Course: CONTROL SYSTEMS

Course Code: 23A0244T

Branch: Electrical and Electronics Engineering

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ANNAMACHARYA INSTITUTE OF TECHNOLOGY AND SCIENCES RAJAMPET
(An Autonomous Institution)

Title of the Course:	Control Systems
Category:	PC
Couse Code:	23A0244T
Branch/es:	EEE
Year:	II
Semester:	II

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	0	0	3

Course Objectives:

1. To provide an introduction to the analysis of linear control systems.
2. To exploit time domain and frequency domain tools.

Course Outcomes:

At the end of the course, the student will be able to

1. Understand the concepts of various mathematical representations of control systems, Time response of first order and second order systems, stability, frequency response and fundamentals of modern control systems.
2. Apply Block diagram reduction, Signal flow graph, Routh criterion, Root locus, Bode, Polar, Nyquist concepts for solving various numerical problems.
3. Analyze time response characteristics, frequency response characteristics, stability analysis of various control systems.
4. Design various compensators and controllers for different control systems by using design procedures.
5. Create suitable control systems for various real time applications.

Unit 1 Control Systems Concepts 12

Open loop and closed loop control systems and their differences- Examples of control systems- Classification of control systems, Feedback characteristics, Effects of positive and negative feedback, Mathematical models – Differential equations of translational and rotational mechanical systems and electrical systems, Analogous Systems, Block diagram reduction methods – Signal flow graphs - Reduction using Mason's gain formula. Principle of operation of DC and AC Servo motor, Transfer function of DC servo motor - AC servo motor, Synchros.

Unit 2 Time Response Analysis 8

Step Response - Impulse Response - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems - Time domain specifications – Steady state response - Steady state errors and error constants, P, PI, PID Controllers.

Unit 3 Stability Analysis in Time Domain 8

The concept of stability – Routh's stability criterion – Stability and conditional stability – limitations of Routh's stability. The Root locus concept - construction of root loci-effects of adding poles and zeros to G(s)H(s) on the root loci.

Unit 4 Frequency Response Analysis 10

Introduction, Frequency domain specifications-Bode Diagrams-Determination of Frequency domain specifications and transfer function from the Bode Diagram-Stability Analysis from Bode Plots. Polar Plots-Nyquist Plots- Phase margin and Gain Margin-Stability Analysis. Compensation techniques – Lag, Lead, and Lag-Lead Compensator design in frequency Domain.

Unit 5 State Space Analysis of Continuous Systems 10

Concepts of state, state variables and state model, state models - differential equations & Transfer function models - Block diagrams. Diagonalization, Transfer function from state model, Solving the Time invariant state Equations- State Transition Matrix and it’s Properties. System response through State Space models. The concepts of controllability and observability, Duality between controllability and observability.

Prescribed Textbooks:

1. Modern Control Engineering by Katsuhiko Ogata, Prentice Hall of India Pvt. Ltd., 5th edition, 2010.
2. Control Systems Engineering by I. J. Nagrath and M. Gopal, New Age International (P) Limited Publishers, 5th edition, 2007.

Reference Books:

1. Control Systems Principles & Design by M. Gopal, 4th Edition, Mc Graw Hill Education, 2012.
2. Automatic Control Systems by B. C. Kuo and Farid Golnaraghi, John wiley and sons, 8th edition, 2003.
3. Feedback and Control Systems, Joseph J Distefano III, Allen R Stubberud & Ivan J Williams, 2nd Edition, Schaum's outlines, Mc Graw Hill Education, 2013.
4. Control System Design by Graham C. Goodwin, Stefan F. Graebe and Mario E. Salgado, Pearson, 2000.
5. Feedback Control of Dynamic Systems by Gene F. Franklin, J.D. Powell and Abbas Emami-Naeini, 6th Edition, Pearson, 2010.

CO-PO Mapping:

Course Outcomes	Engineering Knowledge	Problem Analysis	Design/Development of solutions	Conduct investigations-of complex problems	Modern tool usage	The engineer and society	Environment and sustainability	Ethics	Individual and team work	Communication	Project management and finance	Life-long learning	**PSO1**	**PSO2**	**PSO3**
1	3	3	3	-	2	1	-	-	-	-	3	3	3	3	
2	3	3	2	-	3	1	-	-	-	-	3	3	3	3	
3	3	2	3	-	3	3	1	-	-	-	3	3	3	3	
4	3	3	3	-	3	3	3	-	-	-	3	3	3	3	
5	3	3	2	-	3	3	3	-	-	-	3	3	3	3	

Unit-1

Introduction

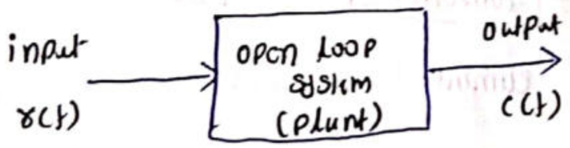
* Control System :-

When a number of elements (or) components are connected in a sequence to perform a specific function, the group thus formed is called a "System."

In a system when the output quantity is controlled by varying the input quantity, then the system is called "control system."

* Open Loop System :-

Any physical system which does not automatically correct the variations in its output, is called an open loop system (or) control system in which the output quantity has no effect upon the input quantity is called open loop control system. This means that the output is not feedback to the input for correction.



In open loop system the o/p can be varied by varying the i/p. But due to external disturbances the system o/p may change. When the o/p changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the i/p manually.

Advantages :-

→ Such systems are simple in construction.
→ very much convenient when o/p is difficult to measure.
→ Such systems are easy from maintanance point of view.
→ Such systems are simple to design and hence economical.
→ The open loop systems are stable.

Disadvantages :-

→ The open loop systems are inaccurate and unreliable.

→ The changes in the o/p due to External disturbances are not corrected automatically.

To overcome all the above disadvantages, generally in practice closed loop systems are used.

* Applications of open loop system in Real time :-

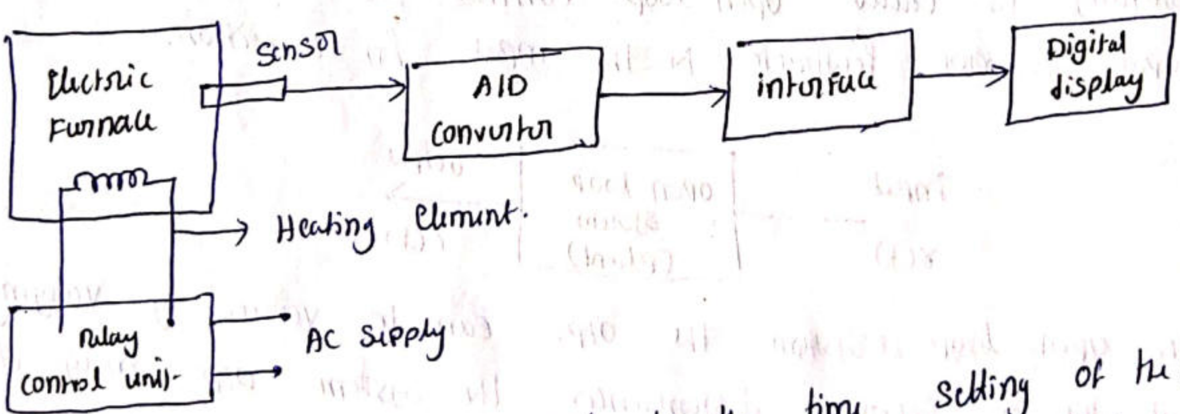
a) Temperature control system :-

✓ The electric furnace shown in fig is an open loop system.

✓ The output in the ~~temperature~~ system is the desired temperature.

✓ The temperature of the system is raised by heat generated by the heating element.

✓ The output temperature depends on the time during which the supply to heater remains "ON".



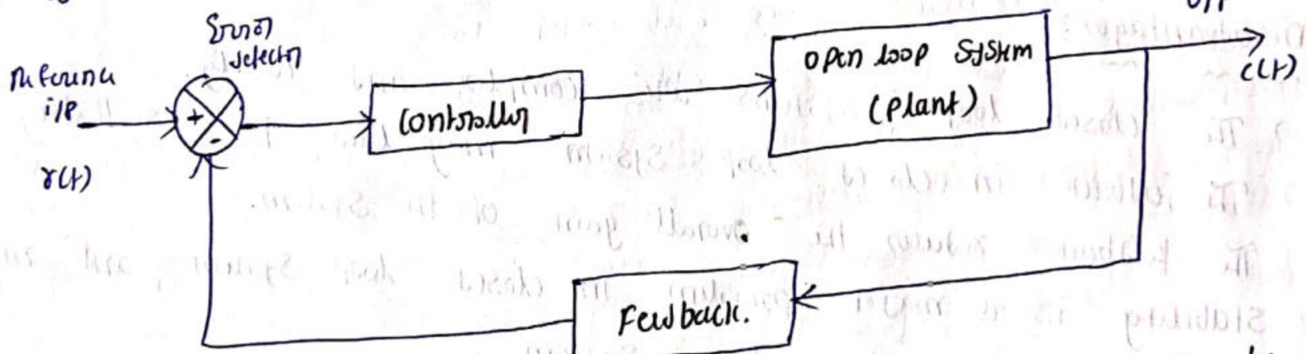
The ON and OFF of supply is governed by the time setting of the relay. The temperature is measured by the sensor, which gives an analog value corresponding the temperature of the furnace. The analog signal is converted to digital signal by A/D converter. The digital signal is given to the digital display. For this system if there is any change in output temp, then the time setting of the relay is not altered automatically.

b) Traffic control System :- (ii)

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slots does not changes according to traffic density, the system is open loop system.

* Closed loop System :-

Control systems in which the o/p has an affect upon the i/p quantity in order to maintain the desired o/p value are called "closed-loop systems".



→ The open loop system can be modified as closed loop system by providing a feedback.

→ The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called "automatic control system."

→ The general block diagram of an automatic control system is shown in fig.

→ It consists of an error detector, a controller, plant (open loop system) and feedback path elements.

→ The reference signal (or input signal) corresponds to desired output.

→ The feedback path elements samples the o/p and converts it to a signal of same type as that of reference signal.

→ The feedback signal is proportional to o/p signal and it is fed to the error detector.

→ The error signal generated by the error detector is the difference between reference signal and feedback signal.
→ The controller modifies and ~~amplifies~~ amplifies the error signal to produce better control action.
→ The modified error signal is fed to the plant to correct its output.

Advantages :-

→ The closed loop systems are accurate.
→ such system senses environmental changes, as well as internal disturbances and accordingly modifies the error.
→ The sensitivity of the systems may be made small to make the system more stable.
→ The closed loop systems are less affected by noise.

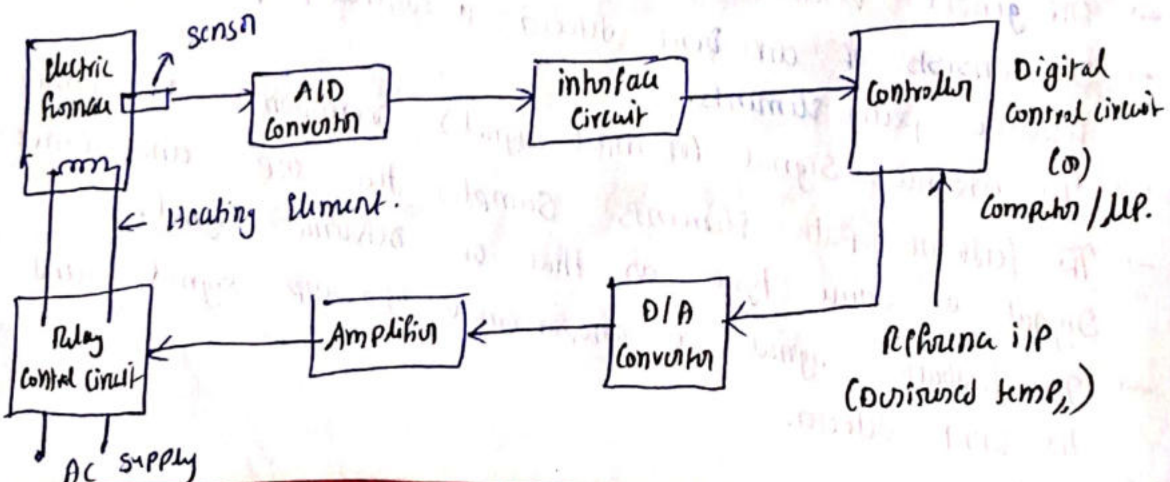
Disadvantages :-

→ The closed loop systems are complex and costly.
→ The feedback in closed loop system may lead to oscillatory response
→ The feedback reduces the overall gain of the system.
→ stability is a major problem in closed loop system and more care is needed to design closed loop system.

Applications of closed loop system in real time :-

a) Temperature control system :-

The electric furnace shown in fig is closed loop system. The O/P of the system is the desired temperature.



The switching ON and OFF of the relay is controlled by a controller which is a digital system (or) computer. The desired temperature is input to the system through key board (or) as a signal corresponding to desired temperature via ports.

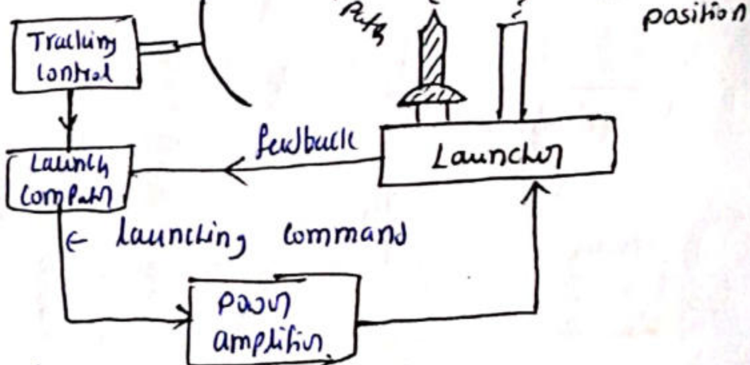
The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in O/P.

b) Traffic control system :-

Traffic control system can be made as a closed loop system if the time slots of the signals are decided based on the density of traffic. In closed loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic. Since the closed loop system dynamically changes the timings, the flow of vehicles will be better than open loop system.

c) Missile launching system :-

This is sophisticated example of military applications of feedback control. The enemy plane is sighted by radar which continuously tracks the path of the aeroplane. The launch computer calculates the firing angle in terms of launch command, which when amplified drives the launcher.



The launcher angular position is the feedback to the launch computer and the missile is triggered when error b/w command signal and missile angle becomes zero.

* comparision of open loop and closed loop control system :-

	Open loop	Closed loop.
1)	Any change in output has no effect on the input i.e, feedback does not exist.	changes in output, affects the input which is possible by use of feedback.
2)	output measurement is not required for operation of system.	output measurement is necessary.
3)	feedback element is absent	feedback element is present.
4)	error detector is absent	error detector is absent.
5)	It is inaccurate and unreliable	Highly accurate and reliable.
6)	Highly sensitive to the disturbances.	Less sensitive to the disturbances.
7)	Highly sensitive to the environmental changes.	Less sensitive to the environmental changes.
8)	Bandwith is small	Bandwidth is large.
9)	Simple to construct and cheap	Complicated to design and hence costly.
10)	Stable in nature	Stability is the main consideration while designing.
11)	Highly affected by nonlinearities	Reduced effect of nonlinearities.

* Classification of control systems :-

Control systems can be classified as,

i) **Natural control systems :-** The biological systems, systems inside human being are of natural type.

Ex: The perspiration system inside the human being is a good example of natural control system. This system activates the secretion glands, secreating sweat and regulates the temperature of human body.

ii) **Man made control systems :-** The various systems, we are using in our day to day life are designed and manufactured by human beings. such systems like vehicles, switches etc are called manmade control systems

Ex: An Automobile system with gears, accelerator, braking system is a good example of man made control system.

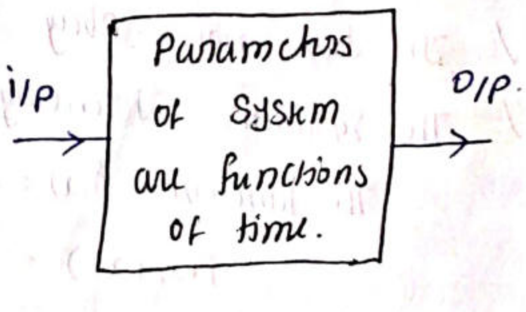
iii) **Combinational control systems :-** Combinational control system is combination of natural and man made together i.e, one, having combination of driver driving a vehicle. In such system, for successful operation of the system, it is necessary that natural systems of driver alongwith systems in vehicles which are manmade must be active.

Out for the Engineering Analysis, control systems can be classified in many different ways. Some of the classifications are given below.

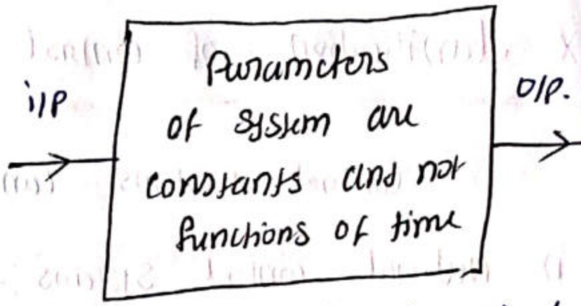
iv) **Time varying and Time - Invariant Systems :-**

Time varying system :-

Time varying control systems are those in which parameters of the systems are varying with time. It is not dependent on whether i/p and o/p are functions of time (or) not.



Time invariant system :-



The inputs and outputs are functions of time but the parameters of system are independent of time, which are not varying with time and are constants, then system is said to be time invariant system.

v) Linear and Non-linear systems :-

Linear systems :- A control system is said to be linear if it is satisfies the following properties.

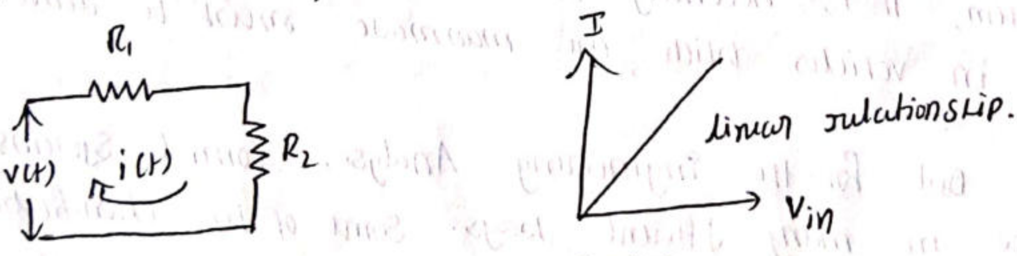
✓ The principle of superposition is applicable to the system.

✓ Additive property which says that for x and y belonging to the domain of the function "f" then we have,

$$f(x+y) = f(x) + f(y)$$

✓ Homogeneous property which says that for any x belonging the domain of function "f" and for any scalar constant "α" we have,

$$f(\alpha x) = \alpha f(x)$$



Example of linear system.

Non-linear systems :- A control system is said to be non-linear if it is satisfies the following properties.

✓ It does not obey the superposition theorem.

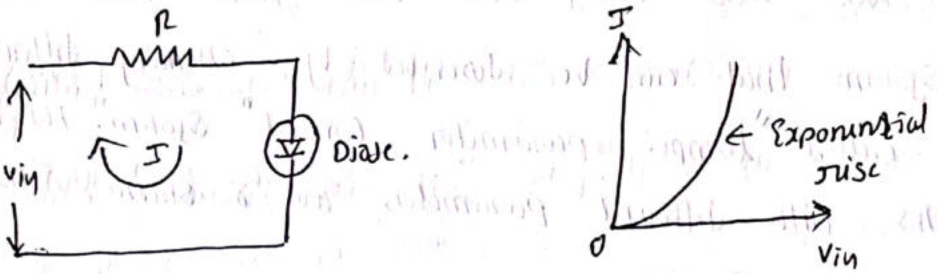
✓ The equations describing the system are non-linear in nature.

The function $f(x) = x^2$ is non linear because

$$f(x_1+x_2) = (x_1+x_2)^2 \neq (x_1)^2 + (x_2)^2$$

$$f(\alpha x) = (\alpha x)^2 \neq \alpha x^2$$ where α is constant.

The output does not vary linearly for non linear systems.



Example of non linear system.

vi) Continuous time and Discrete time control systems :-

In a continuous time control system, all system variables are the functions of a continuous time variable "t".

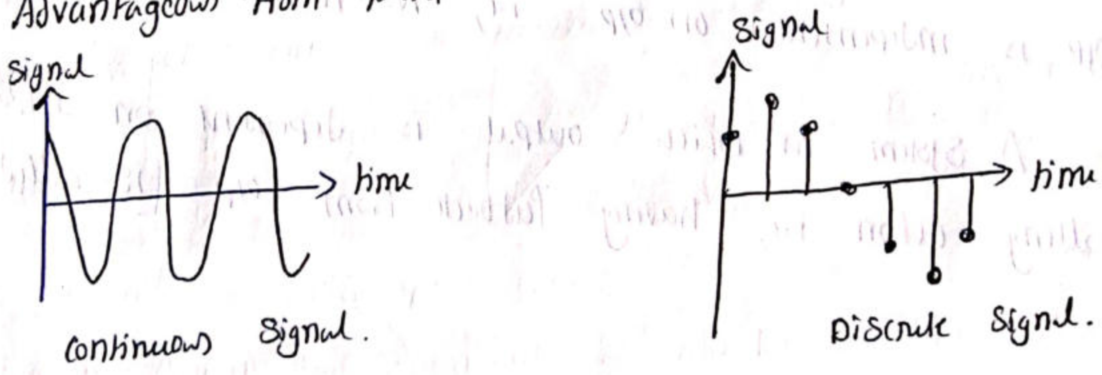
Ex: The speed control of a dc motor using a tachometer feedback an example of continuous data system.

In a discrete time control system one (or) more system variables are known only at certain discrete intervals of time.

Microprocessors (or) computer based systems use such discrete time signals. The reasons for using such signals in digital controllers are,

→ Such signals are less sensitive to noise.

→ Advantageous from point of view of size, memory, flexibility etc.



vii) Single input and Single output (SISO) & Multiple i/p and Multiple O/p (MIMO) Systems :- A system having only one i/p and one o/p is called single i/p and single o/p system. For example a position control system has one i/p (desired position) and one o/p (actual o/p position). Some systems may have multiple type of inputs and multiple o/ps these are called multiple inputs and multiple outputs systems.

viii) Lumped Parameters and Distributed parameter control systems :-

Control systems that can be described by "ordinary differential equations" is called "lumped parameter control system". For example electrical networks with different parameters are resistance, inductance etc are lumped parameter systems.

Control systems that can be described by "partial differential equations" is called "Distributed parameter control system". For example transmission line having its parameters resistance and inductance totally distributed along it.

NOTE:- The lumped parameters are physically separable and can be shown to be located at a particular point while representing the system. The distributed parameters can not be physically separated and hence can not be represented at a particular place.

ix) Open loop and closed loop system :-

A system in which output is dependent on i/p but controlling of o/p is independent on o/p i.e, No feedback from o/p. is called "OLCS"

A system in which output is dependent on input and also controlling action i.e, having feedback from o/p is called "CLCS"

* Feedback Charachristics :-

* The Charachristics of feedback are as fallows.

→ Accuracy in tracking steady state value.

→ Rejection of disturbance signals.

→ Low sensitivity to parameter variations.

→ Reduction in gain at the expense of better stability.

* Effects of positive feedback and negative feedback :-

The positive feedback increases the error signal and drives the o/p to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals (or) parameters.

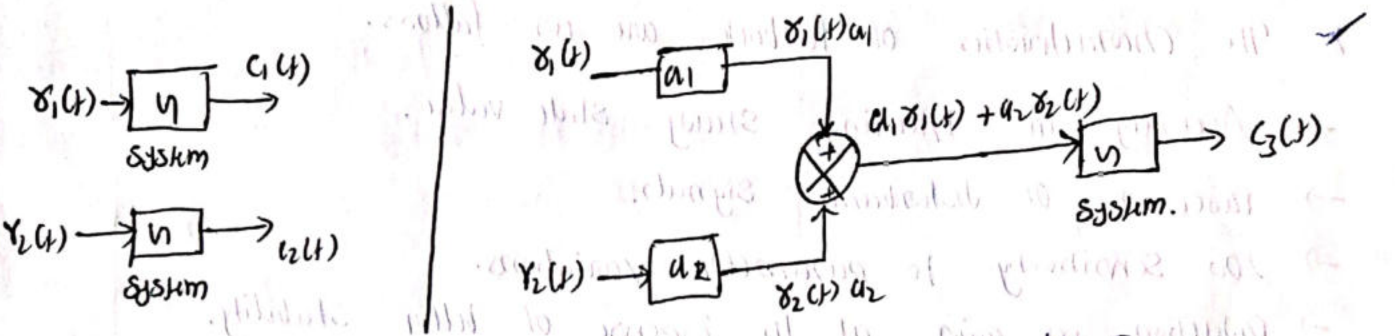
The negative feedback decreases the error signal and drives the o/p to stability. It increases accuracy of the system.

* Mathematical Models of Control Systems :-

A control system is a collection of physical objects (components) connected together to serve an objective. The input-output relations of various physical components of system are governed by differential equations. The mathematical model of control system constitutes a set of differential equations. The output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogenity. This principle implies that if a system model has responses $y_1(t)$ and $y_2(t)$ to any inputs $x_1(t)$ and $x_2(t)$ respectively, then the system response to the linear combination of these inputs $a_1x_1(t) + a_2x_2(t)$ is given by linear combination of individual outputs $a_1y_1(t) + a_2y_2(t)$, where a_1, a_2 are constants.

The principle of superposition can be Explained mathematically as shown in below.



If $c_3(t) = a_1 c_1(t) + a_2 c_2(t)$ then system G is linear.

→ A mathematical model will be linear if the differential Equations describing the system has constant co-efficients.

→ If the co-efficients of the differential Equation describing the system are constants then the model is linear time invariant.

→ If the co-efficients of the differential Equations governing the system are functions of time then the model is linear time varying.

→ The differential Equations of the linear time invariant can be reshaped into different form for the convenience of analysis. One such model for single input and single output system is Transfer function of the system.

(2-Marks)

→ The "transfer function" of the system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero Initial conditions.

$$\text{Transfer function} = \left.\frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}\right|_{\text{with zero Initial conditions.}}$$

* Mechanical translational Systems :-

The model of the mechanical translational systems can be obtained by using three basic elements mass, spring, and dash-pot. These three elements represents three essential phenomena which occur in various ways in mechanical systems.

→ The weight of the mechanical system is represented by the element "mass" and it is assumed to be concentrated at the center of the body.
→ The elastic deformation of the body can be represented by a "spring"
→ The friction existing in mechanical system can be represented by the "dash-pot". The dash-pot is a piston moving inside a cylindrical filled with viscous fluid.

When a force applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a body are governed by "Newton's second law of motion". For translation system it states that the sum of forces acting on a body is zero (or Newton's second law states that the sum of applied forces is equal to sum of opposing forces on a body).

List of symbols used in mechanical translational system :-

x = Displacement, m
v = velocity $\left(\frac{dx}{dt}\right)$, m/sec

Note: Lower case letters are functions of time.

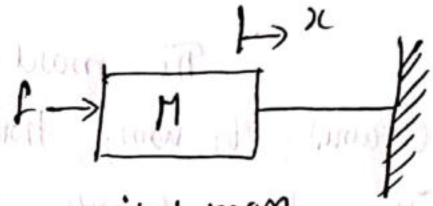
$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration m/sec^2

f = applied force, N (Newtons)
f_m = opposing force offered by mass of the body, N
f_b = opposing force offered by friction of the body, N (dash-pot)
f_k = opposing force offered by elasticity of the body, N (spring)

M = Mass, kg
K = stiffness of spring, N/m
B = viscous friction co-efficient, N-sec/m

Force balanced Equations of idealized elements :-

i) Consider an ideal mass element shown in fig. Which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.



ideal mass element.

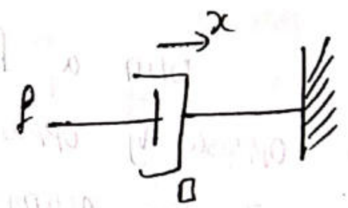
Let f = Applied force

f_m = Opposing force due to mass

Here $f_m \propto \frac{d^2x}{dt^2} \Rightarrow f_m = M\frac{d^2x}{dt^2}$

By Newton's second Law $\left[f = f_m = M\frac{d^2x}{dt^2} \right] \rightarrow \langle 1 \rangle$

ii) Consider an ideal frictional element dash pot shown in fig which has negligable mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.



ideal dashpot with one end fixed to reference.

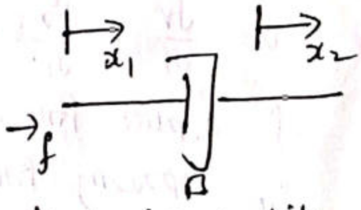
Let f : applied force

f_b : opposing force due to friction by dash-pot

Here $f_b \propto \frac{dx}{dt} \Rightarrow f_b = B\frac{dx}{dt}$

By newtons second Law $\left[f = f_b = B\frac{dx}{dt} \right] \rightarrow \langle 2 \rangle$

When the dashpot has displacement at both ends. as shown in fig, the opposing force is proportional to differential velocity



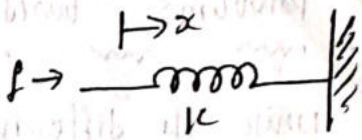
ideal daspot with displacement at both ends.

$f_b \propto \frac{d}{dt}(x_1 - x_2)$ (or) $f_b = B\frac{d}{dt}(x_1 - x_2)$

$\left[f = f_b = B\frac{d}{dt}(x_1 - x_2) \right] \rightarrow \langle 3 \rangle$

(8)

iii) Consider an ideal spring which has negligible mass and friction. Let a force applied on it. The spring will offer opposing force which is proportional to displacement of the body.



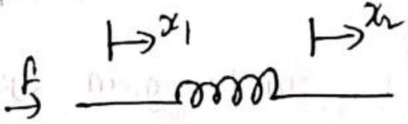
ideal spring with one end fixed to reference.

Let f = applied force

f_K = Opposing force due to elasticity

here $f_K \propto x$ (or) $f_K = Kx$

By newton's second law $\left[f = f_K = Kx \right]$ $\rightarrow$ (4)



ideal spring with displacement at both ends.

here $f_K \propto (x_1 - x_2)$ (or) $f_K = K(x_1 - x_2)$

$$\left[f = f_K = K(x_1 - x_2) \right] \rightarrow (5)$$

Guideline to determine the Transfer function of Mechanical Translational system

→ Draw the free body diagram of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass

→ The sum of opposing forces is equal to applied can be write this equation in the form of differential equation.

→ The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

→ Take Laplace transform of differential equations to convert them to algebraic equations. Then rearranging the s-domain equation to eliminate unwanted variables and obtain the ratio b/w o/p variable and i/p variable. This ratio is the transfer function of the system.

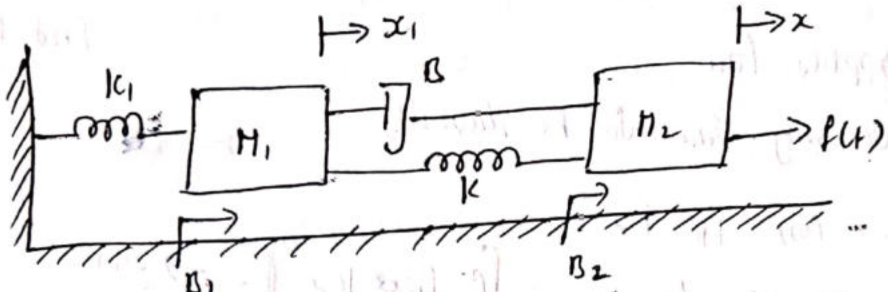
$$L\{x(t)\} = X(s)$$

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) \quad \text{(with zero initial conditions)}$$

$$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) \quad (\quad " \quad)$$

* Problems based on Mechanical translational systems :-

1) Write the differential equations governing mechanical system shown in fig. and determine the transfer function.



Sol, In the given system, applied force "f(t)" is the input and displacement "x" is the output.

Let Laplace transform of $f(t) = \mathcal{L}[f(t)] = F(s)$

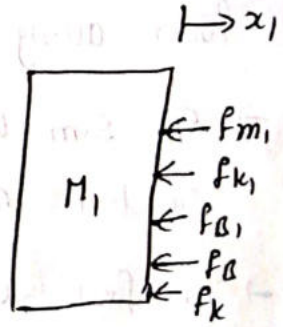
Laplace transform of $x_1(t) = \mathcal{L}[x_1(t)] = X_1(s)$

Laplace transform of $x(t) = \mathcal{L}\{x(t)\} = X(s)$

Hence the required transfer function is $\dfrac{\mathcal{L}\{x(t)\}}{\mathcal{L}\{f(t)\}} \Rightarrow \dfrac{X(s)}{F(s)}$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance eqn. at these nodes

Case 1:- The free body diagram of mass M_1 is shown in fig.



The opposing force acting on mass M_1 are marked as f_{m_1}, f_{k_1}, f_{B_1}, f_B, and f_k

here $f_{m_1} = M_1 \dfrac{d^2x_1}{dt^2}$ $f_{k_1} = K_1 x_1$, $f_{B_1} = B_1 \dfrac{dx_1}{dt}$

$f_B = B \dfrac{d}{dt}(x_1 - x)$, $f_k = K(x_1 - x)$

By Newton's second law,

$$f_{m_1} + f_{k_1} + f_{B_1} + f_B + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + K_1 x_1 + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K(x_1 - x) = 0 \quad \rightarrow (1)$$

on taking Laplace transform of above equation with zero initial conditions we get,

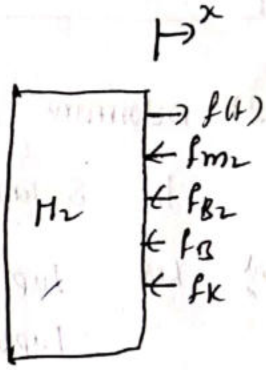
$$M_1 S^2 X_1(s) + K_1 X_1(s) + B_1 S X_1(s) + BS\{X_1(s) - X(s)\} + K\{X_1(s) - X(s)\} = 0 \quad \langle a \rangle$$

$$\Rightarrow M_1 S^2 X_1(s) + K_1 X_1(s) + B_1 S X_1(s) + BS X_1(s) - BS X(s) + K X_1(s) - K X(s) = 0$$

$$\Rightarrow X_1(s)\{M_1 S^2 + K_1 + B_1 S + BS + K\} - X(s)\{BS + K\} = 0$$

$$\Rightarrow X_1(s)\{M_1 S^2 + (B_1 + B)S + (K_1 + K)\} = X(s)\{BS + K\}$$

$$\Rightarrow \left[X_1(s) = X(s) \cdot \frac{BS+K}{M_1 S^2 + (B_1+B)S + (K_1+K)} \right] \rightarrow (2)$$



Case : 2 :- The free body diagram of mass M_2 is shown in fig. The opposing forces acting on mass M_2 are marked as f_{m2}, f_{B2}, f_B, f_K

here $f_{m2} = M_2 \frac{d^2x}{dt^2}$ $\quad f_{B2} = B_2 \frac{dx}{dt}$ $\quad f_B = B \frac{d}{dt}(x - x_1)$

$$f_K = K(x - x_1)$$

By Newton's second Law:

$$f_{m2} + f_{B2} + f_B + f_K = f(t)$$

$$\therefore M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d(x-x_1)}{dt} + K(x-x_1) = f(t) \rightarrow (3)$$

on taking Laplace transform of above Equation with zero initial conditions we get

$$M_2 S^2 X(s) + B_2 S X(s) + BS\{X(s) - X_1(s)\} + K\{X(s) - X_1(s)\} = F(s)$$

$$M_2 S^2 X(s) + B_2 S X(s) + BS X(s) - BS X_1(s) + K X(s) - K X_1(s) = F(s)$$

$$X(s)\{M_2 S^2 + B_2 S + BS + K\} - X_1(s)\{BS + K\} = F(s)$$

$$X(s)\{M_2 S^2 + (B_2 + B)S + K\} - X_1(s)\{BS + K\} = F(s) \rightarrow \langle 4 \rangle$$

Substitute Equation (2) in Equation (4) we get

$$X(s)\{M_2 S^2 + (B_2+B)S + K\} - X(s) \cdot \frac{(BS+K)}{M_1 S^2 + (B_1+B)S + (K_1+K)} \cdot (BS+K) = F(s)$$

$$X(s)\left[M_2 S^2 + (B_2+B)S + K - \frac{(BS+K)^2}{M_1 S^2 + (B_1+B)S + (K_1+K)}\right] = F(s)$$

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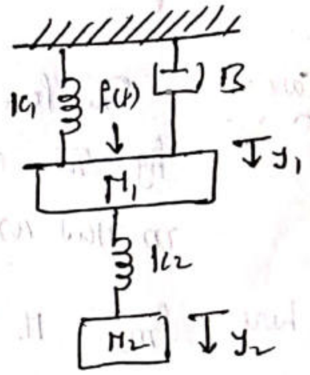
$$X(s)\left[\frac{(M_2s^2+(B_2+B)s+K)(M_1s^2+(B_1+B)s+(K_1+K))-(Bs+K)^2}{M_1s^2+(B_1+B)s+(K_1+K)}\right]=F(s)$$

∴ The transfer function for the given system.

$$\boxed{\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{(M_2s^2+(B_2+B)s+K)(M_1s^2+(B_1+B)s+(K_1+K))-(Bs+K)^2}}$$

2) Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig.



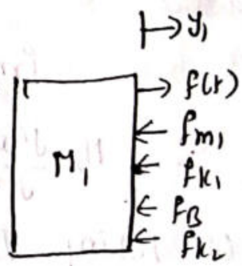
Sol: Let Laplace transform of $f(t) = L\{f(t)\} = F(s)$

Laplace transform of $y_1 = L\{y_1\} = Y_1(s)$

Laplace transform of $y_2 = L\{y_2\} = Y_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance equation at these nodes.

Case 1:- The free body diagram of mass M_1 is shown in fig. The opposing forces on mass M_1 are marked as f_{m_1}, f_B, f_{K_1}, f_{K_2}



here $f_{m_1} = M_1\frac{d^2y_1}{dt^2}$; $f_B = B\frac{dy_1}{dt}$ $f_{K_1} = K_1y_1$, $f_{K_2} = K_2(y_1-y_2)$

By newton's second law.

$$f_{m_1}+f_B+f_{K_1}+f_{K_2} = f(t)$$

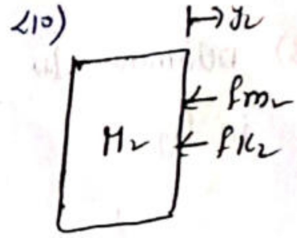
$$M_1\frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1y_1 + K_2(y_1-y_2) = f(t) \rightarrow (1)$$

on taking Laplace transform to above equation at zero initial conditions, we get.

$$M_1s^2Y_1(s) + BsY_1(s) + K_1Y_1(s) + K_2(Y_1(s)-Y_2(s)) = F(s)$$

$$Y_1(s)[M_1s^2+Bs+K_1+K_2] - Y_2(s)K_2 = F(s) \rightarrow (2)$$

case 2 :- The free body diagram is shown in fig. The opposing forces acting on mass M_2 are marked as f_{m_2}, f_{K_2}



here $f_{m_2} = M_2 \frac{d^2 y_2}{dt^2}$ $\quad f_{K_2} = K_2 (y_2 - y_1)$.

By newton's second law

$$f_{m_2} + f_{K_2} = 0$$

$$\therefore \quad M_2 \frac{d^2 y_2}{dt^2} + K_2 (y_2 - y_1) = 0 \rightarrow (3)$$

on taking Laplace transform to above Equation with zero initial conditions we get

$$M_2 s^2 Y_2(s) + K_2 Y_2(s) - K_2 Y_1(s) = 0$$

$$Y_2(s)\left[M_2 s^2 + K_2\right] = K_2 Y_1(s)$$

$$Y_1(s) = \frac{M_2 s^2 + K_2}{K_2} Y_2(s) \rightarrow (4)$$

Substitute Equ. (4) in Equ. (2)

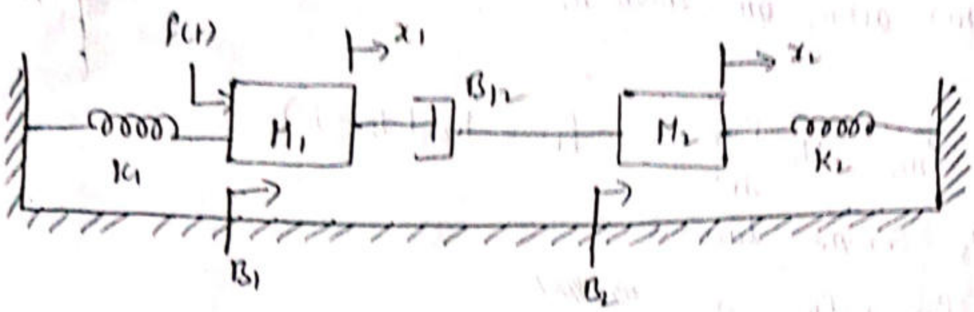
$$Y_2(s) \frac{M_2 s^2 + K_2}{K_2} \cdot \left[M_1 s^2 + Bs + K_1 + K_2\right] - Y_2(s) K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2)}{K_2} - K_2\right] = F(s)$$

$$Y_2(s)\left[\frac{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2}{K_2}\right] = F(s)$$

$$\boxed{\frac{Y_2(s)}{F(s)} = \frac{K_2}{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2}}$$

3) Determine the transfer function, $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$ for the system shown in fig.

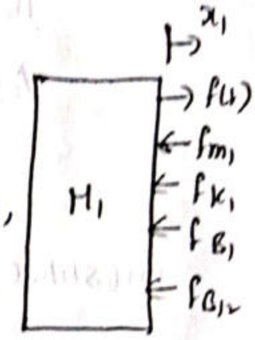


Sol: Let Laplace transform of $f(t) = \mathcal{L}[f(t)] = F(s)$

Laplace transform of $x_1 = \mathcal{L}[x_1] = X_1(s)$

Laplace transform of $x_2 = \mathcal{L}[x_2] = X_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential eqn. of governing system is given by force balance equations.



Case: 1 :- The free body diagram of mass M_1 is shown in fig.

The opposing forces on mass M_1 are marked as $f_{m_1}, f_{B_1}, f_{B_{12}}, f_{K_1}$.

here, $f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$, $f_{B_1} = B_1 \frac{dx_1}{dt}$ $f_{B_{12}} = \frac{d}{dt}(x_1 - x_2)$

$f_{K_1} = K_1 x_1$

By newton's second law

$$f_{m_1} + f_{B_1} + f_{B_{12}} + f_{K_1} = f(t)$$

$$\therefore \quad M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1 x_1 = f(t) \rightarrow (1)$$

on taking Laplace transform to the above equation with zero initial conditions we get,

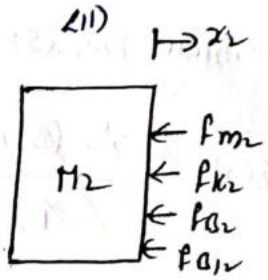
$$M_1 s^2 X_1(s) + B_1 s X_1(s) + B_{12} s [X_1(s) - X_2(s)] + K_1 X_1(s) = F(s)$$

$$X_1(s) [M_1 s^2 + B_1 s + B_{12} s + K_1] - X_2(s) B_{12} s = F(s)$$

$$X_1(s) [M_1 s^2 + (B_1 + B_{12}) s + K_1] - B_{12} s X_2(s) = F(s) \longrightarrow (2)$$

case 2:-

The free body diagram of mass M_2 as shown in fig. The opposing forces acting on mass M_2 are marked as f_{m_2}, f_{B_2}, $f_{B_{12}}$, f_{k_2}



here $f_{m_2} = M_2 \dfrac{d^2x_2}{dt^2}$, $f_{B_2} = B_2 \dfrac{dx_2}{dt}$, $B_{12} = B_{12}\dfrac{d}{dt}(x_2 - x_1)$, $f_{k_2} = k_2 x_2$

By newton's second Law,

$$f_{m_2} + f_{B_2} + f_{B_{12}} + f_{k_2} = 0$$

$$\therefore \quad M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12}\frac{d}{dt}(x_2 - x_1) + k_2 x_2 = 0 \quad \rightarrow \langle 3 \rangle$$

on taking Laplace transform to above Equation with zero initial conditions we get

$$M_2 s^2 X_2(s) + B_2 s X_2(s) + B_{12} s\left(X_2(s) - X_1(s)\right) + k_2 X_2(s) = 0$$

$$X_2(s)\left[M_2 s^2 + B_2 s + B_{12} s + k_2\right] = B_{12} s X_1(s)$$

$$X_2(s) = X_1(s) \frac{B_{12} s}{M_2 s^2 + (B_2 + B_{12})s + k_2} \quad \rightarrow \langle 4 \rangle$$

$$X_1(s) = X_2(s) \frac{M_2 s^2 + (B_2 + B_{12})s + k_2}{B_{12} s} \quad \rightarrow \langle 5 \rangle$$

Substitute Equation $\langle 4 \rangle$ in equation $\langle 2 \rangle$ we get.

$$X_1(s)\left[M_1 s^2 + (B_1 + B_{12})s + k_1\right] - B_{12} s \cdot \frac{B_{12} s}{M_2 s^2 + (B_2 + B_{12})s + k_2} \cdot X_1(s) = F(s)$$

$$X_1(s)\left[M_1 s^2 + (B_1 + B_{12})s + k_1 - \frac{(B_{12} s)^2}{M_2 s^2 + (B_2 + B_{12})s + k_2}\right] = F(s)$$

$$X_1(s)\left[\frac{(M_1 s^2 + (B_1 + B_{12})s + k_1)(M_2 s^2 + (B_2 + B_{12})s + k_2) - (B_{12} s)^2}{M_2 s^2 + (B_2 + B_{12})s + k_2}\right] = F(s)$$

$$\boxed{\frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_{12})s + k_2}{(M_1 s^2 + (B_1 + B_{12})s + k_1)(M_2 s^2 + (B_2 + B_{12})s + k_2) - (B_{12} s)^2}}$$

Substitute Equ...(5) in Equ. (2) we get

$$X_2(S) \cdot \cancel{\frac{M_2S^2 + (B_2+B_{12})S + K_2}{B_{12}S}} - \cancel{B_{12}S\,X_2(S)}$$

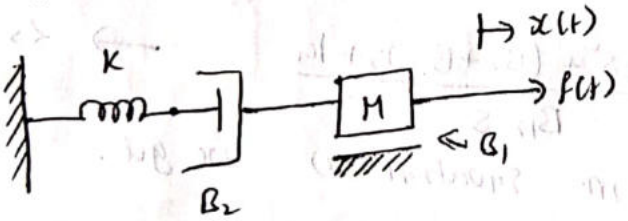
$$X_2(S) \cdot \frac{(M_2S^2 + (B_2+B_{12})S + K_2)}{B_{12}S} \cdot (M_1S^2 + (B_1+B_{12})S + K_1) - B_{12}S\,X_2(S) = F(S)$$

$$X_2(S)\left[\frac{(M_2S^2 + (B_2+B_{12})S + K_2)(M_1S^2 + (B_1+B_{12})S + K_1)}{B_{12}S} - B_{12}S\right] = F(S)$$

$$X_2(S)\left[\frac{(M_1S^2 + (B_1+B_{12})S + K_1)(M_2S^2 + (B_2+B_{12})S + K_2) - (B_{12}S)^2}{B_{12}S}\right] = F(S)$$

$$\boxed{\frac{X_2(S)}{F(S)} = \frac{B_{12}S}{(M_1S^2 + (B_1+B_2)S + K_1)(M_2S^2 + (B_2+B_{12})S + K_2) - (B_{12}S)^2}}$$

4) Write the Equations of motion in S-domain for the system shown in fig. Determine the transfer function of the system



Sol, Let Laplace transform of $x(t) = L[x(t)] = X(S)$

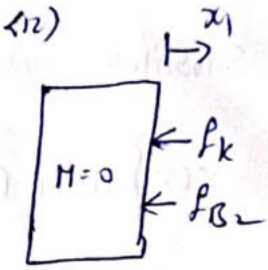
Laplace transform of $f(t) = L[f(t)] = F(S)$

Let X_1 be the displacement at the meeting point of spring and dash pot.

Laplace transform of $x_1(t) = L[x_1] = X_1(S)$

The system has two nodes and they are mass M and meeting point of spring and dash pot (Not having mass i.e, $M=0$). The differential equations of the governing system can be obtained from force balance equation.

Case-1 :- The free body diagram of meeting point of Spring and dash pot as shown in fig. and it has no mass. The opposing forces acting on this point marked as f_{B_2}, f_k.



here $f_{B_2} = B_2 \frac{d}{dt}(x_1 - x)$ $\quad f_k = k x_1$

By newton's second law

$$f_{B_2} + f_k = 0$$

$$\therefore \quad B_2 \frac{d}{dt}(x_1 - x) + k x_1 = 0 \quad \rightarrow (1)$$

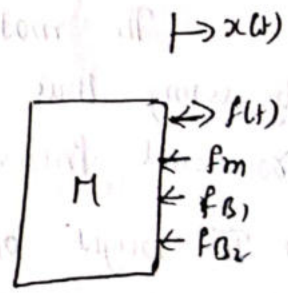
on taking Laplace transform to above Equation with zero initial conditions, we get

$$B_2 s\left[X_1(s) - X(s)\right] + k X_1(s) = 0$$

$$X_1(s)\left[B_2(s) + k\right] = B_2 s\, X(s)$$

$$X_1(s) = \frac{B_2 s}{B_2 s + k} \cdot X(s) \quad \rightarrow (2)$$

Case:2 :- The free body diagram of mass M as shown in fig. The opposing forces acting on mass M, are marked as f_m, f_{B_1}, f_{B_2}.



here $f_m = M \frac{d^2x(t)}{dt^2}$ $\quad f_{B_1} = B_1 \frac{dx(t)}{dt}$, $\quad f_{B_2} = B_2 \frac{d}{dt}(x - x_1)$,

By newton's second Law.

$$f_m + f_{B_1} + f_{B_2} = f(t)$$

$$\therefore \quad M \frac{d^2x(t)}{dt^2} + B_1 \frac{dx(t)}{dt} + B_2 \frac{d}{dt}(x - x_1) = f(t) \quad \rightarrow (3)$$

on taking Laplace transform to above Equation with zero initial conditions we get

$$Ms^2X(s) + B_1 s X(s) + B_2 s X(s) - B_2 s X_1(s) = F(s)$$

$$X(s)\left[Ms^2 + (B_1 + B_2)s\right] - B_2 s\, X_1(s) = F(s) \quad \rightarrow (4)$$

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Substitute Equation (2) in Equation (4) we get.

$$X(s)\left[Ms^2+(B_1+B_2)s\right]-B_2 s\cdot\frac{B_2 s}{B_2 s+K}\cdot X(s)=F(s)$$

$$X(s)\left[(Ms^2+(B_1+B_2)s)-\frac{(B_2 s)^2}{B_2 s+K}\right]=F(s)$$

$$X(s)\left[\frac{(Ms^2+(B_1+B_2)s)(B_2 s+K)-(B_2 s)^2}{B_2 s+K}\right]=F(s)$$

$$\boxed{\frac{X(s)}{F(s)}=\frac{B_2 s+K}{(Ms^2+(B_1+B_2)s)(B_2 s+K)-(B_2 s)^2}}$$

* Mechanical rotational Systems :-

The model of mechanical rotational systems can be obtained by using three elements. moment of inertia [J] of mass, dash-pot [B] with rotational frictional co-efficient, and torsional spring [K] with stiffness.

