + y_1^2)$

$f_3(s_3) = \text{Min}(y_3^2 + y_2^2 + y_1^2)$

Stage (2): $f_2(s_2) = y_2^2 + y_1^2$

$$= y_2^2 + (s_2 - y_2)^2$$

$$= 2y_2^2 - 2s_2 y_2 + s_2^2$$

Differentiate $f_2(s_2)$ w.r.t y_2 & equate to zero. to get minimum of $f_2(s_2)$.

$$\frac{\partial f_2(s_2)}{\partial y_2} = 4y_2 - 2s_2 = 0$$

$$y_2 = \frac{s_2}{2}$$

Again differentiate $f_2(s_2)$ w.r.t. y_2 to confirm the minimum value.

$$\frac{\partial^2 f_2(s_2)}{\partial y_2^2} = 4$$

(If it is +ve the function value of first order derivative is minimum.)

$$\therefore f_2(s_2) = y_2^2 + (s_2 - y_2)^2$$

$$= \frac{s_2^2}{4} + \left(s_2 - \frac{s_2}{2}\right)^2 = \frac{s_2^2}{2}$$

$$\text{Min}\therefore f_2^*(s_2) = \frac{s_2^2}{2}$$

Stage (3):

$$f_3(s_3) = y_3^2 + y_2^2 + y_1^2$$

$$= y_3^2 + f_2^*(s_2)$$

$$= y_3^2 + \frac{s_2^2}{2}$$

$$f_3(S_3) = y_3^2 + \frac{(s_3 - y_3)^2}{2} \qquad \text{w.k.t} \quad s_2 = s_3 - y_3$$

$$= \frac{3}{2} y_3^2 + \frac{s_3^2}{2} - s_3 y_3$$

Differentiating $f_3(s_3)$ gives optimum value of y_3 when ~~it~~ equal to zero.

$$\frac{\partial f_3(s_3)}{\partial y_3} = 3y_3 - s_3 = 0$$

$$y_3 = \frac{s_3}{3} = \frac{10}{3}$$

2nd differentiation of $f_3(s_3)$ confirms the minimum value

$$\frac{\partial^2 f_3(s_3)}{\partial y_3^2} = 3 \qquad \text{(minimum)}$$

$\therefore$ ~~$f_3(s_3)$~~ $s_2 = s_3 - y_3$

$$= 10 - \frac{10}{3} = \frac{20}{3}$$

$$y_2 = \frac{s_2}{2} = \frac{20}{3} \times \frac{1}{2} = \frac{10}{3}$$

$$y_1 = s_2 - y_2 = \frac{20}{3} - \frac{10}{3} = \frac{10}{3}$$

$$\therefore y_1 = y_2 = y_3 = \frac{10}{3}$$

$$z = \left(\frac{10}{3}\right)^2 + \left(\frac{10}{3}\right)^2 + \left(\frac{10}{3}\right)^2$$

$$\therefore \text{Min. } z^* = \frac{100}{3} = 33.33\ldots$$

Decision variables are Non-Negative Integers :

(1) Min. $Z = y_1^2 + y_2^2 + y_3^2$

s.t. $y_1 + y_2 + y_3 = 10$

y_1, y_2, y_3 are non-negative integers.

Sol: The problem can be splitted & solved in three stages.

Stage ①: $f_1(x_1) = \underset{0 \le y_1 \le 10}{\text{Min.}} (y_1^2) = y_1^2$

Stage ②: $f_2(x_2) = \underset{0 \le y_2 \le 10}{\text{Min.}} [y_2^2 + f_1^*(x_1)]$

y_2	$\rightarrow y_1$ / $\downarrow y_2^2$	0	1	2	3	4	5	6	7	8	9	10
	y_1^2	0	1	4	9	16	25	36	49	64	81	100
0	0	0*	1*	4	9	16	25	36*	49	64	81	100
1	1	1*	2*	5*	10	17	26	37	50	65	82	
2	4	4	5*	8*	13*	20	29	40	53	68		
3	9	9	10	13*	18*	25*	34	45	58			
4	16	16	17	20	25*	32*	41*	52				
5	25	25	26	29	34	41*	50*					
6	36	36*	37	40	45	52						
7	49	49	50	53	58							
8	64	64	65	68								
9	81	81	82									
10	100	100										