→ The weight of the rotational mechanical system is represented by the moment of inertia of the mass.

→ The elastic deformation of the body can be represented by a spring.

→ The friction existing in ~~rotated~~ rotational mechanical system can be represented by dash-pot.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by "Newton's second law of motion" for rotational systems. It states that the sum of torque acting on a body is zero. (or Newton's law states that the sum of applied torques is equals to the sum of opposing torques on a body.

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List of Symbols used in mechanical rotational System :-

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque N-m

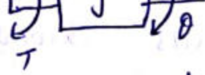
J = moment of inertia, kg-m^2/rad

B = Rotational frictional co-efficient, N-m/(rad/sec)

K = stiffness of Spring N-m/rad.

Torque balanced Equations of idealized Elements :-

i) consider an ideal mass element as shown in fig. which has negligible friction and elasticity. The "opposing torque due to moment of inertia" is directly proportional to angular acceleration.



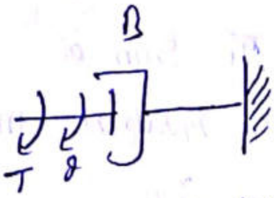
ideal rotational mass element.

$$T_j \propto \frac{d^2\theta}{dt^2} \quad (or) \quad T_j = J\frac{d^2\theta}{dt^2}$$

By newton's second law

$$\left[T = T_j = J\frac{d^2\theta}{dt^2}\right]$$

ii) consider an ideal frictional dash-pot element is shown in fig, which has negligible moment of inertia and elasticity. The dash pot will offer an opposing torque is directly proportional to angular velocity of the body.



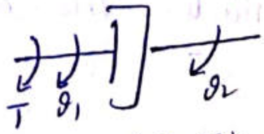
ideal rotational dash pot with one end fixed.

$$T_B \propto \frac{d\theta}{dt} \quad (or) \quad T_B = B\frac{d\theta}{dt}$$

By newton's second Law

$$\left[T = T_B = B\frac{d\theta}{dt}\right]$$

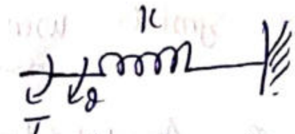
iii) when the dash pot has angular displacement at both ends as shown in fig. the opposing torque is proportional to differential angular velocity.



ideal dash-pot with angular displacement at both ends.

$$T_b \propto \frac{d}{dt}(\theta_1-\theta_2) \quad (or) \quad \left[T = T_B = B\frac{d}{dt}(\theta_1-\theta_2)\right]$$

iv) Consider an ideal elastic element, torsional spring which has negligible moment of inertia and friction. The spring will offer opposing torque is proportional to angular displacement.



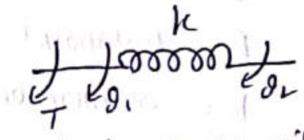
ideal spring with one end fixed to reference.

$$T_K \propto \theta \quad (\text{or}) \quad T_K = K\theta$$

By newton's second law,

$$\left[T = T_K = K\theta\right]$$

v) When a spring has angular displacement at both ends as shown in fig. the opposing torque is proportional to differential angular displacement.



ideal spring with angular displacement at both ends.

$$T_K \propto (\theta_1 - \theta_2) \quad (\text{or}) \quad T_K = K(\theta_1 - \theta_2)$$

By newton's second law

$$\left[T = T_K = K(\theta_1 - \theta_2)\right]$$

Guide to determine the Transfer function of mechanical rotational system:-

→ Draw the free body diagram of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that mass.

→ The sum of opposing torque is equal to applied can be write its equation in the form of differential equation.

→ The first derivative of the angular displacement is angular velocity and second derivative of the angular displacement is angular acceleration.

→ Take Laplace transform of differential equations to convert them to algebric equations. Then rearranging the s-domain equation to eliminate unwanted variables and obtain the ratio between o/p variable and i/p variable. This ratio is the transfer function of the system.

$$L[\theta] = \theta(s)$$

$$L\left[\frac{d\theta}{dt}\right] = s\theta(s) \quad (\text{with zero initial conditions})$$

$$L\left[\frac{d^2\theta}{dt^2}\right] = s^2\theta(s) \quad (\quad " \quad)$$

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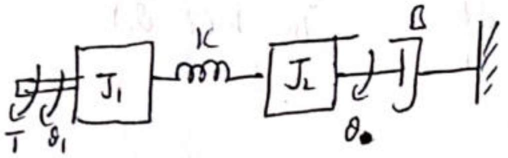
Problems based on mechanical rotational system :-

1) Write the differential equations governing the mechanical rotational system shown in fig. Obtain the transfer function of the system.



sol In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of $T = L[T] = T(s)$

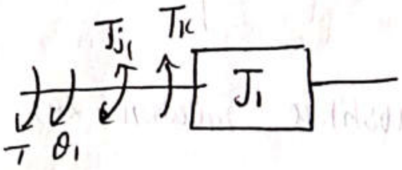
Laplace transform of $\theta = L[\theta] = \theta(s)$

Laplace transform of $\theta_1 = L[\theta_1] = \theta_1(s)$

Hence required transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 & J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Case 1 :- The free body diagram of mass with moment of inertia J_1 is shown in fig.



The opposing torque acting on J_1 are T_{J_1} and T_K

here $T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}$ $\quad T_K = K(\theta_1 - \theta)$

By newton's second law

$$T_{J_1} + T_K = T$$

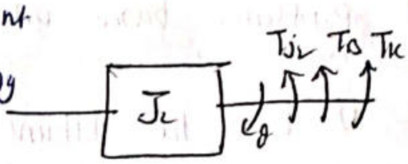
$$\therefore J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T \rightarrow (1)$$

on taking Laplace transform to above equation with zero initial conditions we have,

$$J_1 s^2 \theta_1(s) + K(\theta_1(s) - \theta(s)) = T(s)$$

$$\left[\theta_1(s)\left[J_1 s^2 + K\right] - K\theta(s) = T(s)\right] \rightarrow (2)$$

Case : 2 :- The free body diagram of mass with moment of inertia J_2 shown in fig. The opposing torques acting on J_2 are marked as T_{j2}, T_B, T_K



$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \qquad T_B = B\frac{d\theta}{dt} \qquad T_K = K(\theta - \theta_1)$$

By newton's second law

$$T_{j2} + T_B + T_K = 0$$

$$\therefore \; J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0 \rightarrow (3)$$

on taking Laplace transform to above Equation with zero initial conditions, we get

$$J_2 s^2\theta(s) + Bs\,\theta(s) + K\theta(s) = K\theta_1(s)$$

$$\theta(s)\left[J_2 s^2 + Bs + K\right] = K\theta_1(s)$$

$$\left[\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s)\right] \rightarrow (4)$$

Substitute Equation (4) in Equ. (2) we get

$$\theta(s)\,\frac{(J_2 s^2 + Bs + K)}{K}\cdot(J_1 s^2 + K) - K\theta(s) = T(s)$$

$$\theta(s)\left[\frac{(J_1 s^2 + K)(J_2 s^2 + Bs + K)}{K} - K\right] = T(s)$$

$$\theta(s)\left[\frac{(J_1 s^2 + K)(J_2 s^2 + B_2 s + K) - K^2}{K}\right] = T(s)$$

$$\boxed{\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)(J_2 s^2 + B_2 s + K) - K^2}}$$

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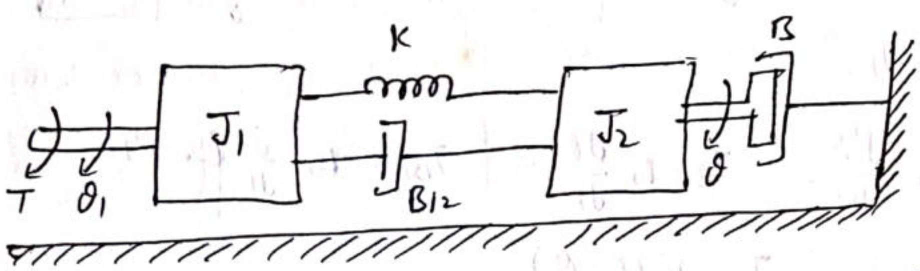
2) Write the differential equations governing the mechanical rotational system shown in fig. and determine the transfer function $\theta(s)/T(s)$ (15)



sol In the given system Torque T is input and angular displacement θ is the output.

Let Laplace transform of $T = L[T] = T(s)$

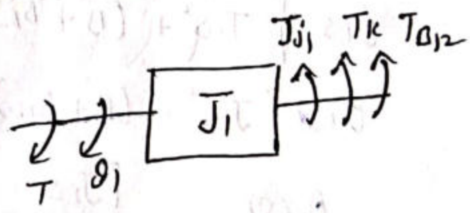
Laplace transform of $\theta_1 = L[\theta_1] = \theta_1(s)$

Laplace transform of $\theta = L[\theta] = \theta(s)$

Hence the required transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Case 1:- The free body diagram of J_1 as shown in fig. The opposing forces are marked as T_{J1}, T_K, T_{B12}



here $T_{J1} = J_1 \frac{d^2\theta_1}{dt^2}$ $T_{B12} = B_{12} \, d/dt \, (\theta_1 - \theta)$, $T_K = K(\theta_1 - \theta)$

By Newton's second Law:

$$T_{J1} + T_{B12} + T_K = T$$

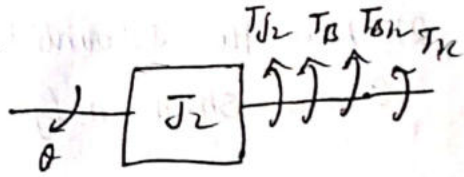
$$\therefore J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T \rightarrow ①$$

on taking Laplace transform to above equation with zero initial conditions we get

$$J_1 s^2 \theta_1(s) + B_{12} s(\theta_1(s) - \theta(s)) + K(\theta_1(s) - \theta(s)) = T(s)$$

$$\theta_1(s)[J_1 s^2 + B_{12} s + K] - \theta(s)[B_{12} s + K] = T(s) \rightarrow ②$$

Case:2:- The free diagram of J_2 is shown in fig. The opposing forces are acting on J_2 are marked as T_{J2}, T_B, T_{B12}, T_K.



here $T_{J2} = J_2 \frac{d^2\theta}{dt^2}$ $\quad T_B = B\frac{d\theta}{dt}$ $\quad T_{B12} = B_{12}\frac{d}{dt}(\theta - \theta_1)$

$$T_K = K(\theta - \theta_1)$$

By newton's second law.

$$T_{J2} + T_{B12} + T_B + T_K = 0$$

$$\therefore J_2\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + B_{12}\frac{d}{dt}(\theta-\theta_1) + K(\theta-\theta_1) = 0 \quad \rightarrow (3)$$

on taking Laplace transform to above Equation with zero initial conditions we get

$$J_2 s^2\theta(s) + Bs\theta(s) + B_{12}s\theta(s) - B_{12}s\theta_1(s) + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\left[J_2s^2 + (B+B_{12})s + K\right] - \theta_1(s)\left[B_{12}s + K\right] = 0$$

$$\theta(s)\left[J_2s^2 + (B+B_{12})s + K\right] = \theta_1(s)\left[B_{12}s+K\right]$$

$$\theta_1(s) = \frac{\left[J_2s^2 + (B+B_{12})s + K\right]}{(B_{12}s+K)}\theta(s) \quad \rightarrow (4)$$

substitute Equation (4) in Equ. (2) we get

$$\frac{\left[J_2s^2 + (B+B_{12})s + K\right]}{(B_{12}s+K)}\theta(s)\cdot(J_1s^2 + B_{12}s + K) - \theta(s)(B_{12}s+K) = T(s)$$

$$\theta(s)\left[\frac{\left[J_2s^2 + (B+B_{12})s + K\right](J_1s^2 + B_{12}s + K) - (B_{12}s+K)^2}{B_{12}s+K}\right] = T(s)$$

$$\boxed{\frac{\theta(s)}{T(s)} = \frac{sB_{12} + K}{(J_2s^2 + (B+B_{12})s + K)(J_1s^2 + B_{12}s + K) - (B_{12}s+K)^2}}$$

* Electrical systems :-

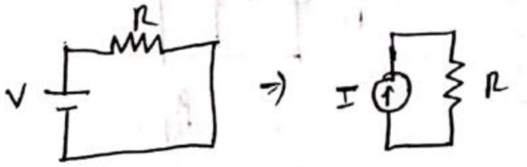
The models of electrical systems can be obtained by using 'resistor', 'inductor' and 'capacitor'. The current - voltage relation of R, L & C are given in Table. For modeling electrical system, the electrical network (or) Equivalent circuit is formed by using R, L and C and "voltage (or) current source."

The differential Equations governing the electrical systems can be formed by writing "KCL" Equations by choosing various nodes in the n/w (or) "KVL" Equations by choosing various closed paths in the network. The transfer function can be obtained by taking Laplace transform of the differential Equations and rearranging them as a ratio of o/p to i/p.

Current - voltage relation of R, L and C :-

Element	Voltage across the element	Current through the element
$i(t)$ R, $v(t)$	$v(t) = R\,i(t)$	$i(t) = \dfrac{v(t)}{R}$
$i(t)$ L, $v(t)$	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = \dfrac{1}{L}\int v(t)\,dt$
$i(t)$ C, $v(t)$	$v(t) = \dfrac{1}{C}\int i(t)\,dt$	$i(t) = C\dfrac{dv(t)}{dt}$

Note :- The differential Equations of electrical network can be found by applying KVL (or) KCL (by using Source transformation Technique).

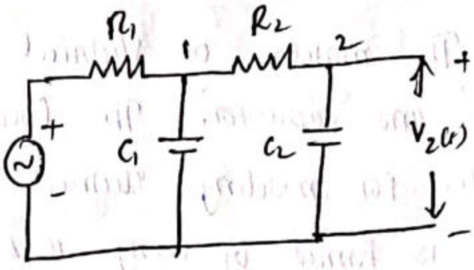


here $I = V/R$.

(V to I)

Problems based on Electrical Systems:-

1) Obtain the transfer function of the electrical network shown in fig.



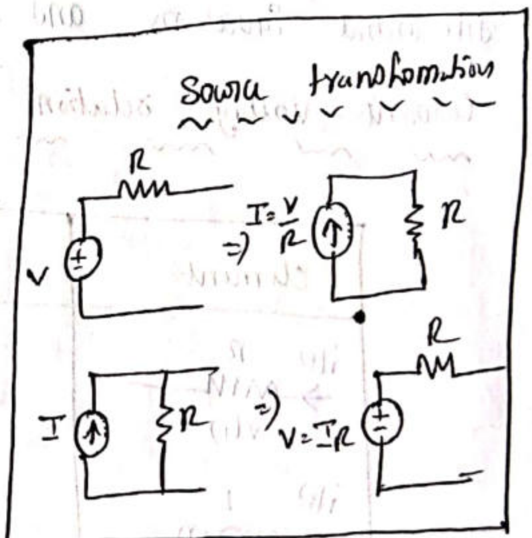
Sol In given network i/p is $e(t)$ and o/p is $V_2(t)$

Let Laplace transform of $e(t) = L[e(t)] = E(s)$

Laplace transform of $v_2(t) = L[v_2(t)] = V_2(s)$

The Transfer function of the network is $\frac{V_2(s)}{E(s)}$

Transform the voltage source in series with resistance R_1 into Equivalent current source as shown in fig.1. The network has two nodes.



Let the node voltages v_1 and v_2. The Laplace transform of node voltages v_1 and v_2 are $V_1(s)$ and $V_2(s)$ respectively. The differential equations governing the network are given by KCL equations at these nodes.

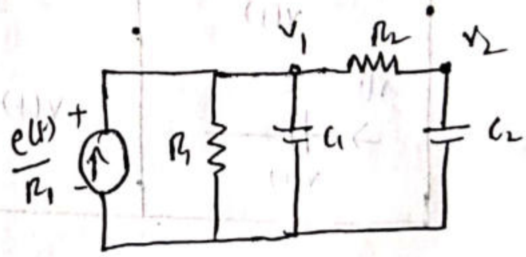


fig.1

Case:1

At node-1 by KCL

from fig.2

$$\frac{e(t)}{R_1} = \frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} \rightarrow (1)$$

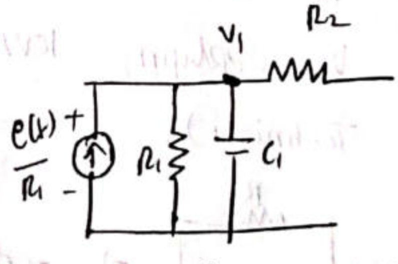


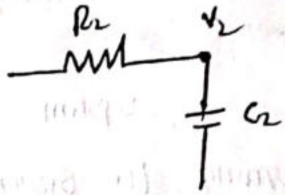
fig.2

On taking Laplace transform to above equation we get

$$\frac{E(s)}{R_1} = \frac{V_1(s)}{R_1} + C_1 s V_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2}$$

$$V_1(s)\left[\frac{1}{R_1} + C_1 s + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \rightarrow (2)$$

Case : 2 :- At node-2 by KCL



From fig (3)

$$\frac{V_2 - V_1}{R_2} + C_2 \frac{dV_2}{dt} = 0 \rightarrow (3)$$

On taking Laplace transform to above Equ. we get

$$\frac{V_2(S)}{R_2} - \frac{V_1(S)}{R_2} + C_2 S V_2(S) = 0$$

$$V_2(S)\left[\frac{1}{R_2} + C_2 S\right] = \frac{V_1(S)}{R_2}$$

$$V_1(S) = R_2\left[\frac{1}{R_2} + C_2 S\right] V_2(S)$$

$$V_1(S) = (1 + SC_2R_2)\, V_2(S) \rightarrow (4)$$

Substitute Equation (4) in Equation (2)

$$V_2(S)(1 + SC_2R_2)\left[\frac{1}{R_1} + C_1 S + \frac{1}{R_2}\right] - \frac{V_2 S}{R_2} = \frac{E(S)}{R_1}$$

$$V_2(S)\left[(1 + SC_2R_2)\left(\frac{1}{R_1} + C_1S + \frac{1}{R_2}\right) - \left[\frac{1}{R_2}\right]\right] = \frac{E(S)}{R_1}$$

$$V_2(S)\left[\frac{(1+SC_2R_2)(R_1+R_2+SC_1R_1R_2) - R_1}{\cancel{R_1} R_2}\right] = \frac{E(S)}{\cancel{R_1}}$$

$$\boxed{\frac{V_2(S)}{E(S)} = \frac{R_2}{(1+SC_2R_2)(R_1+R_2+SC_1R_1R_2) - R_1}}$$

* Electrical Analogous of mechanical Translational System :-

System remain analogous as long as the differential Equations governing the systems. The Electrical analogue of any other kind of System is of greater importance. Since it is Easier to construct Electrical models and analyse them.

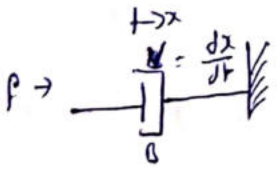
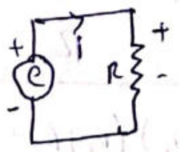
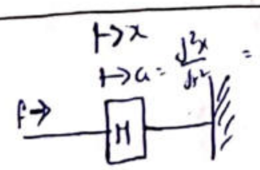
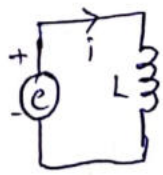
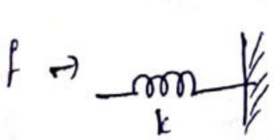
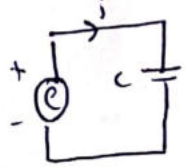
The three basic Elements mass, dash-pot, spring that are using in modelling mechanical translational systems are analogous to resistance, inductance capacitance of Electrical System.

The i/p force in mechanical System is analogous to either voltage source (or) current source of Electrical Systems. The o/p velocity in mechanical system is analogous to either current (or) voltage in an element in Electrical System. Since the Electrical systems has two types of inputs either voltage (or) current source, there are two types of analogies

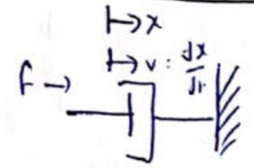
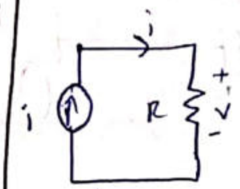
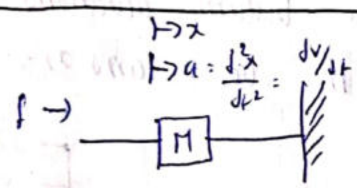
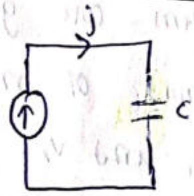
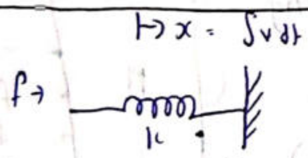
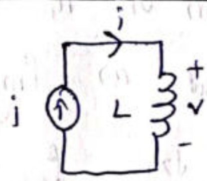
→ Force - voltage analogy

→ Force - current analogy.

Analogous Elements in Force - voltage Analogy :- (Elements in Series with Source)

Mechanical System	Electrical System.
Input : Force Output : velocity	Input : voltage source Output : current through an element
 $v = \frac{dx}{dt}$, $f = B\frac{dx}{dt} = Bv$	 $e = v$ and $v = Ri$ $e = Ri$
 $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$ $F = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $e = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $x = \int v\,dt$ $F = Kx = K\int v\,dt$	 $e = v$ and ~~$e = L\frac{di}{dt}$~~ $e = \frac{1}{C}\int i\,dt$

Analogous Elements in Force - current Analogy :- (elements in (are) parallel with source)

Mechanical System	Electrical System.
Input : force output : velocity	input : current source. output : Voltage across the elements
 $f = B\frac{dx}{dt} \Rightarrow f = Bv$	 $i = V/R$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $i = C\frac{dv}{dt}$
 $f = Kx = K\int v\,dt$	 $i = \frac{1}{L}\int v\,dt$

Summary :-

force - voltage :- (Series elements) (Equations interms of current)

velocity (v) = current (i)
force (f(t)) = Voltage e(t)
Dash pot (B) : Resistance (R)
Mass (M) = Inductance (L)
Spring (K) : $\frac{1}{\text{Capacitance}}$ (1/C)

force - current : (parallel elements) (Equations interms of voltage)

velocity (v) : voltage (v)
force f(t) : current i(t)
Dash pot (B) : $\frac{1}{\text{Resistance}}$ (1/R)
Mass (M) : Capacitance (C)
Spring (K) = $\frac{1}{\text{inductance}}$ (1/L).

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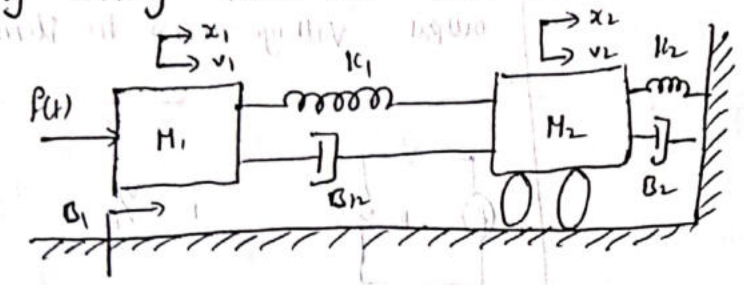
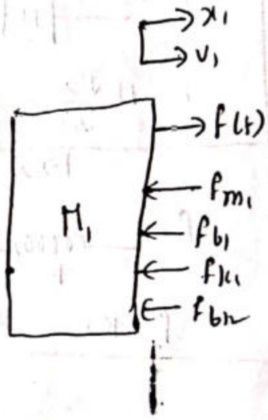
Problems based Analogous of Mechanical translational System :-

1) Write the differential Equations governing the mechanical System shown in fig. Draw the force-voltage and force-current electrical Analogous circuits and verify by writing mesh and node Equations.



Sol, The given mechanical System has two nodes (masses). The differential Equations governing the System are given by force balance Equations at these nodes. Let the displacement of masses M_1 and M_2 be x_1 and x_2. The corresponding velocities be v_1 and v_2.



Case 1 :- The free body diagram of M_1 is shown in fig
The opposing forces are marked as $f_{m_1}, f_{b_1}, f_{b_{12}}, f_{k_1}$

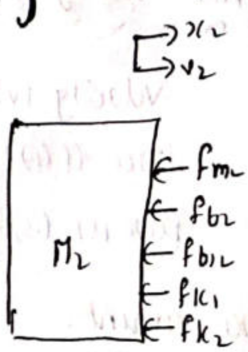
here $f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$ $\quad f_{b_1} = B_1 \frac{dx_1}{dt}$ $\quad f_{B_{12}} = B_{12} \frac{d}{dt}(x_1 - x_2)$

$$f_{k_1} = K_1(x_1 - x_2)$$

By newton's Second law,

$$f_{m_1} + f_{b_1} + f_{b_{12}} + f_k = f(t)$$

$$\left[M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \right] \rightarrow (1)$$



Case 2 :- The free body diagram of M_2 is shown in fig.
The opposing forces are marked as $f_{m_2}, f_{b_2}, f_{b_{12}}, f_{k_1}, f_{k_2}$

here $f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$, $\quad f_{b_2} = B_2 \frac{dx_2}{dt}$ $\quad f_{b_{12}} = B_{12} \frac{d}{dt}(x_2 - x_1)$

$$f_{k_1} = K_1(x_2 - x_1) \qquad f_{k_2} = K_2 x_2$$

By newton's Second law.

$$f_{m_2} + f_{b_2} + f_{b_{12}} + f_{k_1} + f_{k_2} = 0$$

$$\left[M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) + K_2 x_2 = 0 \right] \rightarrow (2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get.

$$\left[\frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \ , \quad x = \int v\,dt\right]$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\,dt = f(t) \quad \rightarrow (3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\,dt + K_2 \int v_2\,dt = 0 \quad \rightarrow (4)$$

Force voltage Analogous Circuit :-

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit have two meshes (Loops)

The force applied to mass M_1 is represent by a voltage source in first mesh. The elements M_1, B_{12}, K_1 and B_{12} are connected together in mesh-1 forming a closed path. The elements M_2, B_{12} K_1 K_2 B_2 are connected together in mesh-2 forming a closed path.

The elements K_1 and B_{12} are common b/w M_1 and M_2 so they are represented by analogous elements as common elements b/w two meshes.

$f(t) \rightarrow e(t)$	$B_1 = R_1$	$M_1 = L_1$	$K_1 = 1/C_1$
velocity $(v_1) \rightarrow i_1$	$B_2 = R_2$	$M_2 = L_2$	$K_2 = 1/C_2$
velocity $(v_2) \rightarrow i_2$	$B_{12} = R_{12}$		

The Equ. (3) and (4) can be write in electrical forms.

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = e(t) \quad \rightarrow (5)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt + \frac{1}{C_2}\int i_2\,dt = 0 \quad \rightarrow (6)$$

The electrical analogous circuits for Equation (5) and Equation (6) as shown in fig.1 and fig.2 and the force-voltage electrical analogous circuit is shown in fig.3

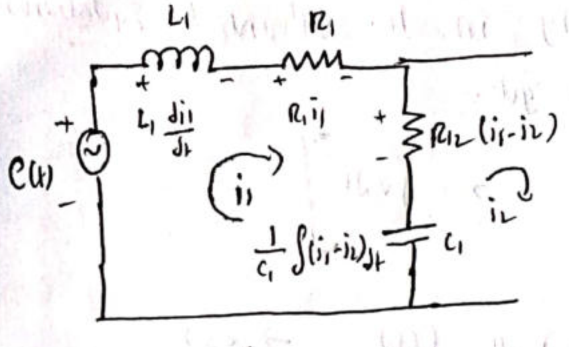


fig-1

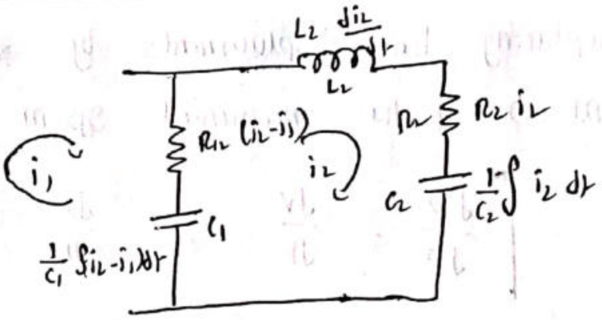


fig (2)

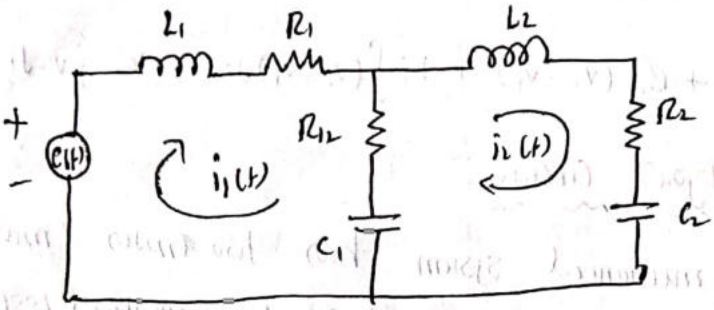


fig (3).

[Note 1. Force - voltage from LOOPS]

Force - current Analogous circuit :-

The given mechanical system has two nodes (masses). Hence force - current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source in analogous electrical circuit. The Elements M_1, B, K_1 & B_{12} connected to node-1 in analogous electrical circuit are connected to node-2 are connected to node-1 & The Elements $M_2, B_{12}, B_2, K_1, K_2$ by analogous elements.

The Elements K_1 and B_{12} are common b/w mass-1 and mass-2. So they are represented by analogous elements as common element b/w two nodes in analogous circuit.

$f(t) \rightarrow i(t)$ $\quad B_1 = 1/R_1$ $\quad M_1 = C_1$ $\quad K_1 = 1/L_1$

velocity $v_1 \rightarrow v_1$ (voltage) $\quad B_{12} = 1/R_2$ $\quad M_2 = C_2$ $\quad K_2 = 1/L_2$

velocity $v_2 \rightarrow v_2$ (voltage)

The Equations (3) and (4) can be represent in electrical form.

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{R_{12}} (v_1 - v_2) + \frac{1}{L_1} \int (v_1 - v_2) dt = i(t) \rightarrow (7)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{R_{12}} (v_2 - v_1) + \frac{1}{L_1} \int (v_2 - v_1) dt + \frac{1}{L_2} \int v_2 dt = 0 \rightarrow (8)$$

The Electrical analogous circuits for Equation (7) and Equation (8) as shown in fig.4 and fig.5 and The force current electrical analogous circuit is shown in fig.6.

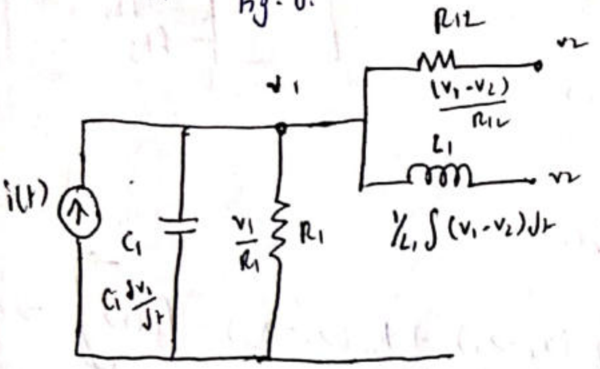


fig (4)

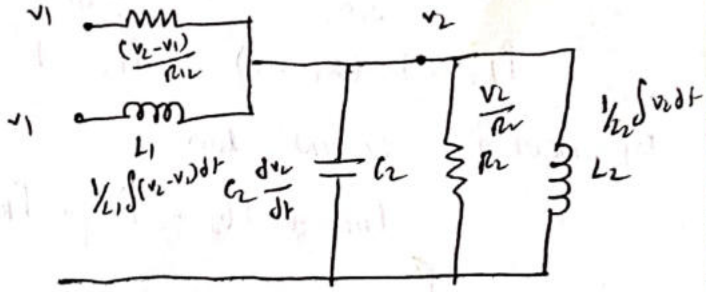


fig (5)

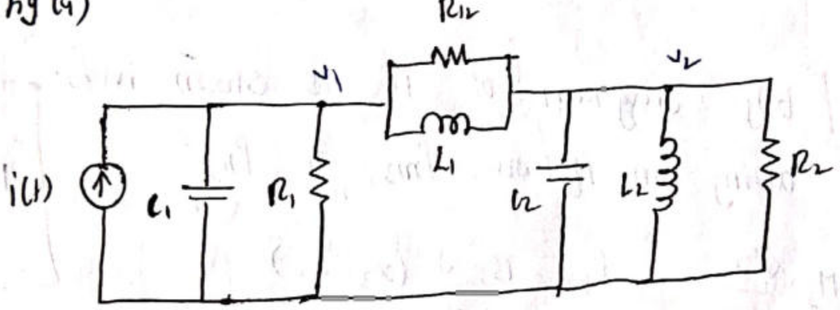
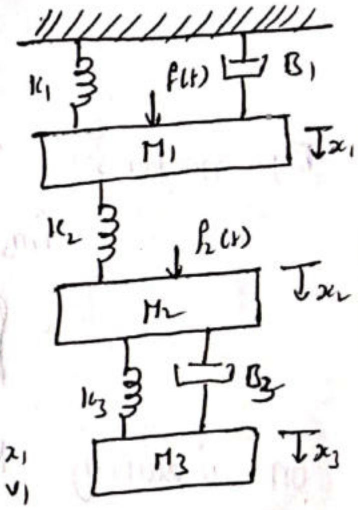
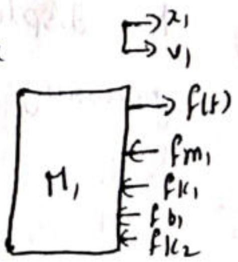


fig (6).

2) Write the differential Equations governing the System shown in fig. Draw the force - voltage and force - current electrical Analogous circuits and verify by writing mesh and node Equations.



Sol Case : 1 :- The free body diagram of mass M_1 is shown in fig. The opposing forces on M_1 are marked as f_{m_1}, f_{b_1}, f_{k_1}, f_{k_2}.



here $f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$ $\quad f_{b_1} = B_1 \frac{dx_1}{dt}$

$f_{k_1} = K_1 x_1$ $\quad f_{k_2} = K_2 (x_1 - x_2)$

by newton's second law.

$$f_{m_1} + f_{b_1} + f_{k_1} + f_{k_2} = f_1(t)$$

$$\therefore \left[M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 + K_2 (x_1 - x_2) = f_1(t) \right] \rightarrow ①$$

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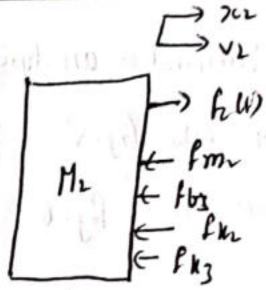
Case : 2 :- The free body diagram of M_2 is shown in fig. The opposing forces on M_2 are marked as $f_{m_2}, f_{b_3}, f_{k_2}, f_{k_3}$



here $f_{m_2} = M_2 \frac{d^2x_2}{dt^2}$ $\quad f_{b_3} = B_3 \frac{d}{dt}(x_2 - x_3)$

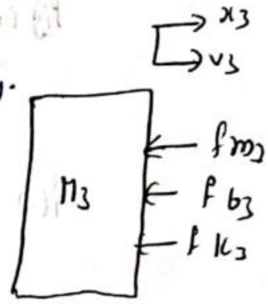
$f_{k_2} = k_2(x_2 - x_1)$ $\quad f_{k_3} = k_3(x_2 - x_3)$

By newton's second law

$$f_{m_2} + f_{b_3} + f_{k_2} + f_{k_3} = f_2(t)$$

$$\therefore \left[M_2 \frac{d^2x_2}{dt^2} + B_3 \frac{d}{dt}(x_2 - x_3) + k_2(x_2 - x_1) + k_3(x_2 - x_3) = f_2(t) \right] \rightarrow (2)$$

Case : 3 :- The free body diagram of M_3 is shown in fig. The opposing forces acting on M_3 are $f_{m_3}, f_{b_3}, f_{k_3}$



here $f_{m_3} = M_3 \frac{d^2x_3}{dt^2}$ $\quad f_{b_3} = B_3 \frac{d}{dt}(x_3 - x_2)$

$f_{k_3} = k_3(x_3 - x_2)$

By newton's second law,

$$f_{m_3} + f_{b_3} + f_{k_3} = 0$$

$$\therefore \left[M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_2) + k_3(x_3 - x_2) = 0 \right] \rightarrow (3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} \qquad \frac{dx}{dt} = v \qquad x = \int v\,dt$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + k_1 \int v_1\,dt + k_2 \int (v_1 - v_2)\,dt = f_1(t) \rightarrow (4)$$

$$M_2 \frac{dv_2}{dt} + B_3(v_2 - v_3) + k_2 \int (v_2 - v_1)\,dt + k_3 \int (v_2 - v_3)\,dt = f_2(t) \rightarrow (5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_2) + k_3 \int (v_3 - v_2) = 0 \rightarrow (6)$$

Force - voltage analogous circuit :-

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes (Loops). The force applied to mass M_1 is represented by a voltage source in first mesh and force applied to mass M_2 is represented by a voltage source in second mesh.

The elements M_1, B_1, K_1, K_2 are connected in Loop-1, the elements M_2, B_3, K_2 K_3 are connected in Loop-2 and the elements M_3, B_3, K_3 are connected in Loop-3 in the form of electrical components.

The element K_2 is common for loop-1 and loop-2

The elements K_3, B_3 are common for loop-2 and loop-3.

The electrical analogous circuits for Eq(7), (8), (9) are shown in fig.1 fig.2 and fig.3. The force-voltage electrical analogous circuit is shown in fig.4

Force $f(t) \rightarrow v(t)$ voltage

velocity $v_1 = i_1$

velocity $v_2 = i_2$

velocity $v_3 = i_3$

$B_1 = R_1$

$B_3 = R_3$

$M_1 \Rightarrow L_1$

$M_2 \rightarrow L_2$

$M_3 \rightarrow L_3$

$K_1 = 1/C_1$

$K_2 = 1/C_2$

$K_3 = 1/C_3$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1\,dt + \frac{1}{C_2}\int (i_1 - i_2)\,dt = e_1(t) \rightarrow (7)$$

$$L_2 \frac{di_2}{dt} + R_3 (i_2 - i_3) + \frac{1}{C_2}\int (i_2 - i_1)\,dt + \frac{1}{C_3}\int (i_2 - i_3) = e_2(t) \rightarrow (8)$$

$$L_3 \frac{di_3}{dt} + R_3 (i_3 - i_2) + \frac{1}{C_3}\int (i_3 - i_2) = 0 \rightarrow (9).$$

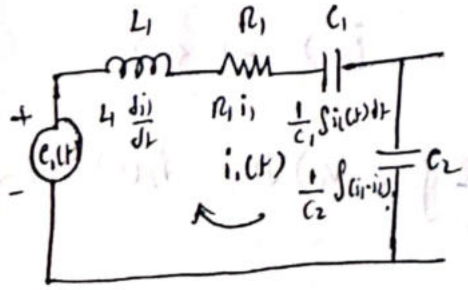


fig. 1

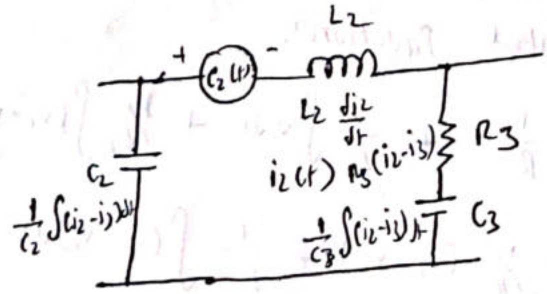


fig. 2

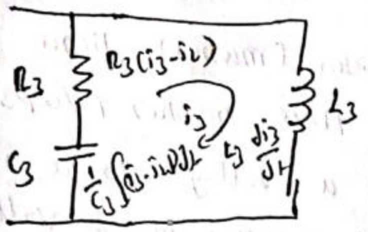


fig-3

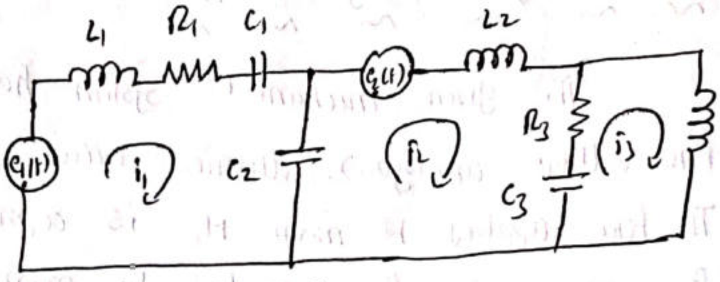


fig (4)

Force - current Analogous circuit :-

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The force applied to mass M_1 is represented as a current source and connected to node-1 in analogous electrical circuit. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electric circuit.

The elements M_1, K_1, B_1, K_2 are connected to node-1, M_2, K_2, K_3, B_3 are connected to node-2 and M_3, B_3, K_3 are connected to node-3 in the form of electrical components. The element K_2 is common b/w node-1 & node-2. The elements B_3, K_3 are connected b/w node-2 and node-3.

$f(t) \rightarrow i(t)$ current

velocity $v_1 \rightarrow v_1$ voltage

$v_2 \rightarrow v_2$

$v_3 \rightarrow v_3$

$B_1 = 1/R_1$

$B_3 = 1/R_3$

$M_1 = C_1$

$M_2 = C_2$

$M_3 = C_3$

$K_1 = 1/L_1$

$K_2 = 1/L_2$

$K_3 = 1/L_3$

The differential equations (1) (2) (3) can be represented in force-current analogous differential equations.

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{L_1}\int v_1 dt + \frac{1}{L_2}\int(v_1-v_2)\,dt = i_1(t) \rightarrow \langle 10\rangle$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_3}(v_2-v_3) + \frac{1}{L_2}\int(v_2-v_1)\,dt + \frac{1}{L_3}\int(v_2-v_3)\,dt = i_2(t) \rightarrow \langle 11\rangle$$

$$C_3\frac{dv_3}{dt} + \frac{1}{R_3}(v_3-v_2) + \frac{1}{L_3}\int(v_3-v_2) = 0 \rightarrow \langle 12\rangle$$

The electrical analogous circuits for eq. ⟨10⟩, ⟨11⟩ ⟨12⟩ are ⟨22⟩ shown in fig.5 fig.6 and fig.7. The force ~~voltage~~ current electrical analogous circuit is shown in fig.8.

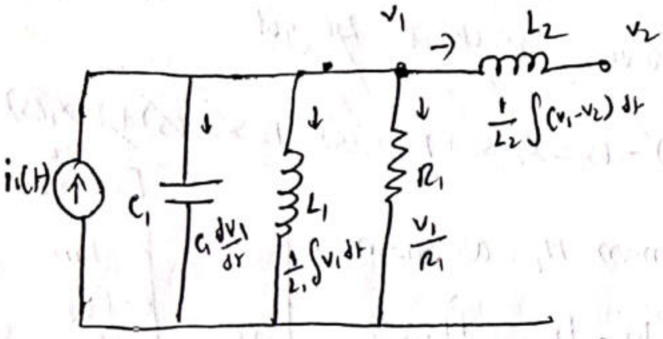


fig.5

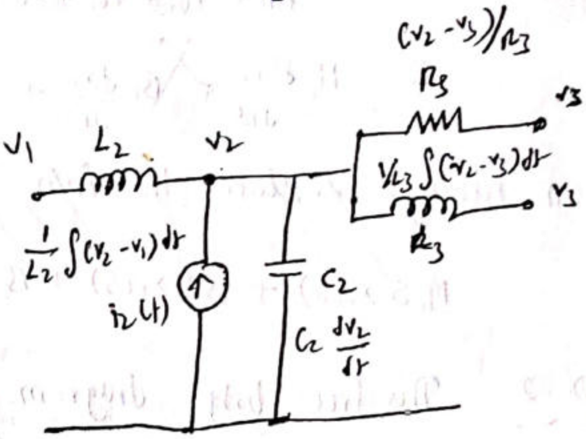


fig.6

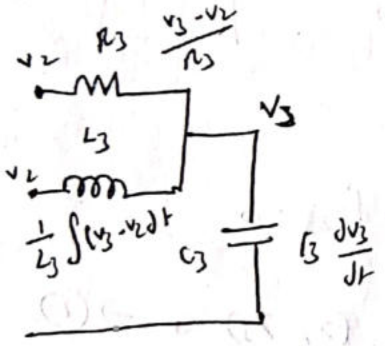


fig.7

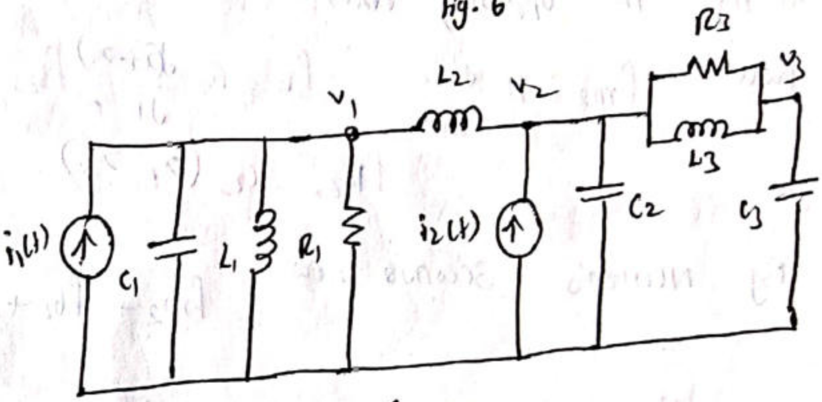
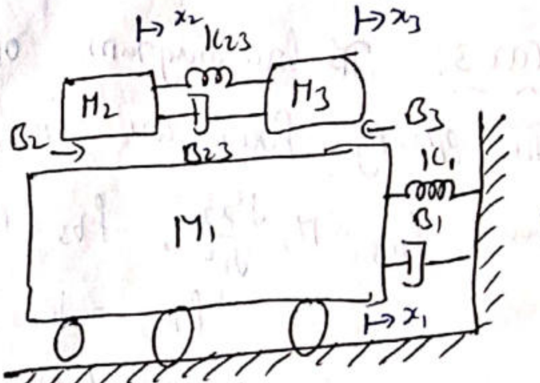
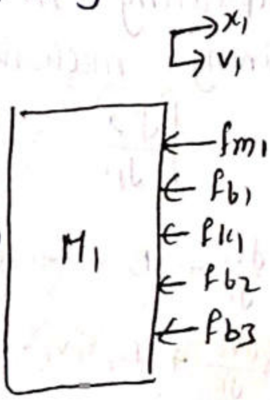


fig.8

3) Write differential equations governing mechanical system shown in fig. Draw force-voltage and force current electrical analogous circuits and verify by writing mesh and node equations.



Sol: The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at all nodes. Let the displacements of masses M_1, M_2, and M_3 are x_1, x_2 and x_3 respectively. The corresponding velocities v_1 v_2 v_3.



Case 1: The free body diagram of M_1 as shown in fig.

The opposing forces are f_{m1}, f_{b1}, f_{b2}, f_{b3}, f_{k1}

Here $f_{m1} = M_1\frac{d^2x_1}{dt^2}$ $\quad f_{b1} = B_1\frac{dx_1}{dt}$ $\quad f_{b2} = B_2\frac{d}{dt}(x_1-x_2)$

$f_{b3} = B_3\frac{d}{dt}(x_1-x_3)$ $\quad f_{k1} = K_1 x_1$

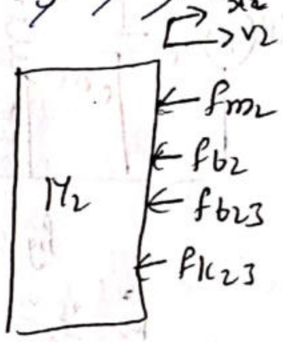
By Newton's second law

$$f_{m_1} + f_{b_1} + f_{b_2} + f_{b_3} + f_{k_1} = 0 \rightarrow (1)$$

$$\therefore \; M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_2 \frac{d}{dt}(x_1 - x_2) + B_3 \frac{d}{dt}(x_1 - x_3) + K_1 x_1 = 0$$

on ~~taking Laplace transform to above equation we get~~

$$\cancel{M_1 s^2 X_1(s) + B_1 s X_1(s) + B_2 s X_1(s) - B_2 s X_2(s) + B_3 s X_1(s) - B_3 s X_3(s) + K_1 X_1(s) = 0}$$



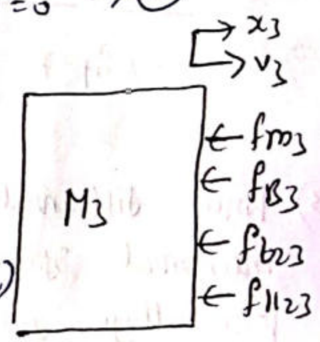
case 2: The free body diagram of mass M_2 as shown in fig. The opposing forces are f_{m_2}, f_{b_2}, $f_{b_{23}}$, $f_{k_{23}}$.

here $f_{m2} = M_2 \frac{d^2x_2}{dt^2}$ $\quad f_{b_2} = B_2 \frac{d(x_2 - x_1)}{dt}$, $f_{b_{23}} = B_{23} \frac{d}{dt}(x_2 - x_3)$

$$f_{k_{23}} = K_{23}(x_2 - x_3)$$

By Newton's second law $\quad f_{m_2} + f_{b_2} + f_{b_{23}} + f_{k_{23}} = 0$.

$$\therefore \; M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d}{dt}(x_2 - x_1) + B_{23} \frac{d}{dt}(x_2 - x_3) + K_{23}(x_2 - x_3) = 0 \rightarrow (2)$$



case 3: The free diagram of mass M_3 as shown in fig.

The opposing forces are f_{m_3}, f_{b_3}, $f_{b_{23}}$, $f_{k_{23}}$.

here $f_{m_3} = M_3 \frac{d^2x_3}{dt^2}$, $f_{b_3} = B_3 \frac{d}{dt}(x_3 - x_1)$, $f_{b_{23}} = B_{23} \frac{d}{dt}(x_3 - x_2)$

$$f_{k_{23}} = K_{23}(x_3 - x_2)$$

By Newton's second law

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B_{23} \frac{d}{dt}(x_3 - x_2) + K_{23}(x_3 - x_2) = 0 \rightarrow (3)$$

on replacing the displacements by velocities in the diff. equations 1, 2, 3 governing mechanical system, we get,

$$\left[\frac{d^2x}{dt^2} = \frac{dv}{dt} \qquad \frac{dx}{dt} = v \qquad x = \int v\, dt\right]$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_2(v_1 - v_2) + B_3(v_1 - v_3) + K_1 \int v_1\, dt = 0 \rightarrow (4)$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B_{23}(v_2 - v_3) + K_{23} \int (v_2 - v_3)\, dt = 0 \rightarrow (5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_1) + B_{23}(v_3 - v_2) + K_{23} \int (v_3 - v_2)\, dt = 0 \rightarrow (6)$$

Force-voltage ~~meshons~~ Analogous Circuit (Loops)

The given system has three nodes (masses). Hence force voltage analogous electrical circuit will have three meshes (Loops).

- → M_1, B_1, B_2, B_3, K_1 are connected in first mesh (Loop-1)
- → M_2 B_2, B_{23}, K_{23} are connected in second mesh (Loop-2)
- → M_3, B_3, B_{23}, K_{23} are connected in third mesh (Loop-3)
- → B_2 is common between loop-1 and loop-2
- → B_3 is common between loop-1 and loop-3
- → B_{23} and K_{23} are common between loop-2 and loop-3.

The electrical analogous elements for the elements of mechanical system are given below.

velocity $(v_1) = i_1$ $\quad B_1 = R_1$ $\quad M_1 = L_1$ $\quad K_1 = 1/C_1$

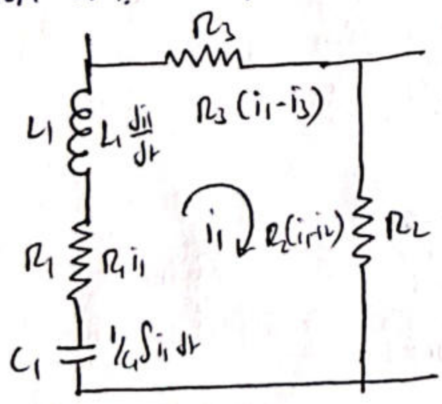
" $v_2 = i_2$ $\quad B_2 = R_2$ $\quad M_2 = L_2$ $\quad K_{23} = 1/C_{23}$

$v_3 = i_3$ $\quad B_3 = R_3$ $\quad M_3 = L_3$

$B_{23} = R_{23}$

The equations (4) (5) and (6) can be represented in electrical force-voltage analogous system circuit by the replacement of mechanical elements by electrical elements in fig.1 fig.2, fig.3 and fig.4.

From equ, (7) ↓



Loop-1

fig-1

The equations (4) (5) (6) can be write in the form of electrical analogous. We get

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_2(i_1-i_2) + R_3(i_1-i_3) + \frac{1}{C_1}\int i_1 dt = 0 \rightarrow (7)$$

$$L_2 \frac{di_2}{dt} + R_2(i_2-i_1) + R_{23}(i_2-i_3) + \frac{1}{C_{23}}\int (i_2-i_3)dt = 0 \rightarrow (8)$$

$$L_3 \frac{di_3}{dt} + R_3(i_3-i_1) + R_{23}(i_3-i_2) + \frac{1}{C_{23}}\int (i_3-i_2)dt = 0 \rightarrow (9)$$

Loop-2 from Eqn. (8)

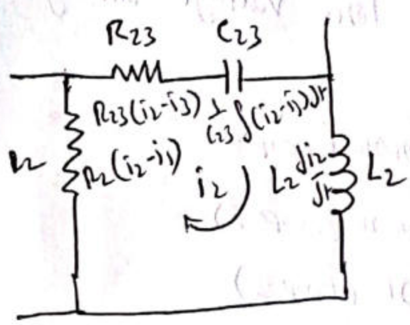


fig. 2

Loop-3 from Eqn. (9)

$= L_3 \Rightarrow L_3 \frac{di_3}{dt}$

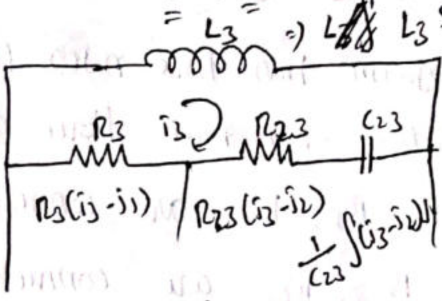


fig. 2

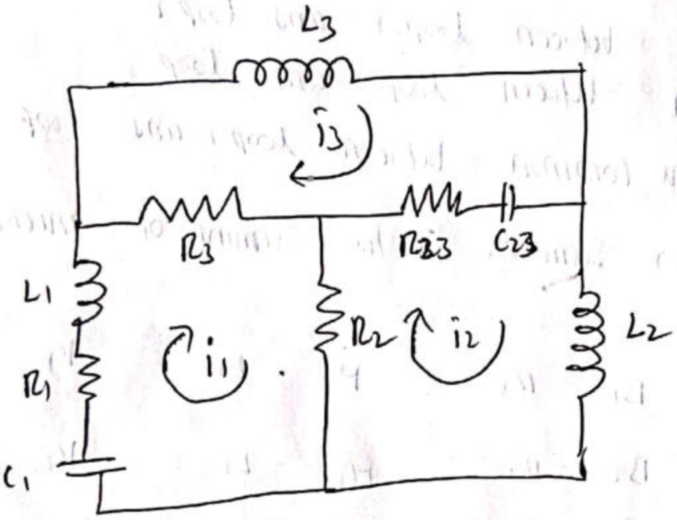


fig. 4

Force-current Analogous circuit :-

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

→ M_1, B_1, B_2, B_3, K_1 are connected to node-1

→ M_2, B_2, B_{23}, K_{23} are connected to node-2

→ M_3, B_3, B_{23}, K_{23} are connected to node-3

→ B_2 is common between node-1 and node-2

→ B_3 is common between node-1 and node-3

→ B_{23}, K_{23} are common between node-2 and node-3.

The electrical analogous elements for the elements of mechanical system are given below.

velocity v_1 = voltage V_1

$v_2 = V_2$

$v_3 = V_3$

$B_1 = 1/R_1$ $\quad M_1 = C_1$ $\quad K_1 = 1/L_1$

$B_2 = 1/R_2$ $\quad M_2 = C_2$ $\quad K_{23} = 1/L_{23}$

$B_3 = 1/R_3$ $\quad M_3 = C_3$

$B_{23} = 1/R_{23}$

The Equations (4) (5) (6) can be write in Electrical Analogous Equations (24), we will get,

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1}\int v_1 dt + \frac{1}{R_2}(v_1 - v_2) + \frac{1}{R_3}(v_1 - v_3) = 0 \rightarrow (10)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_2}(v_2 - v_1) + \frac{1}{L_{23}}\int (v_2 - v_3) dt + \frac{1}{R_{23}}(v_2 - v_3) = 0 \rightarrow (11)$$

$$C_3 \frac{dv_3}{dt} + \frac{1}{R_3}(v_3 - v_1) + \frac{1}{L_{23}}\int (v_3 - v_2) dt + \frac{1}{R_{23}}(v_3 - v_2) = 0 \rightarrow (12)$$

The Electrical Analogous Circuits for Equations 10, 11 and 12 as shown in fig. 5, fig. 6, fig. 7 and fig. 8.

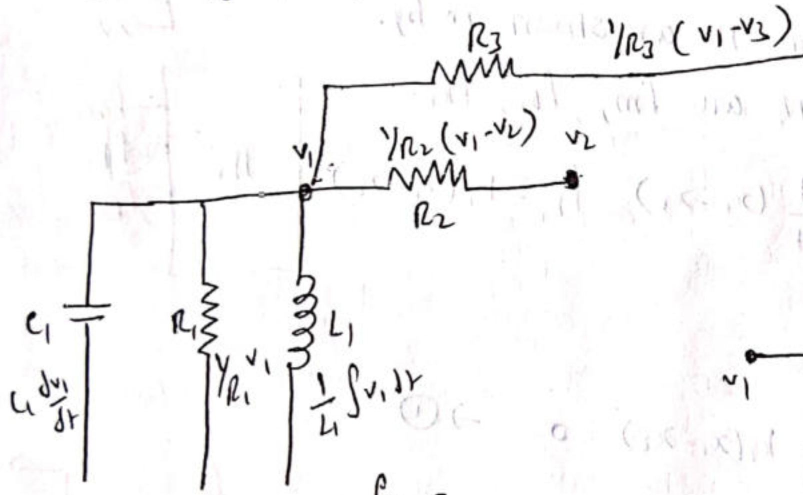


fig. 5

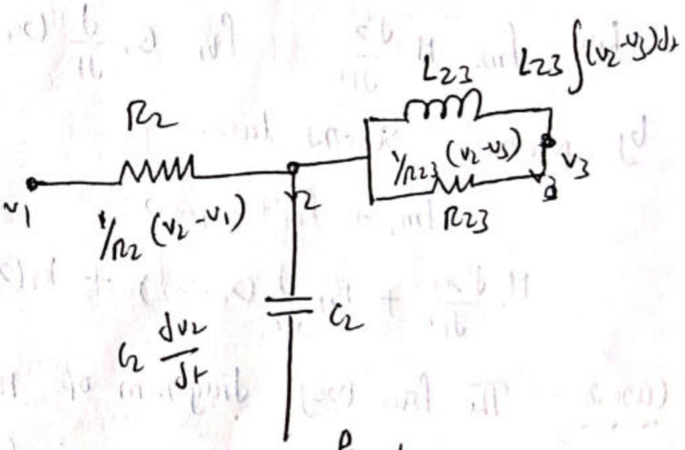


fig. 6

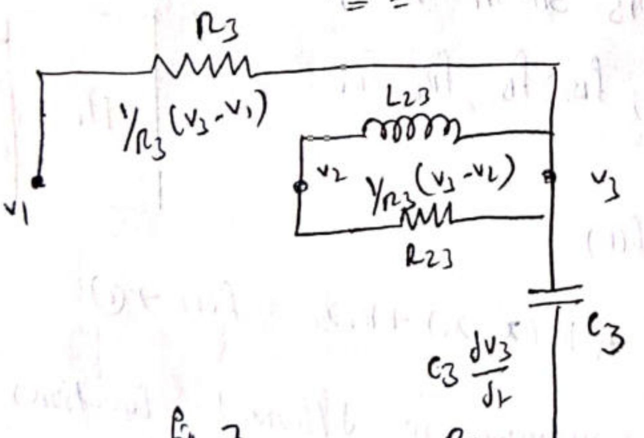


fig. 7

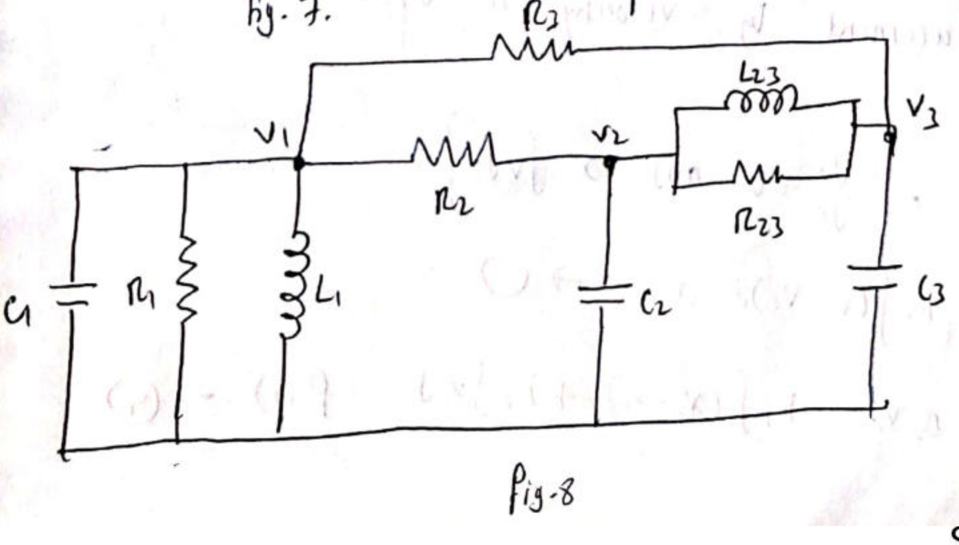
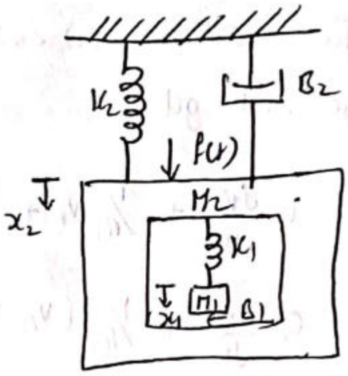


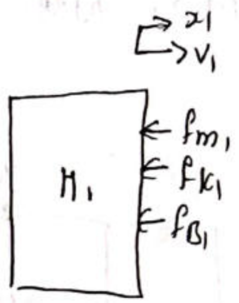
fig. 8

4) Write the differential equations governing the mechanical system shown in fig. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.



Sol: The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations. Let displacements of M_1 and M_2 are x_1 and x_2. The corresponding velocities are v_1 and v_2

Case 1: The free body diagram of M_1 as shown in fig.



The opposing forces acting on M_1 are f_{m_1}, f_{b_1}, f_{k_1}

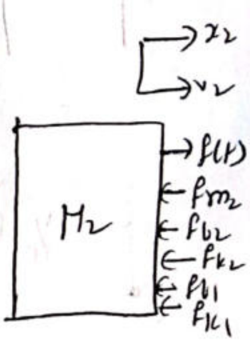
here $f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$ $\quad f_{b_1} = B_1 \frac{d}{dt}(x_1 - x_2)$, $f_{k_1} = K_1(x_1 - x_2)$

by newton's second law.

$$f_{m_1} + f_{b_1} + f_{k_1} = 0$$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = 0 \rightarrow (1)$$

Case 2:- The free body diagram of M_2 is shown in fig.



The opposing forces acting on M_2 are f_{m_2}, f_{B_2}, f_{B_1}, f_{k_1}, f_{k_2}

by Newton's second law,

$$f_{m_2} + f_{B_1} + f_{B_2} + f_{k_1} + f_{k_2} = f(t)$$

$$M_2 \frac{d^2x_2}{dt^2} + B_1 \frac{d}{dt}(x_2 - x_1) + B_2 \frac{dx_2}{dt} + K_1(x_2 - x_1) + K_2 x_2 = f(t) \rightarrow (2)$$

on replacing the displacement by velocity in differential equations (1) & (2) we get

$$\text{here} \left[\frac{d^2x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt\right]$$

$$M_1 \frac{dv_1}{dt} + B_1(v_1 - v_2) + K_1 \int (v_1 - v_2)\,dt = 0 \rightarrow (3)$$

$$M_2 \frac{dv_2}{dt} + B_1(v_2 - v_1) + B_2 v_2 + K_1 \int (x_2 - v_1)\,dt + K_2 \int v_2\,dt = f(t) \rightarrow (4)$$

i) Force - voltage Analogous circuit :-

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit has two meshes (loops). The force applied to mass M_2 is represented by a voltage source in second mesh (loop)

→ The elements M_1, B_1 and K_1 are connected in loop-1

→ The elements M_2, B_2, K_2, K_1, B_1 are connected in loop-2

→ The loop-2 having voltage source.

→ The elements B_1 and K_1 are common for loop-1 and loop-2

The electrical analogous elements for elements of mechanical system are given below.

$f(t)$ force → $e(t)$ (or) $v(t)$ (voltage source)

v_1 (velocity) − i_1 $\quad B_1 = R_1 \quad M_1 = L_1 \quad K_1 = 1/C_1$

v_2 (velocity) − i_2 $\quad B_2 = R_2 \quad M_2 = L_2 \quad K_2 = 1/C_2$

The mesh basic equations using KVL for the circuit are given by.

$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = 0 \rightarrow (5)$$

$$L_2 \frac{di_2}{dt} + R_1(i_2 - i_1) + R_2 i_2 + \frac{1}{C_1}\int (i_2 - i_1)\,dt + \frac{1}{C_2}\int i_2\,dt = e(t) \rightarrow (6)$$

The fig. 1, 2 and 3 shows force - voltage electrical analogous circuits for given mechanical system.

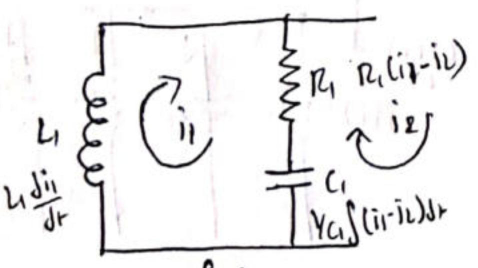
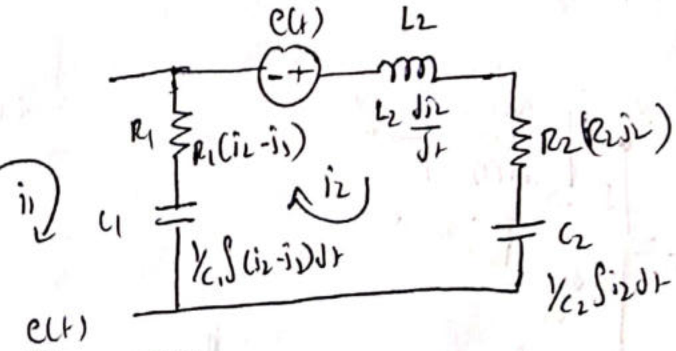


fig. 1



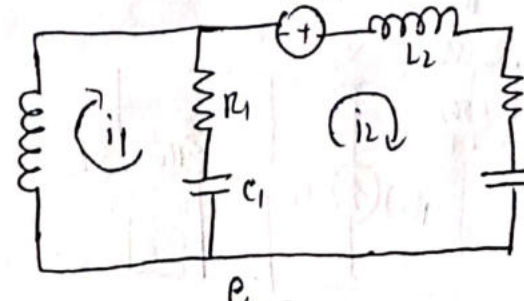


fig-3

2) Force- Current Analogous circuit :-

The given mechanical system has two nodes (masses). Hence the force - current analogous electrical circuits has two nodes. The force applied to mass M_2 is represented by current source connected to node-2

→ The elements M_1, B_1, K_1 are connected to node-1

→ The elements M_2, B_1, B_2, K_1, K_2 are connected to node-2

-) The current source is connected to node-2

-) The elements B_1, K_1 are common for node-1 and node-2

The electrical ~~other~~ analogous elements for elements of mechanical system are given below,

$f(t)$ force → $i(t)$ current source

velocity (v_1) – voltage v_1

v_2 – v_2

$B_1 = 1/R_1$ $M_1 \to C_1$ $K_1 = 1/L_1$

$B_2 = 1/R_2$ $M_2 \to C_2$ $K_2 = 1/L_2$

The nodal equations using KCL for the circuit are given by,

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\, dt = 0 \quad \to (7)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{R_2} v_2 + \frac{1}{L_1}\int (v_2 - v_1) + \frac{1}{L_2}\int v_2\, dt = i(t) \quad \to (8)$$

The fig 4, 5 and 6 shows force - current analogous electrical circuits for given mechanical system.

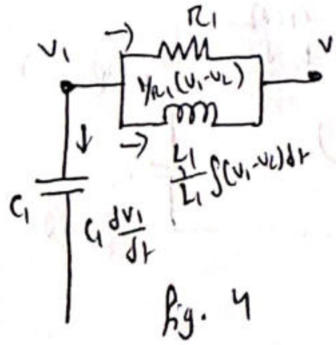


fig. 4

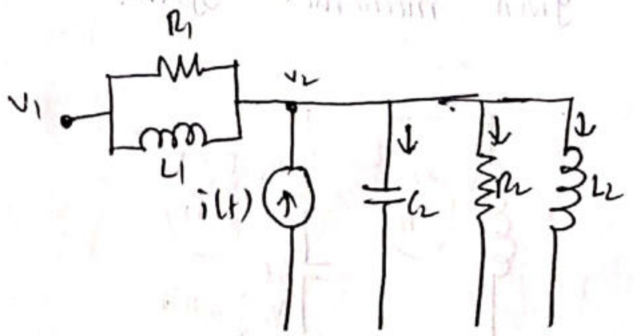


fig. 5

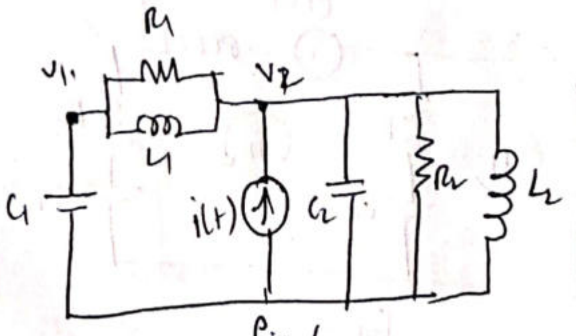


fig. 6

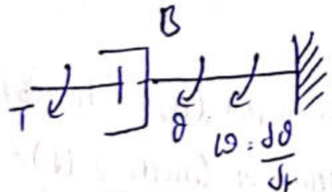
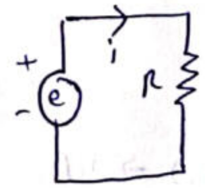
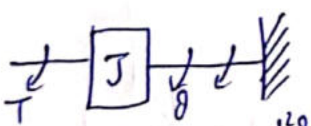
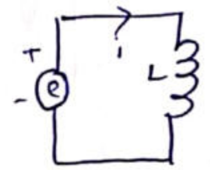
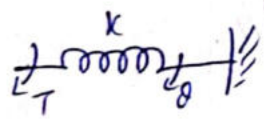
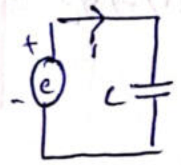
* Electrical Analogous of mechanical Rotational System :- (26)

The three basic elements "moment of inertia", "rotational dashpot" and "torsional spring" that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical system.

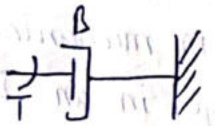
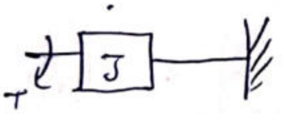
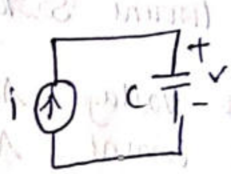
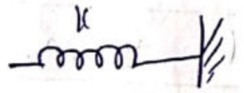
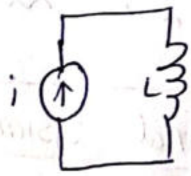
The input torque in mechanical system is analogous to either voltage source (or) current source in electrical systems. The output angular velocity (first derivative of angular displacement) in mechanical rotational system is analogous to either current (or) voltage in an element in electrical system. Since the electrical system has two types of inputs either voltage (or) current source, there are two types of analogies.

→ Torque voltage Analogy
→ Torque current Analogy.

Analogous elements in Torque-Voltage Analogy :-

Mechanical rotational system	electrical system.
Input :- Torque Output :- Angular velocity	Input :- voltage source Output :- current through the element
 $T = B\frac{d\theta}{dt} = B\omega$	 $e = v$ $v = iR$ $e = iR$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\quad a = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = v$ $e = L\frac{di}{dt}$
 $T = K\theta \Rightarrow K\int \omega\, dt$	 $e = v$ $e = \frac{1}{C}\int i\, dt$

Analogous Elements in Torque - Current Analogy :-

Mechanical Rotational System	Electrical System
Input : Torque Output :- Angular velocity	Input :- Current source Output :- Voltage across the element.
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $i = v/R$ $i = e/R$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$	 $i = C\cdot\frac{dv}{dt}$
 $T = K\theta = K\int \omega\, dt$	 $i = \frac{1}{L}\int v\, dt$

Summary :-

Torque - Voltage :-

Torque (T) - Voltage (v) → i/p

angular velocity (ω) = Current (i) → o/p.

Dash pot (B) = Resistance (R)

moment of (inertia) (J) = inductance (L)

Spring (K) = $\frac{1}{\text{Capacitance}}$ (1/C)

Torque - Current :-

Torque (T) - Current (i) → i/p

angular velocity (ω) - Voltage (v) → o/p

Dash pot (B) = 1/R

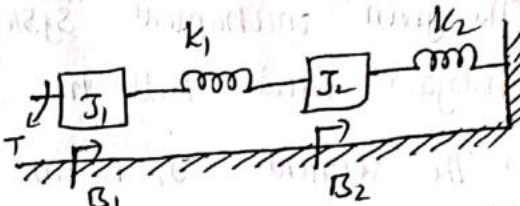
moment of inertia = C

Spring = 1/L.

(27)

* Problems based on Analogous of Mechanical rotational System :-

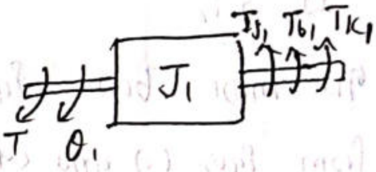
1) Write the differential equations governing the mechanical rotational System shown in fig. Draw the torque-voltage and torque-current electrical Analogous circuits and verify by writing mesh and node equations.



Sol: The mechanical System has two nodes (moment of inertia of masses). The governing Equations of the mechanical rotational System can be obtained by Torque-balanced Equations.

Let Angular displacements of J_1 and J_2 are θ_1 and θ_2. The corresponding angular velocities are ω_1 and ω_2.

Case:1 :- The free body diagram of J_1 as shown in fig. The opposing forces acting on J_1 are marked as $T_{j_1}, T_{b_1}, T_{k_1}$

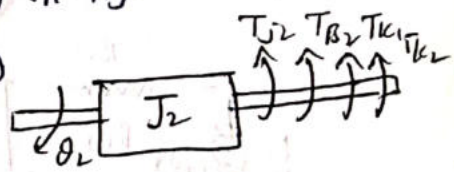


here, $T_{j_1} = J_1 \frac{d^2\theta_1}{dt^2}$ $\quad T_{b_1} = B_1 \frac{d\theta_1}{dt}$ $\quad T_k = K(\theta_1 - \theta_2)$

By newton's second law

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K(\theta_1 - \theta_2) = T \rightarrow ①$$

Case 2:- The free body diagram of J_2 as shown in fig. The opposing forces acting on J_2 are marked as $T_{j_2}, T_{B_2}, T_{K_1}, T_{K_2}$



here, $T_{j_2} = J_2 \frac{d^2\theta_2}{dt^2}$ $\quad T_{B_2} = B_2 \frac{d\theta_2}{dt}$

$T_{K_1} = K_1(\theta_2 - \theta_1)$ $\quad T_{K_2} = K_2\theta_2$

By newton's second law

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_1(\theta_2 - \theta_1) + K_2\theta_2 = 0 \rightarrow ②$$

on replacing angular displacement by angular velocity in the differential Equations ① and ② we get $\left[\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}, \frac{d\theta}{dt} = \omega, \theta = \int \omega\, dt\right]$

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)\,dt = T \rightarrow (3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_1\int(\omega_2 - \omega_1)\,dt + K_2\int \omega_2\, dt = 0 \rightarrow ④$$

Torque-voltage analogous circuit :-

The given mechanical system has two masses (J_1 and J_2). Hence Torque-voltage Analogous circuit will have two meshes (Loops)

→ The elements J_1 B_1 and K_1 are connected in first loop.

→ The elements J_2, B_2, K_1 and K_2 are connected in second loop.

→ The element K_1 is common for both loop-1 and loop-2

The electrical analogous elements for the elements of mechanical rotational system

$T \to e(t)$ $\quad B_1 \to R_1$ $\quad J_1 \to L_1$ $\quad K_1 \to 1/C_1$

$\omega_1 \to i_1$ $\quad B_2 \to R_2$ $\quad J_2 \to L_2$ $\quad K_2 \to 1/C_2$

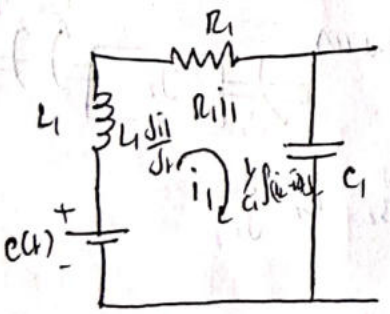
$\omega_2 \to i_2$

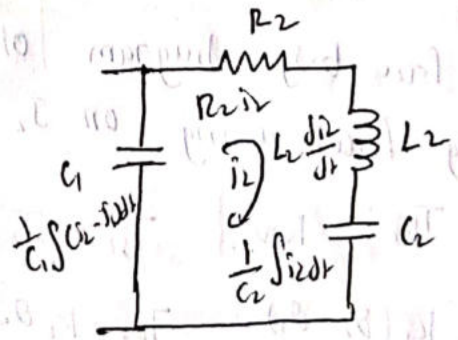
The mesh basic Equations using KVL for the circuit are given below from Equ: (3) and (4)

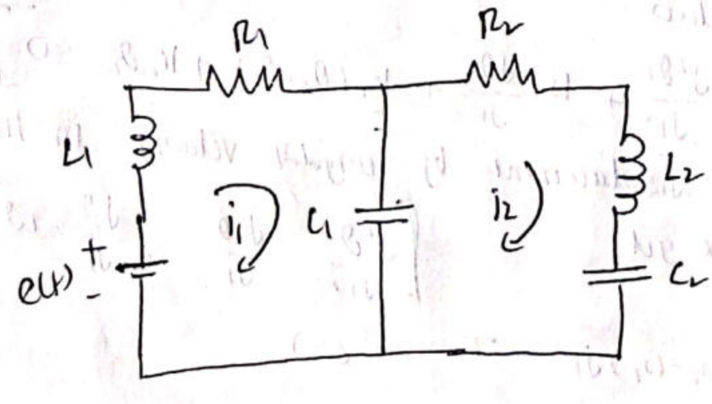
$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int (i_1 - i_2)\,dt = e(t) \quad \to (5)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_1}\int (i_2 - i_1)\,dt + \frac{1}{C_2}\int i_2\,dt = 0 \quad \to (6)$$

The torque voltage analogous circuit as shown in below.







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Torque-Current Analogous circuit:-

The given mechanical system has two nodes (J_1 and J_2). Hence Torque-Current Analogous circuit will have two nodes.

→ The elements J_1, B_1 and K_1 are connected to node-1

→ The elements J_2, B_2, K_1 K_2 are connected to node-2

→ The element K_1 is common for both node-1 and node-2

The electrical analogous elements for the elements of mechanical rotation system

$T \to i(t)$ $\quad B_1 \to 1/R_1$ $\quad J_1 = C_1$ $\quad K_1 = 1/L_1$

$\omega_1 \to v_1$ $\quad B_2 \to 1/R_2$ $\quad J_2 \to C_2$ $\quad K_2 = 1/L_2$

$\omega_2 \to v_2$

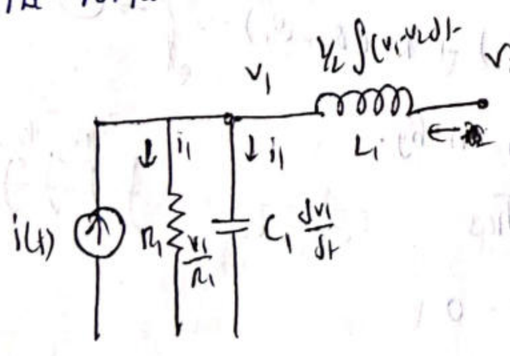
The node basic equations using KCL for the circuit are given below.

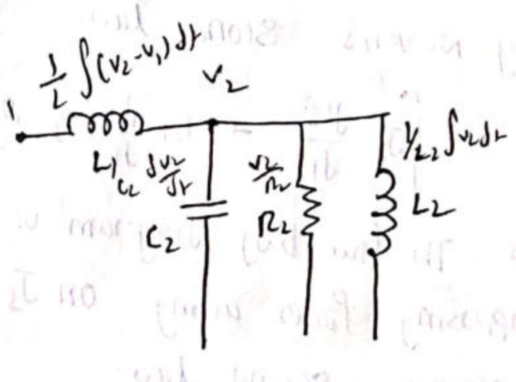
from equ. (3) and (4)

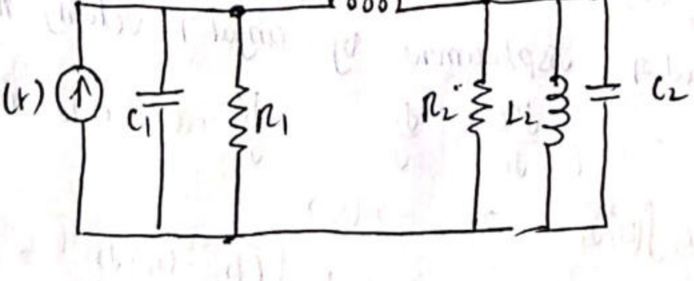
$$C_1 \frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{1}{L_1}\int (v_1 - v_2)\,dt = i(t) \rightarrow (7)$$

$$C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_1}\int (v_2 - v_1)\,dt + \frac{1}{L_2}\int v_2\,dt = 0 \rightarrow (8)$$

The torque-current Analogous circuit as shown in below.







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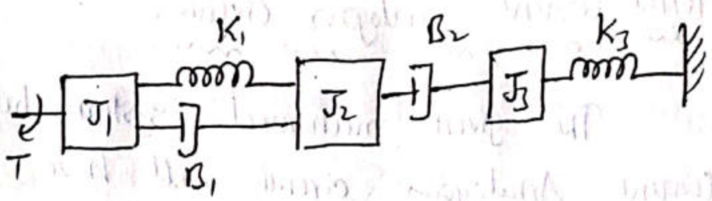
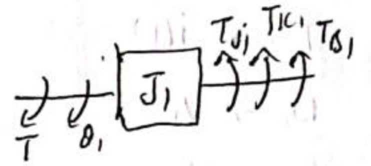
2) Write the differential Equations governing the mechanical system shown in fig. Draw the torque-voltage and torque-current Electrical analogous circuits and verify by writing mesh and node equations.



Sol The given mechanical rotational system has three nodes (moment of inertia of masses). The differential governing equations can be obtained by torque-balance equations.

The angular displacements of J_1, J_2 and J_3 are θ_1, θ_2 and θ_3. The corresponding angular velocities be ω_1, ω_2 and ω_3.

Case:1:- The free body diagram of J_1 as shown in fig.



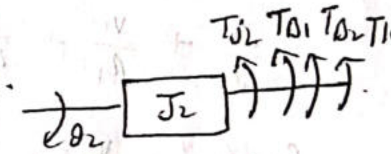
The opposing Torques acting on J_1 are T_{J_1}, T_{B_1}, T_{K_1}

here $T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}$ $\quad T_{B_1} = B_1 \frac{d(\theta_1-\theta_2)}{dt}$ $\quad T_{K_1} = K_1(\theta_1-\theta_2)$

By Newton's second law.

$$\left[J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d(\theta_1-\theta_2)}{dt} + K_1(\theta_1-\theta_2) = T\right] \rightarrow (1)$$

Case 2:- The free body diagram of J_2 as shown in fig.

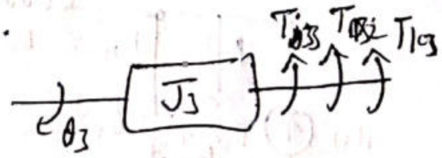


The opposing forces acting on J_2 are T_{J_2}, T_{B_1}, T_{B_2}, T_{K_1}

By Newton's second law. $\quad T_{J_2} + T_{B_1} + T_{B_2} + T_{K_1} = 0$

$$\left[J_2 \frac{d^2\theta_2}{dt^2} + B_1 \frac{d}{dt}(\theta_2-\theta_1) + B_2 \frac{d}{dt}(\theta_2-\theta_3) + K_1(\theta_2-\theta_1) = 0\right] \rightarrow (2)$$

Case 3:- The free body diagram of J_3 as shown in fig.



The opposing forces acting on J_3 are T_{J_3}, T_{B_2}, T_{K_3}

By Newton's second law $\quad T_{J_3} + T_{B_2} + T_{K_3} = 0$

$$\left[J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d}{dt}(\theta_3-\theta_2) + K_3\theta_3 = 0\right] \rightarrow (3)$$

on replacing the angular displacement by angular velocity then the Eqs 1, 2, 3 becomes.

$$\left[\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}, \quad \frac{d\theta}{dt} = \omega \text{ and } \theta = \int \omega\, dt\right]$$

$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1-\omega_2) + K_1\int(\omega_1-\omega_2)dt = T \rightarrow (4)$$

$$J_2 \frac{d\omega_2}{dt} + B_1(\omega_2-\omega_1) + B_2(\omega_2-\omega_3) + K_1\int(\omega_2-\omega_1)dt = 0 \rightarrow (5)$$

$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3-\omega_2) + K_3\int\omega_3\, dt = 0 \rightarrow (6)$$

Torque - voltage Analogous circuit :-

The given system has three nodes (moment of inertia of body). Hence the torque - voltage Analogous electric circuit has three meshes (Loops).

- → The elements, J_1, B, K_1 are connected in Loop-1
- → The elements J_2, B_1, B_2 K_1 are connected in Loop-2
- → The elements J_3, B_2, K_3 are connected in Loop-3
- → The elements B_1, K_1 are common for both loop-1 and loop-2
- → The element B_2 is common for both loop-2 and loop-3

The electrical Analogous elements for elements of mechanical rotational system are given below.

$T \to v(t) = e(t)$ $\quad B_1 = R_1$ $\quad J_1 = L_1$ $\quad K_1 = 1/C_1$

$\omega_1 \to i_1(t)$ $\quad B_2 = R_2$ $\quad J_2 = L_2$ $\quad K_2 = 1/C_3$

$\omega_2 \to i_2(t)$ $\quad J_3 = L_3$

The Equations (4), (5), (6) can be represented in the form of mesh Equations.

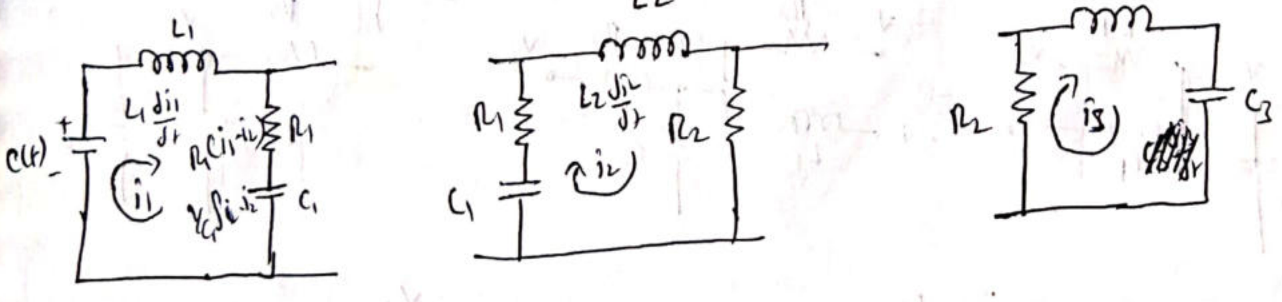
are,

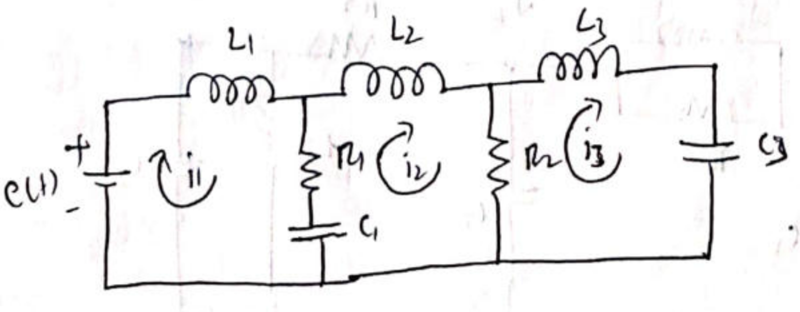
$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + 1/C_1 \int (i_1 - i_2) dt = e(t) \rightarrow (7)$$

$$L_2 \frac{di_2}{dt} + R_1(i_2 - i_1) + R_2(i_2 - i_3) + 1/C_1 \int (i_2 - i_1) dt = 0 \rightarrow (8)$$

$$L_3 \frac{di_3}{dt} + R_2(i_3 - i_2) + 1/C_3 \int i_3 dt = 0 \rightarrow (9)$$

The torque- voltage Analogous circuits as shown in below.





Torque- current Analogous circuit :-

The given system has three nodes (moment of inertia of body). Hence the torque current Analogous electric circuit has three nodes.

- → The elements J_1, B_1 K_1 are connected to node-1
- → The elements J_2, B_1, B_2 K_1 are connected to node-2
- → The elements J_3, B_2, K_3 are connected to node-3
- → The elements B_1, K_1 are common to node-1 & node-2
- → The elements B_2 is common to node-2 & node-3

The electrical Analogous elements for elements of mechanical rotational system are given below,

$T \to i(t)$ $\quad B_1 = 1/R_1$ $\quad J_1 = C_1$ $\quad K_1 = 1/L_1$

$\omega_1 \to v_1$ $\quad B_2 = 1/R_2$ $\quad J_2 = C_2$ $\quad K_3 = 1/L_3$

$\omega_2 \to v_2$ $\quad J_3 = C_3$

$\omega_3 \to v_3$

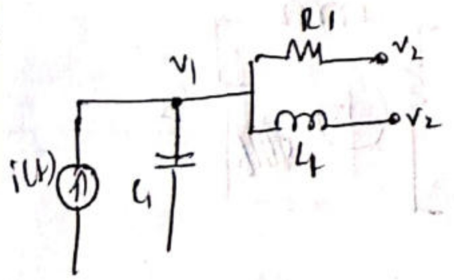
The representation of Equ. (4) (5) (6) in electrical Analogous Equations

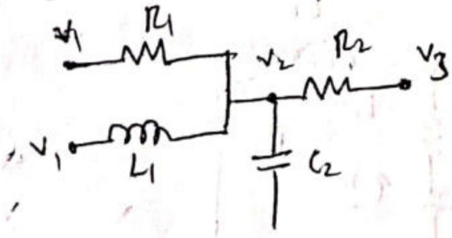
$$C_1 \frac{dv_1}{dt} + \frac{(v_1 - v_2)}{R_1} + \frac{1}{L_1}\int (v_1 - v_2)\,dt = i(t)$$

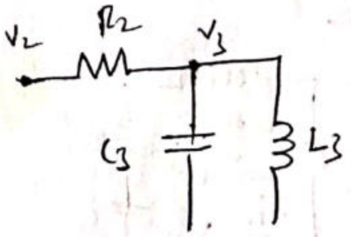
$$C_2 \frac{dv_2}{dt} + \frac{(v_2 - v_1)}{R_1} + \frac{(v_2 - v_3)}{R_2} + 1/L_1 \int (v_2 - v_1)\,dt = 0$$

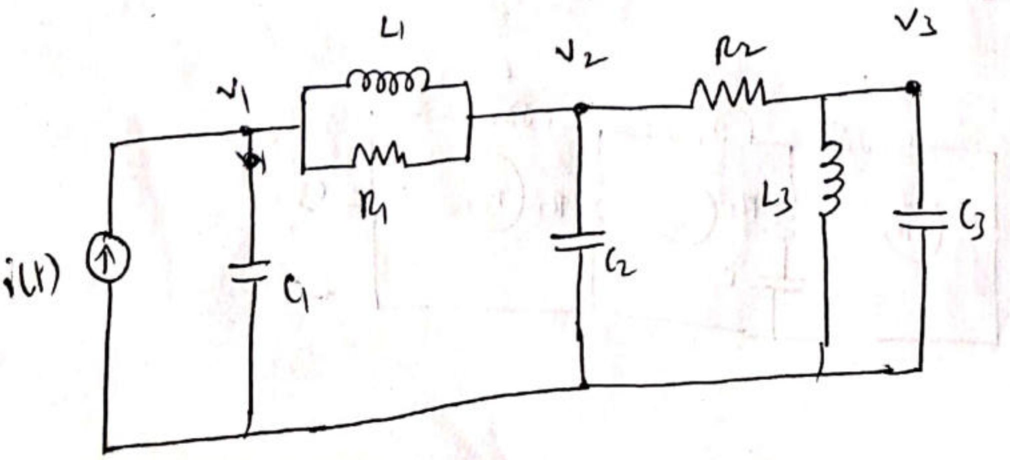
$$C_3 \frac{dv_3}{dt} + \frac{(v_3 - v_2)}{R_2} + 1/L_3 \int v_3\,dt = 0$$

The torque- current Analogous circuits as shown in below,









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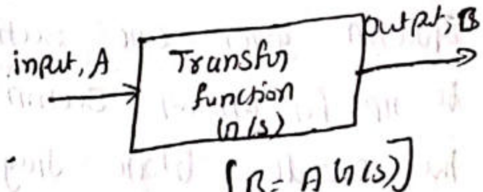
* BLOCK DIAGRAMS :-

A control system may consist of a number of components. In control Engineering to show the functions performed by Each component, we commonly use a diagram is called "Block diagram".

The Block diagram shows interrelationships that exists among the various components. The Elements of block diagram are Block, branch-point and summing point.



Block:-

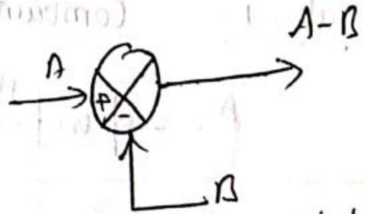
→ In block diagram all system variables are linked to each other through functional blocks.

→ The functional block (or) simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.

→ The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicates the direction of the flow of signals.

→ The arrowhead pointing towards the block indicates the input and the arrowhead leading away from the block represents the output.

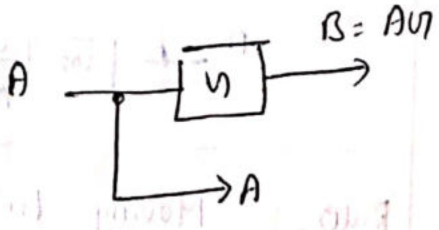
→ The output signal from the block is given by the product of input signal and transfer function in the block.



Summing point.

Summing point:-

Summing point is used to add two (or) more signals in the system. The plus (or) minus sign at each arrowhead indicates whether the signal is to be added (or) substracted.



Branch point:- A branch point is a point from which the signal from a block goes concurrently to other blocks (or) summing points.

Constructing Block diagram for control system :-

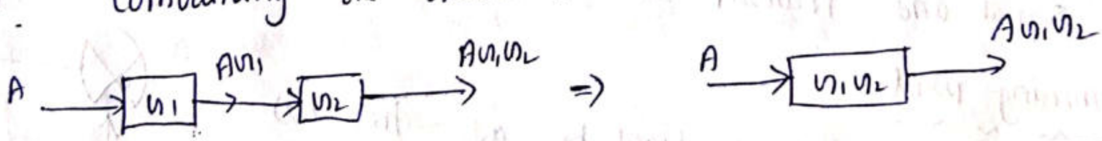
A control system can be represented diagramatically by block diagram. The differential equations are governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebric equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The o/p of one section will be i/p for another section. The various sections are interconnected to obtain the overall block diagram of the system.

* Block diagram Reduction :-

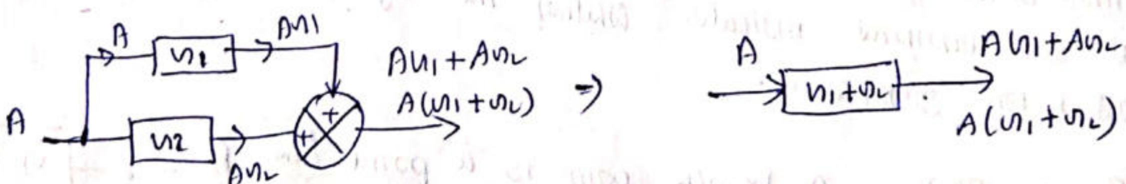
Block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction. The rules are framed such that any modification made on the diagram does not alter the i/p - o/p relation.

RULES OF BLOCK DIAGRAM ALGEBRA

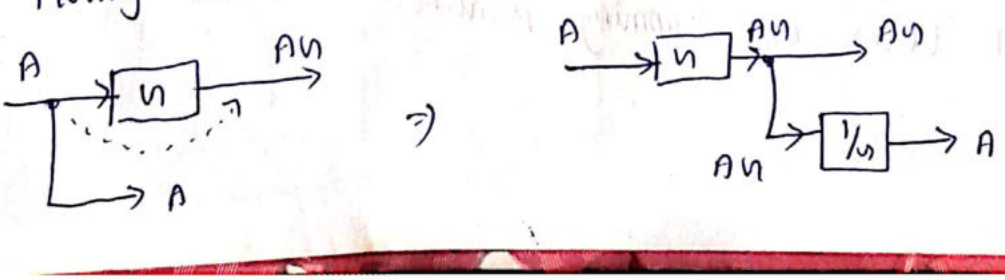
Rule : 1 Combaining the blocks in cascade



Rule : 2 Combaining parallel blocks (or Combaining feed forward paths)

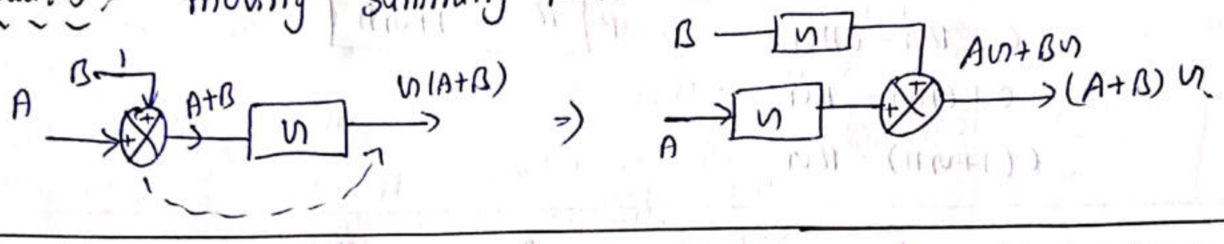


Rule 3 :- Moving the branch point ahead of the block.

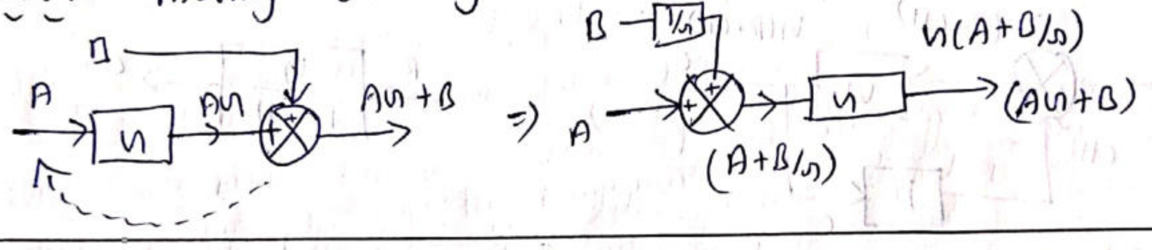


Rule : 4 :- moving branch point before the block.

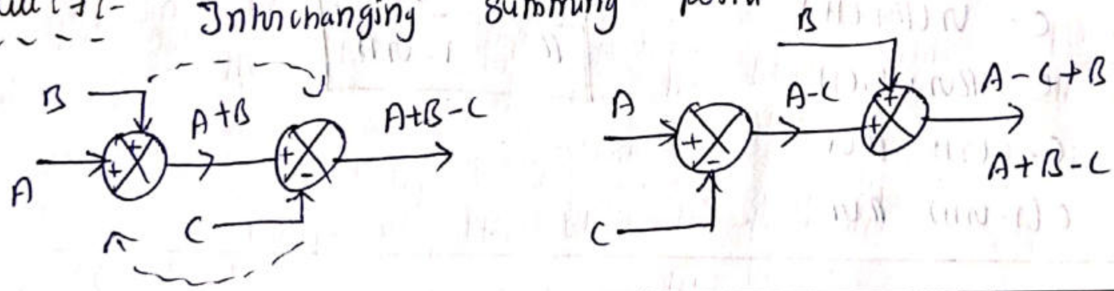
Rule : 5 :- moving summing point ahead of the block.