Stage-3: $f_3(x_3) = \min \{ y_3^2 + f_2(x_2) \}$

$$= \min [y_3^2 + f_2(x_3 - y_3)^2]$$

y_3	0	1	2	3	4	5	6	7	8	9	10
y_3^2	0	1	4	9	16	25	36	49	64	81	100
$(x_3 - y_3)$	10	9	8	7	6	5	4	3	2	1	0
$f_2(x_3 - y_3^2)$	50	41	32	$\boxed{25}$	18	13	10	5	2	1	0
$f_2(x_3)$	50	42	36	$\boxed{34^*}$	34^*	38	46	54	66	82	100

$\therefore f_3(x_3) = 34$

$\therefore y_3 = 3$ (or) 4

for $y_3 = 3$; $f_2(x_2) = 25$ for which $y_1 = 3$ (or) 4 &
$y_2 = 4$ (or) 3.

for $y_3 = 4$; $f_2(x_2) = 18$ for which $y_1 = y_2 = 3$.

$\therefore$ The minimum value of 34 corresponds to (y_1, y_2, y_3) at $(3,4,3)$, $(4,3,3)$ & $(3,3,4)$.

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② Max. $Z = u_1^2 + u_2^2 + u_3^2$

subject to $u_1 \cdot u_2 \cdot u_3 = 6$

where u_1, u_2, u_3 are +ve integers.

Sol: let $x_1 = u_1 = \frac{x_2}{u_2}$

$x_2 = u_1 . u_2 = \frac{x_3}{u_3} = \frac{6}{u_3}$

$x_3 = u_3 . u_2 . u_1 = 6$

Stage ①: $x_1 = u_1$ and u_1 can vary from 1 to 6

$$f_1(x_1) = \underset{0 \le u_1 \le 6}{\text{Max.}} (u_1^2)$$

u_1	1	2	3	4	5	6
$f_1(x_1)$	1	4	9	16	25	36

Stage ②: $x_2 = u_2 . u_1$

=

$$f_2(x_2) = \underset{0 \le u_2 \le 6}{\text{Max.}} \left[u_2^2 + u_1^2\right] = \underset{0 \le u_2 \le 6}{\text{Max.}} \left[u_2^2 + f_1^*(x_1)\right]$$

u_2	u_2^2	u_1: 1 / $f_1(x_1)$: 1	u_1: 2 / $f_1(x_1)$: 4	u_1: 3 / $f_1(x_1)$: 9	u_1: 6 / $f_1(x_1)$: 36
1	1	2*	5*	10*	37*
2	4	5*	8	13	
3	9	10*	13		
6	36	37*			

Linear Programming Problems : (only standard - form problems)

① Max. $Z = 2x_1 + 5x_2$

Subject to $2x_1 + x_2 \leq 430$

$2x_2 \leq 460$

$x_1, x_2 \geq 0$

sol

As no. of variables are '2' no of stages are also '2'.

Stage ①: Max. $[2x_1]$

Constraints: ① → $2x_1 + x_2 \leq 430$ ② → $0x_1 + 2x_2 \leq 460$

$x_1 \leq \frac{430 - x_2}{2}$ $x_1 \leq \frac{460 - 2x_2}{0}$

$x_1 \leq \infty$.

$\therefore$ Max. $(x_1) = \text{Min.} \left[\frac{430 - x_2}{2}, \infty\right]$

Lower limit for $x_1 = 0$

Upper limit for $x_1 = b_1$

if $x_2 = 0$; $x_1 = \frac{430 - x_2}{2} = \frac{430 - 0}{2} = 215$

$\text{Min}(215, \infty) = 215$.