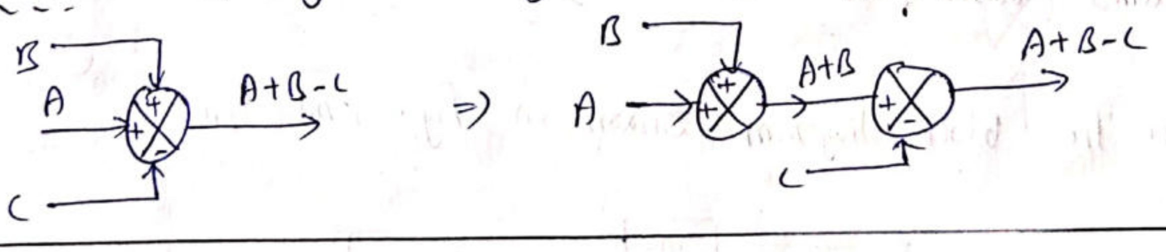
Rule : 6 :- moving summing point before the block.



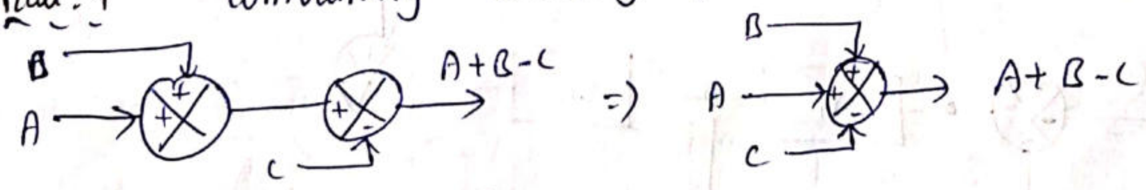
Rule : 7 :- Interchanging summing point.



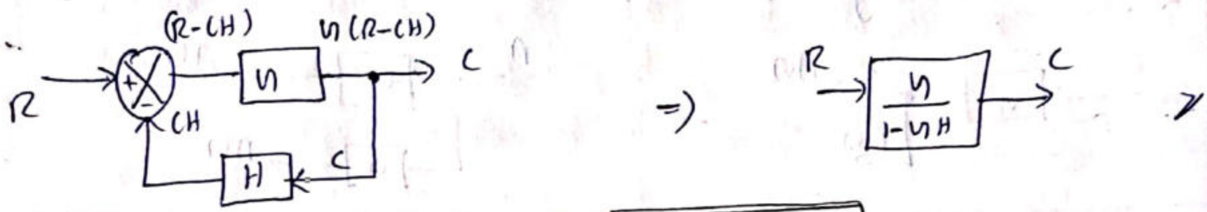
Rule : 8 : splitting summing points.



Rule : 9 combaining summing points.



Rule 10: Elimination of (negative) feedback loop

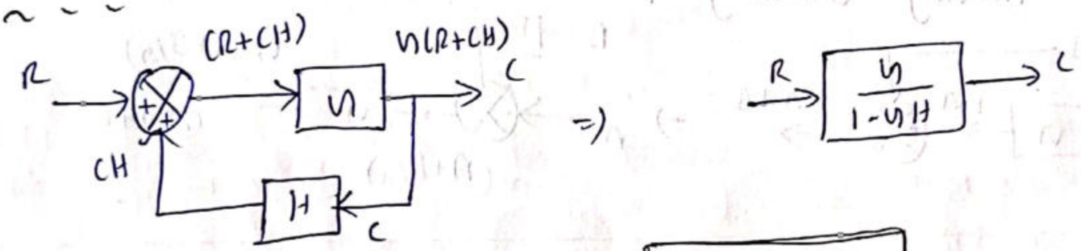


Proof:

$$C = (R-CH)G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1+GH) = RG$$

$$\frac{o/p}{i/p} \quad \boxed{\frac{C}{R} = \frac{G}{1+GH}}$$

Rule 11: Elimination of (positive) feedback loop.



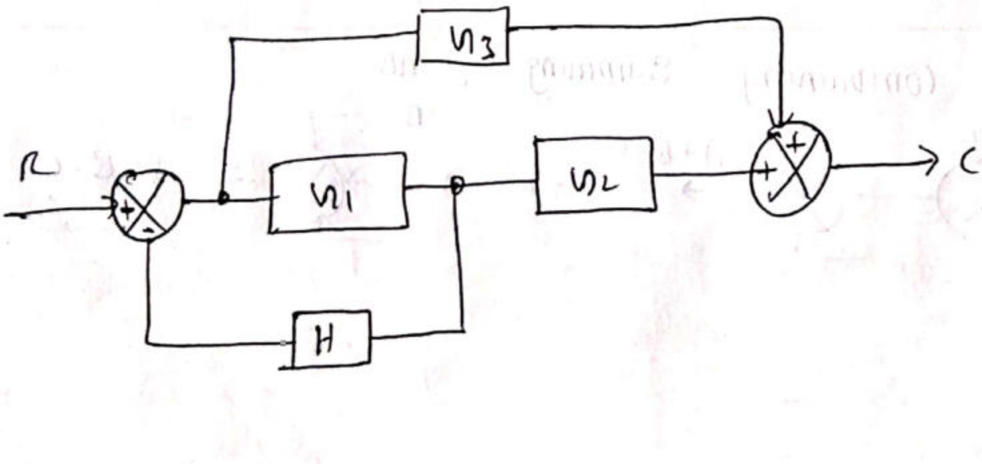
Proof:

$$C = G(R+CH)$$
$$C = RG + GCH$$
$$C - CGH = RG$$
$$C(1-GH) = RG$$

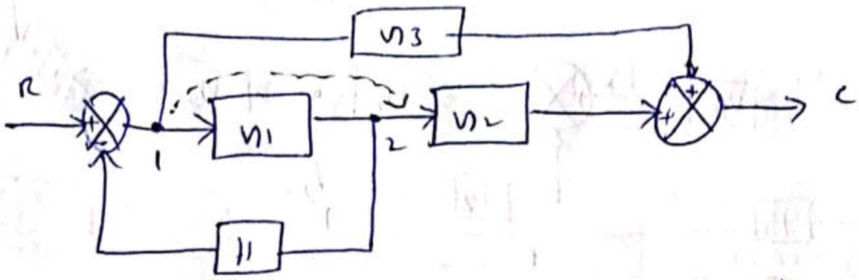
$$\boxed{\frac{C}{R} = \frac{G}{1-GH}}$$

* Problems based on Block diagram reduction :-

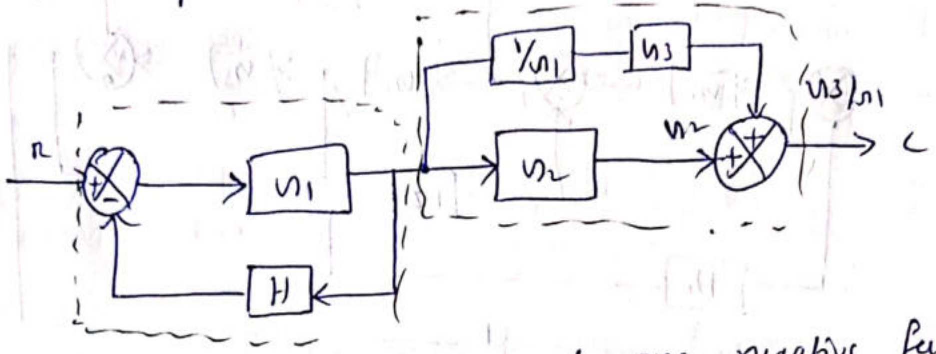
1) Reduce the block diagram shown in fig. Find C/R.



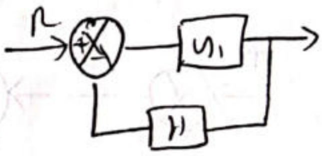
Sol.



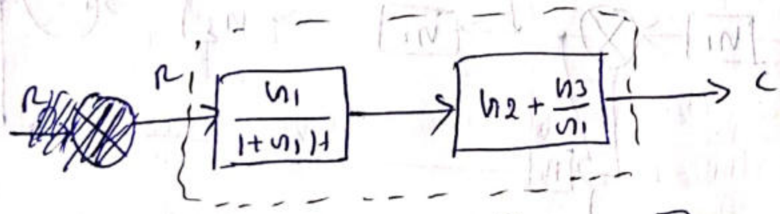
Move branch point ahead of block G_1 then the above fig becomes



The above block diagram consists of one negative feedback loop and one feed forward loop then above diagram becomes.



$$\left[\frac{G_1}{1+G_1H}\right]$$

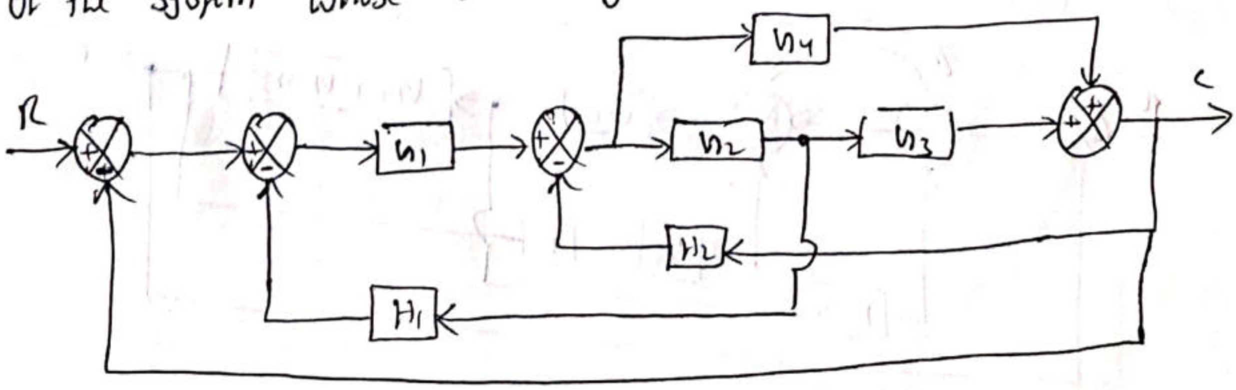


Output $$C = R\left[\frac{\cancel{G_1}}{1+G_1H}\right]\left[\frac{G_1G_2+G_3}{\cancel{G_1}}\right]$$

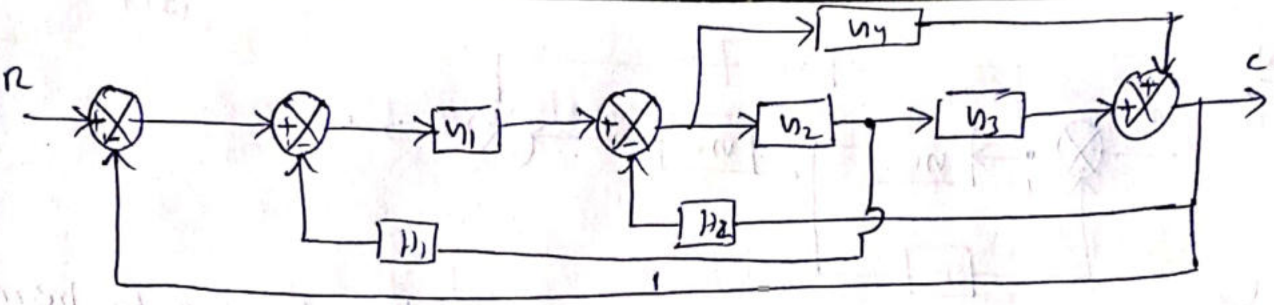
$$\frac{C}{R} = \frac{G_1G_2+G_3}{1+G_1H}$$

The overall transfer function of the system $\frac{C}{R} = \frac{G_1G_2+G_3}{1+G_1H}$

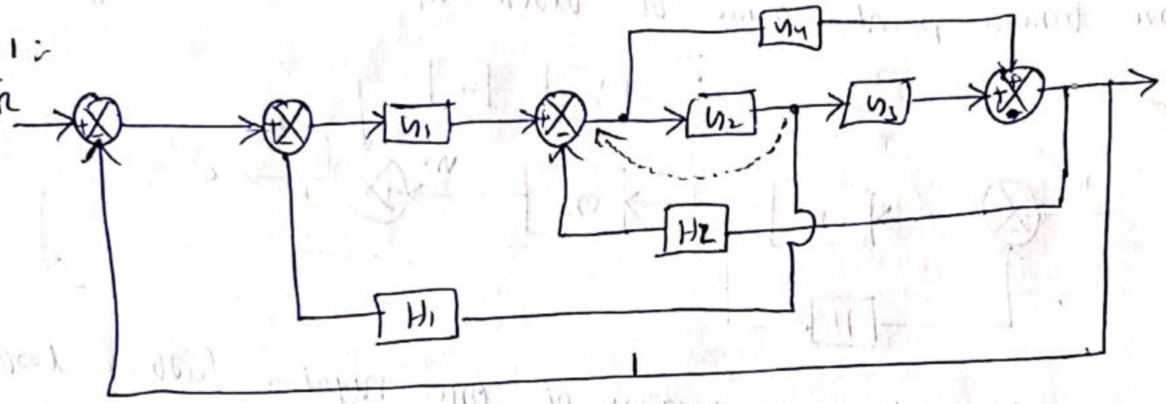
2) Using block diagram reduction technique, find closed loop T.F of the system whose block diagram shown in fig.



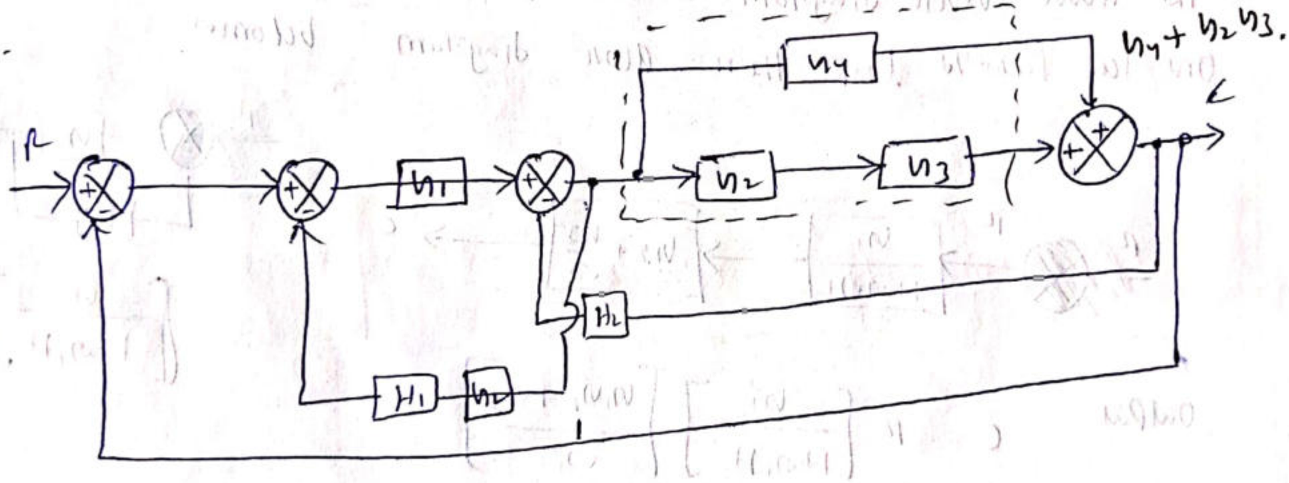
Sol



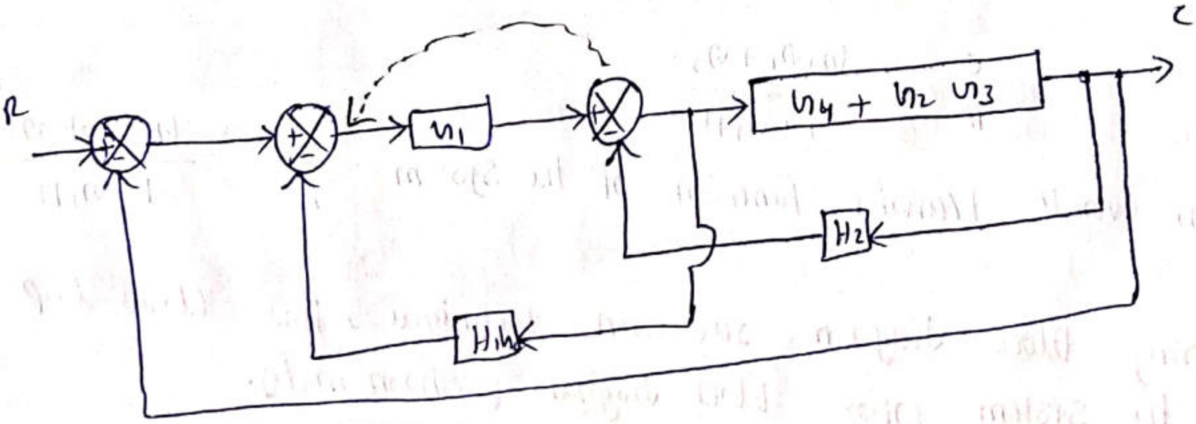
Step 1:-



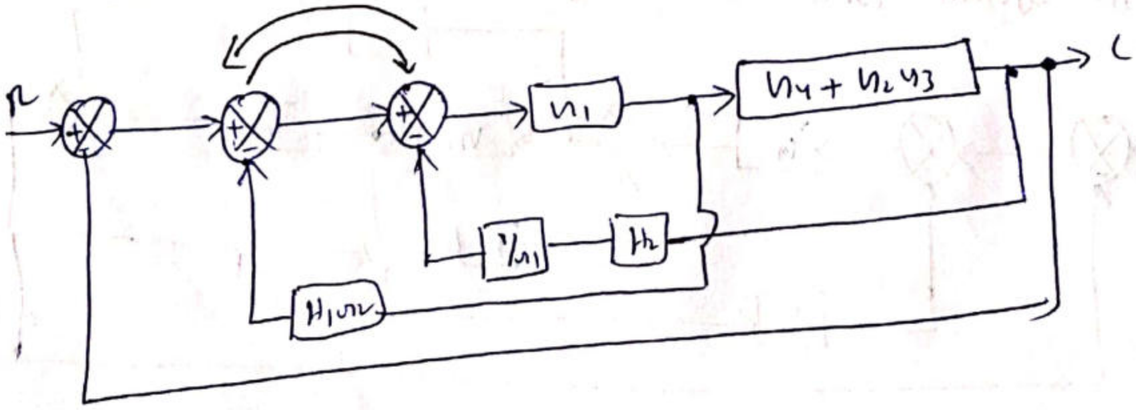
Step 2:-



Step 3:



Step 4:-



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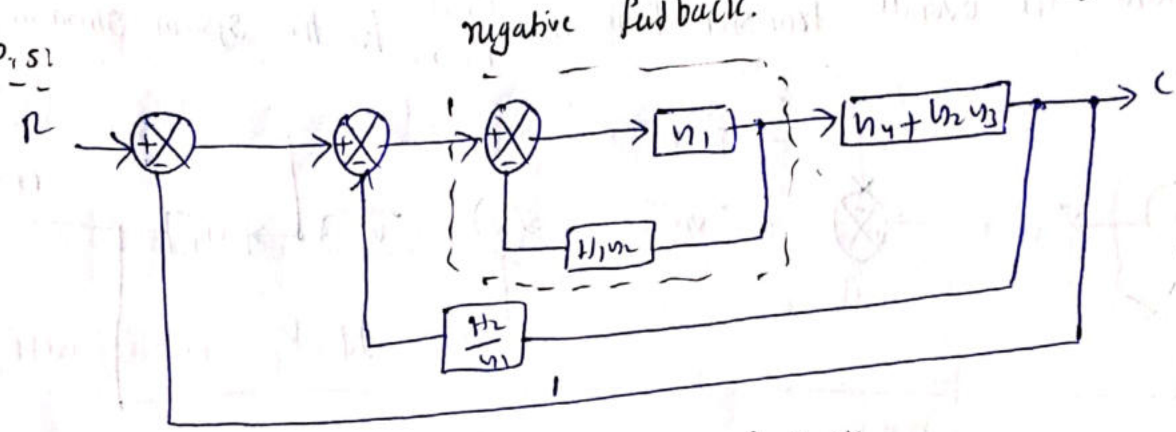
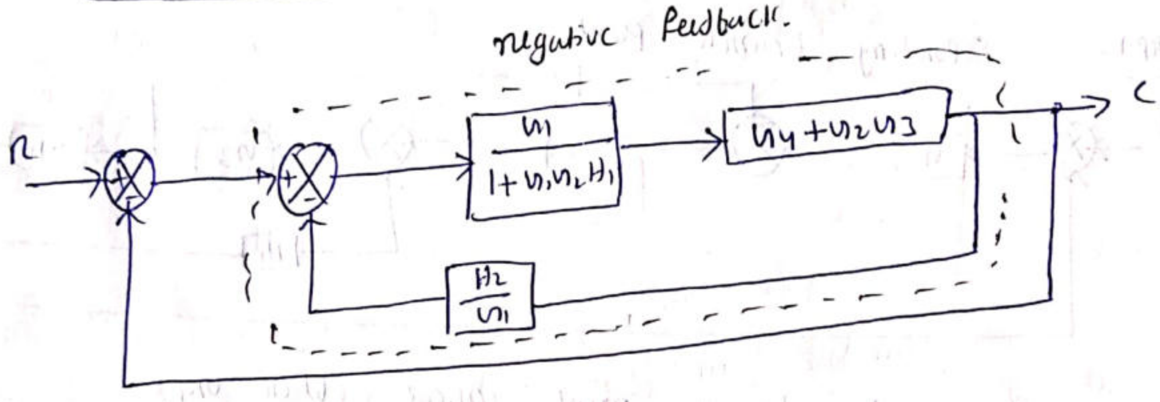
Step 5:

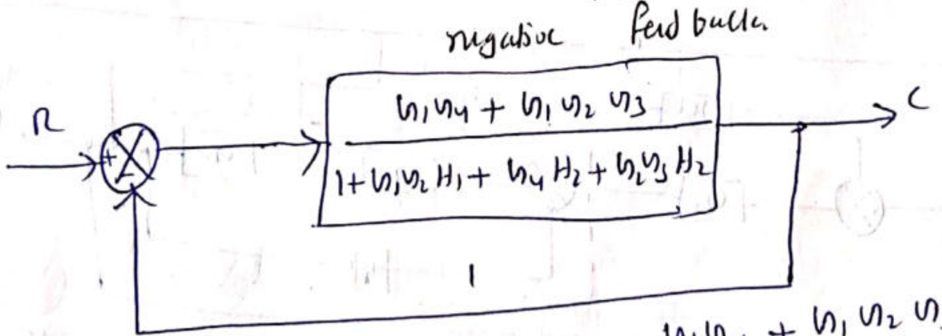


Step 6:



Step 7:

$$-\text{ve feedback} = \frac{\dfrac{G_1(G_4+G_2G_3)}{1+G_1G_2H_1}}{1+\dfrac{\cancel{G_1}(G_4+G_2G_3)}{1+G_1G_2H_1}\cdot\dfrac{H_2}{\cancel{G_1}}} = \frac{G_1(G_4+G_2G_3)}{1+G_1G_2H_1+(G_4+G_2G_3)H_2}$$



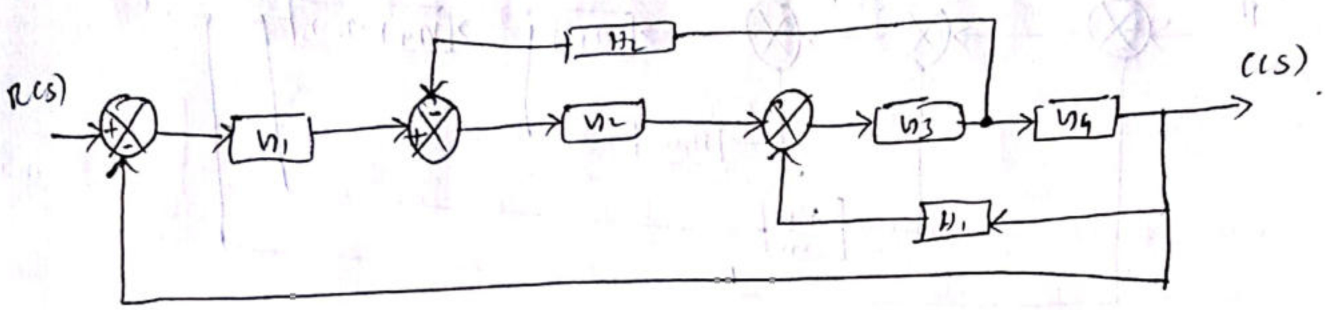
Step 8:

$$\text{Transfer function } \frac{C}{R} = \frac{\dfrac{G_1G_4+G_1G_2G_3}{1+G_1G_2H_1+G_4H_2+G_2G_3H_2}}{1+\dfrac{G_1G_4+G_1G_2G_3}{1+G_1G_2H_1+G_4H_2+G_2G_3H_2}\cdot 1}$$

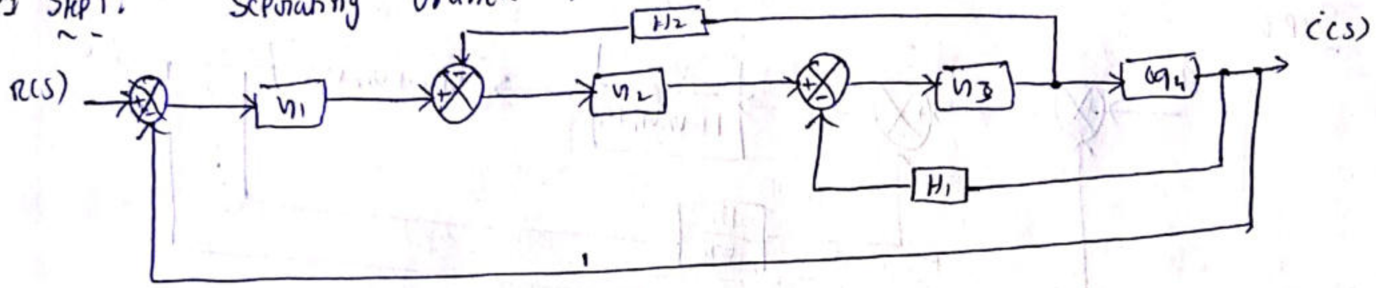
The overall transfer function is given by.

$$\boxed{\frac{C}{R} = \frac{G_1G_4+G_1G_2G_3}{1+G_1G_2H_1+G_4H_2+G_2G_3H_2+G_1G_4+G_1G_2G_3}}$$

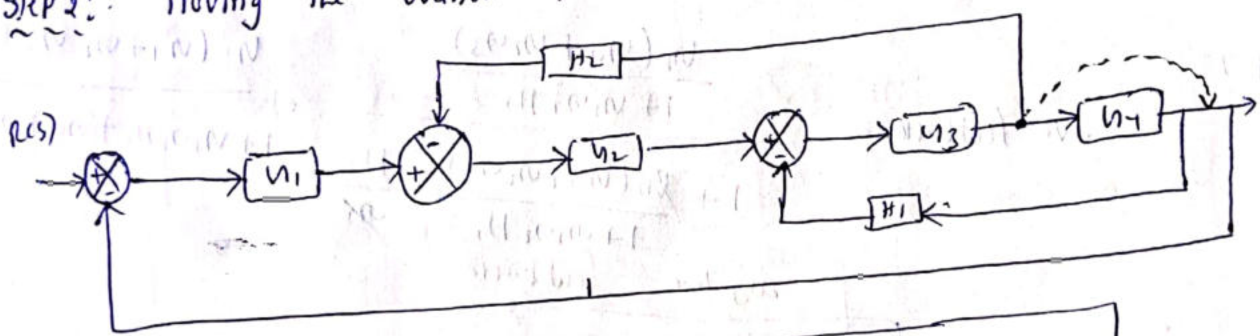
3) Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig.

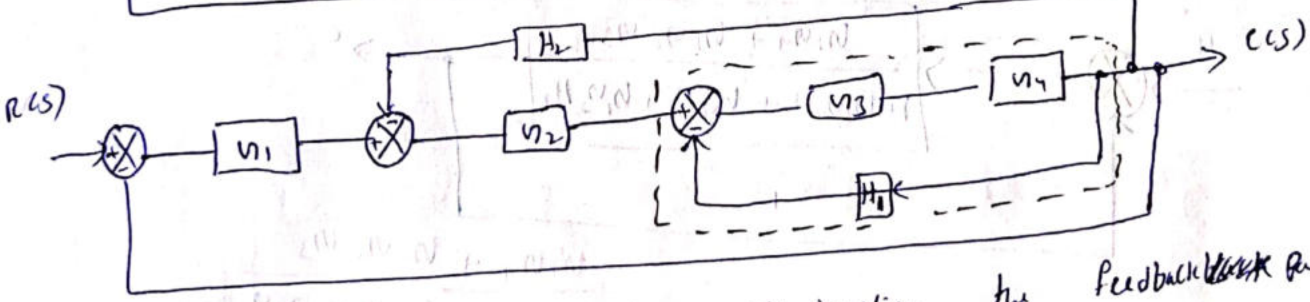


Sol Step 1: Separating branch point

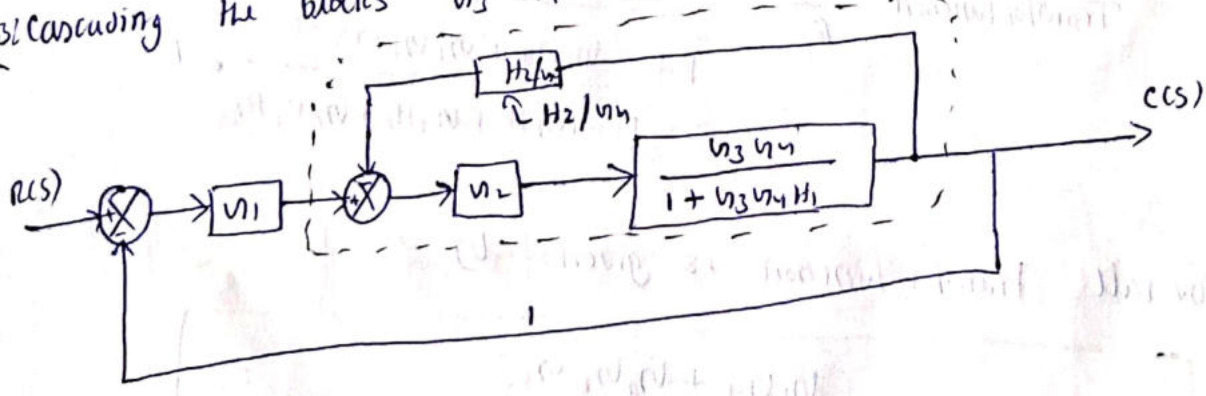


Step 2: Moving the branch point ahead block G_4.





Step 3: Cascading the blocks G_3 G_4 and eliminating the feedback path

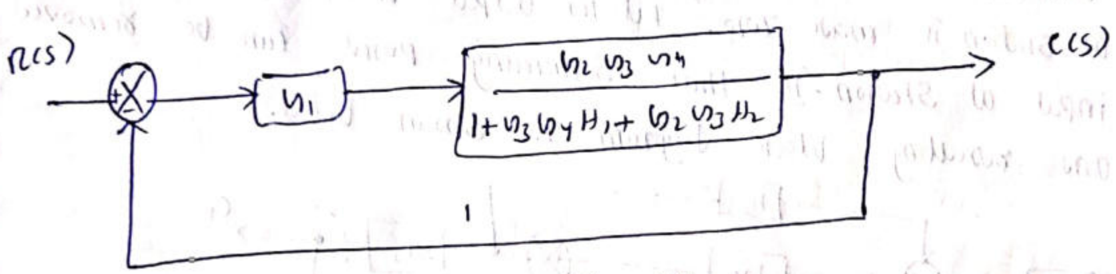


Step 4: Eliminating the feedback path.

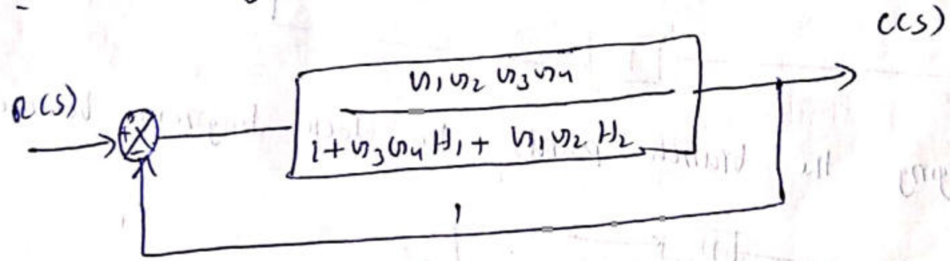
$$G(s) = \frac{G_2 G_3 G_4}{1+G_3 G_4 H_1} \qquad H(s) = H_2/G_4$$

For negative feedback $\dfrac{G(s)}{1+G(s)H(s)}$

$$\Rightarrow \frac{\dfrac{G_2 G_3 G_4}{1+G_3 G_4 H_1}}{1+\dfrac{G_2 G_3 G_4}{1+G_3 G_4 H_1} \times \left(\dfrac{H_2}{G_4}\right)} \Rightarrow \frac{G_2 G_3 G_4}{1+G_3 G_4 H_1 + G_2 G_3 H_2}$$



Step 5: Adding cascade elements.



Step 6: eliminating feedback path.

here, $G(s) = \dfrac{G_1 G_2 G_3 G_4}{1+G_3 G_4 H_1 + G_1 G_2 H_2}$ $\qquad H(s) = 1$

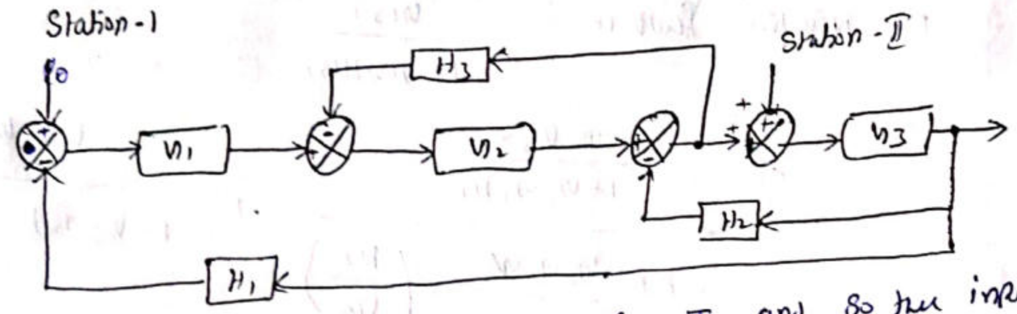
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{G_1 G_2 G_3 G_4}{1+G_3 G_4 H_1 + G_1 G_2 H_2}}{1+\dfrac{G_1 G_2 G_3 G_4}{1+G_3 G_4 H_1 + G_1 G_2 H_2}}$$

$\therefore$ The overall Transfer function.

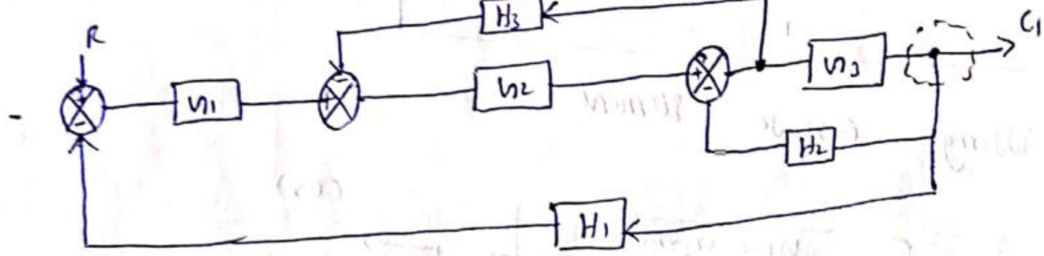
$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1+G_3 G_4 H_1 + G_1 G_2 H_2 + G_1 G_2 G_3 G_4}}$$

4) For the system represented by the block diagram shown in fig. Evaluate the closed loop transfer function when the input R is (i) at station-I (ii) at station-II.

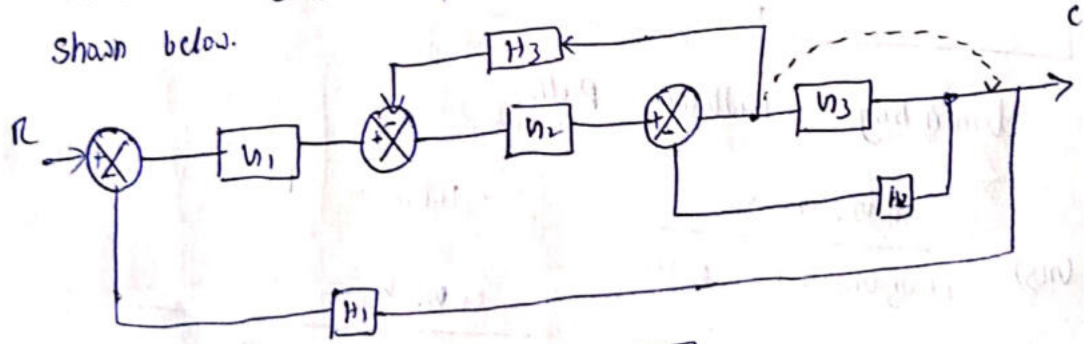
Sol:- ~~At station-I~~



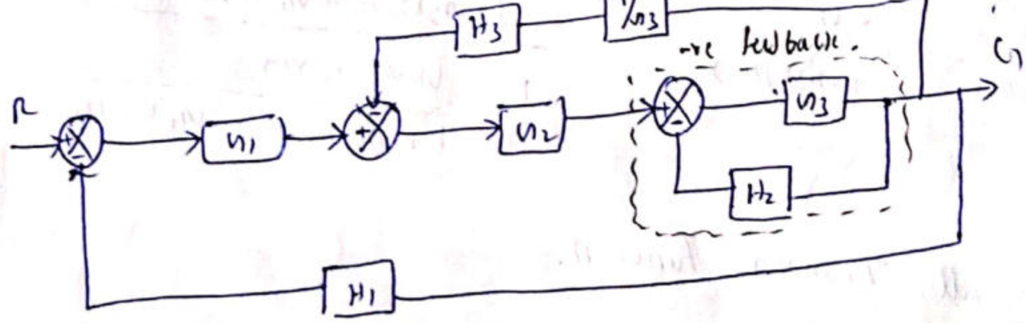
1) At Station-I:- Consider the input R is at station-I and so the input at station-II make zero. Let the output be C_1. Since there is no input at station-II that summing point can be removed and resulting block diagram is shown below.



Step: 1:- After rearranging the branch points the block diagram becomes shown below.



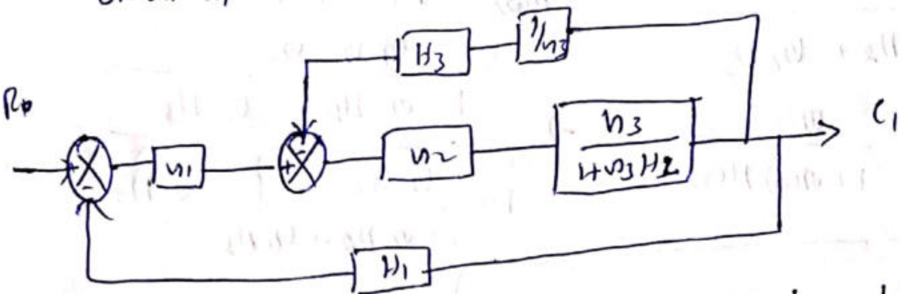
Step 2:



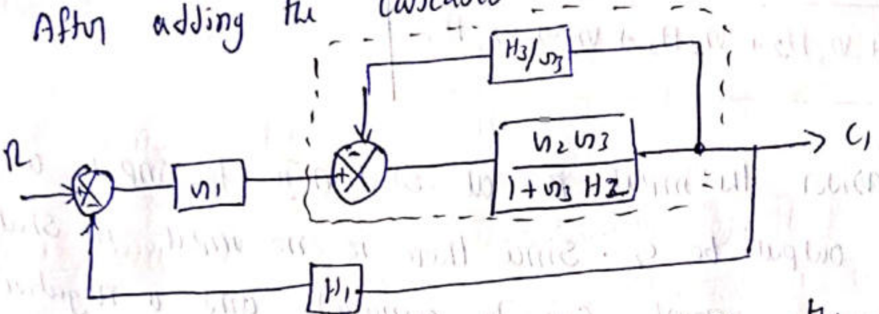
After moving the branch point ahead block G_3 as shown in above.

Step 3:- After eliminating the feedback loop the block diagram becomes shown in below.

$$\boxed{-ve\ feedback = \frac{G_3}{1+G_3H_2}}$$



Step 4:- After adding the cascaded blocks the block diagram becomes

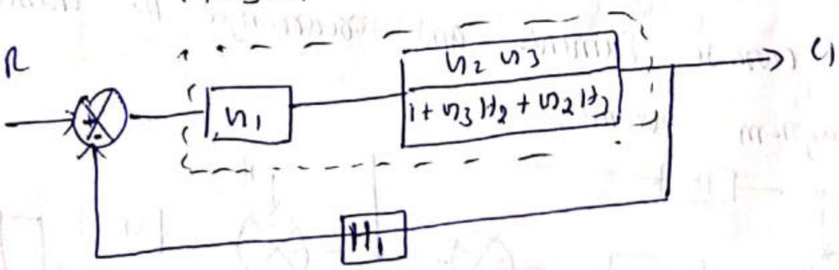


Step 5:- After eliminating the feedback path the block diagram becomes.

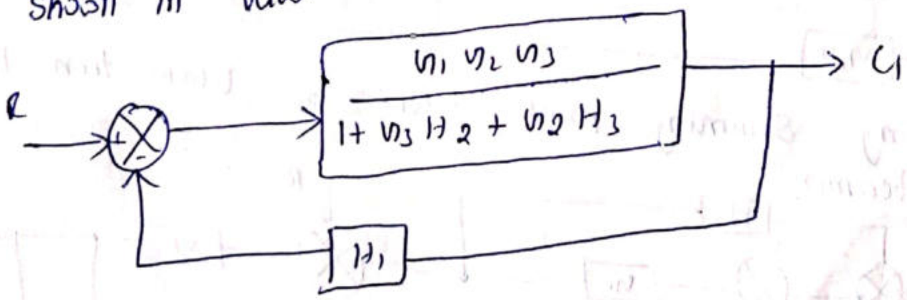
$$G(s) = \frac{G_2G_3}{1+G_3H_2} \qquad H(s) = H_3/G_3$$

for Negative feedback path $= \dfrac{G(s)}{1+G(s)\cdot H(s)} = \dfrac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2\cancel{G_3}}{1+G_3H_2}\times\dfrac{H_3}{\cancel{G_3}}}$

$$\Rightarrow \frac{G_2G_3}{1+G_3H_2+G_2H_3}.$$



Step 6:- After adding the cascaded elements the block diagram becomes shown in below.



Step 2: After eliminating feedback path, the T.F $\frac{C_1(s)}{R}$ becomes

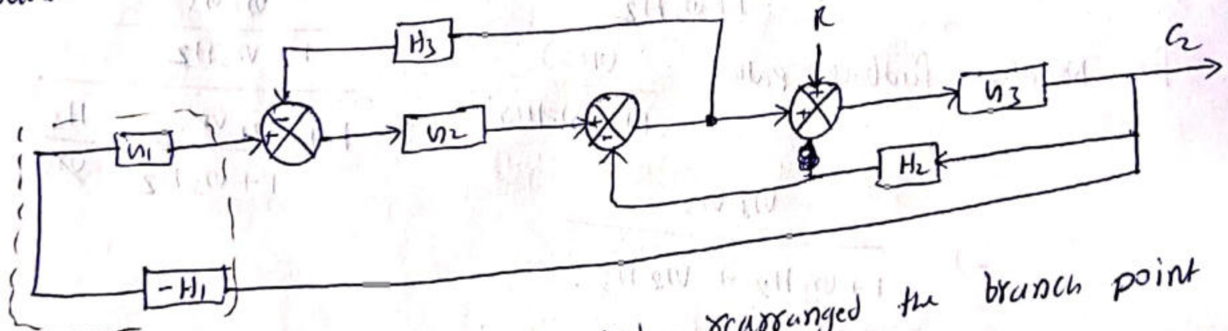
here $G(s) = \frac{G_1 G_2 G_3}{1+G_3 H_2 + G_2 H_3}$ $\quad H(s) = H_1$

For negative feedback $= \frac{G(s)}{1+G(s)H(s)}$ $\Rightarrow \frac{\frac{G_1 G_2 G_3}{1+G_3 H_2 + G_2 H_3}}{1+\frac{G_1 G_2 G_3}{1+G_3 H_2 + G_2 H_3} \times H_1}$

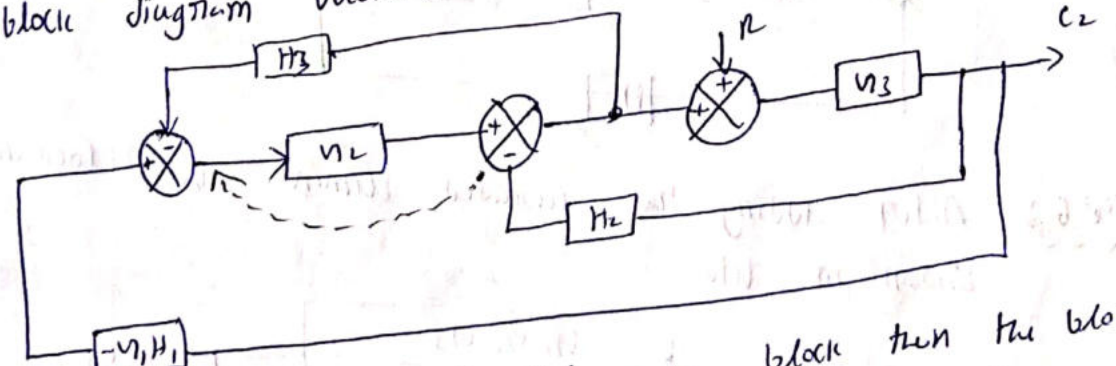
$$\boxed{\therefore \frac{C_1(s)}{R(s)} = \frac{G_1 G_2 G_3}{1+G_3 H_2 + G_2 H_3 + G_1 G_2 G_3 H_1}}$$

2) At Station-II :- Consider the input R at station-II, the input at station-I in station-I that made zero. Let output be c_2. Since there is no input in station-I that corresponding summing point can be removed and a negative sign can attached to the feedback path gain H_1. The resulting block diagram is shown in below.

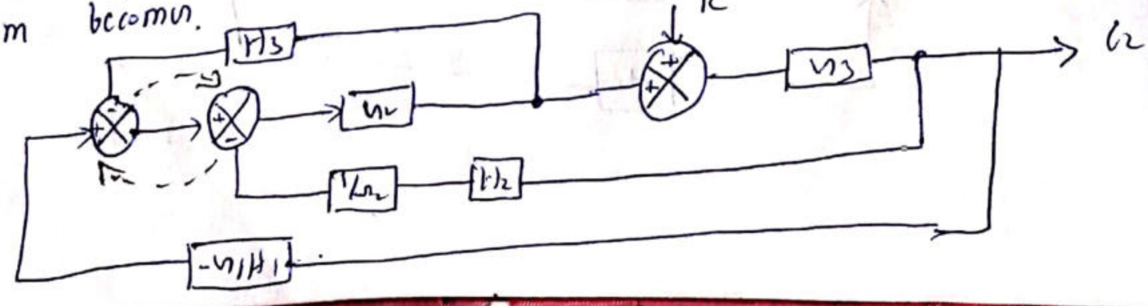
Step 1:-



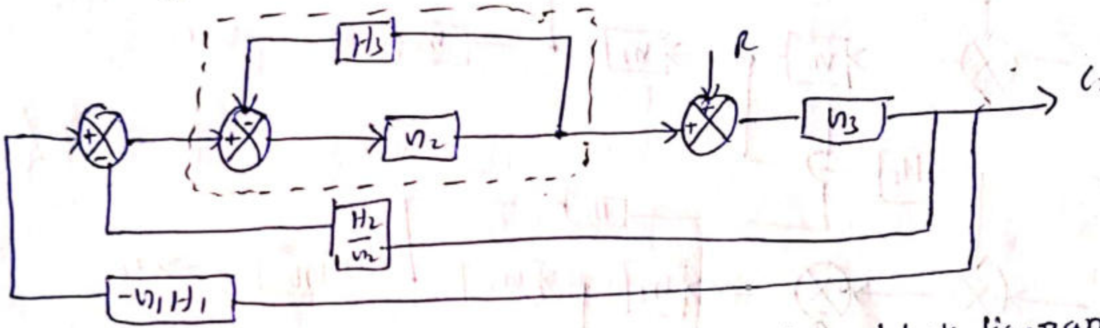
Step 2:- After Adding cascade elements and rearranged the branch point the block diagram becomes.



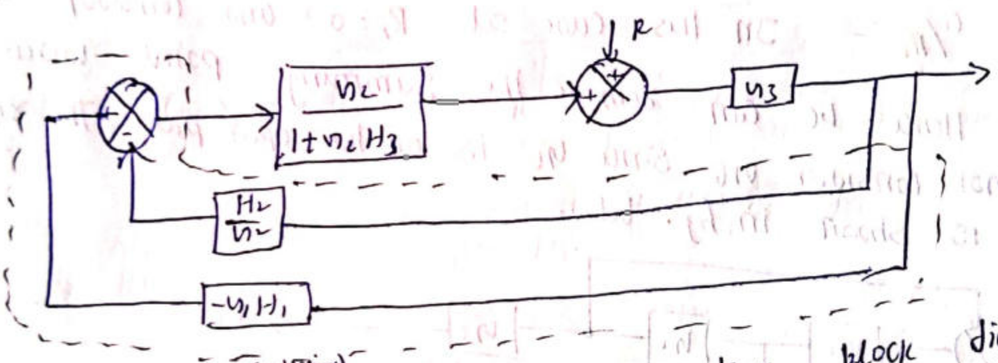
Step 3:- After moving summing point before G2 block then the block diagram becomes.



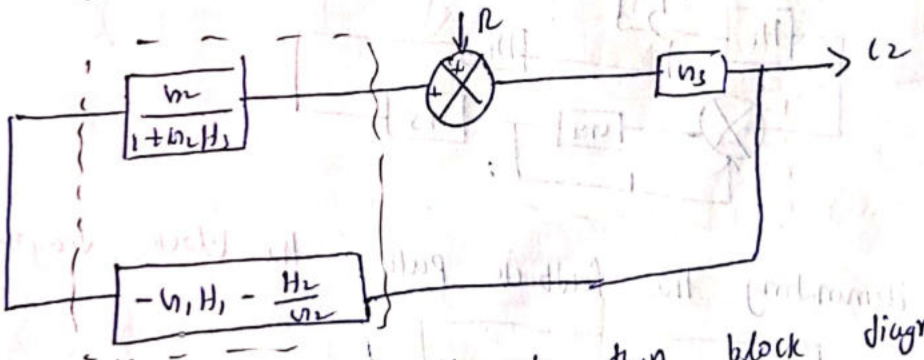
Step 4:- After changing the summing points and adding (3 L) cascade element the block diagram becomes.



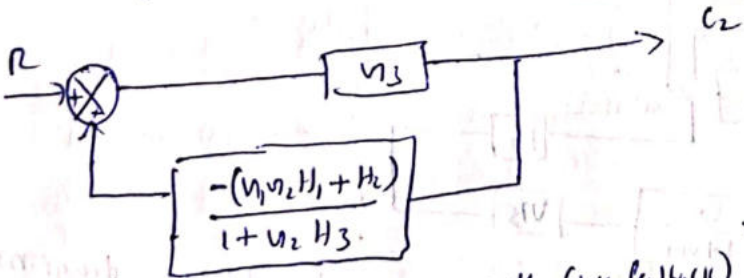
Step 5:- After eliminating the feedback path the block diagram becomes



Step 6: combaining the (Adding) ~~[illegible]~~ elements then block diagram becomes.



Step 7: combaining the cascade elements then block diagram becomes.



$$\left(\frac{G_2}{1+G_2H_3}\right)\times\left(-G_1H_1-\frac{H_2}{G_2}\right)$$

$$\Rightarrow \left(\frac{\cancel{G_2}}{1+G_2H_3}\right)\left(\frac{-G_1G_2H_1-H_2}{\cancel{G_2}}\right)$$

$$\Rightarrow -\left(\frac{G_1G_2H_1+H_2}{1+G_2H_3}\right)$$

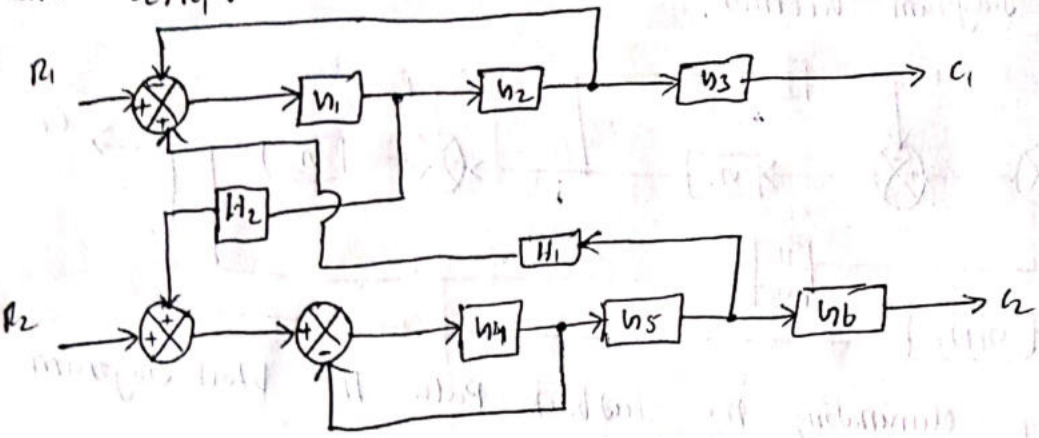
Step 8:- eliminating the feedback path (+ve feedback)

$$\frac{C_2}{R} = \frac{G_3}{1-\left[\frac{-(G_1G_2H_1+H_2)}{1+G_2G_3}\right]\cdot G_3}$$

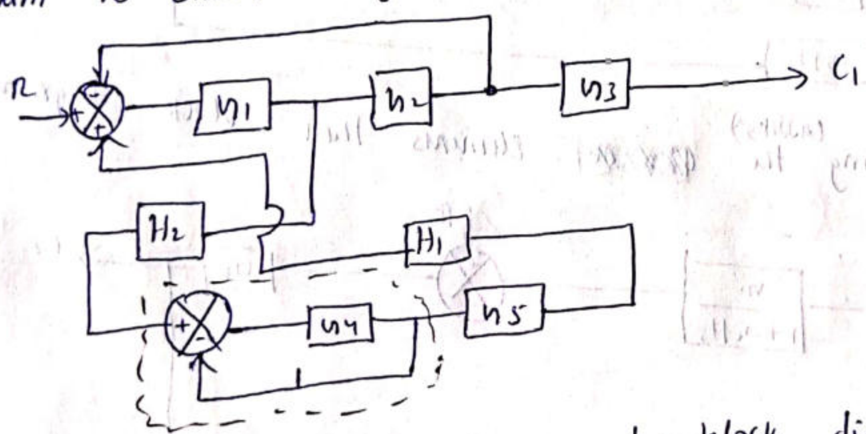
$$\boxed{\frac{C_2}{R} \Rightarrow \frac{G_3(1+G_2G_3)}{1+G_2H_3+G_1G_2G_3H_1+G_3H_2}}$$

Transfer function at station-II

5) For the system represented by the block diagram shown in fig. determine C_1/R_1 and C_2/R_1.

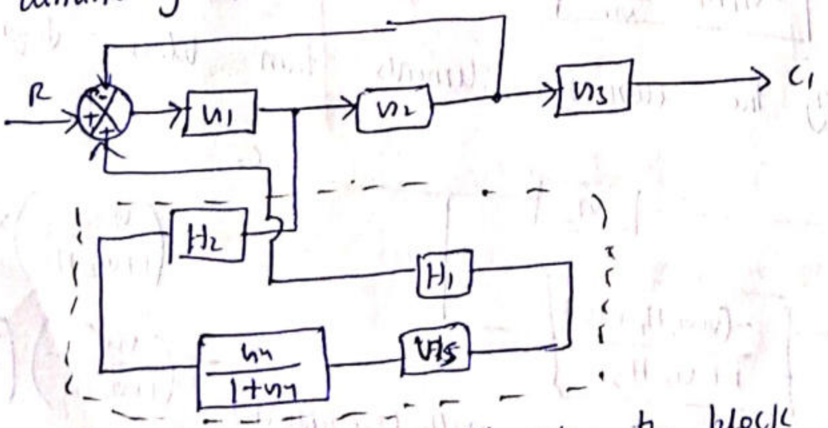


Sol Case (i) To find C_1/R_1 :- In this case set $R_2 = 0$ and consider only one output C_1. Hence we can remove the summing point which adds R_2 and need not consider G_6 since G_6 is on the open path. The resulting block diagram is shown in fig. below.

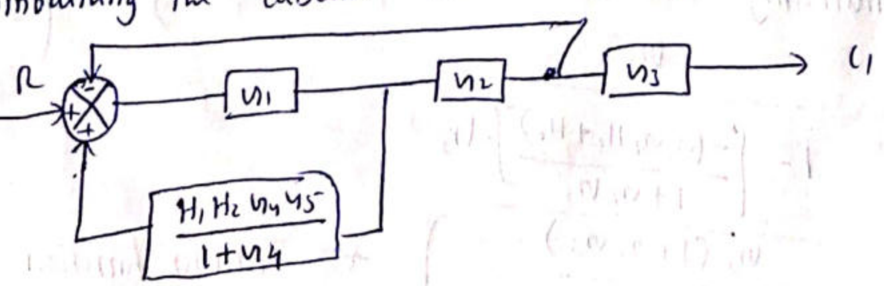


$$\frac{G_4}{1+G_4}$$

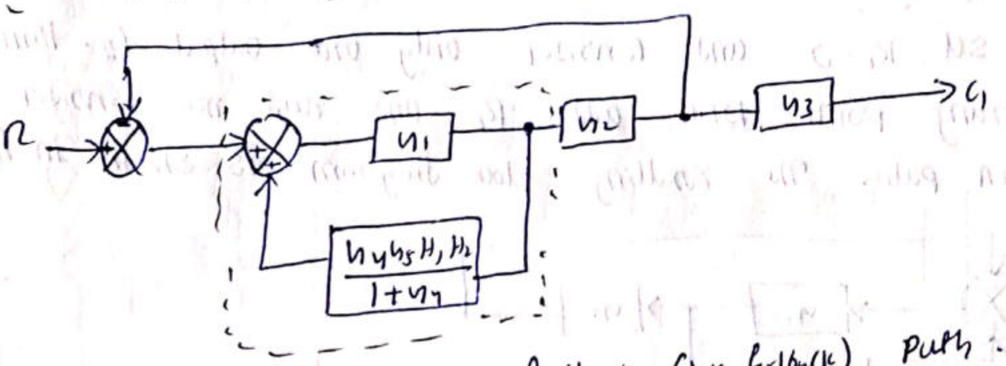
Step 1: After eliminating the feedback path the block diagram becomes



Step 2: After combining the cascade elements the block diagram becomes

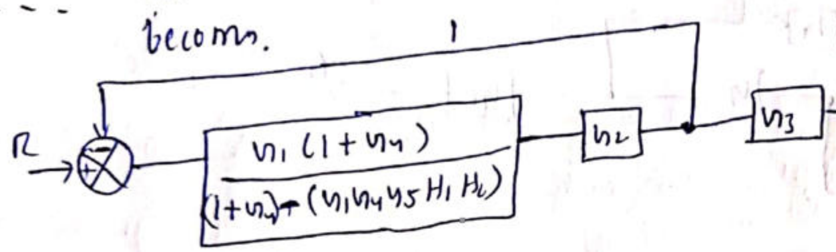


Step 3:- After splitting the summing point the block diagram becomes



Step 4:- After eliminating the feedback (+ve feedback) path the block diagram becomes.

$$\frac{G(s)}{1+G(s)H(s)} = \frac{G_1}{1 + \frac{G_1G_4G_5H_1H_2}{1+G_4}}$$



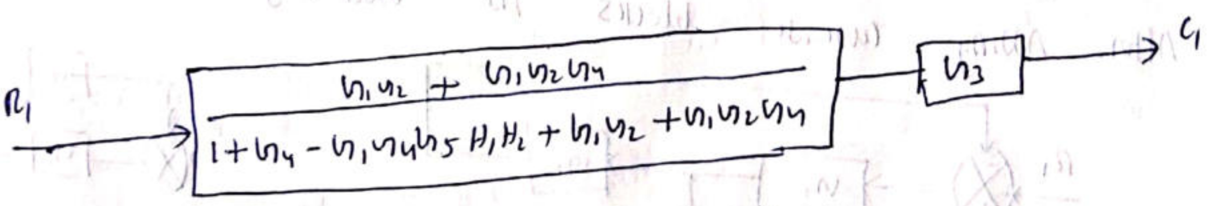
$$\Rightarrow \frac{G_1(1+G_4)}{1+G_4 + G_1G_4G_5H_1H_2}$$

Step 5:- After adding cascade block and eliminating the negative feedback the block diagram becomes.

$$H(s) = 1$$

here $$G(s) = \frac{G_1G_2 + G_1G_2G_4}{1+G_4 - G_1G_4G_5H_1H_2}$$

$$\frac{G(s)}{1+G(s)H(s)} \Rightarrow \frac{G_1G_2 + G_1G_2G_4}{1+G_4 - G_1G_4G_5H_1H_2 + G_1G_2 + G_1G_2G_4}$$

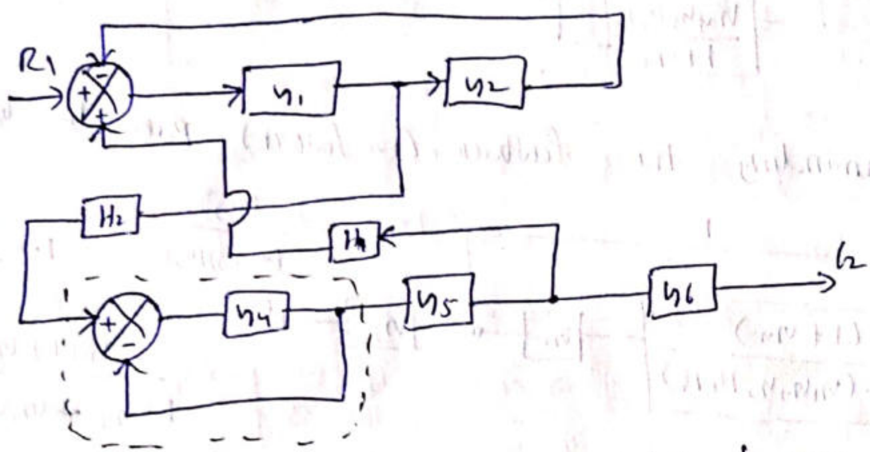


Step 6: The overall transfer function C_1/R_1

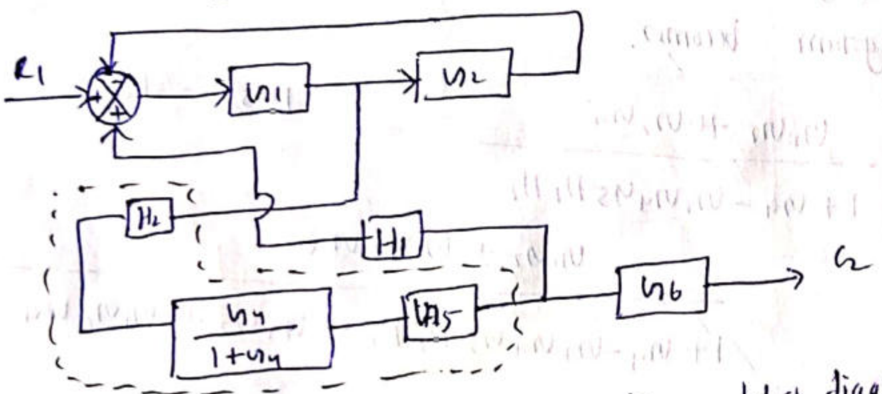
$$\boxed{\frac{C_1}{R_1} = \frac{G_1G_2G_3 + G_1G_2G_3G_4}{1+G_4 - G_1G_4G_5H_1H_2 + G_1G_2 + G_1G_2G_4}}$$

Case-2 :- To find $\frac{C_2}{R_1}$:-

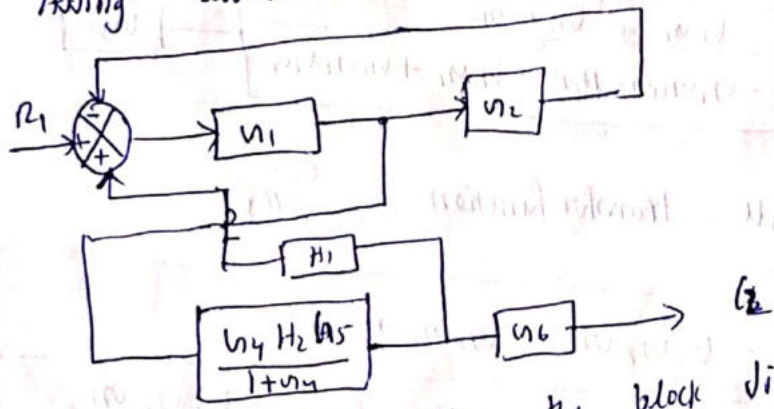
In this case set $R_2 = 0$ and consider only one output C_2. Hence we can remove the summing point which adds R_2 and need not consider G_3 and since G_3 is open path. The resulting block diagram is shown in fig.

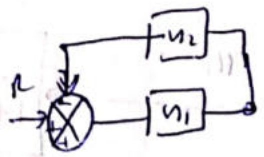


Step 1:- After eliminating feedback path the block diagram becomes $\frac{G_4}{1+G_4}$

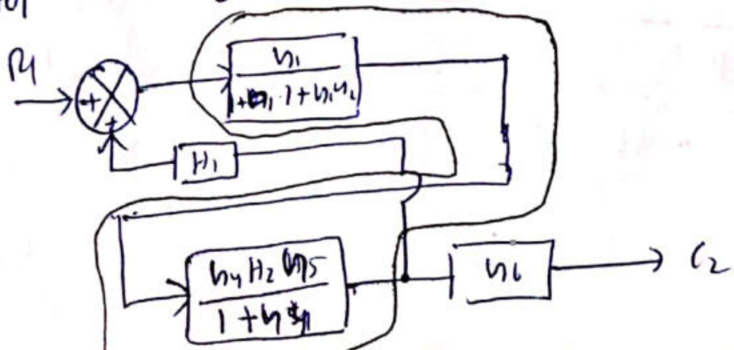


Step 2:- After adding cascade blocks the block diagram becomes.

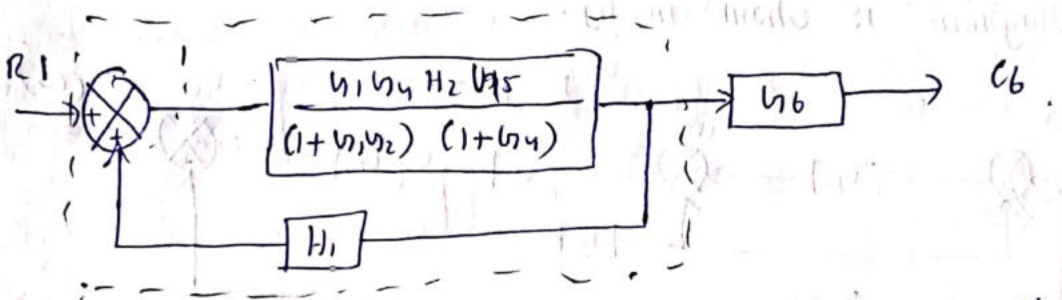




Step 3: After eliminating feedback path the block diagram becomes. $\frac{G_1}{1+G_1G_2}$



Step 4: After Adding cascade elements the block diagram becomes



Step 5:- After eliminating feedback back and adding cascade elements then the overall Transfer function becomes

$$G(s) = \frac{G_1 G_4 H_2 H_5}{(1+G_1G_2)\,(1+G_4)} \qquad H(s) = H_1$$

(+ve feedback)

$$\frac{G(s)}{1-G(s)H(s)} = \cancel{\frac{G_1 G_4 H_2 H_5}{(1+G_1G_2)(1+G_4) - H_1(1+G_1G_2)(1+G_4)}}$$

$$\cancel{\frac{G_1 G_4 H_2 H_5}{(1+G_1G_2)(1+G_4) - G_1 G_4 G_5 H_1 H_2}}$$

$$\frac{\dfrac{G_1 G_4 G_5 H_2}{(1+G_1G_2)\,(1+G_4)}}{1-\dfrac{G_1G_4G_5 H_1 H_2}{(1+G_1G_2)\,(1+G_4)}} \Rightarrow \frac{G_1 G_4 G_5 H_2}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

$$\boxed{\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6 H_2}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_1H_2}}$$

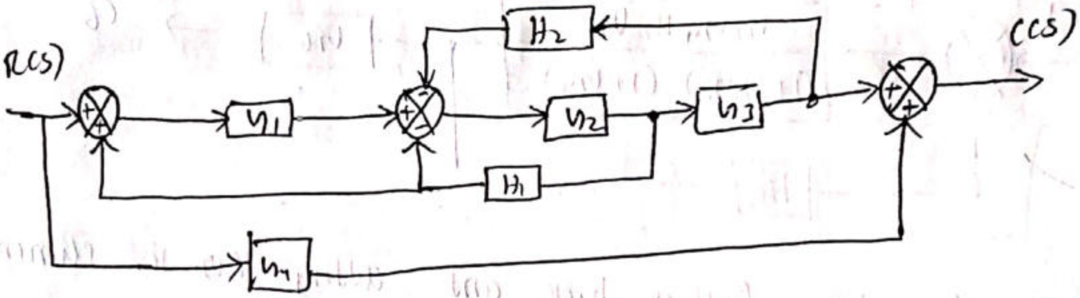
Result: The T.F of the system when the i/p and o/p are R_1 and C_1 is given by

$$\frac{C_1}{R_1} = \frac{G_1G_2G_3 + G_1G_2G_3G_4}{1+G_4 - G_1G_4G_5H_1H_2 + G_1G_2 + G_1G_2G_4}$$

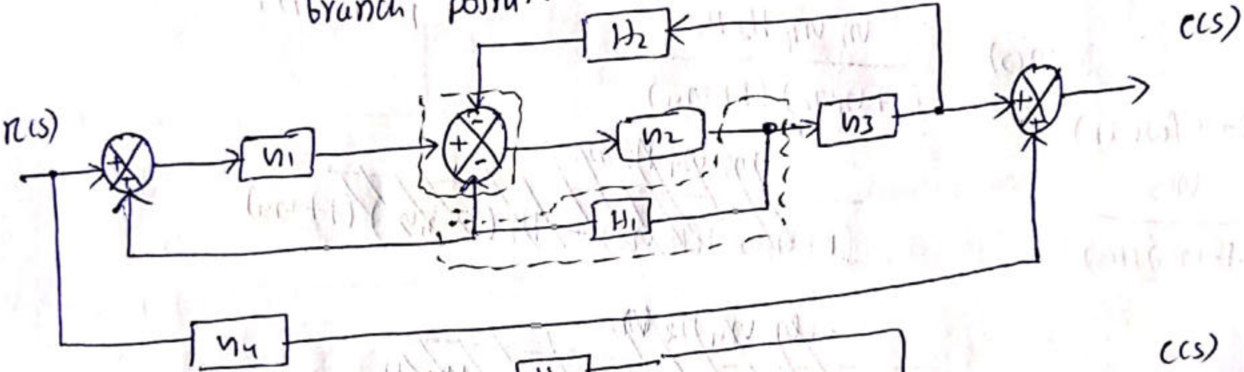
The T.F of the System when the i/p and o/p are R_1 and C_2 is given by

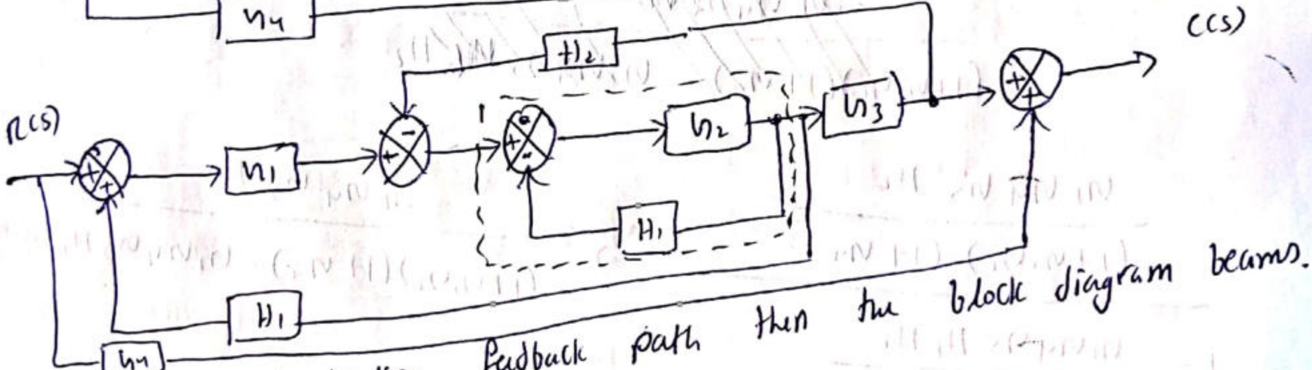
$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

6) Obtain the closed loop transfer function $C(s)/R(s)$ of the system whose block diagram is shown in fig.



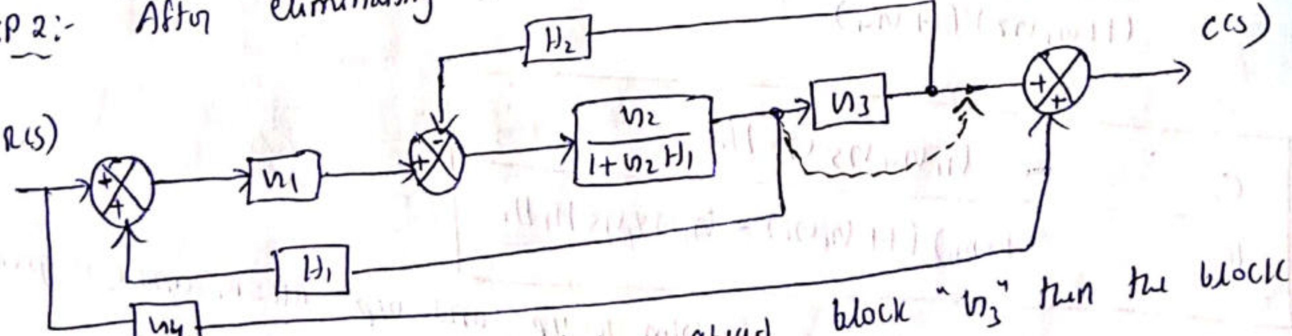
Sol: Step-1: Splitting the summing point and rearranging the branch points.



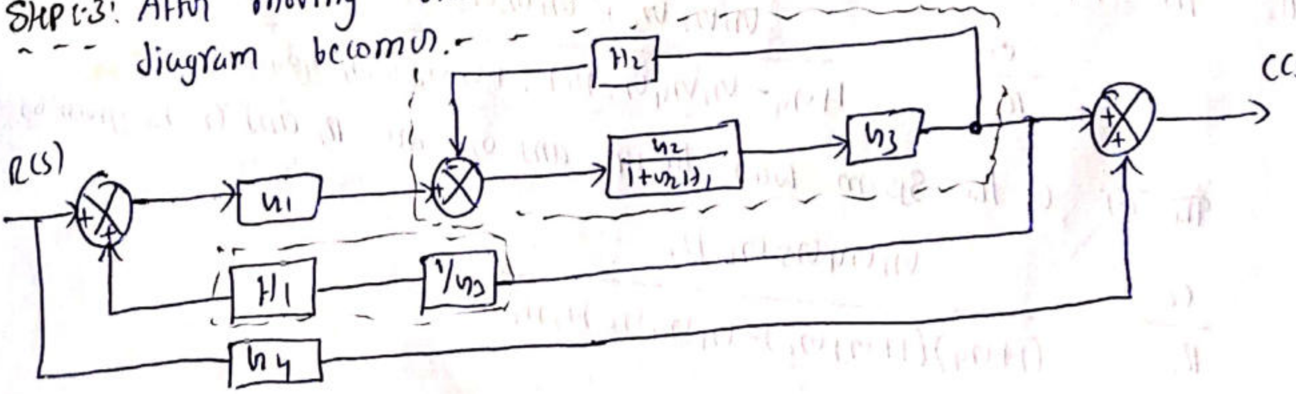


After eliminating feedback path then the block diagram becomes.

Step 2:-



Step-3: After moving branch point ahead block "G_3" then the block diagram becomes.-



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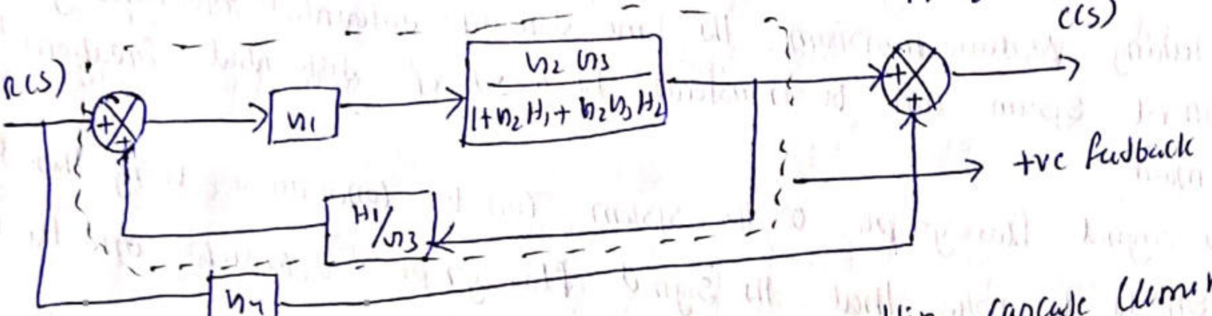
Step - 4 :- After eliminating the feedback path and adding cascade elements then block diagram becomes.

$$G(s) = \frac{G_2 G_3}{1+G_2 H_1} \qquad H(s) = H_2$$

For negative feedback $\Rightarrow \dfrac{G(s)}{1+G(s)H(s)} \Rightarrow \dfrac{\dfrac{G_2 G_3}{1+G_2 H_1}}{1+\dfrac{G_2 G_3 H_2}{1+G_2 H_1}} \Rightarrow \dfrac{G_2 G_3}{1+G_2 H_1 + G_2 G_3 H_2}$



Step 6:- After eliminating feedback path and adding cascade elements, then the overall Transfer function becomes.

where $G(s) = \dfrac{G_1 G_2 G_3}{1+G_2 H_1 + G_2 G_3 H_2}$

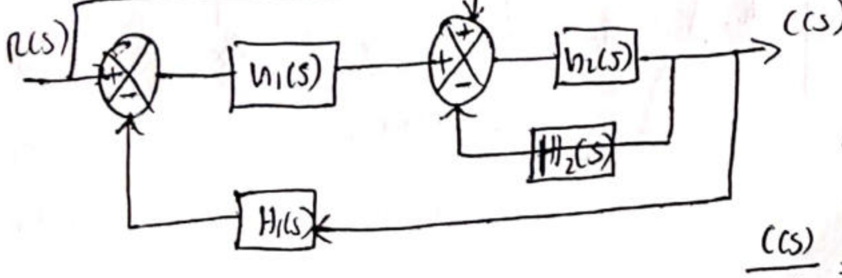
For positive feedback $\Rightarrow \dfrac{G(s)}{1-G(s)H(s)}$ $\qquad H(s) = H_1/G_3$

$$\Rightarrow \frac{G_1 G_2 G_3/(1+G_2 H_1 + G_2 G_3 H_2)}{1-\dfrac{G_1 G_2 \cancel{G_3}}{1+G_2 H_1 + G_2 G_3 H_2} \times \dfrac{H_1}{\cancel{G_3}}} \Rightarrow \frac{G_1 G_2 G_3}{1+G_2 H_1 + G_2 G_3 H_2 - G_1 G_2 H_1}$$

The overall Transfer function.

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_4}{1+G_2 H_1 + G_2 G_3 H_2 - G_1 G_2 H_1}}$$

Ex: The block diagram of a closed loop system is shown in fig. using the block diagram reduction technique determine the closed loop transfer function $C(s)/R(s)$



Ans

$$\frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)H_2(s)+G_1(s)G_2(s)H_1(s)}$$

* Signal flow graph :-

The Signal flow graph is used to represent the control system graphically and it was developed by "S.J. Mason."

→ A signal flow graph is a diagram that represents a set of simultaneous linear algebraic Equations.

→ By taking Laplace transform, the time domain differential Equations governing a control system can be transformed to a set of differential Equations in S-domain.

→ The signal flow graph of the system can be constructed using those Equations.

→ It should be noted that the signal flow graph approach and the block-diagram approach yeild the same information.

→ The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed Easily. This method is simpler than the block diagram reduction technique.

→ In a signal flow graph the flow of signals from one point of a system to another and given relations among the signals.

→ A signal flow graph consists of a network in which nodes are connected by directed branches.

→ Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier.

→ Each branch has a gain (or) transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch.

→ In a signal flow graph, the signal flows in only one direction. The direction of signal flow indicated by an arrow placed on the branch and the gain (multiplication factor) is indicated along the branch.

Explanation of terms used in Signal flow graph :-

Node : A node is a point representing a variable (or) Signal

Branch : A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of Signal flow and the gain of a branch is the transmittance.

Transmittance :- The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real (or) complex.

Input node (Source) :- It is a node that has only outgoing branch.

Output node (Sink) :- It is a node that has only incoming branches.

Mixed node :- It is a node that has both incoming and outgoing branches.

Path : A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.

Open path :- A open path starts at a node and ends at another node.

Closed path : closed path starts and ends at same node.

Forward path :- It is a path from an input node to an output node that does not cross any node more than once.

Forward path gain :- It is the product of the branch transmittances (gains) of a forward path.

Individual loop :- It is a closed path starting from a node and after passing through a certain part of a graph arrives at a same node without crossing any node more than once.

Loop gain : It is a product of the branch transmittances (gains) of a loop.

Non-touching loops : If the loops does not having common node then they are said to be non-touching loops.

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* Properties of Signal flow graph :-

The basic properties of signal flow graph are the following

i) The algebraic Equations which are used to construct signal flow graph must be in the form of cause and Effect relationship.

ii) Signal flow graph is applicable to linear system only.

iii) A node in the Signal flow graph represents the variable (or) signal.

iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

v) A mixed node which has both incoming and outgoing signals can be treated as an o/p node by adding an outgoing branch of unity transmittance.

vi) A branch indicates functional dependence of one signal on the other.

vii) A signal travel along branches only in the marked direction and when it travels it gets multiplied by the gain (or) transmittance of the branch.

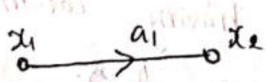
viii) The signal flow graph of system is not unique. By rearranging the system Equations different types of signal flow graphs can be drawn for a given system.

* Signal flow graph Algebra :-

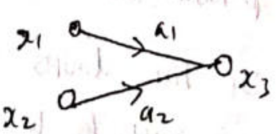
Signal flow graph for a system can be reduced to obtain the T.F of the system using the following rules.

Rule-1 :- Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Ex:

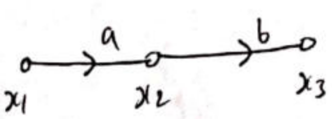


$x_2 = a_1x_1$

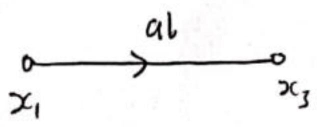


$x_3 = a_1x_1 + a_2x_2$

Rule : 2 :- Cascade branches can be combined to give a single branch whose transmittance is equal to product of individual branch transmittance.

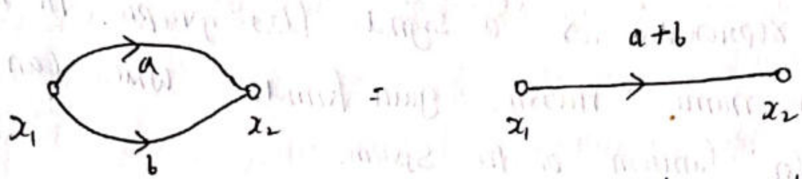


$\Rightarrow$



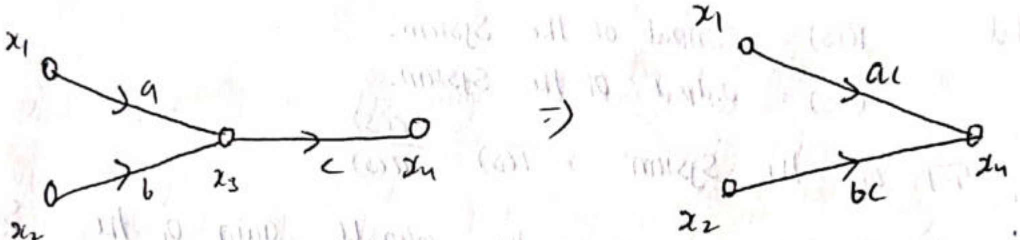
Rule :- 3 :- Parallel branch may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Ex.



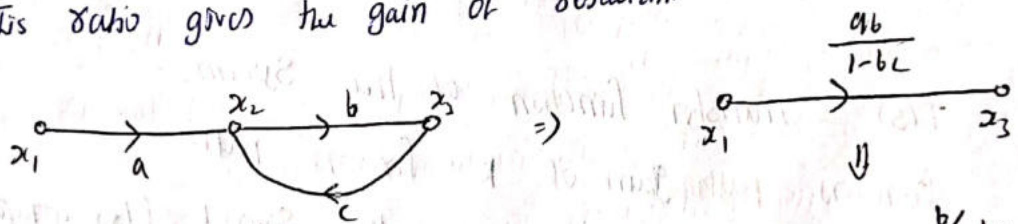
Rule :- 4 :- A mixed node can be eliminated by multiplying of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

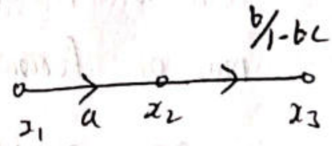
Ex.



Rule 5 :- A loop may be eliminated by writing Equations at the i/p and o/p node and rearranging the Equations to find the ratio of o/p to i/p. This ratio gives the gain of resultant branch.

Ex.





Proof :- $x_2 = ax_1 + cx_3$; $x_3 = bx_2$

$x_3 = b(ax_1 + cx_3) \Rightarrow abx_1 + bcx_3$

$x_3 - bcx_3 = abx_1 \Rightarrow x_3(1-bc) = abx_1$

$$\Rightarrow \boxed{x_3 = \frac{ab}{1-bc} \cdot x_1}$$

* Signal flow graph Reduction :-

The signal flow graph of a system can be reduced either by using the rules of "signal flowgraph algebra" (or) by using "Mason's gain formula".

For signal flow graph reduction using the rules of signal flow graph, write Equations at every node and then rearrange these Equations the get the ratio of output and input (transfer function).

The Signal flow graph reduction by above method will be time consuming and tedious. "S. J. Mason" has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name "Mason's gain formula" which can be directly used to find the transfer function of the system.

Mason's gain formula :- The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let $R(s)$ = Input of the system.

$C(s)$ = Output of the system.

Now, T.F of the system, $T(s) = \frac{C(s)}{R(s)}$

Mason's gain formula states the overall gain of the system (T.F) as follows,

$$\boxed{\text{overall gain (T.F)}, \ T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K}$$

Where

$T = T(s)$ = Transfer function of the system.

P_K = forward path gain of K^{th} forward path

K = No. of forward paths in the signal flow graph.

Δ = 1 − (Sum of individual loop gains)

+ (Sum of gain products of all possible combinations of two non-touching loops)

− (Sum of gain products of all possible combinations of three non-touching loops)

+ . . . . .

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path.

* Constructing Signal flowgraph for control systems :-

A Control system can be represented diagramatically by singnal-flow graph. The differential Equations governing the system are used to construct the signal flow graph. The following procedure can be used to construct the signal flow graph of a system.

i) Take Laplace transform of the differential Equations governing the system in order to convert them to algebric Equations in s-domain.

ii) The constants and variables of the s-domain Equations are identified.

iii) From the working knowledge of the system, the variables are identified as input, output and intermediate variables.

iv) For each variable a node is assigned in signal flow graph and constants are assigned as the gain of branches connected to the nodes.

v) For each Equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

~~* Procedure for converting block diagram to signal flow graph :-~~

~~The signal flow graph and block diagram of a system provide the same information but there is no standard procedure~~

~~* problems based on signal flow graph :-~~

~~1) Find the overall Transfer function of the system whose signal flow graph is shown in fig.~~

Steps involving to solve signal flow graph :-

i) Find forward path gains

ii) Individual loop gain

iii) Gain products of non touching loops

iv) Calculation of Δ and Δ_K.

here $\Delta = 1 -$ (individual loop gains) + (Gain products of non touching loops)

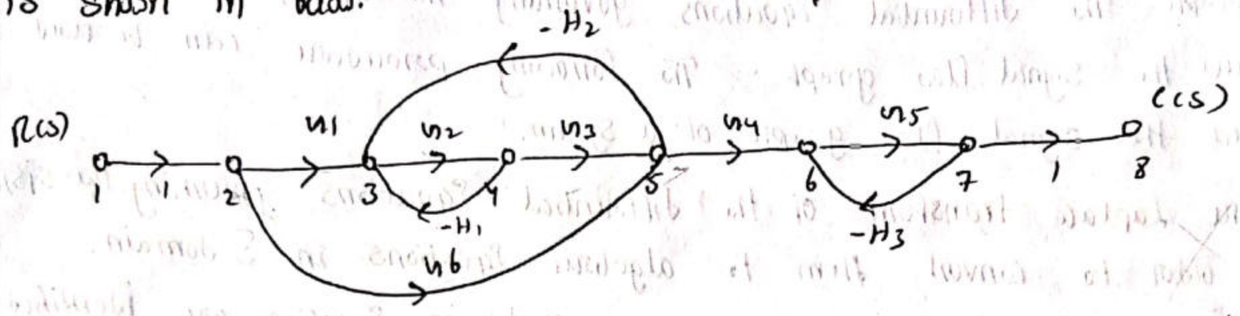
Δ_K = No. of loops not touching forward paths.

$K = 1, 2 \dots n \rightarrow K$ = No. of forward path gains.

v) Find Transfer function using Mason's gain formula.

Problems based on Signal flow graph :-

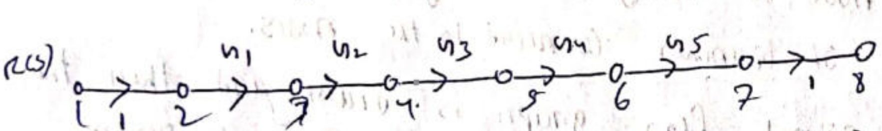
1) Find the overall Transfer function of the system whose signal flow graph is shown in below.



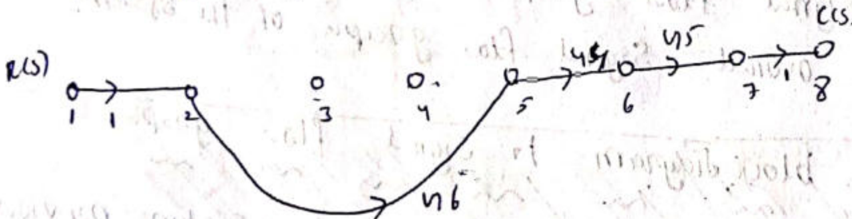
Sol: Step 1: Forward path gain :-

There are two forward paths $\therefore$ $K = 2$ [$\because$ all gains are in cascaded connection]

Let forward path gains be P_1 and P_2

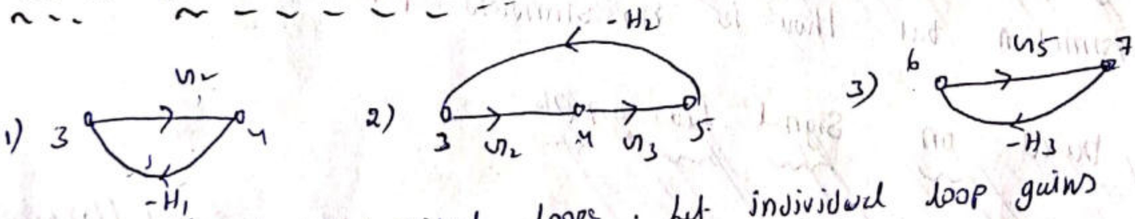


$\Rightarrow [P_1 = G_1 G_2 G_3 G_4 G_5]$



$\Rightarrow [P_2 = G_4 G_5 G_6]$

Step 2: Individual loop gains :- (Depends on feedback terms $-H$ (or) -1) H_1, H_2, H_3



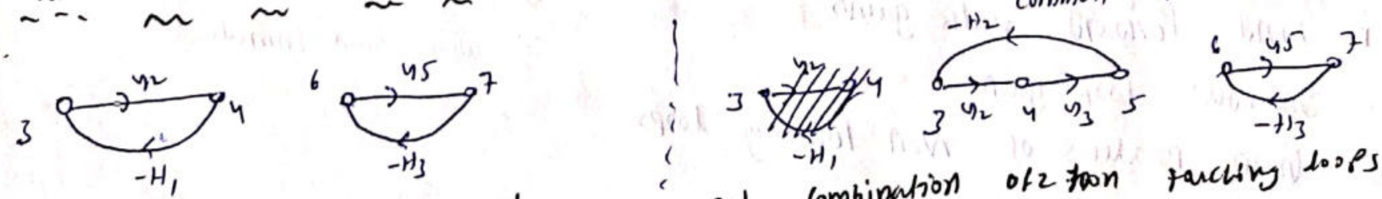
There are three individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31}

Loop gain of individual Loop-1 $\Rightarrow$ $P_{11} = -G_2 H_1$

Loop gain of individual loop-2 $\Rightarrow$ $P_{21} = -G_2 G_3 H_2$

Loop gain of individual loop-3 $\Rightarrow$ $P_{31} = -G_5 H_3$

Step 3: Gain products of Two Non-Touching loop :- (any two loops not having common nodes).



1st combination of 2-non touching loops | 2nd combination of 2 non touching loops.

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}

Gain product of first combination of two non-touching loops $\}$ $P_{12} = (-G_2 H_1)(-G_5 H_3) = G_2 G_5 H_1 H_3$

Gain product of second combination of two non-touching loops $\}$ $P_{22} = (-G_2 G_3 H_2)(-G_5 H_3) = G_2 G_3 G_5 H_2 H_3$

Step 4: Calculation of Δ and Δk :-

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2 H_1 - G_2 G_3 H_2 - G_5 H_3) + (G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3)$$

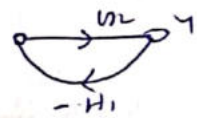
$$\Rightarrow 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3$$

$\Delta k \Rightarrow$ here k is no. of forward paths, since $k = 2$

Δ_1 and Δ_2.

$\Delta_1 = 1$, since, there is no part of graph which is not touching with 1st forward path (1st forward path has all touching loops)

$\Delta_2 =$ Only one loop is not touching forward path - 2 i.e, Loop-1



$\Rightarrow 1 - P_{11} \Rightarrow 1 + G_2 H_1$

Step 5: Transfer function, T :-

By Mason's gain formula the transfer function T is given by

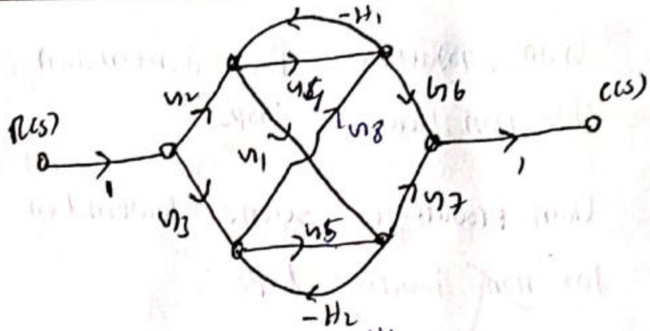
$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K \Rightarrow \frac{1}{\Delta}(P_1 \Delta_1 + P_2 \Delta_2)$$ (no. of forward paths $\Rightarrow 2$) $k = 2$

$$T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

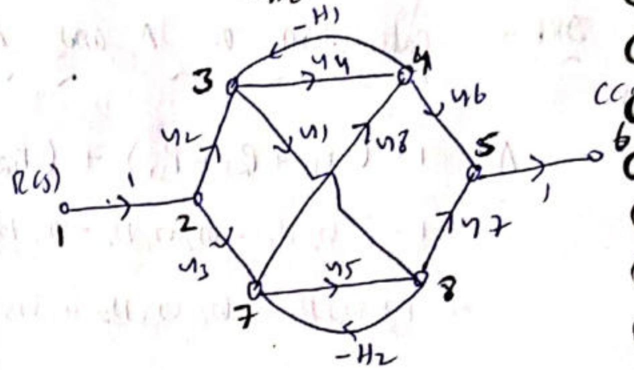
$$T = \frac{(G_1 G_2 G_3 G_4 G_5) \times 1 + (G_4 G_5 G_6)(1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

$$T = \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 + G_2 G_4 G_5 G_6 H_1}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$$

2) Find the overall gain of the system whose signal flow graph is shown in fig.

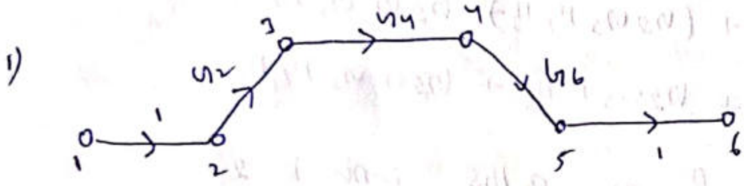


Sol: Let us number the nodes shown in fig.