$\therefore$ Limits of x_1 are $0 \leq x_1 \leq 215$.

$\therefore f_1(x_1) = 2 \cdot \text{Min.} \left[\frac{430 - x_2}{2}, \infty\right]$

$0 \leq x_1 \leq 215$

Stage ②: Max. $[5x_2 + 2x_1]$

$\Rightarrow$ Max. $\left[5x_2 + 2 \cdot \text{Min.} \left(\frac{430 - x_2}{2}, \infty\right)\right]$

Constraints:

① → $2x_1 + x_2 \leq 430$ ② → $0x_1 + 2x_2 \leq 460$

$x_2 \leq 430 - 2x_1$ $x_2 \leq \frac{460}{2}$

$\therefore x_2 \leq 230$

Max. $(x_2) = \text{Min.} [430 - 2x_1, 230]$

if $x_1 = 0$; $\text{Min}(430 - 2(0), 230) = 230$.

$\therefore$ lower limit for $x_2 = 0$

$\therefore$ Upper limit for $x_2 = 230$.

$$f_2(x_2) = \text{Max.}\left[5x_2 + 2\,\text{Min.}\left(\frac{430-x_2}{2}, \infty\right)\right]$$
$$0 \le x_2 \le 230$$

W.K.T $\text{Min}\left(\frac{430-x_2}{2}, \infty\right) = \frac{430-x_2}{2}$

$$\therefore f_2(x_2) = \text{Max.}\left[5x_2 + \cancel{2}\left(\frac{430-x_2}{\cancel{2}}\right)\right]$$
$$= \text{Max.}\left[5x_2 + 430 - x_2\right]$$
$$= \text{Max.}\left[4x_2 + 430\right]$$

if $\boxed{x_2 = 230} \Rightarrow f_2(x_2) = \boxed{1350 = Z_{max}}$

from stage ①: $x_1 = \text{Min.}\left[\frac{430-230}{2}, \infty\right] = \boxed{100 = x_1}$

② $\text{Max. } Z = 8x_1 + 7x_2$

subject to $2x_1 + x_2 \le 8$ —①

$5x_1 + 2x_2 \le 15$ —②

$x_1, x_2 \ge 0$

(May 2018, Supply)

Sol: As the number of decision variables are two the stages in the problem could be two.

Stage ①: $f_1(S_1) = \text{Max.}(8x_1)$

$$= 8.\,\text{Max}(x_1)$$
$$0 \le x_1 \le b.$$

Constraints:

① $2x_1 + x_2 \le 8$

$x_1 \le \frac{8-x_2}{2}$

② $5x_1 + 2x_2 \le 15$

$x_1 \le \frac{15-2x_2}{5}$

$\therefore$ The maximum value of x_1 can be assumed as 'b'

$\therefore b = \text{Min}\left[\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right]$

if $x_2 = 0$; $\quad b_1 = \min\left(\frac{8-0}{2}, \frac{15-2(0)}{5}\right)$
(x_2 is present in stage (1)) $\quad = \min(4, 3) = 3$

$\therefore \quad f_1^*(S_1) = 8 \cdot \underset{0 \le x_1 \le 3}{\text{Min}} \left(\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right)$

Stage (2): $f_2(S_2) = \text{Max.}\left(7x_2 + 8x_1\right)$ $\quad f_1^*(S_1)$

$= \underset{0 \le x_2 \le b_2}{\text{Max}} \left[7x_2 + 8 \cdot \text{Min}\left(\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right)\right]$

Constraints:

(1) $2x_1 + x_2 \le 8$
$x_2 \le 8 - 2x_1$

(2) $5x_1 + 2x_2 \le 15$
$x_2 \le \frac{15-5x_1}{2}$

The maximum value of x_2 can be assumed as b_2.