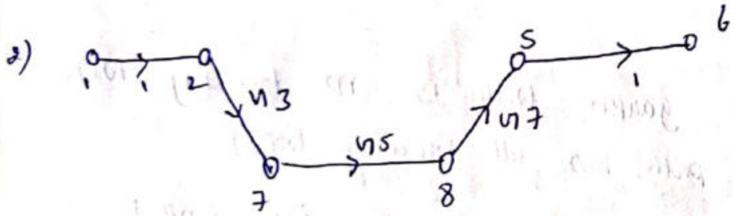


Case: I :- Forward path gains :-

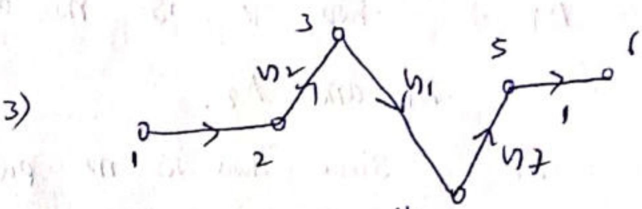
1)



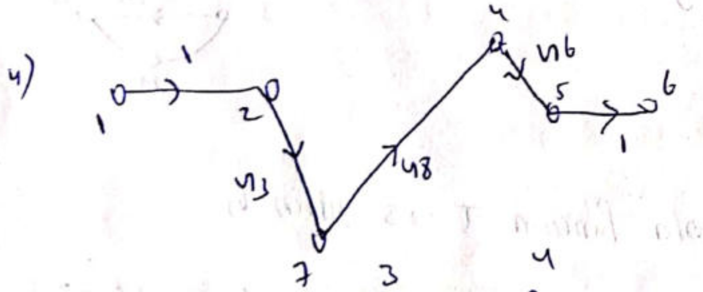
2)



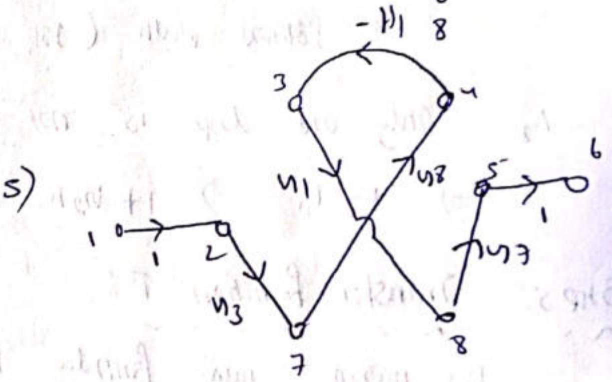
3)



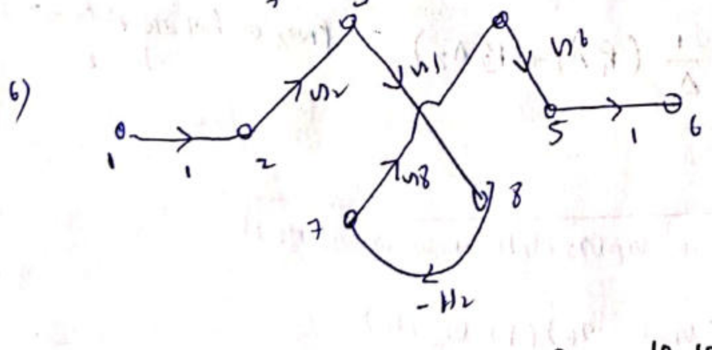
4)



5)



6)



There are six forward paths $\therefore K = 6$

Let the forward path gains be $P_1, P_2, P_3, P_4, P_5, P_6$

Gain of forward path-1 $\Rightarrow$ $P_1 = G_2 G_4 G_6$

Gain of forward path-2 $\Rightarrow$ $P_2 = G_3 G_5 G_7$

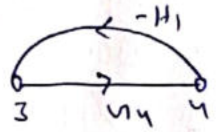
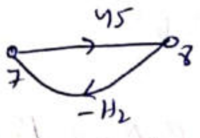
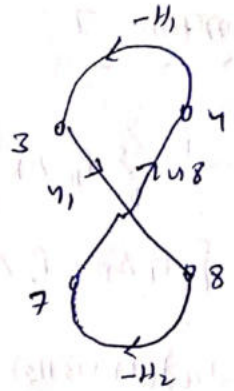
Gain of forward path-3 $\Rightarrow$ $P_3 = G_1 G_2 G_7$

Gain of forward path-4 $\Rightarrow$ $P_4 = G_3 G_6 G_8$

Gain of forward path-5 $\Rightarrow$ $P_5 = -G_1 G_3 G_7 G_8 H_1$

Gain of forward path-6 $\Rightarrow$ $P_6 = -G_1 G_2 G_6 G_8 H_2$

Case 2: Individual loop gains :-

1)  2)  3) 

There are three individual loops

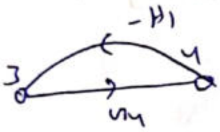
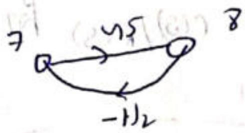
Let the individual loop gains P_{11}, P_{21}, P_{31}

Loop gain of individual loop-1 $\Rightarrow$ $P_{11} = -G_4 H_1$

Loop gain of individual loop-2 $\Rightarrow$ $P_{21} = -G_5 H_2$

Loop gain of individual loop-3 $\Rightarrow$ $P_{31} = G_1 G_8 H_1 H_2$

Case 3: Gain products of two non-touching loops :-

There is only one combination of two non-touching loops. Let gain product of two non-touching loops be P_{12}

gain product of 2 Non-touching Loops $\Rightarrow$ $P_{12} = (-G_4 H_1)(-G_5 H_2)$

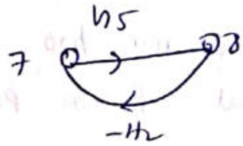
$= G_4 G_5 H_1 H_2$

Case 4: calculation of Δ and Δk :-

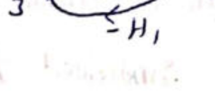
$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4 H_1 - G_5 H_2 + G_1 G_8 H_1 H_2) + G_4 G_5 H_1 H_2$

$\Rightarrow 1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2$

$\Delta k =$ here "6" forward paths here $k = 6$

$\Delta_1 =$ forward path-1 having one non-touching loop i.e, 

$= 1 + G_5 H_2$

$\Delta_2 =$ forward path-2 having one non-touching loop. i.e, 

$= 1 + G_4 H_1$

$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$

Step:5 : Transfer function, T :-

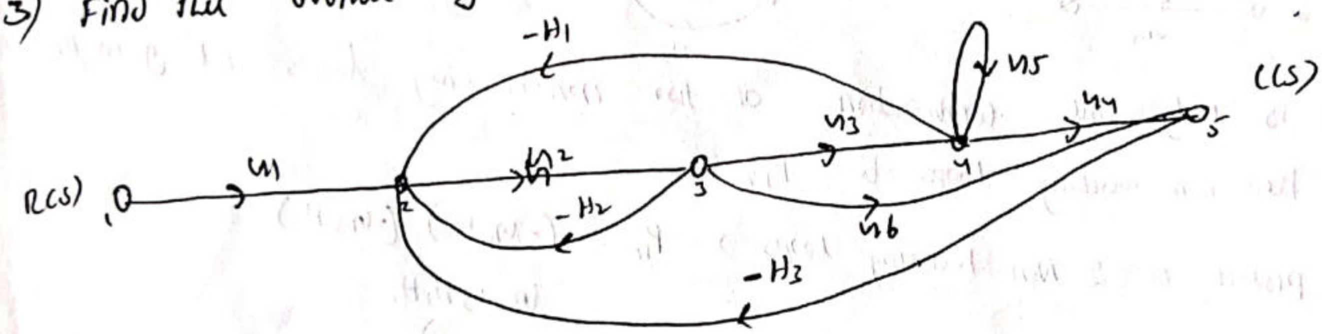
$$T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K \qquad K = 6$$

$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta} \left[P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6 \right]$$

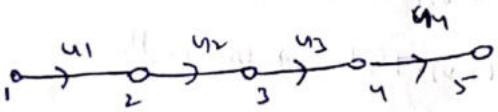
$$\Rightarrow \frac{(G_2G_4G_6)(1+G_5H_2) + (G_3G_5G_7)(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8}{1 + G_4H_1 + G_5H_2 + G_4G_5H_1H_2 - G_1G_8H_1H_2}$$

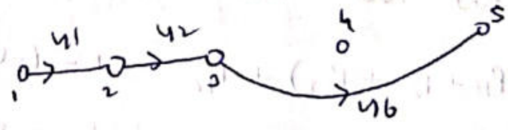
$$\boxed{T = \frac{C(s)}{R(s)} = \frac{(G_2G_4G_6)(1+G_5H_2) + (G_3G_5G_7)(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_1}{1 + G_4H_1 + G_5H_2 + G_4G_5H_1H_2 - G_1G_8H_1H_2}}$$

3) Find the overall gain C(s)/R(s) for the signal flow graph shown in fig



Step(case)1: Forward path gains, K :-

1)


2)


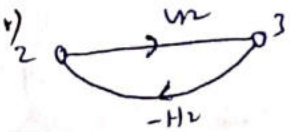
There are two forward paths $K = 2$

Let forward path gains are P_1, P_2

Gain of forward path-1 $\Rightarrow$ $P_1 = G_1G_2G_3G_4$

Gain of forward path-2 $\Rightarrow$ $P_2 = G_1G_2G_6$

Case 2: Individual loop gains:

1)


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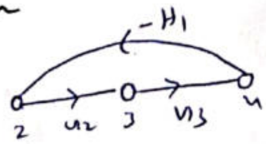
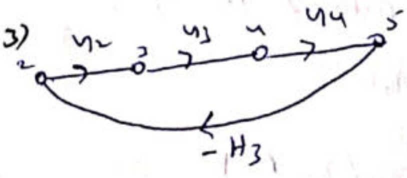


3)


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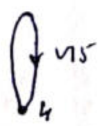

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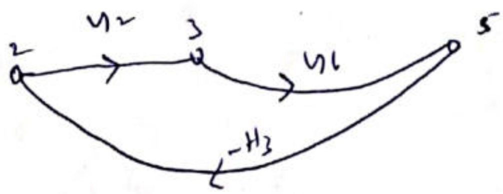


There are five individual loops.

Let the individual loop gains be $P_{11}, P_{21}, P_{31}, P_{41}, P_{51}$

Loop gain of individual loop-1 $\Rightarrow P_{11} = -G_2 H_2$

Loop gain of individual loop-2 $\Rightarrow P_{21} = -G_2 G_3 H_1$

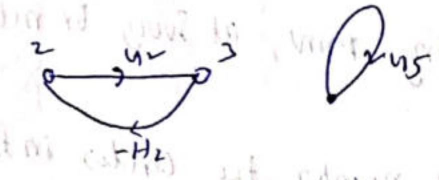
Loop gain of individual loop-3 $\Rightarrow P_{31} = -G_2 G_3 G_4 H_3$

Loop gain of individual loop-4 $\Rightarrow P_{41} = G_5$

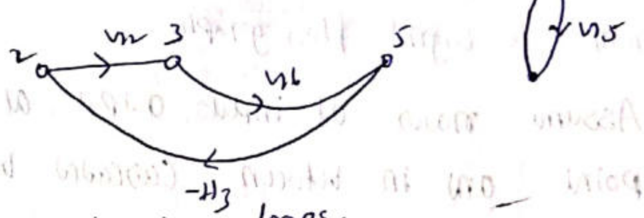
Loop gain of individual loop-5 $\Rightarrow P_{51} = -G_2 G_6 H_3$

Case 3: gain products of two Non-touching loops:-

1)



2)



There are two combination of two non-touching loops.

Let the gain products of two non-touching loops be P_{12}, P_{22}

gain product of first combination of two non-touching loops $\Big\}\ P_{12} = -G_2 G_5 H_2$

gain product of second combination of two non-touching loops $\Big\}\ P_{22} = -G_2 G_5 G_6 H_3$

Case 4: Calculation of Δ and Δk:-

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) + (P_{12} + P_{22})$$

$$\Rightarrow 1 + G_2 H_2 + G_2 G_3 H_1 + G_2 G_3 G_4 H_3 + G_2 G_6 H_3 - G_5 - G_2 G_5 H_2 - G_2 G_5 G_6 H_3$$

$\Delta k =$ it has 2 forward paths i.e, $K = 2$

$\Delta_1 = 1$

$\Delta_2 = 1 - G_5$

Case 5: Transfer function, T:

$$T = \frac{1}{\Delta}[P_1 \Delta_1 + P_2 \Delta_2].$$

$$\boxed{T = \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 + G_1 G_2 G_6 - G_1 G_2 G_5 G_6}{1 + G_2 H_2 + G_2 G_3 H_1 + G_2 G_3 G_4 H_3 + G_2 G_6 H_3 - G_5 - G_2 G_5 H_2 - G_2 G_5 G_6 H_3}}$$

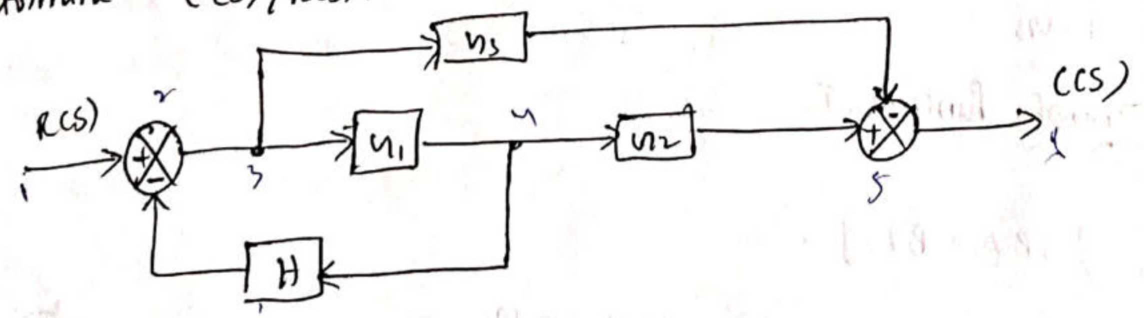
* Procedure for converting block diagram to signal flow graph :-

The signal flow graph and block diagram of the system provides same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rules to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and "Mason's gain formula" is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

- → Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
- → Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4 ... etc.
- → From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
- → Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
- → Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

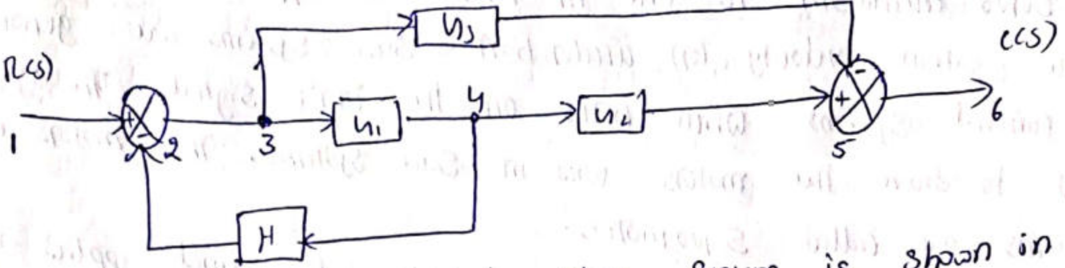
Problems based on conversion of block diagram into signal flow graph :-

1) Convert the given block diagram to signal flow graph and determine $C(s)/R(s)$.

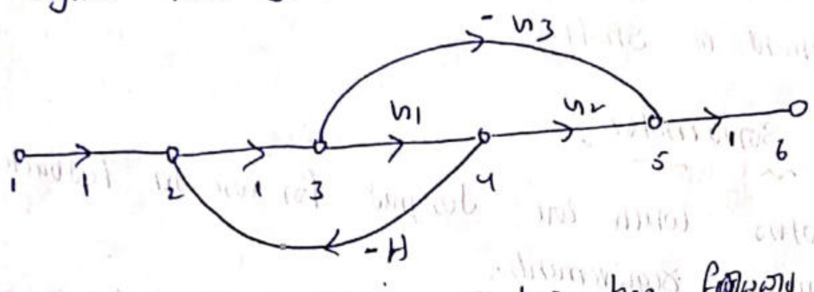


(4b)

Sol The nodes are assigned at i/p, o/p, at every summing point & at every branch point as shown in fig.

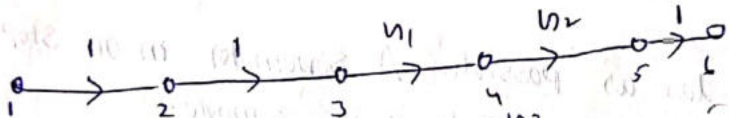


The Signal flow graph of the above system is shown in fig.

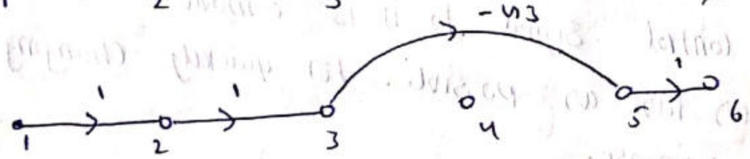


Case 1: Forward path gains :- It has two forward paths i.e, $K=2$ (P_{1}, P_{2})

1)



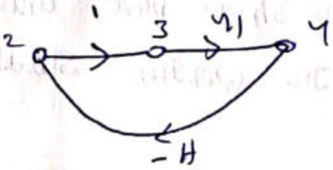
2)



gain of forward path -1 $\Rightarrow$ $P_{1} = G_1 G_2$

gain of forward path-2 $\Rightarrow$ $P_{2} = -G_3$

Case 2: Individual loop gains :-



It has only one individual loop.

gain of individual loop $\Rightarrow$ $P_{11} = -G_1 H$

Case 3: Gain product of two non-touching loops :-

There is no combination of non-touching loops

Case 4: Calculation of Δ and Δ_K :-

$\Delta = 1 - P_{11} \Rightarrow 1 + G_1 H$

$(K=2)$

$\Delta_1 = 1$, $\Delta_2 = 1$

Case 5: Transfer function, T :- $T = \frac{1}{\Delta} \sum_K P_K \Delta_K$

$$\boxed{T = \frac{G_1 G_2 - G_3}{1 + G_1 H}}$$

Scanned with CamScanner

* Servo Motors :-

The servo mechanism is one in which the o/p is some mechanical variable like position, velocity (or) acceleration. Such systems are generally automatic control systems which works on the error signals. The error signals are amplified to drive the motors used in such systems. These motors are used in servo systems are called servomotors.

These motors are used to convert electrical signal applied, into the angular velocity (or) moment of shaft.

Requirements of Good Servo motor :-

The servomotors which are designed for use in feedback control systems must have following requirements.

i) Linear relationship between electrical control signal and the rotor speed over a wide range.

ii) Inertia of rotor should be low as possible. A servomotor must stop running without any time delay, if control signal to it is removed.

iii) Its response should be as fast as possible. For quickly changing error signals it must react with good response.

iv) It should be easily reversible.

v) It should have linear torque - speed char/.

vi) Its operation should be stable without any oscillations (or) overshoots.

Types of Servomotors :-

The servomotors are basically classified depending upon the nature of the electric supply to be used for its operation.

The types of servomotors are shown in below,

- Servo motors
  - DC Servomotors
    - Armature controlled
    - Field controlled.
  - AC Servomotors
  - Special Servomotors

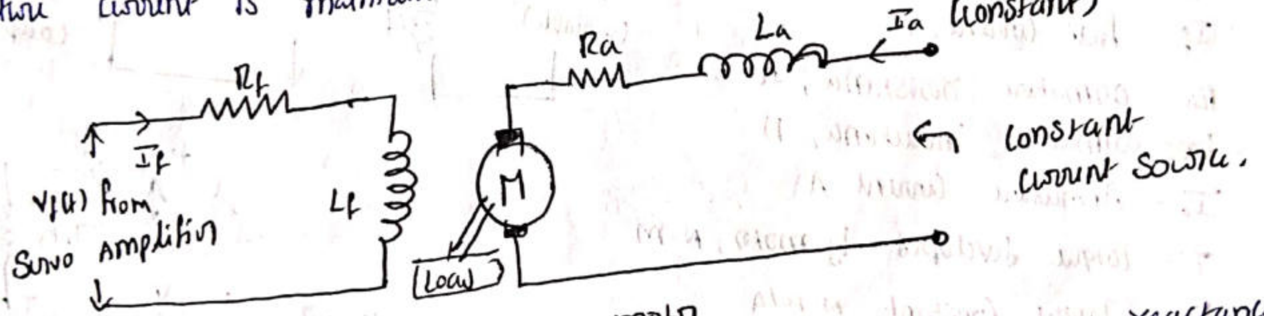
i) DC Servo motor :-

Basically DC servomotor is more (or) less same as normal DC motor. There are some minor differences b/w the two. All dc servomotors are essentially excited type. This ensures linear torque-speed characteristics. The control of dc servomotor can be from field side (or) from armature side. Depending on this, these are classified as

a) Field controlled DC servo motor
b) Armature controlled DC servo motor.

a) Field controlled DC servo motor :-

In this motor, the controlled signal obtained from the servoamplifier is applied to field winding. With the help of constant current source, the armature current is maintain constant. The arrangement is shown in below.



Field control DC servo motor

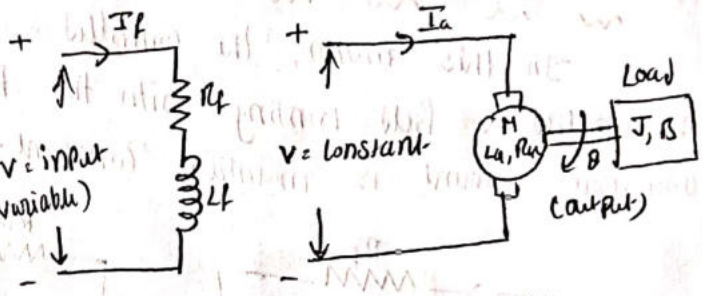
This type of motor has large L_f/R_f ratio where L_f is reactance and R_f is resistance of field winding. Due to this the time constant of motor is high. This means it can not give rapid response to the quick changing control signals hence this is uncommon in practice.

Features :-

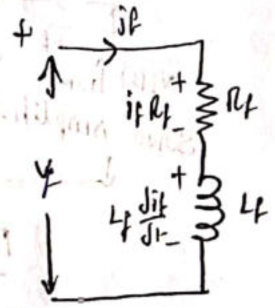
-> Preferred for small rated motors
-> It has large time constant
-> control ckt is simple to design.
-> It is open loop system.

Transfer function of field controlled DC Servo motor:-

In field controlled DC servo motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since the flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load is connected to the shaft of the motor. The field controlled DC servo motor speed control system is shown in fig.



Let
R_f = field resistance, Ω
L_f = field inductance, H
I_f = field current, A
R_a = armature resistance, Ω
L_a = armature inductance, H
I_a = Armature current, A
T = Torque developed by motor, N-m
K_{tf} = Torque constant, N-m/A
J = Moment of inertia of the rotor and load $kg\text{-}m^2/rad$
B = frictional co-efficient of rotor and load N-m/(rad/sec)

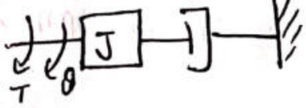


The Equivalent circuit of field as shown in fig,
By KVL, we can write.

$$V_f = i_f R_f + L_f \frac{di_f}{dt} \quad \longrightarrow (1)$$

The torque of motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

$$T \propto i_f \qquad \therefore \text{Torque } T = K_{tf}\, i_f \quad \rightarrow (2)$$



The mechanical system of the motor is shown in fig.
The differential equation governing the mechanical system of the motor is given by

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \rightarrow (3)$$

On taking Laplace Transform of the Equ. (1) (2) (3) with zero initial condition we get

$$V_f(s) = R_f I_f(s) + L_f S I_f(s) \rightarrow (4)$$

$$T(s) = K_{ff} I_f(s) \rightarrow (5)$$

$$T(s) = J s^2 \theta(s) + B S \theta(s) \rightarrow (6)$$

Equating Equation (5) and (6)

$$K_{ff} I_f(s) = J s^2 \theta(s) + B S \theta(s)$$

$$I_f(s) = s \frac{(Js + B)}{K_{ff}} \theta(s) \rightarrow (7)$$

The Equation (4) can be written as

$$V_f(s) = I_f(s) (R_f + SL_f) \rightarrow (8)$$

Substitute Equ. (7) in Equ. (8)

$$(R_f + SL_f) \cdot s \frac{(Js+B)}{K_{ff}} \theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{ff}}{s(R_f + SL_f)(Js+B)} \Rightarrow \frac{K_{ff}}{sR_f\left(1+\frac{SL_f}{R_f}\right) B\left(1+\frac{SJ}{B}\right)}$$

$$\boxed{\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(1+sT_f)(1+sT_m)}}$$

where, $K_m = K_{ff}/R_f B$ = Motor gain constant

$T_f = L_f/R_f$ = field time constant

$T_m = J/B$ = mechanical time constant

Block diagram of field controlled DC Servo motor:-

The differential Equations governing the field controlled dc motor are

$$V_f = R_f i_f + L_f \frac{di_f}{dt} \rightarrow ①$$

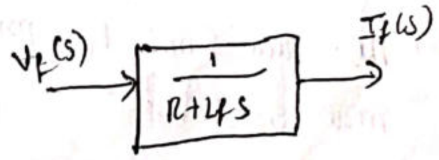
$$T = K_{ff} I_f \rightarrow ②$$

$$T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow ③$$

On taking Laplace Transform to Equ. ① we get

$$V_f(s) = R_f I_f(s) + L_f S I_f(s)$$

$$V_f(s) = I_{fs} (R + L_f S)$$



$$(o/p)\ I_f(s) = \frac{1}{R + L_f s} V_f(s)\ (i/p)$$

On taking Laplace transform to equation (1) we get

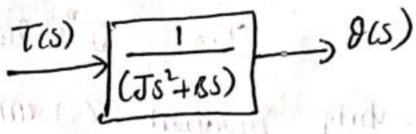
$$T(s) = K_{tf}\, I_f(s)$$

On taking Laplace transform to equation (3) we get

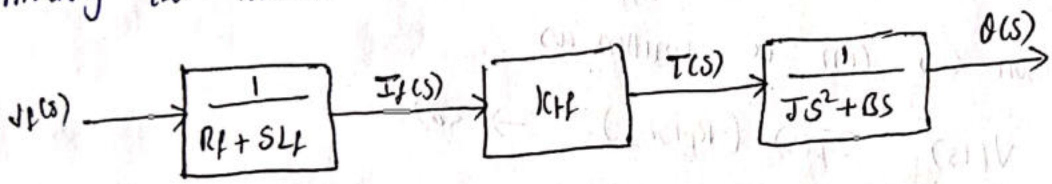
$$T(s) = Js^2\theta(s) + Bs\,\theta(s)$$

$$T(s) = \theta(s)\,[Js^2 + Bs]$$

$$\theta(s) = \frac{1}{(Js^2+Bs)}\,T(s)$$

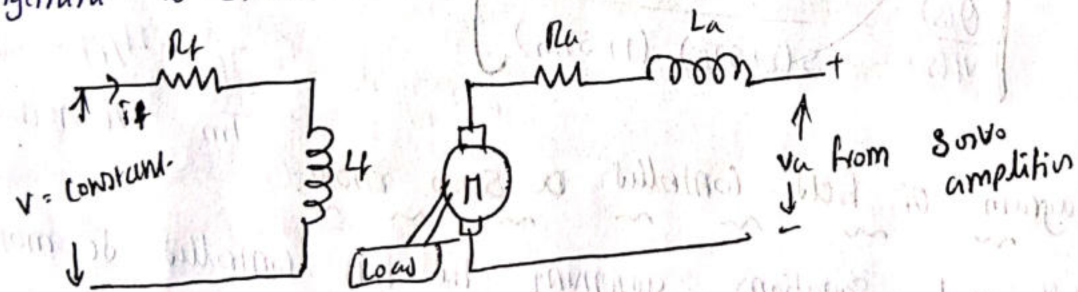


The overall block diagram of field controlled DC servo motor is obtained by connecting the various sections. The overall block diagram is shown below.



b) Armature controlled DC Servo motor :-

In this type of motor, the i/p voltage is 'Va' applied to the armature with a resistance of Ra and inductance La. The field winding is supplied by constant current 'If'. Thus armature i/p voltage controls the motor shaft output. The arrangement is shown in fig.

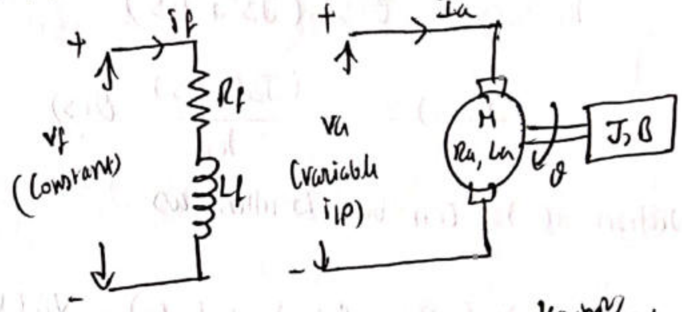


Features :-

-) Suitable for large rated motors.
-) It has small time constant hence its response is fast to the control signal
-) It is closed loop system.
-) The back e.m.f Eb provides internal damping which makes motor operation more stable.

Transfer function of Armature controlled DC servo motor:-

In armature controlled DC servo motor the desired speed is obtained by varying the armature voltage. This speed control system is an electro-mechanical control system. The electrical system consists of the armature and the field circuit but for analysis purpose, only the armature circuit is considered because the field is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The armature controlled DC servo motor speed control system is shown in below.

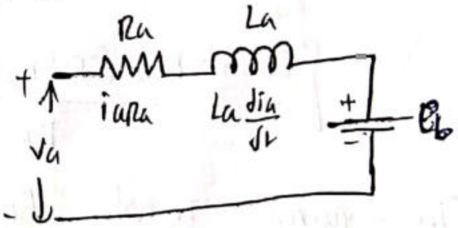


Let

R_f = field resistance, Ω
L_f = field inductance, H
I_f = field current, A
V_f = voltage applied to field
R_a = armature resistance, Ω
L_a = armature inductance, H
I_a = armature current, A
V_a = voltage applied to armature,

J = moment of inertia kg-m²/rad
B = frictional co-efficient N-m/(rad/sec)
K_b = Back emf constant
K_t = Torque constant.

The Equivalent circuit of armature is shown in fig. By KVL, we can write.

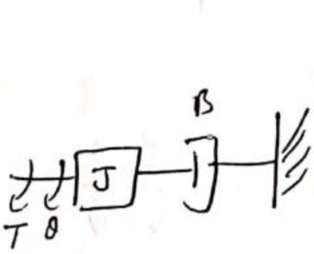


$$V_a = i_aR_a + L_a\frac{di_a}{dt} + e_b \rightarrow (1)$$

The torque of DC motor is proportional to product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.

$$T \propto i_a \qquad \therefore T = K_t i_a \rightarrow (2)$$

The mechanical system of the motor is shown in fig. The differential equation governing the mechanical system of motor is given by,



$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \rightarrow (3)$$

The back emf of DC machine is proportional to speed ~~dt~~ (angular velocity) of the shaft

$$e_b \propto \frac{d\theta}{dt} \qquad (or) \quad \text{Back emf} \quad e_b = K_b\frac{d\theta}{dt} \rightarrow (4)$$

On taking Laplace transform to Equ). 1, 2, 3, 4 with zero initial conditions, we get.

$$V_a(s) = R_a I_a(s) + L_a s I(s) + E_b(s) \rightarrow (5)$$

$$T(s) = K_t I_a(s) \rightarrow (6)$$

$$T(s) = Js^2\theta(s) + Bs\theta(s) \rightarrow (7)$$

$$E_b(s) = K_b s \theta(s) \rightarrow (8)$$

On Equating Equations 6 & 7 we get

$$K_t I_a(s) = \theta(s)(Js^2 + Bs)$$

$$I_a(s) = \frac{(Js^2+Bs)}{K_t}\theta(s) \rightarrow (9)$$

Equation (5) can be written as

$$I_a(s)(R_a + sL_a) + E_b(s) = V_a(s) \rightarrow (10)$$

Sub/. Equ/. (8) & (9) in Equ). (10)

$$(R_a+sL_a) \cdot \frac{(Js^2+Bs)}{K_t} \cdot \theta(s) + K_b s\theta(s) = V_a(s)$$

$$\left[\frac{(R_a+sL_a)(Js^2+Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required Transfer function

$$\boxed{\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a+sL_a)(Js^2+Bs) + K_b K_t s}}$$

(or)

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t/R_aB}{(1+sL_a)(s+Js^2) + K_b K_t s}$$

$$\Rightarrow \frac{K_t/R_aB}{s\left[\left(1+\frac{sL_a}{R_a}\right)\left(1+\frac{sJ}{B}\right) + \frac{K_bK_t}{R_aB}\right]}$$

$$\boxed{\frac{\theta(s)}{V_a(s)} \Rightarrow \frac{K_t/R_aB}{s\left[(1+sT_a)(1+sT_m) + \frac{K_bK_t}{R_aB}\right]}}$$

Block diagram for Armature controlled DC Servo motor :- 250)

The differential Equations governing the armature controlled dc servo motor are

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \rightarrow ①$$

$$T = K_t i_a \rightarrow ②$$

$$T = J\frac{d\omega}{dt} + B\omega \rightarrow ③$$

$$e_b = K_b \omega \rightarrow ④$$

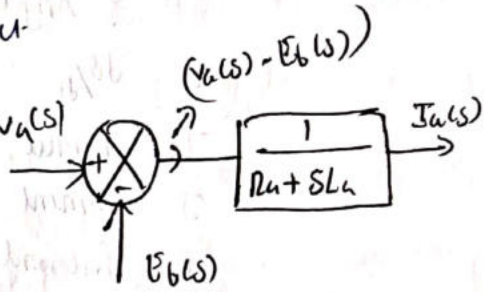
$$\omega = d\theta/dt \rightarrow ⑤$$

on taking Laplace transform to Eq. ① we get

$$V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$$

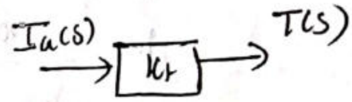
$$V_a(s) - E_b(s) = I_a(s)\,[R_a + sL_a]$$

$$\therefore I_a(s) = \frac{1}{(R_a+sL_a)}(V_a(s) - E_b(s))$$



on taking Laplace transform to Eq. (2) we get

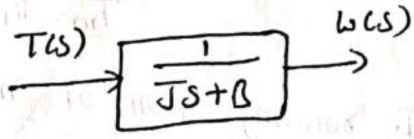
$$T(s) = K_t I_a(s)$$



on taking Laplace transform to Eq. (3) we get.

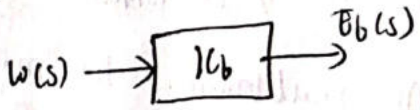
$$T(s) = Js\,\omega(s) + B\omega(s)$$

$$\omega(s) = \frac{1}{Js+B}\cdot T(s)$$



on taking Laplace transform to Eq. (4) we get

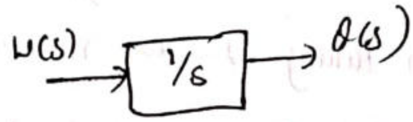
$$E_b(s) = K_b\,\omega(s)$$



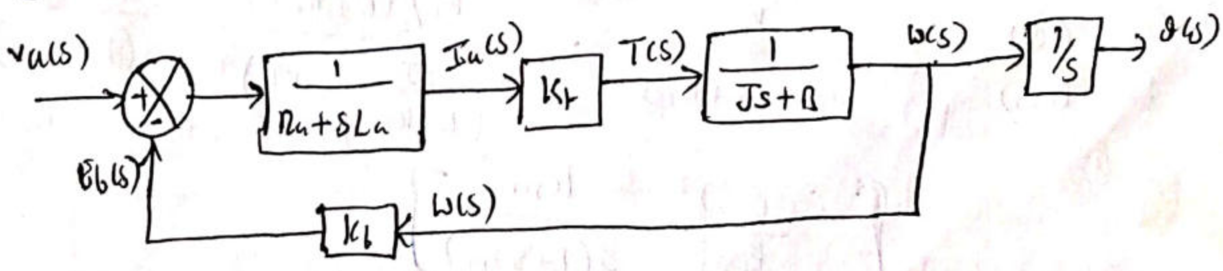
on taking Laplace transform to Eq. (5) we get.

$$\omega(s) = s\,\theta(s)$$

$$\theta(s) = \frac{1}{s}\,\omega(s)$$



Combining all blocks, The overall block diagram becomes.



2) AC Servo motor :-

Most of the servomotors used in low power servomechanisms are a.c servomotors. The a.c servomotor is basically two phase induction motor. The O/P power of ac servomotor varies from fraction of watt to few hundred watts. The operating frequency is 50Hz to 400 Hz.

Transfer function of AC Servo motor :-

Let

T_m = Torque developed by servomotor.

θ = Angular displacement of rotor.

$\omega = d\theta/dt$ = Angular speed (or) Angular velocity

T_L = Torque required by the load.

J = Moment of inertia.

B = frictional co-efficient

K_1 = voltage - Torque constant

K_2 = Speed - Torque constant.

$e_c = E$ = rated i/p voltage.

Torque developed by motor

$$T_m = K_1 e_c - K_2 \frac{d\theta}{dt} \rightarrow (1)$$

The rotating part of motor and load can be modelled by Equ, (2) given below.

Load Torque, $$T_L = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \rightarrow (2)$$

At Equilibrium the motor torque is Equal to load torque

$$\therefore J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_1 e_c - K_2\frac{d\theta}{dt} \rightarrow (3)$$

on taking Laplace transform to Equ, (3) with zero initial conditions we get,

$$Js^2\theta(s) + Bs\theta(s) = K_1 E_c(s) - K_2 s\theta(s)$$

$$[Js^2 + Bs + K_2 s]\theta(s) = K_1 E_c(s)$$

$$\therefore \frac{\theta(s)}{E_c(s)} = \frac{K_1}{Js^2 + Bs + K_2 s} = \frac{K_1/(B+K_2)}{s\left(\frac{J}{B+K_2}\cdot s + 1\right)} = \frac{K_m}{s(1+sT_m)}$$

$$\left[\therefore \frac{\theta(s)}{E_c(s)} = \frac{K_m}{s(1+sT_m)}\right]$$

* SYNCHRO :-

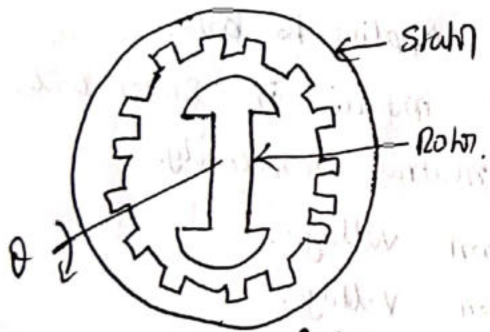
A Synchro is an Electromagnetic device (or) transducer, which converts the angular position of the shaft into electric signal. Synchro consists of synchro transmitter and reciever.

Synchro Transmitter and Reciever :-

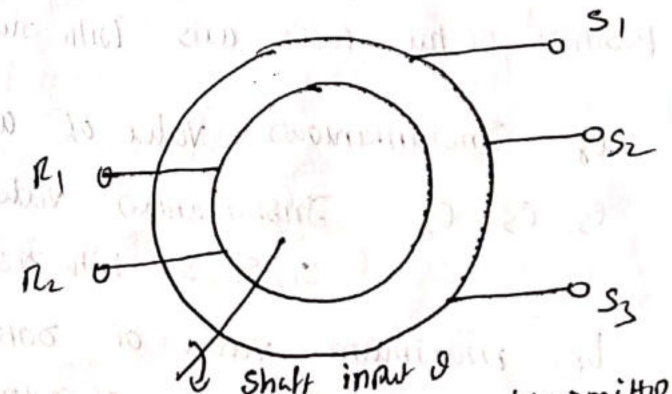
The synchro transmitter and Reciever form a synchro pair. A synchro transmitter is a unit where the mechanical i/p of the rotor is converted into an electrical signal in the stator. A synchro reciever is a unit where the electrical i/p of the stator is converted into angular displacement in the rotor. It is also known as "Control Transformer".

Synchro Transmitter :-

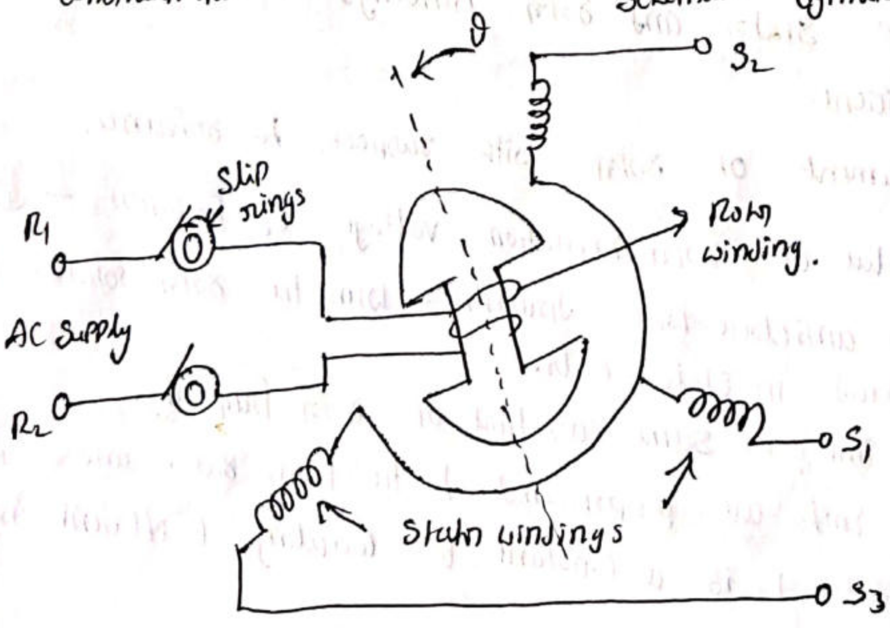
Construction :- The constructional features, electrical circuit and a schematic symbol of synchro transmitter are shown below.



Constructional features.



Schematic symbol of a synchro transmitter



electrical circuit.

The two major parts of synchro transmitter are stator and rotor. The stator is identical to the stator of the three phase Alternator. It is made of laminated silicon steel and slotted on the inner periphery to accommodate a balanced three phase winding. The stator winding is concentric type with the axis of three coils 120° apart. The stator winding is star connected (Y-connection)

The rotor is of dumb bell construction with a single winding. The ends of rotor windings are terminated on two slip rings. A single phase ac excitation voltage is applied to rotor through slip rings.

Working principle:-

→ When the rotor is excited by ac voltage, the rotor current flows, and a magnetic field is produced.

→ The rotor magnetic field induces an emf in the stator coils by transformer action.

→ The effective voltage induced in any stator coil depends upon the angular position of the coil's axis with respect to the rotor axis.

Let

e_r = Instantaneous value of ac voltage applied to rotor.

e_{s1}, e_{s2}, e_{s3} = Instantaneous value of emf induced in stator coils S_1, S_2, S_3 with respect to neutral respectively.

E_r = maximum value of rotor excitation voltage.

ω = Angular frequency of rotor excitation voltage.

K_t = Turns ratio of stator and rotor windings.

K_c = Coupling co-efficient.

θ = Angular displacement of rotor with respect to reference.

Let the instantaneous value of rotor excitation voltage $e_r = E_r \sin\omega t \rightarrow (1)$

→ Let the rotor rotates in anticlock wise direction. When the rotor rotates by an angle θ emf's are induced in stator coils.

→ The frequency of induced emf is same as that of rotor frequency.

→ The magnitude of induced emfs are proportional to the turns ratio and coupling co-efficient. The turns ratio, K_t is a constant, but coupling co-efficient is function of rotor angular position.

∴ Induced emf in stator coil = $K_t K_c E_r \sin\omega t \rightarrow (2)$

→ Let e_{s2} be reference vector. With reference in fig, when $\theta = 0$, the flux linkage of coil S_2 is maximum and when $\theta = 90°$, the flux linkage of coil S_2 is zero.

→ Hence the flux linkage of coil S_2 is function of $\cos\theta$ (i.e, $K_c = K_1 \cos\theta$ for coil S_2). The flux linkage of coil S_3 will be maximum after a rotation of 120° in anticlockwise direction and that of S_1 after a rotation of 240°

∴ Coupling co-efficient K_c for $S_2 = K_1 \cos\theta$

Coupling co-efficient K_c for $S_3 = K_1 \cos(\theta - 120°)$

Coupling co-efficient K_c for $S_1 = K_1 \cos(\theta - 240°)$

→ Hence the emf's of stator coil with respect to neutral can be expressed as follows.

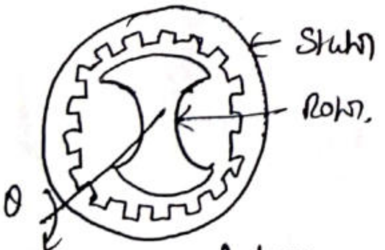
$$e_{s2} = K_t K_1 \cos\theta \, E_r \sin\omega t = K E_r \cos\theta \sin\omega t$$

$$e_{s3} = K_t K_1 \cos(\theta - 120°) E_r \sin\omega t = K E_r \cos(\theta - 120°) \sin\omega t$$

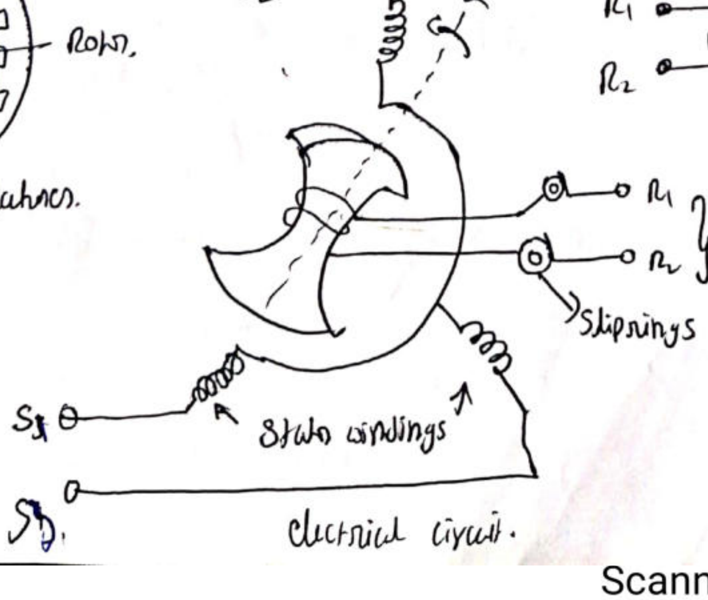
$$e_{s1} = K_t K_1 \cos(\theta - 240°) E_r \sin\omega t = K E_r \cos(\theta - 240°) \sin\omega t$$

Synchro Receiver (or) Control Transformer:-

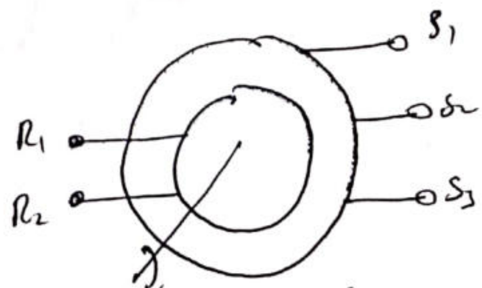
Construction:- The constructional features of synchro receiver is similar to that of synchro transmitter, except the shape of rotor. The rotor of synchro receiver is made cylindrical so that the air gap is practically uniform. This feature of the synchro reciever minimizes the changes in the rotor impedance with the rotation of the shaft. The constructional features, electrical circuit and a schematic symbol of synchro reciever are shown below.



constructional features.



electrical circuit.



Working :-

→ The generated emf of the synchro transmitter is applied as i/p to the stator coils of synchro reciever.

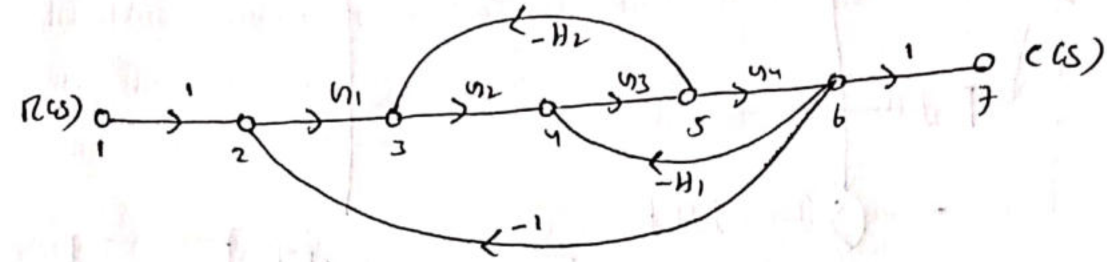
→ The rotor shaft is connected to the load whose position has to be maintained at the desired value.

→ Depending on the current position of the rotor and the applied emf on the stator, an emf is induced on the rotor winding.

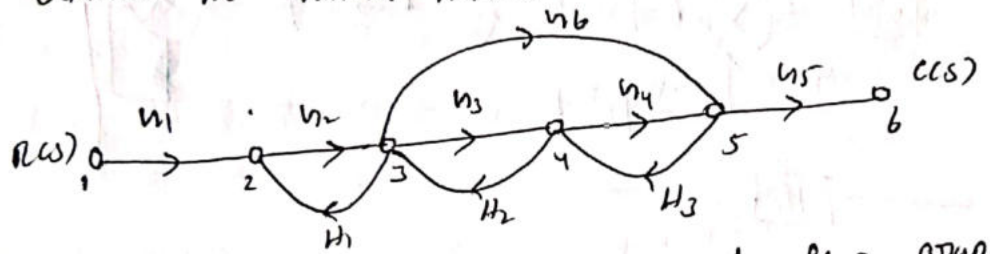
→ This emf can be measured and used to drive a motor so that the position of the load is corrected.

Exercise problems

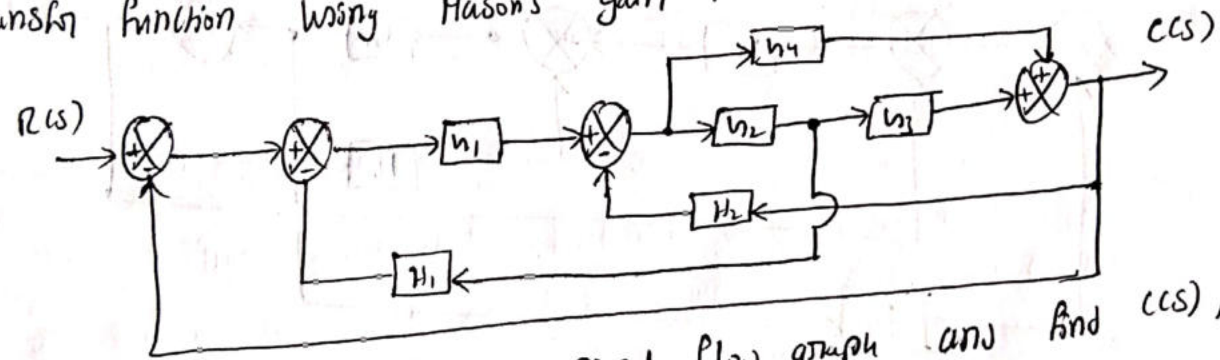
1) Find the overall gain $C(s)/R(s)$ for the signal flow graph shown only.



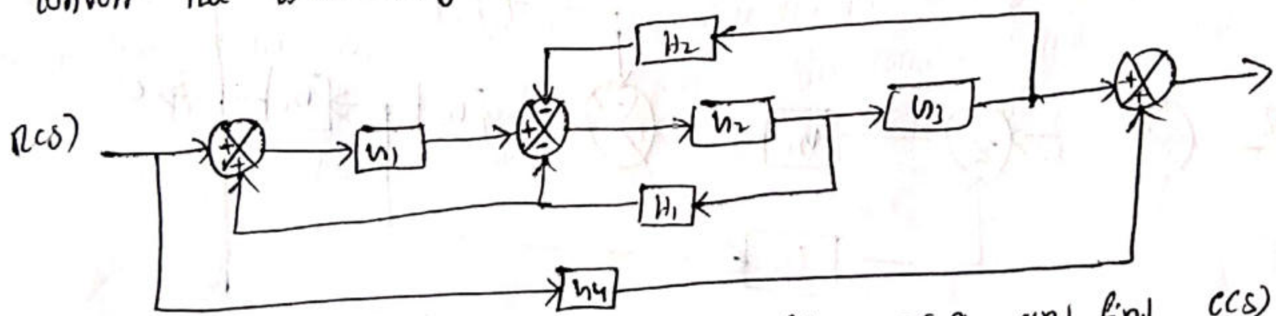
2) Determine the Transfer function $C(s)/R(s)$



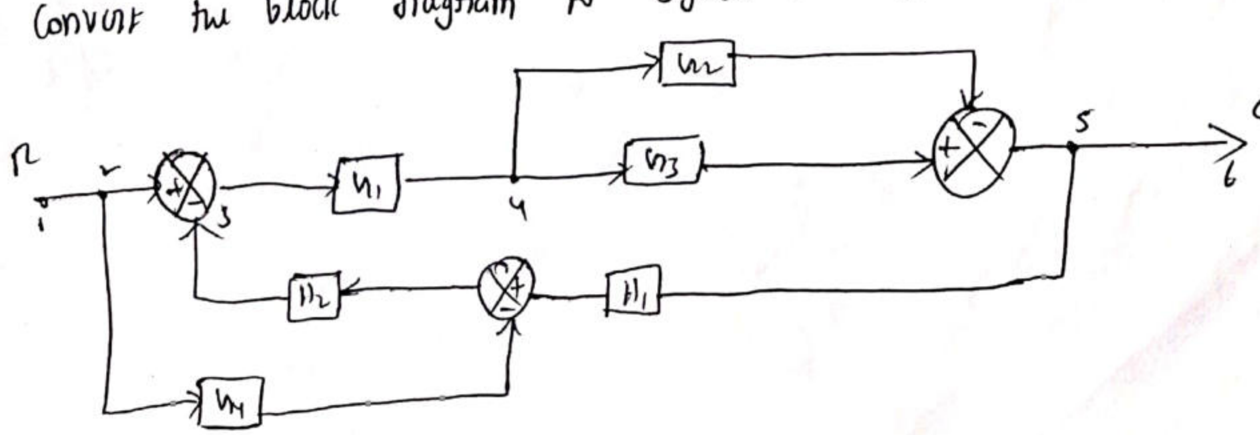
3) Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.



4) Convert the block diagram to signal flow graph and find $C(s)/R(s)$

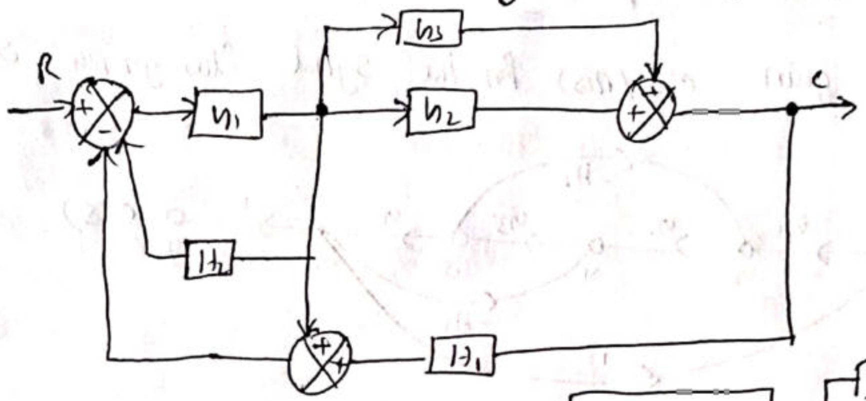


5) Convert the block diagram to signal flow graph and find $C(s)/R(s)$

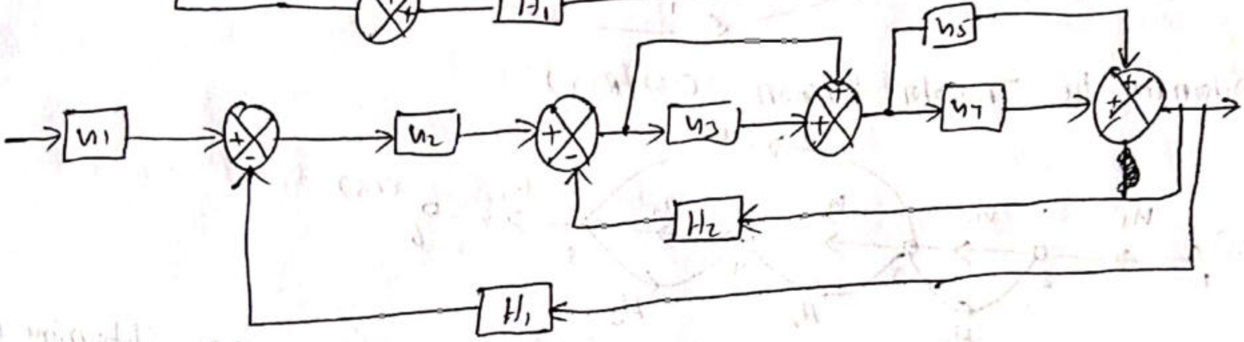


6) Find the Transfer functions using block diagram reduction.

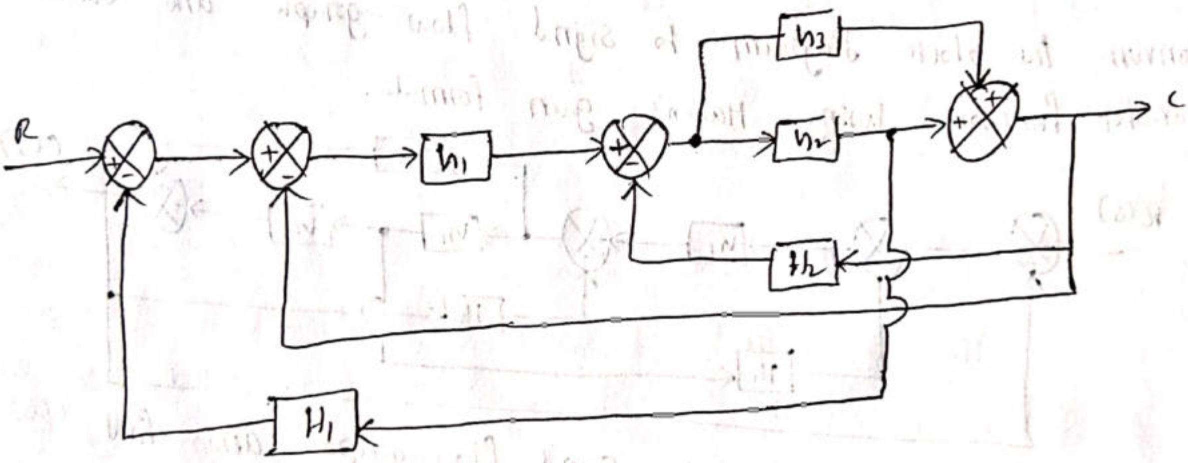
a)



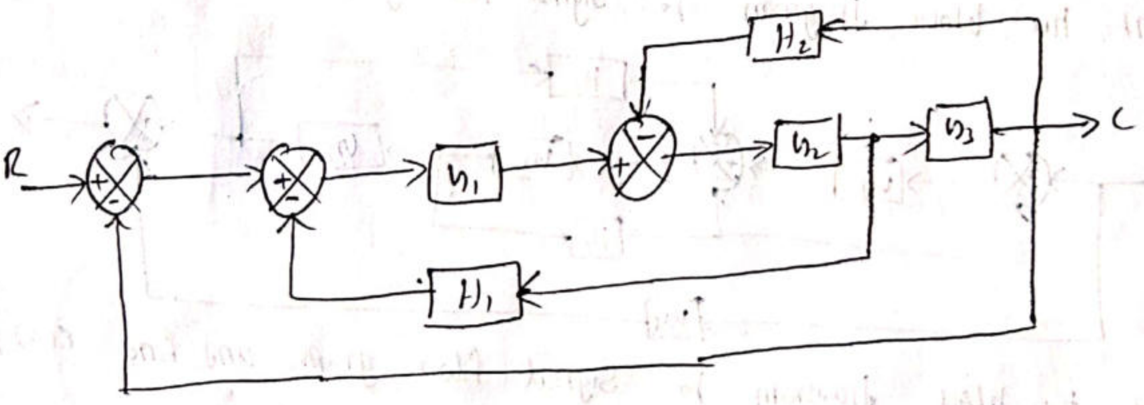
b)



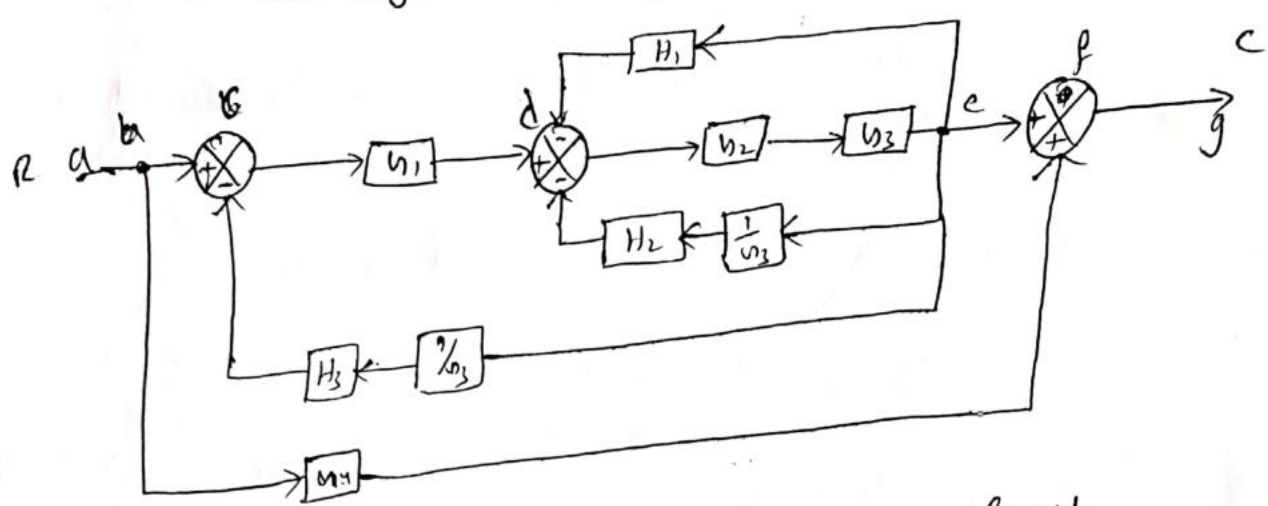
c)



d)

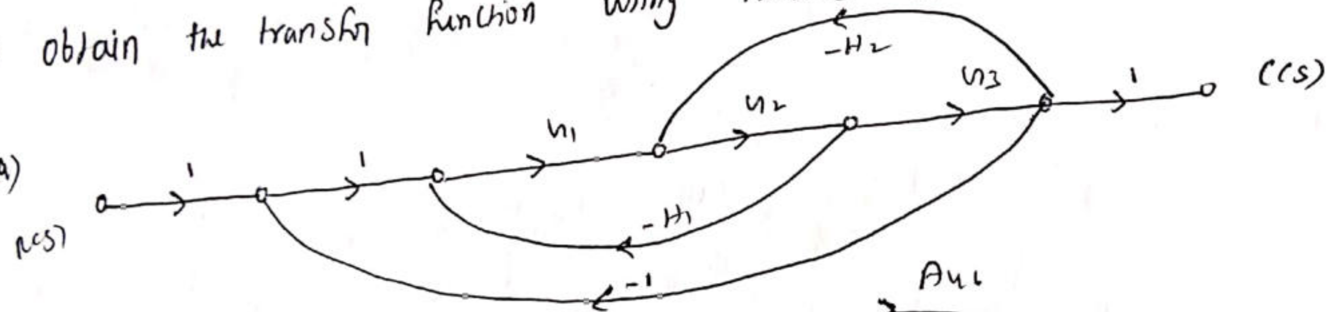


7) Convert the block diagram into signal flow graph and find T.F

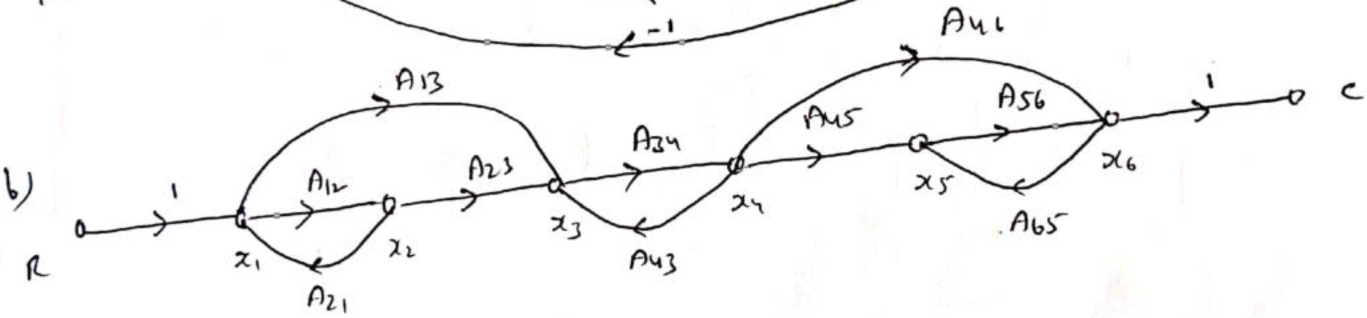


8) obtain the transfer function using Mason's gain formula.

a)



b)



Unit - II

Time Response Analysis

* Time Response

Time response of a control system means, how output behaves with respect to time. so it can be defined as below.

Defination :- "The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t)"

The time response can be obtained by solving the differential Equation governing the system. Alternatively, the time response c(t) can be obtained from the "transfer function" of the system and "input" of the system.

[The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = M(s)$]

The output (or) response in s-domian C(s) is given by the product of the transfer function and the input R(s). on taking inverse Laplace transform of this product the time response c(t) can be obtained.

response in s-domain ⇒ $C(s) = R(s) \cdot M(s)$

Response in time domain → $c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\{R(s) \cdot M(s)\}$

Where $M(s) = \frac{G(s)}{1+G(s)H(s)}$.

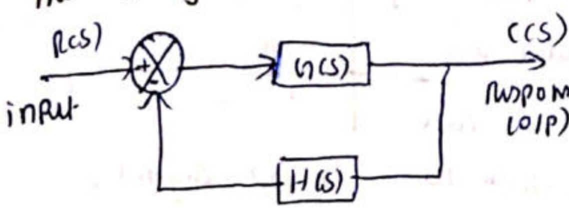
The time response of control system consists of two parts

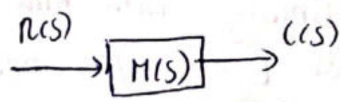
→ Transiant Response

→ Steady State response

* The transiant response is the response of the system when the input change from one state to another.

* The steady state response is the response as time 't' approaches to infinity.





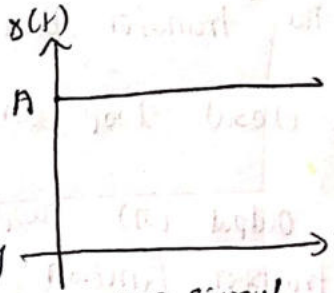
$M(s) = \frac{G(s)}{1+G(s)H(s)}$

closed - Loop system

* Test Signals :-

In practice, many signals are available which are the functions of the time and can be used as reference inputs for the various control systems. These signals are step, ramp, sawtooth type, square wave, triangular etc. But while analysing the systems it is highly impossible to consider each one of it as an input and study the response.

Hence from the Analysis point of view, those signals which are most commonly used as reference inputs are defined as "standard Test inputs". The evaluation of the system can be done on the basis of the response given by the system to the standard test inputs. once system behaves satisfactorily to a test input, its time response to actual input is assumed to be upon on the mark. These standard test signals are,



step signal

i) Step input (position function) :-

The step signal is a signal whose value changes from 0 to A at $t=0$ and remains constant at A for $t>0$. The step signal resembles an actual steady input to a system. A special case of step signal is "unit step" in which A is unity.

The mathematical representation of step signal is

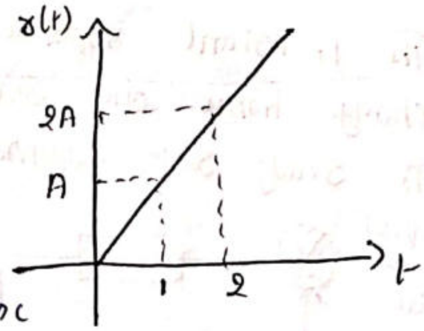
$$r(t) = A \quad \text{for } t \geq 0$$
$$r(t) = 0 \quad \text{for } t < 0$$

If $A=1$, then it is called "unit step function" and it is denoted by $u(t)$.

Laplace transform of such i/p is $A/s \Rightarrow$ step input

$1/s \Rightarrow$ unit step input.

2) Ramp input (velocity function) :-



The ramp signal is a signal whose value increases linearly with time from an initial value of zero at $t=0$. The ramp signal resembles a constant velocity i/p to the system. A special case of ramp signal is "unit ramp signal" in which the value of A is unity.

Ramp signal.

The mathematical representation of ramp signal is,

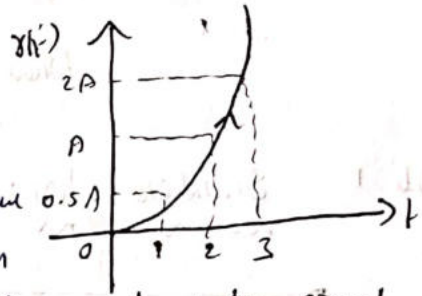
$$r(t) = At \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

If $A = 1$, then it is called "unit ramp function" and it is denoted by $r(t)$.

Laplace transform of such i/p is $A/s^2 \Rightarrow$ ramp i/p

$1/s^2 \Rightarrow$ unit ramp i/p.

3) Parabolic input (Acceleration function):-



In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at $t = 0$. The graph of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration i/p to the system. A special case of parabolic signal is "unit parabolic signal" in which A is unity.

The mathematical representation of parabolic signal is,

A

A

At

A

$$r(t) = At^2 \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

$At^2/2$

If $A = 1$, then it is called "unit parabolic function".

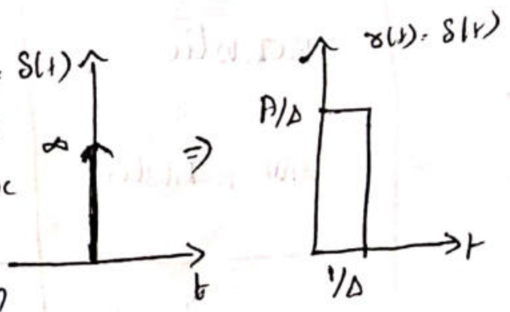
$t^2/2$

Laplace transform of such i/p is $A/s^3 \Rightarrow$ parabolic i/p

$\delta(t)$

$1/s^3 \Rightarrow$ unit parabolic i/p.

4) Impulse signal :-



Impulse signal.

A signal of very large magnitude which is available for very short duration is called Impulse signal. Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unit.

The impulse signal is denoted by $\delta(t)$ and mathematically it is expressed as

$$\delta(t) = \infty \quad ; \quad t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0 \quad ; \quad t \neq 0$$

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width with Area A. Mathematically and impulse signal is the derivative of a step signal. Laplace transform of impulse function is "unity".

$$\text{Impulse response } \delta(t) = L^{-1}\{R(s)\,M(s)\} = L^{-1}\{M(s)\}$$

$$\text{Where } M(s) = \frac{G(s)}{1+G(s)H(s)} \qquad R(s) = 1 \text{ for impulse.}$$

Table: 1 Standard test signals:-

Name of the Signal	Time domain Equation of Signal $r(t)$	Laplace transform of the Signal $R(s)$
Step unit step	A 1	A/s $1/s$
ramp unit ramp	At t	A/s^2 $1/s^2$
parabolic unit parabolic	$At^2/2$ $t^2/2$	A/s^3 $1/s^3$
Impulse	$\delta(t)$	1

* Review of partial fraction Expansion :-

The time response of a system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer functions are encountered. They are,

Case 1 :- Functions with separate poles

Case 2 :- Functions with multiple poles

Case 3 :- Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example.

Case 1 :- When the transfer function has distinct poles.

Let $T(S) = \dfrac{K}{S(S+P_1)(S+P_2)}$

By partial fraction expansion $T(s)$ can be expressed as

$$T(S) = \frac{K}{S(S+P_1)(S+P_2)} = \frac{A}{S} + \frac{B}{S+P_1} + \frac{C}{S+P_2}$$

$A = T(S) \times S|_{S=0}$, $B = T(S) \times (S+P_1)|_{S=-P_1}$ $C = T(S) \times (S+P_2)|_{S=-P_2}$

Ex: Let $T(S) = \dfrac{2}{S(S+1)(S+2)}$

By applying partial fraction to above T.F we get

$$T(S) = \frac{2}{S(S+1)(S+2)} = \frac{A}{S} + \frac{B}{S+1} + \frac{C}{S+2} \rightarrow ①$$

$$\Rightarrow \frac{2}{\cancel{S(S+1)(S+2)}} = \frac{A(S+1)(S+2) + BS(S+2) + CS(S+1)}{\cancel{S(S+1)(S+2)}}$$

$$\Rightarrow 2 = A(S+1)(S+2) + BS(S+2) + CS(S+1) \rightarrow ②$$

Put $S=0$ $\Rightarrow$ $2 = A(1)(2)$ $\Rightarrow$ $[A=1]$

Put $S=-1$: $\Rightarrow$ $2 = B(-1)(-1+2)$ $\Rightarrow$ $[B=-2]$

Put $S=-2$ $\Rightarrow$ $2 = C(-2)(-2+1)$ $\Rightarrow$ $[C=1]$

$$\left[T(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \right]$$

Case 2:- When the transfer function has multiple poles:-

$$\text{Let } T(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$$

By partial fraction expansion $T(s)$ can be written as

$$T(s) = \frac{K}{s(s+P_1)(s+P_2)^2} = \frac{A}{s} + \frac{B}{(s+P_1)} + \frac{C}{(s+P_2)} + \frac{D}{(s+P_2)^2}$$

The residues, A, B, C and D are given by

$$A = T(s) \times s\Big|_{s=0} \qquad B = T(s) \times (s+P_1)\Big|_{s=-P_1} \qquad C = T(s) \times (s+P_2)\Big|_{s=-P_2}$$

$$D = T(s) \times (s+P_2)^2\Big|_{s=-P_2}.$$

Ex: Let $T(s) = \dfrac{2}{s(s+1)(s+2)^2}$

By partial fraction expansion $T(s)$, can be written as

$$T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+2)^2} \quad \longrightarrow (1)$$

$$\frac{2}{s(s+1)\cancel{(s+2)^2}} = \frac{A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1)(s+2) + Ds(s+1)}{s(s+1)\cancel{(s+2)^2}}$$

$$2 = A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1)(s+2) + Ds(s+1)$$

Put $s=0$:-

$$2 = A(1)(4) \Rightarrow A = \tfrac{1}{2} = 0.5 \Rightarrow [A = 0.5]$$

Put $s=-1$:- $2 = B(-1)(-1+2)^2 \Rightarrow 2 = B(-1) \Rightarrow [B = -2]$

Put $s=-2$:- $2 = D(-2)(-2+1) \Rightarrow 2 = D(-2)(-1) \Rightarrow [D = 1]$

Let consider s^3 terms

$$A + B + C = 0$$
$$0.5 - 2 + C = 0 \Rightarrow -1.5 + C = 0 \Rightarrow [C = 1.5]$$

Substitute A, B, C and D in Equ. (1) will get.

$$T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{0.5}{s} - \frac{2}{(s+1)} + \frac{1.5}{(s+2)} + \frac{1}{(s+2)^2}$$

Case 3: When the transfer function has complex conjugate poles :-

$$\text{Let } T(s) = \frac{K}{(s+P_1)(s^2+bs+c)}$$

By partial fraction expansion, T(s) can be written as,

$$T(s) = \frac{K}{(s+P_1)(s^2+bs+c)} = \frac{A}{(s+P_1)} + \frac{Bs+C}{(s^2+bs+c)}$$

$$\Rightarrow \frac{A}{(s+P_1)} + \frac{Bs}{(s^2+bs+c)} + \frac{C}{(s^2+bs+c)} \rightarrow (1)$$

The residue A is given by

$$A = T(s) \times (s+P_1)\Big|_{s=-P_1}$$

The residues B and C are solved by cross multiplying the Equ. (1) and then Equating the co-efficients of like power of 's'.

Ex: Let $T(s) = \frac{1}{(s+2)(s^2+s+1)}$

By partial fraction expansion, T(s) can be written as,

$$T(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{A}{(s+2)} + \frac{Bs+C}{(s^2+s+1)} \rightarrow (1)$$

$$\frac{1}{(s+2)(s^2+s+1)} \Rightarrow \frac{A}{(s+2)} + \frac{Bs}{(s^2+s+1)} + \frac{C}{(s^2+s+1)}$$

$$\frac{1}{(s+2)\cancel{(s^2+s+1)}} = \frac{A(s^2+s+1) + Bs(s+2) + C(s+2)}{(s+2)\cancel{(s^2+s+1)}}$$

$$1 = A(s^2+s+1) + Bs(s+2) + C(s+2)$$

Put $s=-2$:- $1 = A(4-2+1) \Rightarrow \boxed{A = 1/3}$

Put $s=0$:- $1 = A(1) + 2C \Rightarrow 1 = A + 2C$

$$\Rightarrow 1 = 1/3 + 2C \Rightarrow 2C = 1 - 1/3$$

$$\Rightarrow \cancel{2}C = \cancel{2}/3 \Rightarrow \boxed{C = 1/3}$$

Let consider s^2 terms ;

$$0 = A + B$$

$$0 = \frac{1}{3} + B \Rightarrow \boxed{B = -\frac{1}{3}}$$

Substitute A, B and C values in Equ. ① will get,

$$T(s) = \frac{1}{(s+1)(s^2+s+1)} \Rightarrow \frac{\frac{1}{3}}{(s+1)} - \frac{\frac{1}{3}s}{(s^2+s+1)} + \frac{\frac{1}{3}}{(s^2+s+1)}$$

$$\Rightarrow \frac{1}{3(s+1)} - \frac{s}{3(s^2+s+1)} + \frac{1}{3(s^2+s+1)}$$

Time domain $f(t)$	$\underset{\text{I.L.P.T}}{\overset{\text{L.P.T}}{\rightleftharpoons}}$	s-domain $F(s)$
$u(t)$	-	$\frac{1}{s}$
$t\,u(t)$	-	$\frac{1}{s^2}$
Impulse function δ	-	1
e^{at}	-	$\frac{1}{s-a}$
e^{-at}	-	$\frac{1}{s+a}$
te^{at}	-	$\frac{1}{(s-a)^2}$
te^{-at}	-	$\frac{1}{(s+a)^2}$
$\sin at$	-	$\frac{a}{s^2+a^2}$
$\cos at$	-	$\frac{s}{s^2+a^2}$
$\sinh at$	-	$\frac{a}{s^2-a^2}$
$\cosh at$	-	$\frac{s}{s^2-a^2}$
$e^{at}\sin bt$	-	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	-	$\frac{s-a}{(s-a)^2+b^2}$

$e^{at}\sinh bt \quad - \quad \frac{b}{(s-a)^2-b^2}$

$e^{at}\cosh bt \quad - \quad \frac{s-a}{(s-a)^2-b^2}$

$e^{at}t^n \quad = \quad \frac{n!}{(s-a)^{n+1}}$

* ~~Time / Response of First / order Systems~~ :-

Order of a system :-

The input and output relationship of a control system can be expressed by n^{th} order differential equation shown in below.

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \cdots + a_{n-1} \frac{d}{dt} p(t) + a_n p(t) =$$

$$b_0 \frac{d^m}{dt^m} q(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}} q(t) + \cdots + b_{m-1} \frac{d}{dt} q(t) + b_m q(t) \rightarrow (1)$$

Where $p(t)$ = output / response , $q(t)$ = input/response.

The order of the system can be determined by from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system. When Applying L.P.T to Eq. (1) will get

Transfer function

$$T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n} \rightarrow (2)$$

Where $P(s)$ = Numerator polynomial, $Q(s)$ = Denominator polynomial.

" The order of the system is given by the max. power of "s" in the denominator polynomial $Q(s)$ ".

Here $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$.

Now n is order of the system.

When $n=0$, the system is zero order system.

$n=1$, the system is first order system.

$n=2$, the system is second order system.

The numerator and denominator polynomial of Equ. ② can be expressed in the factorized form as shown in below.

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $z_1, z_2 \ldots z_m$ are zeros of the system

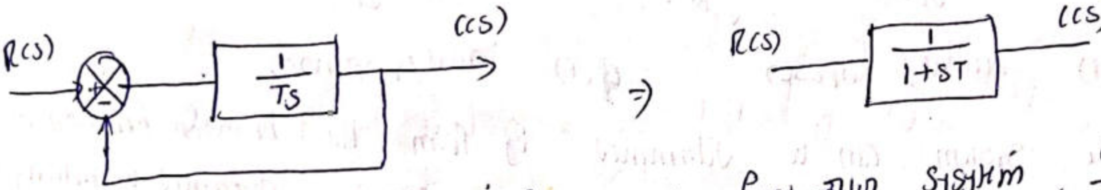
$p_1, p_2 \ldots p_n$ are poles of the system.

Now, the value of n gives the number of poles in the Transfer function.

* Time response of first order systems :-

Response of first order system for unit step input :

The closed loop order system with unity feed back is shown below



The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+ST}$

If the input is unit step, then $r(t) = 1$, then $R(s) = 1/s$.

The response in s-domain

$$C(s) = R(s) \cdot \frac{1}{1+ST} \Rightarrow \frac{1}{s} \cdot \frac{1}{(1+sT)} \Rightarrow \frac{1}{s} \cdot \frac{1}{T(s+1/T)}$$

$$\Rightarrow \frac{1/T}{s(s+1/T)}$$

$$\therefore \boxed{C(s) = \frac{1/T}{s(s+1/T)}} \leftarrow \text{response in s-domain}$$

By applying partial fraction expansion, the above Equ. can be written as.

$$C(s) = \frac{1/T}{s(s+1/T)} = \frac{A}{s} + \frac{B}{(s+1/T)} \rightarrow ①$$

$$\Rightarrow \frac{\frac{1}{T}}{\cancel{s(s+1/T)}} = \frac{A(s+1/T) + Bs}{\cancel{s(s+1/T)}}$$

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$$\frac{1}{T} = A(s+1/T) + Bs$$

Put $s = 0$:- $\quad 1/T = A(1/T)$

$$\boxed{A = 1}$$

Put $s = -1/T$:- $\quad 1/T = B(-1/T) \Rightarrow \boxed{B = -1}$

The values of A and B are substitute in Eqn. ① will get

$$C(s) = \frac{1/T}{s(s+1/T)} = \frac{1}{s} - \frac{1}{s+1/T}$$

The response in time domain will get by taking inverse Laplace transform to above equation.

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

$$L^{-1}\{C(s)\} = L^{-1}[1/s] - L^{-1}\left[\frac{1}{s+1/T}\right]$$

$$\boxed{c(t) = (1 - e^{-t/T})}$$

The above equation is the first closed loop first order system with unit step response.

Note: For closed loop first order system, unit step response $= 1 - e^{-t/T}$

Step response $= A(1 - e^{-t/T})$

when $t = 0$, $\quad c(t) = 1 - e^{0} = 0$

when $t = 1T$, $\quad c(t) = 1 - e^{-1} = 0.632$

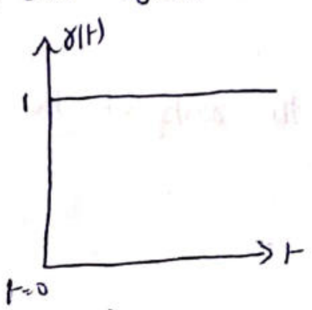
when $t = 2T$, $\quad c(t) = 1 - e^{-2} = 0.865$

when $t = 3T$, $\quad c(t) = 1 - e^{-3} = 0.95$

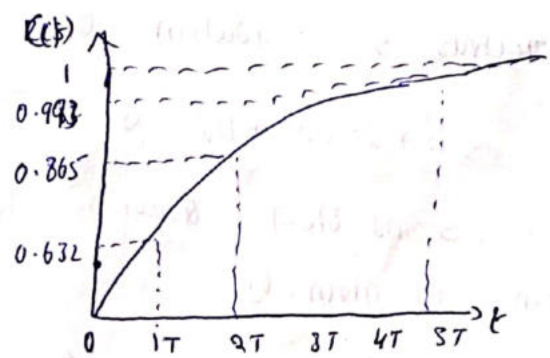
when $t = 4T$, $\quad c(t) = 1 - e^{-4} = 0.982$, when $t = 5T$, $c(t) = 1 - e^{-5} \Rightarrow 0.993$.

when $t = \infty$, $\quad c(t) = 1 - e^{-\infty} = 1$

Here T is called Time constant of the system. In a time 5T, the system is assumed to have attained steady state. The i/p and o/p signal of the first order system is shown below.



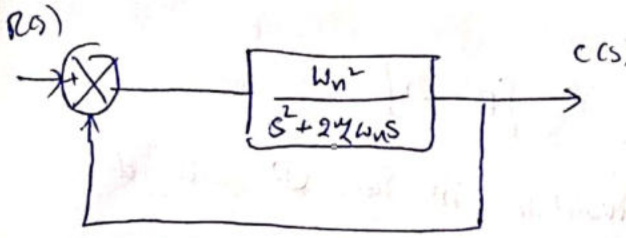
unit step input



Response for unit step input

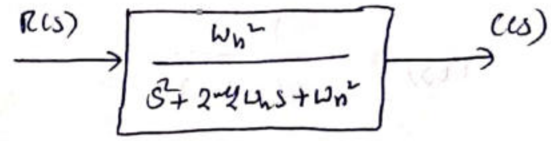
* Second order system :-

The closed loop second order system is shown below.



$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}{1+\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \Rightarrow$$



The standard form of closed loop transfer function of second order system is given by

$$\left[\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

Where ω_n = undamped angular frequency, rad/sec

ζ = Damping ratio.

The damping ratio is defined as the ratio of actual damping to critical damping. The response c(t) of second order system depends on the value of ζ (Damping ratio). Depending upon the value of ζ, the system can be classified into following four cases.

Case 1:- undamped system, $\zeta = 0$

Case 2:- underdamped system, $0 < \zeta < 1$

Case 3:- Critically damped system, $\zeta = 1$

Case 4:- over damped system, $\zeta > 1$

The characteristics equation of second order system is given by

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

It is a second order quadratic equation and the roots of the equation is given by

here $a = 1$, $b = 2\zeta\omega_n$ $c = \omega_n^2$

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The formula used to find roots of quadratic equation is given by.

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\Rightarrow \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2-1)}}{2} \Rightarrow \frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{(\zeta^2-1)}}{2}$$

$$\left[s_1, s_2 \Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right]$$

when $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$ — roots are purely imaginary and the system is undamped.

when $\zeta = 1$, $s_1, s_2 \Rightarrow -\omega_n$; — roots are real and equal and the system is critically damped.

when $\zeta > 1$, $s_1, s_2 \Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ — roots are real and unequal and system is over damped

when $0 < \zeta < 1$, $s_1, s_2 \Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2-1)}$

$$\Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$\Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{(-1)} \cdot \sqrt{(1-\zeta^2)}$$

$$\Rightarrow -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$$

$$\Rightarrow -\zeta\omega_n \pm j\omega_d$$

— roots are complex conjugate and the system is under damped.

where $\boxed{\omega_d = \omega_n\sqrt{(1-\zeta^2)}}$

Here ω_d is called "damped frequency" of oscillation of the system and its unit is rad/sec.

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I: Response of undamped second order system for unit step input:-

The standard form of closed loop Transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For undamped system, $\zeta = 0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2} \rightarrow (1)$$

when the input is unit step $r(t) = 1$ and $R(s) = 1/s$.

$\therefore$ The response in s-domain, $C(s) = R(s) \cdot \frac{\omega_n^2}{s^2+\omega_n^2}$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2+\omega_n^2} \Rightarrow \frac{\omega_n^2}{s(s^2+\omega_n^2)} \rightarrow (2)$$

By partial fraction Expansion.

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} \rightarrow (3)$$

$$\Rightarrow \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A(s^2+\omega_n^2)+Bs}{s(s^2+\omega_n^2)}$$

$$\Rightarrow \omega_n^2 = A(s^2+\omega_n^2) + Bs \rightarrow (4)$$

Put $s = 0$,

$$\omega_n^2 = A\omega_n^2$$

$$\boxed{A = 1}$$

Now substitute A value in Equ. (4)

$$\Rightarrow \cancel{\omega_n^2} = s^2 + \cancel{\omega_n^2} + Bs$$

$$\Rightarrow s^2 + Bs = 0$$

$$\Rightarrow s(s+B) = 0$$

$$\Rightarrow s+B = 0 \Rightarrow \boxed{B = -s}$$

now substitute A and B values in Equation (3)

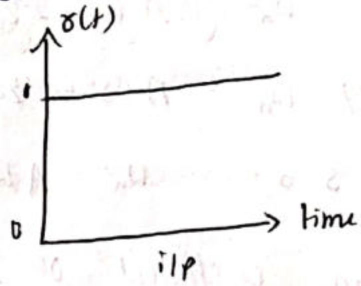
$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2} \rightarrow (5)$$

The above Equation shows response in s-domain. To obtain in time domain. Applying Laplace inverse transform to Equ. (5)

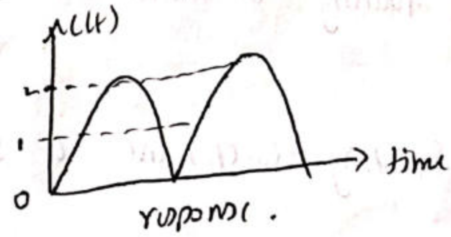
$$L^{-1}\{C(s)\} = L^{-1}[1/s] - L^{-1}\left[\frac{s}{s^2+\omega_n^2}\right]$$

$$\left[c(t) = 1 - \cos\omega_n t\right] \rightarrow (6)$$

using Equation (6), the response of undamped second order system for unit step input is sketched in below and observed that response is completely oscillatory.



Note: Every practical system has some amount of damping. Hence undamped system does not exist in practice.



For closed loop undamped second order system is

$$\left[\begin{array}{l} \text{unit step response} = 1 - \cos\omega_n t \\ \text{step response} = A(1-\cos\omega_n t) \end{array}\right]$$

II Response of underdamped second order system for unit step input

The standard form of closed loop T.F of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For under damped system $0<\xi<1$ and roots of the denominator are complex conjugate.

The roots of denominator are, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

$$\Rightarrow -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$s_1, s_2 \Rightarrow -\xi\omega_n \pm j\omega_d$$

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The response in s-domain $C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input $r(t) = 1$ and $R(s) = 1/s$

$$\therefore C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \rightarrow (1)$$

$$\Rightarrow \omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + (Bs + C)s$$

$$\Rightarrow \omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs.$$

Put $s = 0$:- $\omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A = 1}$

Equating co-efficient of s^2 terms $\Rightarrow 0 = A + B$

$$\Rightarrow \boxed{B = -1}$$

Equating co-efficient of s terms $\Rightarrow 0 = A(2\zeta\omega_n) + C$

$$\boxed{C = -2\zeta\omega_n}$$

Substitute A, B and C values in equation (1)

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \rightarrow (2)$$

Let us add and substract $\zeta^2\omega_n^2$ to the denominator of second term in equation (2)

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$\Rightarrow \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$\Rightarrow \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$\Rightarrow \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \rightarrow (3)$$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} \rightarrow (3)$$

Let us multiply and divide by ω_d in the third term of Eqn. (3)

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

To get response in t-domain apply I.L.P.T to above Eqn. will get

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t.$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right] \quad \because \left[\omega_d = \omega_n\sqrt{1-\xi^2}\right]$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right].$$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right].$$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t \times \xi + \cos\omega_d t \times \sqrt{1-\xi^2}\right)$$

Let us express $c(t)$ in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta\right)$$

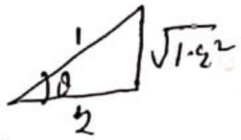
$$\Rightarrow 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) \rightarrow (4)$$

On constructing right angle triangle with ξ and $\sqrt{1-\xi^2}$ we get

$\cos\theta = \xi$

$\sin\theta = \sqrt{1-\xi^2}$

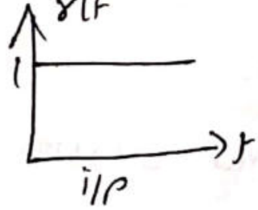
$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$



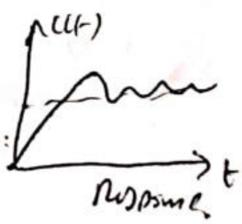
Where $\left[\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$

For closed loop under damped second order system

unit step input $= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$, $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$



step input: $A\left(1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right)$, $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$



The oscillations depends on the value of damping ratio.

III Response of critically damped second order system for unit step input

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For critical damping $\zeta = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} \Rightarrow \frac{\omega_n^2}{(s+\omega_n)^2} \rightarrow (1)$$

when input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

$\therefore$ The response in s-domain

$$C(s) = R(s) \cdot \frac{\omega_n^2}{(s+\omega_n)^2} \Rightarrow C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} \rightarrow (2)$$

By partial fraction expansion, we can write.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2} \rightarrow (3)$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs}{s(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs$$

Put $s=0$;

$$\omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A=1}$$

Put $s=-\omega_n$:-

$$\omega_n^2 = C(-\omega_n) \Rightarrow \boxed{C=-\omega_n}$$

Equating s^2 terms:-

$$0 = A+B$$

$$\boxed{B=-1}$$

Now substitute A, B and C ~~are~~ in Equation (3)

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \rightarrow (4)$$

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Applying inverse Laplace transform to Equation (4) will get

$$L^{-1}\{C(s)\} = L^{-1}\left[\frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}\right]$$

$L[1] = 1/s$

$L[e^{-at}] = \frac{1}{s+a}$

$L[te^{-at}] = \frac{1}{(s+a)^2}$

$$C(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

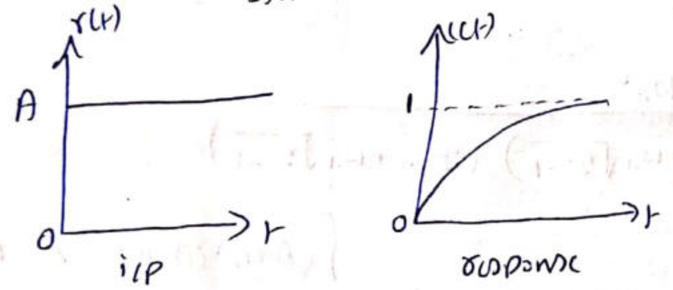
$$\boxed{C(t) = 1 - e^{-\omega_n t}(1-\omega_n t)} \rightarrow \langle 5 \rangle$$

The Equation (4) is the response of critically damped closed loop second order system for unit step input.

∴ For closed loop critically damped second order system.

$$\text{unit step response} = 1 - e^{-\omega_n t}(1-\omega_n t)$$

$$\text{step response} = A\left(1 - e^{-\omega_n t}(1-\omega_n t)\right)$$



This response have no oscillations.

IV **Response of over damped second order system for unit step input**

Transfer function

The standard form of closed loop second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} \Rightarrow -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right]$$

Let $s_1 = -s_a$, $s_2 = -s_b$

$$\therefore\ s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$

The closed loop Transfer function can be written in terms of s_1 and s_2 as shown below.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

For unit step input $r(t) = 1$, $R(s) = 1/s$.

The response in s-domain is given by.

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fraction expansion we can write.

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2} \rightarrow (1)$$

$$\frac{\omega_n^2}{s(s+s_1)\cancel{(s+s_2)}} = \frac{\cancel{A(s+s_1)+Bs} \quad A(s+s_1)(s+s_2)+Bs(s+s_2)+Cs(s+s_1)}{s(s+s_1)\cancel{(s+s_2)}}$$

$$\omega_n^2 = A(s+s_1)(s+s_2) + Bs(s+s_2) + Cs(s+s_1)$$

Put $s=0$:- $\quad \omega_n^2 = A(s_1)(s_2)$

$$A = \frac{\omega_n^2}{(s_1)(s_2)} \Rightarrow \frac{\omega_n^2}{(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})}$$

$$[(A+B)(A-B) = A^2-B^2]$$

$$\Rightarrow \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)}$$

$$\Rightarrow \frac{\omega_n^2}{\zeta^2\omega_n^2 - \zeta^2\omega_n^2 + \omega_n^2} \Rightarrow 1$$

$$\boxed{A = 1}$$

Put $s = -s_1$:- $\quad \omega_n^2 = B(-s_1)(-s_1+s_2)$

$$\omega_n^2 = B(s_1^2 - s_1 s_2)$$

$$B = \frac{\omega_n^2}{s_1(s_1 - s_2)}$$

$$B = \frac{\omega_n^2}{s_1[\cancel{-\zeta\omega_n} - \omega_n\sqrt{\zeta^2-1} + \cancel{\zeta\omega_n} - \omega_n\sqrt{\zeta^2-1}]}$$

$$B = \frac{\omega_n^2}{s_1[-2\omega_n\sqrt{\zeta^2-1}]} \Rightarrow \Rightarrow \frac{-\omega_n}{2\sqrt{\zeta^2-1}} \times \frac{1}{s_1}$$

$$\boxed{B = \left[\frac{1}{s_1} \cdot \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\right]}$$

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Put $S = -S_2$:-

$$\omega_n^2 = C(-S_2)(-S_2+S_1)$$

$$\omega_n^2 = C(S_2^2 - S_1S_2)$$

$$C = \frac{\omega_n^2}{S_2(S_2-S_1)}$$

$$C = \frac{\omega_n^2}{S_2\left[\cancel{\zeta\omega_n}+\omega_n\sqrt{\zeta^2-1}\cancel{-\zeta\omega_n}+\omega_n\sqrt{\zeta^2-1}\right]} \Rightarrow \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_2}$$

$$\boxed{C = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_2}}$$

Substitute A, B and C values in Equation (1), will get,

$$C(s) = \frac{1}{S} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_1}\cdot\frac{1}{(S+S_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_2}\cdot\frac{1}{(S+S_2)}$$

Apply Inverse Laplace Transform to above Equation will get

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_1}e^{-S_1t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\cdot\frac{1}{S_2}e^{-S_2t}$$

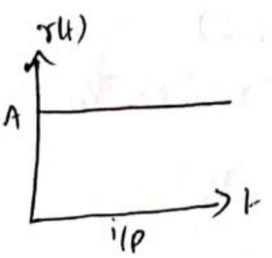
$$\boxed{c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-S_1t}}{S_1} - \frac{e^{-S_2t}}{S_2}\right]}$$

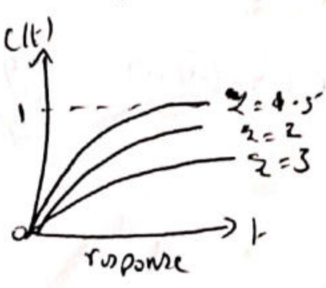
Where $S_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ and $S_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

For closed loop over damped second order system.

$$\text{unit step response} = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-S_1t}}{S_1} - \frac{e^{-S_2t}}{S_2}\right]$$

$$\text{Step response} = A\left[\qquad \text{"} \qquad\right]$$



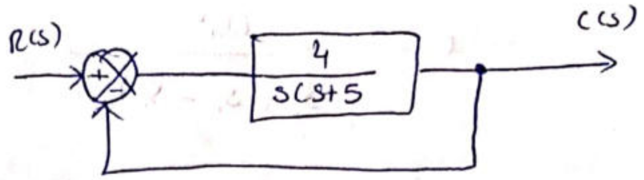


Observed that the response has no oscillations but it takes longer time for the final steady value to reach

Example problems

1) Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the i/p is unit step.

Sol: from given data



The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}}$$

$$\frac{C(s)}{R(s)} = \frac{4}{s(s+5)+4} \Rightarrow \frac{4}{s^2+5s+4} \Rightarrow \frac{4}{(s+1)(s+4)}$$

The response in s-domain

$$C(s) = R(s) \cdot \frac{4}{(s+1)(s+4)}$$

for unit step input $r(t) = 1$ and $R(s) = 1/s$

$$C(s) = \frac{4}{s(s+1)(s+4)}$$

By partial fraction expansion

$$C(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4} \rightarrow (1)$$

$$\Rightarrow \frac{4}{\cancel{s(s+1)(s+4)}} = \frac{A(s+1)(s+4) + Bs(s+4) + Cs(s+1)}{\cancel{s(s+1)(s+4)}}$$

$$\Rightarrow 4 = A(s+1)(s+4) + Bs(s+4) + Cs(s+1)$$

Put $s=0$: $\quad 4 = 4A \Rightarrow \boxed{A=1}$

Put $s=-1$: $\quad 4 = B(-1)(-1+4) \quad 4 = -3B \Rightarrow \boxed{B = -4/3}$

Put $s=-4$: $\quad 4 = C(-4)(-4+1) \quad 4 = C(-4)(-3)$

$4 = 12C \Rightarrow C = \frac{4}{12} \Rightarrow \frac{1}{3}$

$$\boxed{C = 1/3}$$

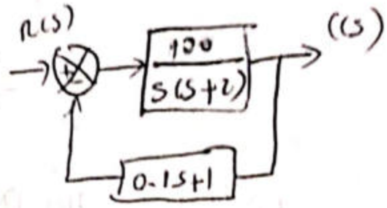
Substitute A, B and C values in Equ. (1)

$$C(s) = \frac{1}{S} - \frac{4/3}{S+1} + \frac{1/3}{S+4}$$

Applying Inverse Laplace transform to above equation, will get

$$\left[c(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t} \right]$$

2) A positional control system with velocity feedback is shown in fig. What is the response of the system for unit step i/p.



Sol The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad \text{here, } G(s) = \frac{100}{S^2+2S}, \quad H(s) = 0.1S+1$$

$$\frac{C(s)}{R(s)} = \frac{100/S^2+2S}{1 + \frac{100(0.1S+1)}{S^2+2S}} \Rightarrow \frac{100}{S^2+2S+10S+100} \Rightarrow \frac{100}{S^2+12S+100}$$

Here $S^2+12S+100$ is characteristics equation. The roots of the chart. equ. are

$$S_1, S_2 = \frac{-12 \pm \sqrt{144-400}}{2} \Rightarrow \frac{-12 \pm j16}{2} \Rightarrow -6 \pm j8.$$

The roots are complex conjugate. The system is underdamped and so the response of the system will have damped oscillations.

The response in S-domain

$$C(S) = R(s) \cdot \frac{100}{S^2+12S+100}$$

For unit step input $r(t) = 1$, $R(s) = 1/s$.

$$\therefore C(S) = \frac{100}{S(S^2+12S+100)} = \frac{A}{S} + \frac{BS+C}{S^2+12S+100} \rightarrow (1)$$

$$\Rightarrow \quad 100 = A(S^2+12S+100) + BS^2 + CS$$

Put S = 0 :- $\quad 100 = A\,100 \Rightarrow \boxed{A=1}$

Consider S^2 terms :- $\quad 0 = A+B \Rightarrow \boxed{B=-1}$

Consider S terms :- $\quad 0 = 12A + C \Rightarrow \boxed{C=-12}$

Substitute A, B and C values in Equ. (1)

$$C(s) = \frac{1}{S} - \frac{S}{S^2+12S+100} - \frac{12}{S^2+12S+100}$$

$$C(S) = \frac{1}{S} - \frac{S+12}{S^2+12S+100}$$

$$C(s) = \frac{1}{s} - \frac{s+12}{s^2+12s+36+64} \Rightarrow \frac{1}{s} - \frac{s+6+6}{(s+6)^2+8^2}$$

$$\Rightarrow \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2}$$

$$\Rightarrow \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\cdot\frac{8}{(s+6)^2+8^2}$$

Applying inverse Laplace Transform to above eqn, will get.

$$c(t) = 1 - e^{-6t}\cos 8t - 6/8\, e^{-6t}\sin 8t$$

$$c(t) = 1 - e^{-6t}\left[\cos 8t + 6/8 \sin 8t\right]$$

3) The response of a servomechanism is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

Sol given that, $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$

on taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - 1.2\frac{1}{(s+10)} \Rightarrow \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$C(s) = \frac{(s+60)(s+10) + 0.2s(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$\Rightarrow \frac{s^2+70s+600 + 0.2s^2 + 2 - 1.2s^2 - 72s}{s^2+60s^2(s+10)}$$

$$\Rightarrow \frac{600}{s(s^2+70s+600)} \Rightarrow \frac{1}{s}\cdot\frac{600}{s^2+70s+600}$$

[Since i/p is unit step, $R(s) = 1/s$]

$$C(R) = R(s)\cdot\frac{600}{s^2+70s+600}$$

∴ The closed loop transfer function of the system, $\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$

The standard form of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$, when compared to both eqn,

$\omega_n^2 = 600$

$\omega_n = \sqrt{600}$

$\omega_n = 24.49$ rad/sec

$2\zeta\omega_n = 70$

$$\zeta = \frac{70}{2\omega_n} \Rightarrow \frac{70}{2\times 24.49} \Rightarrow 1.429$$

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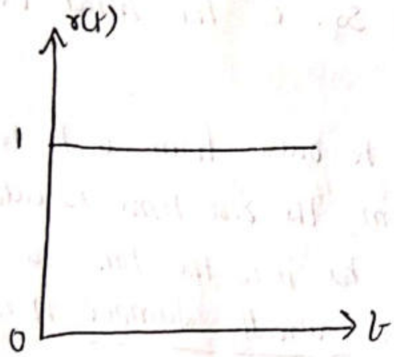
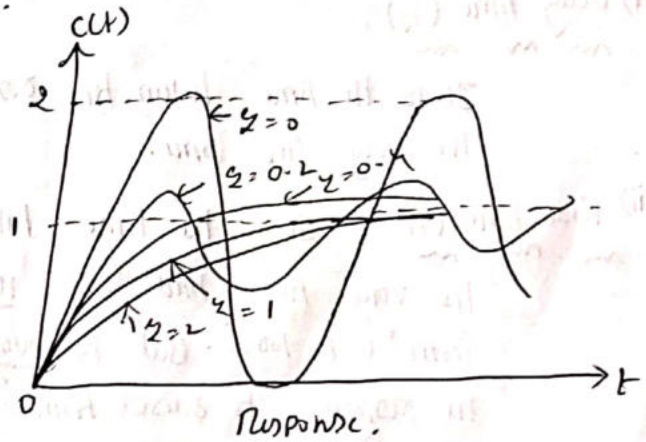
* TIME DOMAIN SPECIFICATIONS :-

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient response, whenever they are subjected to inputs (or) disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of second order system for unit step input with various values of damping ratio is shown in fig.



input



Response.