$\therefore b_2 = \min.\left(8 - 2x_1, \frac{15-5x_1}{2}\right)$

if $x_1 = 0$; $b_2 = \text{Min}\left(8 - 2(0), \frac{15-5(0)}{2}\right)$
$= \frac{15}{2}$.

if $x_1 = 0$; $\text{Min} = \frac{15-5x_1}{2}$ $\quad 7.5$
$x_1 = 1$; $\text{Min} = \frac{15-5x_1}{2}$ $\quad 5$
$x_1 = 2$; $\text{Min} = \frac{15-5x_1}{2}$
$x_1 = 3$; $\text{Min} = \frac{15-5x_1}{2}$ $\quad 0.$

$\therefore f_2(S_2) = \underset{0 \le x_2 \le 7.5 \text{ (or) } \frac{15}{2}}{\text{Max.}} \left[7x_2 + 8 \text{ Min}\left(\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right)\right]$

Put $x_2 = \frac{15}{2}$. in $f_2(S_2)$:

$\text{Max } Z = f_2(S_2) = 7\left(\frac{15}{2}\right) + 8 \cdot \text{Min}\left(\frac{8-\frac{15}{2}}{2}, \frac{15-\cancel{2}\times\frac{15}{\cancel{2}}}{5}\right)$

$= \frac{105}{2} + 8 \text{ Min}\left(\frac{1}{4}, 0\right) = \frac{105}{2}$

$\therefore \text{Max. } Z = 52.5$

From stage (1): $\text{Max}(x_1) = \min\left(\frac{8-x_2}{2}, \frac{15-2x_2}{5}\right)$

Put $x_2 = \frac{15}{2}$ $\quad = \text{Min}\left(\frac{8-\frac{15}{2}}{2}, \frac{15-2\times\frac{15}{2}}{5}\right)$

$x_1 = 0$

$\therefore \boxed{x_1 = 0 \;;\; x_2 = \frac{15}{2} \;;\; \text{Max } Z = \frac{105}{2}}$

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(3) Max. $Z = x_1 + 9x_2$

subject to $2x_1 + x_2 \le 25$

$x_2 \le 11$

$x_1, x_2 \ge 0$.

Sol. Two stage Problem.

Stage (1): $f_1(s_1) = \underset{0 \le x_1 \le b_1}{\text{Max}} (x_1)$

Constraint (1)

$2x_1 + x_2 \le 25$

$x_1 \le \frac{25 - x_2}{2}$

Constraint (2)

$0x_1 + x_2 \le 11$

$x_1 \le \frac{11 - x_2}{0}$

$x_1 \le \infty$

The maximum value of 'x_1' can be

$b_1 = \min\left(\frac{25 - x_2}{2}, \infty\right)$

if $x_2 = 0$; $b_1 = \min\left(\frac{25-0}{2}, \infty\right)$

$= \frac{25}{2}$.

$\therefore f_1^*(s_1) =$ ~~max(x_1)~~ $\min\left(\frac{25 - x_2}{2}, \infty\right)$
$0 \le x_1 \le \frac{25}{2}$

Stage (2):

$f_2(s_2) = \underset{0 \le x_2 \le b_2}{\text{Max}} (9x_2 + x_1) = \underset{0 \le x_2 \le b_2}{\text{Max}} \left[9x_2 + \min\left(\frac{25 - x_2}{2}, \infty\right)\right]$

Constraint (1)

$2x_1 + x_2 \le 25$

$x_2 \le 25 - 2x_1$

Constraint (2)

$x_2 \le 11$

The maximum value of x_2 can be assumed as

$b_2 = \min(25 - 2x_1, 11)$

if $x_1 = 0$; $b_2 = 11$
$x_1 = 1$; $b_2 = 11$
$x_1 = 6$; $b_2 = 11$
$x_1 = \frac{25}{2}$; $b_2 = 0$

Assignment:

(4) Max. $Z = 3x_1 + 9x_2$

subject to $2x_1 + x_2 \le 40$

$2x_1 + x_2 \le 180$

$x_1, x_2 \ge 0$.

Sol: $x_1 = 2.5$ (or) $5/2$

$x_2 = 35$

$Z = 147.5$.

Scanned with CamScanner

(5) $\text{Max. } Z = 3x_1 + 5x_2$

Subject to $x_1 \le 4 \quad \leftarrow (1)$

$x_2 \le 6 \quad - (2)$

$3x_1 + 2x_2 \le 18 \quad - (3).$

$x_1, x_2 \ge 0.$

Sol. Two stage problem.

Stage (1): $f_1(S_1) = \text{Max}(3x_1) = 3.\ \underset{0 \le x_1 \le b_1}{\text{Max}}(x_1)$

Constraint (1)	Constraint (2)	Constraint (3)
$x_1 \le 4$	$0x_1 + x_2 \le 6$ $x_1 \le \frac{6 - x_2}{0}$ $x_1 \le \infty$	$3x_1 + 2x_2 \le 18$ $x_1 \le \frac{18 - 2x_2}{3}$

The maximum value of x_1 can be obtained as

$$b_1 = \min\left(4, \infty, \frac{18 - 2x_2}{3}\right)$$

if $x_2 = 0$; $b_1 = \text{Min}\left(4, \infty, \frac{18 - 2(0)}{3}\right) = 4.$

$$\therefore f_1^*(S_1) = 3.\ \underset{0 \le x_1 \le 4}{\text{Min}}\left(4, \infty, \frac{18 - 2x_2}{3}\right)$$

Stage (2): $f_2(S_2) = \underset{0 \le x_2 \le b_2}{\text{Max}}(5x_2 + 3x_1)$ (or)

$$= \underset{0 \le x_2 \le b_2}{\text{Max}}\left[5x_2 + 3.\min\left(4, \infty, \frac{18 - 2x_2}{3}\right)\right]$$

Constraint (1)	Constraint (2)	Constraint (3)
$x_1 + 0x_2 \le 4$ $x_2 \le \frac{4 - x_1}{0}$ $x_2 \le \infty$	$x_2 \le 6$	$3x_1 + 2x_2 \le 18$ $x_2 \le \frac{18 - 3x_1}{2}$

The maximum value of x_2 can be assumed as.

$$b_2 = \min\left(\infty, 6, \frac{18 - 3x_1}{2}\right)$$

$k_2 \Rightarrow$ Min.

$$f_2(s_2) = \max_{0 \le x_2 \le 6} \left[5x_2 + 3 \cdot \min\left(4, \infty, \frac{18-2x_2}{3}\right) \right]$$

$$\min\left(4, \infty, \frac{18-2x_2}{3}\right) = \begin{matrix} 4 & \text{if } x_2 = 0. \\ 2 & \text{if } x_2 = 6 \end{matrix}$$

$\therefore$ The intermediate point satisfies ① st & 3rd constraint given by

$$\frac{18-2x_2}{3} = 4$$

$$\boxed{x_2 = 3}.$$

$$\min\left(4, \infty, \frac{18-2x_2}{3}\right) = \begin{cases} 4 & \text{if } 0 \le x_2 \le 3 \\ 2 & \text{if } 3 \le x_2 \le 6. \end{cases}$$

$$f_2^*(s_2) = \begin{cases} 5(3) + 3(4) = 27 & \text{if } x_2 = 3 \\ 5(6) + 3(2) = 36 & \text{if } x_2 = 6. \end{cases}$$

$\therefore$ $\boxed{\text{Max } z = 36.}$

From stage ①: $x_1 = \text{Min}\left(4, \infty, \frac{18-2x_2}{3}\right)$ at $x_2 = 6$.

$$= \text{Min}\left(4, \infty, \frac{18-2(6)}{3}\right) = 2$$

$\therefore$ $\boxed{x_1 = 2,\ x_2 = 6\ ;\ z^* = 36}$