Unit step response of second order system.

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t=0$ will be zero. The transient response of a practical control system often exhibits damped oscillations before reaching steady state. A typical damped oscillatory response of a system is shown in fig.

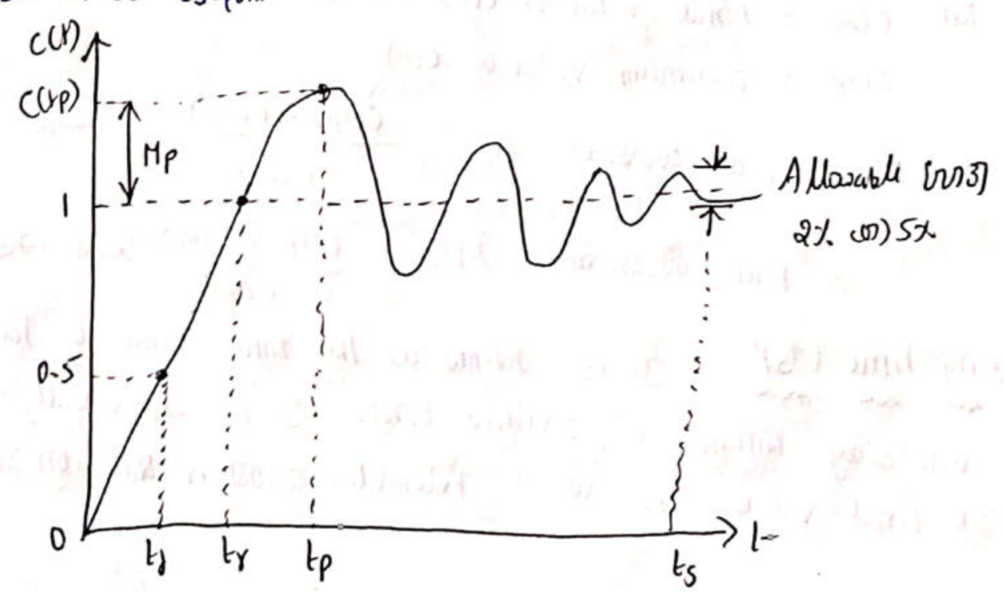


Fig.1 Damped oscillatory response of second order system for unit step input.

The transient response characteristics of a control system to a unit step i/p is specified in terms of the following time domain specifications.

i) Delay time, t_d

ii) Rise time, t_r

iii) peak time, t_p

iv) Maximum overshoot, M_p.

v) settling time, t_s

The time domain specifications are defined as follows.

i) Delay time (t_d) :-

It is the time taken for response to reach 50% of the final value, for the very first time.

ii) Rise time (t_r):- It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from "0 to 100%". But for overdamped system, it is the time taken by the response to raise from "10% to 90%". For critically damped, it is time taken for response to raise from "5% to 95%".

iii) peak time (t_p):- It is the time taken for response to reach the peak value the very first time.

iv) peak overshoot (or) Maximum overshoot (M_p):- It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured is measured from final value.

Let $c(\infty)$ = Final value of $c(t)$

$c(t_p)$ = Maximum value of $c(t)$

Now, peak overshoot, $$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \rightarrow (1)$$

% peak overshoot, $$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \rightarrow (2)$$

v) settling time (T_s):- It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2% (or) 5% of final value.

Expressions for time domain specifications:-

Rise time (t_r):- The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ (Refer fig. 1)

$$c(t_r) \Rightarrow 1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$\therefore -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\xi\omega_n t} \neq 0$, the term $\sin(\omega_d t_r + \theta) = 0$

When

$$\sin(\omega_d t_r + \theta) = 0$$

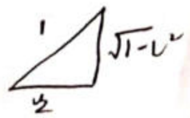
$$\omega_d t_r + \theta = \sin^{-1}(0)$$

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

rise time $\Rightarrow$ $\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$

Note: On constructing right angle triangle with ξ and $\sqrt{1-\xi^2}$ we get



$$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Here $\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$; and $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore \left[\text{Rise time } (t_r) = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}} \text{ in sec} \right]$$

Note: θ (or) $\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$ should be measured in radians.

Peak time (t_p):-

To find the expression for peak time, t_p, differentiate $c(t)$ with respect to t and equate to zero.

$$\text{i.e., } \frac{d}{dt} c(t)\Big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Differentiating $c(t)$ with respect to t.

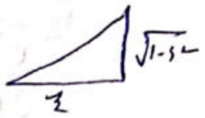
$$\frac{d}{dt}c(t) = (-\xi\omega_n)\cdot -\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) + \left[\frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\right]\cos(\omega_d t+\theta)\,\omega_d$$

$$\Rightarrow \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\xi\omega_n)\sin(\omega_d t+\theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cos(\omega_d t+\theta)\,\omega_d$$

Put $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{d}{dt}c(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\xi\omega_n)\sin(\omega_d t+\theta) - \frac{\omega_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\cos(\omega_d t+\theta)$$

$$\Rightarrow \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left\{\xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right\}$$



$$\Rightarrow \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$\Rightarrow \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin((\omega_d t+\theta)-\theta)\right] \Rightarrow \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\omega_d t$$

at $t = t_p$ and $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t_p}\sin(\omega_d t_p) = 0$$

since $e^{-\xi\omega_n t_p} \neq 0$, the term $\sin(\omega_d t_p) = 0$

$$\omega_d t_p = \sin^{-1}(0)$$

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d} \qquad \boxed{\therefore \text{ Peak time } (t_p) = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Peak overshoot (M_p) (or) Maximum over shoot :-

$$\%\text{ Peak overshoot } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

where $c(t_p)$ = Peak response at $t = t_p$

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = \infty$, $c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) = 1 - 0 = 1$

$\left[t_p = \frac{\pi}{\omega_d} \right]$

At $t = t_p$ $\quad c(t) = c(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right) \qquad \left[\omega_d = \omega_n\sqrt{1-\xi^2} \right]$$

$\sin(\pi+\theta) = -\sin\theta$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi+\theta)$$

$\boxed{\text{from } \Delta^{le} \quad \sin\theta = \sqrt{1-\xi^2}}$

$$\Rightarrow 1 - \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}(-\sin\theta) \Rightarrow 1 + \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\theta$$

$$\Rightarrow 1 + \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\cancel{\sqrt{1-\xi^2}}} \cdot \cancel{\sqrt{1-\xi^2}}$$

$$\left[c(t_p) \Rightarrow 1 + e^{-\xi\pi/\sqrt{1-\xi^2}} \right]$$

% peak overshoot, $\quad \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

$$\Rightarrow \frac{1 + e^{-\xi\pi/\sqrt{1-\xi^2}} - 1}{1} \times 100$$

$$\boxed{\%M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100}$$

settling time (t_s) :-

* settling time, $t_s = 4T$ (for 2% error)
* settling time, $t_s = 3T$ (for 5% error)

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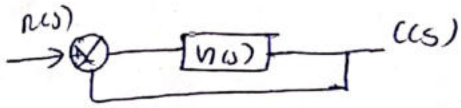
Summary :

Rise time $(t_r) = \frac{\pi - \theta}{\omega_d}$ $\quad\quad$ $\% M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

Peak time $(t_p) = \pi/\omega_d$ $\quad\quad$ $t_s = 4T$ (2% error)
$= 3T$ (5% error).

Q: The unity feedback system is characterized by an open loop transfer function $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that system will have a damping ratio of 0.5 for this value of K. Determine ~~settling time~~, peak overshoot, and time at peak overshoot for a unit step input.



Sol:
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$ here $G(s) = \frac{K}{s(s+10)}$

$H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} \Rightarrow \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

on comparing we get

$\omega_n^2 = K$ $\quad\quad$ $2\zeta\omega_n = 10$ $\quad\quad$ $K = 100$

$\omega_n = \sqrt{K}$ $\quad\quad$ $\zeta\omega_n = 5$ $\quad\quad$ $\omega_n = 10$ rad/sec

$\omega_n = 5/0.5 = 10$

$\sqrt{K} = 10$

The value of gain $K = 100$

Percentage peak over shoot $\% M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$\Rightarrow e^{-0.5\pi/\sqrt{1-(0.5)^2}} \times 100 \Rightarrow e^{-1.571/0.866} \times 100$$

$$\Rightarrow 16.3\%$$

Peak time $(t_p) = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \frac{\pi}{10\sqrt{1-(0.5)^2}} \Rightarrow \frac{\pi}{8.660}$

$t_p = 0.363$ sec

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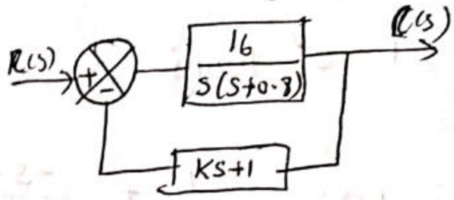
Ex: 2 A positional control system with velocity feedback is shown in fig. What is the response $c(t)$ to the unit step i/p. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

Sol The closed loop transfer function is,



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

given that $G(s) = \dfrac{16}{s(s+0.8)}$ $\quad H(s) = Ks+1$

$$\frac{C(s)}{R(s)} = \frac{16/s(s+0.8)}{1+\dfrac{16(Ks+1)}{s(s+0.8)}} \Rightarrow \frac{16}{s^2+0.8s+16Ks+16}$$

$$\Rightarrow \frac{16}{s^2+(0.8+16K)s+16} \rightarrow ①$$

The value of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{16}{s^2+(0.8+16K)s+16}$$

$\omega_n^2 = 16$ $\quad 2\zeta\omega_n = 0.8+16K$ $\quad (\zeta = 0.5)$

$[\omega_n = 4]$ $\quad 2(0.5)(4) = 0.8+16K \Rightarrow 4-0.8 = 16K$

$\Rightarrow [K = 0.2]$

Given that the damping ratio $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of the characteristics polynomial will be complex conjugate.

The response s-domain $\Rightarrow$ $C(s) = R(s) \cdot \dfrac{16}{s^2+4s+16}$

For unit step input $r(t) = 1$ and $R(s) = 1/s$

$$C(s) = \frac{16}{s(s^2+4s+16)}$$

. By partial fraction expansion.

$$C(s) = \frac{16}{s(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$16 = A(s^2+4s+16) + Bs^2 + Cs$$

Put $s=0$, $16 = 16A \Rightarrow \boxed{A=1}$

Consider "s^2" terms, $0 = A+B \Rightarrow \boxed{B=-1}$

Consider "s^1" terms $0 = 4A+C \Rightarrow \boxed{C=-4}$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} \Rightarrow \frac{1}{s} - \frac{s+4}{s^2+4s+16}$$

$$\Rightarrow \frac{1}{s} - \frac{s+4}{s^2+4s+4+12} \Rightarrow \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12}$$

$$\Rightarrow \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{(s+12)^2+12}$$

$\omega_n^2 = 12$

$\omega_n = \sqrt{12}$

$$\Rightarrow \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}} \cdot \frac{\sqrt{12}}{(s+12)^2+12}$$

By taking I.L.P.T

$$c(t) = 1 - e^{-2t}\cos\sqrt{12}t - \frac{\cancel{2}}{\cancel{2}\sqrt{3}} e^{-2t}\sin\sqrt{12}t$$

$$\Rightarrow 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin\sqrt{12}t + \cos\sqrt{12}t\right]$$

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\xi^2} \Rightarrow 4\sqrt{1-(0.5)^2} = 3.464$ rad/sec

Rise time $t_r = \frac{\pi-\theta}{\omega_d}$ $\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = 1.047$

$$\Rightarrow \frac{\pi - 1.047}{3.464} \Rightarrow 0.605 \text{ sec}$$

Peak time: $t_p = \frac{\pi}{\omega_d} \Rightarrow \frac{\pi}{3.464} \Rightarrow 0.907$ sec

% Max. overshoot: $\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100 \Rightarrow e^{\frac{-0.5\times\pi}{\sqrt{1-(0.5)^2}}} \times 100$

$$\Rightarrow e^{-1.571/0.866} \times 100 \Rightarrow e^{-1.814} \times 100 \Rightarrow 16.3\%$$

Time constant: $T = \frac{1}{\xi\omega_n} \Rightarrow \frac{1}{0.5\times4} = 0.5$ sec.

For 5% error settling time $t_s = 3T = 3\times0.5 \Rightarrow 1.5$ sec

For 2% error settling time $t_s = 4T = 4\times0.5 \Rightarrow 2$ sec

* Type number of control systems :-

The Type number is specified for loop transfer function $G(s)H(s)$. The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if N is the number of poles at the origin then the type number is "N".

The loop transfer function can be expressed as a ratio of two polynomials.

$$G(s)H(s) \Rightarrow K\frac{(s+z_1)(s+z_2)(s+z_3)\cdots}{s^N(s+P_1)(s+P_2)(s+P_3)\cdots}$$

N = number of poles at origin.

If $N=0$, then the system is Type-0 system $(s^0=1)$
If $N=1$, then the system is Type-1 system $(s^1=s)$
If $N=2$, then the system is Type-2 system $(s^2=s^2)$

* Steady state Response :-

→ Steady state occurs after the system becomes settled and at the steady state system working normally.

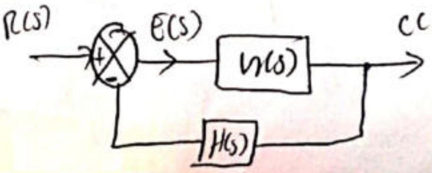
→ Steady state response of control system is a function of input signal and it is also called as forced response.

→ Steady state response of control system gives a clear description of how the system functioning during steady state.

* Steady State Errors :-

The Steady State error is the value of error signal $e(t)$, when t tends to infinity. The Steady State error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system generally judged by its steady state error to unit step, ramp and parabolic inputs.

Consider a closed system shown in fig.



Let $R(s)$ = input signal
$E(s)$ = error signal
$C(s)H(s)$ = feedback signal, $C(s)$ = output signal (or) response.

The error signal $E(s) = R(s) - C(s)H(s) \rightarrow (1)$

The output signal $C(s) = E(s)G(s) \rightarrow (2)$

Sub. (2) in (1)

$$E(s) = R(s) - E(s)G(s)H(s)$$

$$R(s) = E(s) + E(s)G(s)H(s)$$

$$R(s) = E(s)\{1 + G(s)H(s)\}$$

$$\therefore \boxed{E(s) = \frac{R(s)}{1+G(s)H(s)}} \rightarrow (3)$$ error signal in s-domain

Let $e(t)$ = Error signal in time domain

Applying I.L.P.T to Equation (3)

$$e(t) = L^{-1}\left[\frac{R(s)}{1+G(s)H(s)}\right] \rightarrow (4)$$

Let e_{ss} = Steady state error

The steady state error is defined as the value of $e(t)$ when "t" tends to infinity.

$$e_{ss} = \lim_{t\to\infty} e(t) \rightarrow (5)$$

The final value theorem of Laplace transform states that.

If $F(s) = L[f(t)]$ then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \rightarrow (6)$$

Using final value theorem

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{e_{ss} = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)H(s)}} \rightarrow (7)$$

* Static Error Constants :-

When a control system is excited with standard input signal, the steady state error may be zero, constant (or) infinity. The value of steady state error depends on "Type number" and the "input signal."

=) Type-0 System will have constant steady state error when the i/p is step signal

=) Type-1 system will have constant steady state error when the i/p is Ramp signal

=) Type-2 system will have constant steady state error when the i/p is parabolic signal.

For the three cases mentioned above the steady state error associated with one of the constants defined as follows.

Positional error constant, $K_p = \lim_{s\to 0} G(s)H(s)$ (for step input) → ①

velocity error constant, $K_v = \lim_{s\to 0} s\,G(s)H(s)$ (for ramp input) → ②

acceleration error constant, $K_a = \lim_{s\to 0} s^2 G(s)H(s)$ (for parabolic i/p) → ③

The K_p, K_v and K_a are called steady state error constants (or) Static error constants.

a) Steady state error when the input is unit step signal :-

Steady state error, $e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$

for unit step i/p $r(t) = 1$ and $R(s) = 1/s$.

$$e_{ss} = \lim_{s\to 0} \frac{s\,{}^1\!/_s}{1+G(s)H(s)} \Rightarrow \lim_{s\to 0} \frac{1}{1+G(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p}.$$

$$\left[e_{ss} = \frac{1}{1+K_p}\right] \to ① \quad \text{where } K_p = \lim_{s\to 0} G(s)H(s)$$

The constant K_p is called "positional error constant."

Type-0 system:- $K_p = \lim_{s\to 0} G(s)H(s) \Rightarrow \lim_{s\to 0} K.\frac{(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots} = \text{Constant}.$

$e_{ss} = \frac{1}{1+K_p} = \text{Constant}.$

Hence in Type-0 System when the i/p is unit step there will be constant steady state error

Type-1 System: $K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} K.\frac{(s+z_1)(s+z_2)\cdots}{s(s+P_1)(s+P_2)}\cdots \Rightarrow \frac{1}{0} = \infty$

$e_{ss} = \frac{1}{1+K_p} \Rightarrow \frac{1}{1+\infty} = 0$

In Type-1 System, for unit step i/p the value of $K_p = \infty$ and $e_{ss} = 0$

b) Steady state error when the input is unit Ramp signal :-

Steady state error $e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$

When the i/p is unit ramp $r(t) = t$ and $R(s) = 1/s^2$

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}\, 1/s^{\cancel{2}}}{1+G(s)H(s)} \Rightarrow \lim_{s\to 0} \frac{1/s}{1+G(s)H(s)}$$

$$\Rightarrow \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} \Rightarrow \frac{1}{0+\lim_{s\to 0} sG(s)H(s)} = \frac{1}{K_v}$$

$$\boxed{e_{ss} = 1/K_v} \qquad K_v = \lim_{s\to 0} s\,G(s)H(s)$$

The constant K_v is called velocity error constant.

Type-0 :

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} sK \frac{(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots} = 0$$

$$\boxed{K_v = 0}$$

$$\boxed{e_{ss} = 1/0 \Rightarrow \infty}$$

Type-1 :

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \cancel{s}K \frac{(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+P_1)(s+P_2)\cdots} = \text{constant}$$

$$\boxed{e_{ss} = 1/K_v = \text{constant}}$$

Type : 2 :

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \cancel{s} \frac{(s+z_1)(s+z_2)\cdots}{s^{\cancel{2}}(s+P_1)(s+P_2)\cdots} = \infty$$

$$\boxed{K_v = \infty}$$

$$\boxed{e_{ss} = 0}$$

c) Steady state error when the i/p is unit parabolic signal :-

Steady state error $e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$

For unit parabolic i/p $r(t) = t^2/2$ and $R(s) = 1/s^3$

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}\, 1/s^{\cancel{3}}}{1+G(s)H(s)} \Rightarrow \lim_{s\to 0} \frac{1}{s^2+s^2G(s)H(s)}$$

$$\boxed{e_{ss} \Rightarrow \frac{1}{K_a}} \qquad K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

The constant K_a is called acceleration error constant

Type-0:- $K_a = \lim_{s\to 0} s^2 G(s)H(s) \Rightarrow \lim_{s\to 0} s^2 K \frac{(s+z_1)(s+z_2)\ldots}{(s+P_1)(s+P_2)\ldots} = 0$

$[K_a = 0]$

$[e_{ss} = \frac{1}{K_a} = \infty]$

Type-1:- $K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^{\cancel{2}} K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+P_1)(s+P_2)\ldots} = 0$

$[K_a = 0]$

$[e_{ss} = \frac{1}{0} = \infty]$

Type-2:- $K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s^2}(s+P_1)(s+P_2)\ldots}$ = constant.

K_a = constant

$e_{ss} = \frac{1}{K_a}$

Type-3:- $K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^{\cancel{2}} K \frac{(s+z_1)(s+z_2)\ldots}{s^{3}(s+P_1)(s+P_2)\ldots} \Rightarrow \infty$

$K_a = \infty$

$e_{ss} = \frac{1}{\infty} = 0$

* Steady state error constants for various Type number systems

Error constant	Type number of System			
	Type - 0	Type - 1	Type - 2	Type - 3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

* Steady state error for various types of inputs

i/p Signal	Type number of System		
	Type-0	Type - 1	Type - 2
unit step	$\frac{1}{1+K_p}$	0	0
unit Ramp	∞	$\frac{1}{K_v}$	0
unit Parabolic	∞	∞	$\frac{1}{K_a}$

Ex: For a unity feedback control system the open loop transfer function

$G(s) = \dfrac{10(s+2)}{s^2(s+1)}$. Find.

a) the position, velocity and acceleration error constants.

b) the steady state error when the input is $R(s)$, where $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$

Sol:

a) To find error constants:-

given $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$ $\quad H(s) = 1$

Positional error constant $\Rightarrow K_p = \lim\limits_{s\to 0} G(s)H(s) = \infty$

Velocity error constant $\Rightarrow K_v = \lim\limits_{s\to 0} s\,G(s)H(s)$

$\Rightarrow K_v = \lim\limits_{s\to 0} \cancel{s} \cdot \dfrac{10(s+2)}{s^{\cancel{2}}(s+1)} \cdot 1 = \infty$

acceleration error constant $\Rightarrow K_a = \lim\limits_{s\to 0} \cancel{s^2} \cdot \dfrac{10(s+2)}{\cancel{s^2}(s+1)} \Rightarrow \dfrac{20}{1} = 20$

b) To find steady state error:-

Steady state error $e_{ss} = \lim\limits_{s\to 0} s \cdot \dfrac{R(s)}{1+G(s)H(s)} \rightarrow (1)$

To find $\dfrac{R(s)}{1+G(s)H(s)}$ here $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$ $\quad G(s) = \dfrac{10s+20}{s^3+s^2}$ $\quad H(s)=1$

$$\Rightarrow \frac{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}}{1 + \dfrac{10s+20}{s^3+s^2} \cdot 1} \quad \Rightarrow \cancel{\frac{\dfrac{9s^2-6s+3}{3s^3}}{\dfrac{s^3+s^2+10s+20}{s^3(s+1)}}}$$

$$\Rightarrow \frac{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}}{\dfrac{s^2(s+1)+10s+20}{s^2(s+1)}}$$

$$\Rightarrow \left(\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right) \times \frac{s^2(s+1)}{s^2(s+1) + 10(s+2)}$$

$$\Rightarrow \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] \rightarrow (2)$$

Substitute (2) in (1)

$$e_{ss} = \lim_{s \to 0} s\left\{\frac{3}{s}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) - \frac{2}{s^2}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) + \frac{1}{3s^3}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right)\right\}$$

$$e_{ss} = \lim_{s \to 0}\left[\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \lim_{s \to 0}\left[\frac{2s(s+1)}{s^2(s+1)+10(s+2)}\right] + \lim_{s \to 0}\left[\frac{(s+1)}{3s^2(s+1)+30(s+2)}\right]$$

$$e_{ss} = 0 - 0 + \lim_{s \to 0}\left[\frac{(s+1)}{3s^2(s+1)+30(s+2)}\right]$$

$$\Rightarrow 0 - 0 + \left[\frac{1}{0+60}\right]$$

$$\boxed{e_{ss} = 1/60}$$

2) For Servomechanisms with open loop transfer function given below Explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$ b) $G(s) = \frac{10}{(s+2)(s+3)}$ c) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Sol a) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system $H(s) = 1$

The open loop transfer function has one pole at origin. Hence it is Type-1 system. In systems with type-1, the ramp (velocity) input give a constant steady state error.

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The steady state error with unit ramp input $e_{ss} = 1/K_v$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) \Rightarrow \lim_{s\to 0} \cancel{s} \times \frac{20(s+2)}{\cancel{s}(s+1)(s+3)}$$

$$\Rightarrow \frac{40}{3}$$

$$e_{ss} = 1/K_v = 3/40 = 0.075.$$

b) $$G(s) = \frac{10}{(s+2)(s+3)}$$

Let us assume unity feed back $H(s) = 1$

The given open loop T.F has no pole at origin. Hence it is Type-0 system. For type-0 system unit step i/p gives constant steady state error.

$$K_p = \lim_{s\to 0} G(s)H(s)$$

$$e_{ss} = \frac{1}{1+K_p}$$

$$\Rightarrow \lim_{s\to 0} \frac{10}{(s+2)(s+3)} \Rightarrow 10/6$$

$$e_{ss} = \frac{1}{1+10/6} \Rightarrow \frac{\cancel{6}\,3}{\cancel{16}\,8}$$

$$\Rightarrow 0.375.$$

c) $$G(s) = \frac{10}{s^2(s+1)(s+2)}$$

Let us assume unity feed back $H(s) = 1$

The given open loop Transfer function has two poles at origin. Hence it is Type-2 system. For type-2 system unit parabolic i/p gives constant steady state error.

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$e_{ss} = 1/K_a$$

$$\Rightarrow \lim_{s\to 0} \cancel{s^2} \frac{10}{\cancel{s^2}(s+1)(s+2)} \Rightarrow 5$$

$$e_{ss} = 1/5 \Rightarrow 0.2$$

Unit - III

STABILITY

Definition :- The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system response (or) output is predictable, finite and stable for a given input.

Concept of Stability :-

i) For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable (or) unstable under some limited constraints. Such a system is called " limitedly stable ".

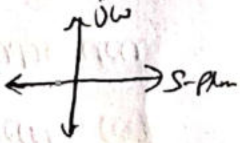
ii) If the system output is stable for all variations of its parameters, then the system is called " absolutely stable system ".

iii) If the system output is stable for a limited range of variations of its parameters, then the system is called " conditionally stable system ".

Location of poles on S-plane for stability :-

The closed loop Transfer function H(s) can be expressed as a ratio of two polynomials in S-domain. The denominator of closed loop T.F is called " characteristic Equation ". The roots of characteristic Equation are called " poles ".

" The requirement for stability can be linked to the location of roots of characteristic Equation in the S-plane "



The closed loop T.F can be expressed as the ratio of two polynomials in 'S' is given by

$$T(S) = \frac{(S+Z_1)(S+Z_2)(S+Z_3)\cdots(S+Z_m)}{(S+P_1)(S+P_2)(S+P_3)\cdots(S+P_m)}$$

The roots of the numerator are "zeros". The roots of denominator are "poles".
By partial fraction expansion we can write,

$$H(s) = \frac{A_1}{s+P_1} + \frac{A_2}{s+P_2} + \frac{A_3}{s+P_3} + \cdots \frac{A_m}{s+P_m}$$

The roots (poles) $P_1, P_2, P_3 \ldots P_m$ may be lying at origin (or) lying on imaginary axis (or) lying on right half of s-plane (or) lying on left half of s-plane.

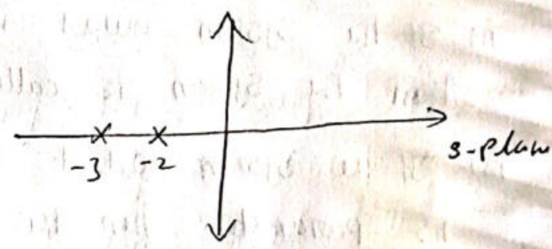
The following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

i) If all the roots of characteristic equation has negative real parts, then the system is stable.

Ex: Consider char. equation

$$s^2+5s+6$$

$$(s+2)(s+3) \Rightarrow s = -2, -3$$



(roots are locating on left half of s-plane) → stable condition.

ii) If any root of the char. equation has a positive real parts (or) if there is a repeated roots on the imaginary axis then the system is unstable.

iii) If the condition (i) is satisfied except for the presence of one (or) more non-repeated roots on the imaginary axis then the system is limitedly (or) marginally stable.

Ex. Consider a char. equation $s^3+s^2+2s+8=0$

roots are $s = -2, \ \frac{1}{2}+j\frac{\sqrt{15}}{2}, \ \frac{1}{2}-j\frac{\sqrt{15}}{2}$

The co-efficients of the polynomial are all positive, but two roots have positive real part and so will lie on right half of s-plane, therefore the system is "unstable".

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* Routh - Hurwitz criterion :-

The R-H stability criterion is an analytical procedure for determining whether all the roots of polynomial have negative real part or not.

" The necessary and sufficient condition for stability is that all the elements in the first column of the routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of routh array corresponds to the no. of roots of char. equation lying in the right half of s-plane."

Note :- If the order of sign of first column elements is +, +, −, + and +. Then + to − is considered as one sign change and − to + as another sign change (Two times sign changes → 2 roots lies on right half of s-plane).

Construction of Routh array :-

Let consider a characteristic polynomial be

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots + a_{n-1} s^1 + a_n s^0$$

The co-efficients of polynomial are arranged in two rows are given below

$$s^n : \quad a_0 \quad a_2 \quad a_4 \quad a_6 \dots$$
$$s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \dots$$

The other rows of routh array upto s^0 row can be formed by following procedure. Each row of routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of routh array as shown below.

$$s^{n-x} : \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \dots$$
$$s^{n-x-1} : \quad y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \dots$$

Let the next row be

$$s^{n-x-2} : \quad z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \dots$$

The elements of S^{n-x-2} row are given by.

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0x_1 - x_0y_1}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0x_2 - x_0y_2}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0x_3 - x_0y_3}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0x_4 - y_4x_0}{y_0}$$

$$z_4 = \frac{(-1)\begin{vmatrix} x_0 & x_5 \\ y_0 & y_5 \end{vmatrix}}{y_0} = \frac{y_0x_5 - x_0y_5}{y_0}$$ and so on.

In this process of constructing routh array the missing terms are considered as zeros. Also all the elements of any row can be multiplied (or) divided by positive constant to simplify the computational work.

In the construction of routh array one may come across the following three cases.

Case-1 : Normal routh array (Non-zero elements in the first column of routh array)

Case-2 :- A row of all zeros.

Case-3 :- First element of a row is zero but other elements are not zero.

Ex:1 Using R-H criterion, determine the stability of the system represented by characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of char. equation.

Sol: The char. equation of the system is $s^4+8s^3+18s^2+16s+5=0$

It is 4th order equation and the highest power is even number, so form the first row of routh array using co-efficients of even powers of s and form the second row using co-efficients of odd powers of s.

$$\begin{array}{llll} s^4 : & 1 & 18 & 5 \quad \text{row-1} \\ s^3 : & 8 & 16 & \quad \text{row-2} \end{array}$$

The elements of row-2 is divided by "8" for easy simplification.

$$\begin{array}{lllll} s^4 : & 1 & 18 & 5 & \text{row-1} \\ s^3 : & 1 & 2 & 0 & \text{row-2} \\ s^2 : & 16 & 5 & 0 & \text{row-3} \\ s^1 : & 1.7 & 0 & 0 & \text{row-4} \\ s^0 : & 5 & & & \end{array}$$

[missing number considered as zero]

column-1

$s^2 : \frac{(1\times18)-(1\times2)}{1}, \frac{(1\times5)-(1\times0)}{1}$

$: 16, 5$

$s^1 : \frac{(16\times2)-(5\times1)}{16}, \frac{(5\times0)-(2\times0)}{16}$

$: 1.7, 0$

The elements of first column of routh array it is observed that all the elements are positive there is no sign change. Hence all the roots are lying on left half of s-plane. and the system is stable.

Result:

i) Stable system

2) All the four roots are lying on left half of s-plane.

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Ex: 2:- Using RH criterion, determine the stability of the system represented by char. eqn. $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$. Comment on the location of the roots of the char. eqn.

Sol The char. eq. of the system is $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$. It is fourth order characteristic equation. and the highest power is even number. so form the first row of the routh array by even power of "s" co-efficients and second row of the routh array by odd power of "s" co-efficients.

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	-6	0	0
s^0	5	0	0

$\frac{6-4}{2} \Rightarrow$

$\frac{4-10}{1} \Rightarrow -6$

$\frac{-6\times 5 - 0}{-6}$

The elements of first column of the routh array is observed that there is sign change by two time from +ve to -ve and -ve to +ve. and hence "two poles are locating at right half of s-plane."

Ex. 3: $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$.

s^5	1	2	3
s^4	1	2	5
s^3			
s^2			
s^1			
s^0			

Ex. $3s^4 + 10s^3 + 5s^2 + 5s + 2 = 0$

Ex.4 Construct Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

Sol The characteristic equation of the system is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.
The given characteristic polynomial is 6th order equation and so it has 6 roots.
since highest power of s is even number, form the first row of routh array using the co-efficients of even powers of "s" and form the second row using the co-efficients of odd power of "s".

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^3	1	3	0	0
s^2	6	16	0	
s^1	0.333	0	0	0
s^0	16	0	0	0

s^4 : $\frac{(2\times8)-(12\times1)}{2}$, $\frac{(2\times20)-(1\times16)}{2}$, $\frac{(2\times16)-(1\times0)}{2}$

2 12 16

s^3 : $\frac{(12\times2)-(12\times2)}{2}$, $\frac{(2\times16)-(16\times2)}{2}$, $\frac{(2\times0)-(0\times2)}{2}$

* When all elements in a first column of routh array will be zero, proceed the next step by considering auxiliary equation. The auxiliary equation can be obtained the above row of the row having all zero elements (since it is even row)

Auxiliary Equation $\Rightarrow$ $2s^4+12s^2+16$ $\rightarrow$ ①

$\Rightarrow$ s^4+6s^2+8 ($\div$ by 2)

differentiating above eq. with respect to s.

$$\frac{dA}{ds} = 4s^3+12s \quad (or) \quad s^3+3s.$$

s^3 | 1 3.

s^2 : $\frac{(1\times12)-(2\times3)}{1}$, $\frac{(1\times16)-(2\times0)}{1}$

: 6 16

s^1 : $\frac{(18)-(16)}{6}$ 0

On observing that all elements of 1st column of routh array, it is observed that there is no sign change. The row with all zeros indicates the possibility of roots on imaginary axis.

Hence the system is marginally stable.

To find the location of poles:

Consider the auxilary polynomial equation.

$$s^4 + 6s^2 + 8 = 0$$

Let $s^2 = x$

$$\therefore x^2 + 6x + 8 = 0$$

The roots of quadratic are, $x = \frac{-6 \pm \sqrt{36-32}}{2} \Rightarrow -3 \pm 1$

$$x \Rightarrow -2 \text{ (or) } -4$$

The roots of auxilary polynomial is

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= \pm j\sqrt{2} \text{ and } \pm j2$$

$$\Rightarrow -j\sqrt{2}, +j\sqrt{2}, -j2, +j2$$

The roots of auxilary equation are also roots of Char. equation. Hence 4 roots are lying on imaginary axis and remaining two roots are lying on the left half of s-plane.

Result:
i) The system is marginally stable.
ii) 4 roots lying on imaginary axis and 2 roots lying on left half of s-plane.

Ex: Construct Routh array and determine the stability of the system represented by the characteristic Equation $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of char. Equation.

Sol: The char. Equation of the system is $s^5+s^4+2s^3+2s^2+3s+5=0$

The given char. Equation is 5^{th} order Equation so it has 5 roots. Since the highest power of "s" is odd number, form the first row of routh array using the co-efficients of odd powers of 's' and form the second row using the co-efficients of Even powers of 's'.

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 5 \\ s^3 & 0 & -2 & \\ s^2 & \infty & 5 & \\ s^1 & -2 & 0 & \\ s^0 & 5 & & \end{array}$$

** If the first element of any row is zero and remaining are non zeros we have the build routh array by any one of the following methods.

Method: 1: Replace s by $1/z$ then char. Equation becomes

$$(1/z)^5+(1/z)^4+2(1/z)^3+2(1/z)^2+3(1/z)+5=0$$

$$5z^5+3z^4+2z^3+2z^2+z+1=0$$

$$\begin{array}{c|ccc} z^5 & 5 & 2 & 1 \\ z^4 & 3 & 2 & 1 \\ z^3 & -1.33 & -0.66 & 0 \\ z^2 & 0.51 & 1 & 0 \\ z^1 & 1.94 & 0 & 0 \\ z^0 & 1 & & \end{array}$$

+ve to -ve

-ve to +ve

z^3: $\frac{6-10}{3}$: $\frac{3-5}{3}$

-1.33 -0.66

z^2: $\frac{(-1.33\times2)-(3\times-0.66)}{-1.33}$: $\frac{(-1.33\times1)-0}{-1.33}$

0.51 1

z^1: $\frac{(0.51\times-0.66)+1.33}{0.51}$, 0

On observing the elements of first column of Routh array it is found that there are two sign changes. Hence two roots are lying on right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

Result:- 1) The system is unstable.

2) Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

Method : 2 On letting ~~s~~ 0 by ε

s^5	1	2	3
s^4	1	2	5
s^3	ε	-2	0
s^2	$\frac{2\varepsilon+2}{\varepsilon}$	5	
s^1	$\frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}$	0	
s^0	5		

$$s^1 : \frac{\left(\frac{2\varepsilon+2}{\varepsilon} \times -2\right) - 5\varepsilon}{\frac{2\varepsilon+2}{\varepsilon}} : 0$$

$$\Rightarrow \frac{\frac{-4\varepsilon-4}{\varepsilon} - 5\varepsilon}{\frac{2\varepsilon+2}{\varepsilon}} \quad 0$$

$$\Rightarrow \frac{-4\varepsilon-4-5\varepsilon^2}{2\varepsilon+2}$$

Apply $\lim_{\varepsilon\to 0}$ $\Rightarrow \lim_{\varepsilon\to 0} \frac{2\varepsilon+2}{\varepsilon} \Rightarrow \frac{2}{0} = \infty$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2} \Rightarrow -2$$

Result:-

1) The system is unstable

2) Two roots are lying on Right half of s-plane and remaining three roots are lying on left half of s-plane.

①

2c. The char. polynomial of a system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

Sol)

The char. eqn. of the system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$

The given char. eqn. is 7th order equation and so it has 7 roots. Since the highest power of "s" is odd number, form the first row of array using the co-efficients of odd powers of "s" and form the second row using the co-efficients of even powers of "s" as shown below.

s^7	1	24	24	23
s^6	9	24	24	15

Divide s^6 row by 3 to simplify the computation.

s^7	1	24	24	23	row-1
s^6	3	8	8	5	row-2
s^5	21.33	21.33	21.33	0	row-3
s^4	5	5	5	0	row-4
s^3	0	0	0		
s^3	4	2	0		row-5
s^2	2.5	5	0		
s^1	-6	0			
s^0	5				

On examing the array the s^3 row contains all zeros at this condition. The we have to consider auxilary equation for forming the array. The auxilary equation can be obtained from above row of s^3 then will get

$$A = 5s^4+5s^2+5$$

$$A = s^4+s^2+1 \Rightarrow \text{Auxilary equation.}$$

Differentiate above eqn. w.r.t. s

$$\frac{dA}{ds} = 4s^3+2s$$

By using above eqn. co-efficient we form s^3 row.

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane. and so system is unstable.

To find roots and location :-

Consider A.E $\Rightarrow$ $s^4 + s^2 + 1 = 0$

Let Put $s^2 = x$ in Auxilary Equation.

$$x^2 + x + 1 = 0$$

The roots of Quadratic Equation $x = \frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$-\frac{1}{2} + j\sqrt{3}/2$, $-\frac{1}{2} - j\sqrt{3}/2$ $\Rightarrow$ $\Rightarrow 1\angle 120^\circ$ (or) $1\angle -120^\circ$

But $s^2 = x$ $\Rightarrow s = \pm\sqrt{x}$ $\Rightarrow \pm\sqrt{1\angle 120^\circ}$ (or) $\pm\sqrt{1\angle -120^\circ}$

$\Rightarrow \pm 1\angle 120^\circ/2$ (or) $\pm 1\angle -120^\circ/2$

$\Rightarrow \pm 1\angle 60^\circ$ (or) $\pm 1\angle -60^\circ$

$\Rightarrow \pm(0.5 + j0.866)$ (or) $\pm(0.5 - j0.866)$

Two roots are lying on the right half of s-plane remaining five roots are lying on left half of s-plane.

Results:-

1) The system is unstable.
2) Two roots are lying on right half of s-plane and five roots are lying left half of s-plane.

(3)

Ex: Determine the range of K for stability of unity feedback system whose open loop transferfunction is $G(s) = \frac{K}{s(s+1)(s+2)}$

Sol: The closed loop Transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}\cdot 1}$

$$\Rightarrow \frac{K}{s(s+1)(s+2)+K}$$

The characteristic Equation is $\Rightarrow s(s+1)(s+2)+K$

$\Rightarrow (s^2+s)(s+2)+K$

$\Rightarrow s^3+2s^2+s^2+2s+K$

$\Rightarrow s^3+3s^2+2s+K$

The construction of routh array is shown below. The highest power of "s" in the characteristic Equation is odd number. Hence from the first row of routh array from co-efficients of odd power of s and second row of routh array from co-efficients of even power of s.

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{K} & 0 \\ s^0 & K & 0 \end{array}$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that first column elements are positive.

from s^0 row, for the system to be stable $K>0$

from s^1 row, for the system to be stable, $\frac{6-K}{3}>0$

for $\frac{6-K}{3}>0$, the value of K should be less than "6"

$\therefore$ The range of K for the system to be stable is $0<K<6$

Result:

The value of K is in the range $0<K<6$ for the system to be stable

Ex: The open loop transfer function of unity feedback control system is given by $G(s) = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies.

Sol: The closed loop Transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

$$\Rightarrow \frac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}} \Rightarrow \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

The characteristic Equation is $\Rightarrow (s+2)(s+4)(s^2+6s+25)+K$

$$\Rightarrow (s^2+6s+8)(s^2+6s+25)+K$$

$$\Rightarrow s^4+6s^3+25s^2+6s^3+36s^2+150s+8s^2+48s+200+K=0$$

$$\Rightarrow s^4+12s^3+69s^2+198s+200+K=0$$

The construction of routh array is shown below. The highest power of characteristic equation is even number. Hence from the first row of routh array using co-efficients of even powers of s and from the second row of routh array using co-efficients of odd powers of s.

s^4	1	69	200+K	→ Row-1
s^3	12	198		→ Row-2

Divide s^3 row by 12 to simplify the computation.

s^4	1	69	200+K
s^3	1	16.5	0
s^2	52.5	200+K	0
s^1	$\dfrac{666.25-K}{52.5}$	0	0
s^0	200+K	0	0

For the system to be stable there should not be any sign change in the elements of first column of routh array. Hence choose the value of k so that the first column elements are positive.

✓ from s^1 row, for the system to be stable $(666.25-K) > 0$

since $(666.25-K) > 0$, K should be less than 666.25.

✓ from s^0 row, for the system to be stable $(200+K) > 0$

since $(200+K) > 0$, K should be greater than -200, but in practice the value of K starts from 0. Hence K should be greater than "0"

∴ The range of K is $0 < K < 666.25$

when $K = 666.25$, the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate when the roots on imaginary axis and no roots on right half of s-plane.

when $K = 666.25$. The auxiliary Eqn. becomes

$$52.5s^2 + 200 + 666.25 = 0$$

$$52.5s^2 = -866.25$$

$$s^2 = -\frac{866.25}{52.5} \Rightarrow -16.5$$

$$s = \pm\sqrt{-16.5}$$

$$s = \pm j4.06$$

The frequency of oscillation, $\omega = 4.06$ rad/sec.

Result:
1) The range of K for the stability $0 < K < 666.25$
2) The system oscillates when $K = 666.25$
3) The frequency of oscillation $\omega = 4.06$ rad/sec (when $K = 666.2$

Ex: The open loop Transfer function of a unity feedback system is given by,

$G(s) = \frac{k(s+1)}{s^3+as^2+2s+1}$. Determine the value of k and a so that the system oscillates at a frequency of 2 rad/sec.

Sol: The closed loop Transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K(s+1)}{s^3+as^2+2s+1}}{1+\frac{K(s+1)}{s^3+as^2+2s+1}}$

$$\Rightarrow \frac{k(s+1)}{s^3+as^2+2s+1+ks+k}$$

The characteristic Equation is $s^3+as^2+2s+ks+k+1 = 0$

$$s^3+as^2+(2+k)s+(k+1)=0$$

The highest power of s is odd number. Hence the first row of routh array is formed by using co-efficient of odd power of "s" and second row is formed by using co-efficient of even power of "s".

s^3	1	$2+k$
s^2	a	$k+1$
s^1	$\frac{a(2+k)-(k+1)}{a}$	0
s^0	$k+1$	0

If the elements of s^1 row are all zeros then there exist an auxiliary equation. The roots of the auxiliary equation are purely imaginary and the system oscillates. Then the roots are lying on imaginary axis and the roots of auxiliary equation. The frequency of oscillation is a roots of auxiliary equation.

From s^2 row, auxiliary equation

$$as^2+k+1=0 \Rightarrow as^2 = -(k+1) \Rightarrow s = \pm j\sqrt{\frac{k+1}{a}}$$

given $s = \pm j2$ $\therefore$ $\cancel{\pm j}2 = \cancel{\pm j}\sqrt{\frac{k+1}{a}} \Rightarrow \frac{k+1}{a} = 4$

$$\boxed{k = 4a-1}$$

From s^1 row,

$$\frac{a(2+k)-(k+1)}{a} = 0 \Rightarrow a(2+k)-(k+1) = 0$$

$$\Rightarrow 2a + ka - k - 1 = 0$$

$$\therefore 2a - 1 + k(a-1) = 0.$$

Put $k = 4a-1$ in above Eqn.

$$2a-1 + (4a-1)(a-1) = 0$$

$$2a-1+4a^2-4a-a+1 = 0$$

$$4a^2-3a = 0$$

$$4a^{\cancel{2}} = 3\cancel{a}$$

$$4a = 3$$

$$\boxed{a = 3/4}$$

$$\therefore k = 4a-1$$

$$k = 4(3/4)-1 \Rightarrow \boxed{k=2}$$

Result: when the system oscillates at a frequency of 2 rad/sec

$$\boxed{k=2} \quad \boxed{a=3/4}$$

Ex: A feedback system has open loop T.F of $G(s) = \frac{ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of k for stability of closed loop system.

So, Generally control systems have very low bandwidth which implies that very low frequency range of operation. Hence for low frequency range the term e^{-s} can be replaced by $1-s$ $(\because e^{-sT} \simeq (1-sT))$

$$G(s) = \frac{ke^{-s}}{s(s^2+5s+9)} = \frac{k(1-s)}{s(s^2+5s+9)}$$

The closed loop T.F $\Rightarrow \frac{G(s)}{1+G(s)H(s)} \Rightarrow \frac{\frac{k(1-s)}{s(s^2+5s+9)}}{1+\frac{k(1-s)}{s(s^2+5s+9)}}$

$$\Rightarrow \frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

The Characteristic Equation is $s(s^2+5s+9)+K(1-s)=0$

$$s^3+5s^2+9s+K-Ks=0$$

$$s^3+5s^2+(9-K)s+K=0$$

The maximum power of Characteristic Equation is 3 (odd number). Hence the first row of routh array can be formed by using co-efficients of odd powers of s and the second row of routh array can be formed by using co-efficients of Even powers of s.

$$\begin{array}{c|cc} s^3 & 1 & (9-K) \\ s^2 & 5 & K \\ s^1 & \frac{5(9-K)-K}{5} & 0 \\ s^0 & K & 0 \end{array}$$

From s^1 row, for stability of the system, $\frac{5(9-K)-K}{5}>0$

If $(9-1.2K)>0$ then $1.2K<9$ $\therefore K<\frac{9}{1.2} = K<7.5$.

From s^0 row, for stability of the system, $K>0$

Finally we can conclude that for stability of the system K should be in the range of $0<K<7.5$

Result:

For stability of the system K should be in the range of

$$0<K<7.5.$$

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* Limitations of RH criterion :-

→ It is valid only if the characteristic equation is algebric.

→ If any co-efficient of characteristic equation is complex (or) contains power of "e" this criterion can not be applied.

→ It gives information about how many roots are lying in RHS of s-plane values of the roots are not available. Also it cannot distinguish between real & complex roots.

* Root Locus :-

The Root Locus technique in control system was introduced in the year of 1948 by "Evans". Any physical system is represented by a transfer function in the form of

$$\left[G(s) = K * \frac{\text{numerator of } s}{\text{denominator of } s} \right]$$

We can find the poles and zeros from G(s). The location of poles and zeros are crucial keeping view stability, transiant response and error Analysis. In root locus technique we will evaluate the position of roots, their locus of movement and associated information. These information is used to find system Perfomance.

Advantages :-

→ Root locus technique in control system is easy to implement as compared to other methods.

→ With the help of root locus we can easily predict the performance of whole system.

→ Root locus provides the better way to indicate the parameters.

Consider the open loop Transfer function of system $G(s) = \frac{K}{s(s+P_1)(s+P_2)}$

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s+P_1)(s+P_2)+K}.$$

The denominator polynomial of $C(s)/R(s)$ is the Char. Equation of the system.

The Char. Equation is given by,

$$s(s+P_1)(s+P_2)+K=0$$

The roots of characteristic equation is a function of open loop gain K. when the gain K is varied from "0 to ∞", the roots of char. equation will take different values.

When K=0, the roots are given by open loop poles.

When K→∞ the roots will takes the value open loop zeros.

Root loci:- The path taken by the roots of char. equation when open loop gain K is varied from 0 to ∞ are called root loci.

~~Construction rules for root locus~~

* What is Root-Locus ?

sol It is plotting of system's dynamic characteristics.

Ex If system with unity feedback is given by $G(s) = \frac{K}{s(s+2)}$

→ characteristic equation of above system is

$$1+G(s)H(s) = 0$$

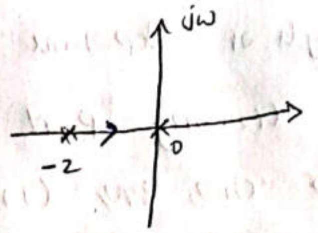
$$1+\frac{K}{s(s+2)}=0 \Rightarrow s^2+2s+K=0.$$

The position of poles are changing with respect to value of "K" case 1 :- If $K = 0$

$s^2 + 2s = 0$

$s(s+2) = 0$

$s = 0, \quad s = -2$

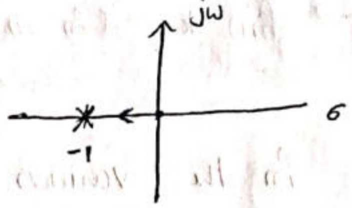


case 2 : If $K = 1$

$s^2 + 2s + 1 = 0$

$(s+1)^2 = 0$

$s = -1, -1$

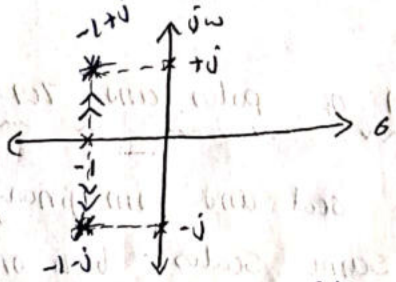


case 3 : If $K = 2$

$s^2 + 2s + 2 = 0$

$(s+1)^2 + 1 = 0$

$s = -1 + j, \quad -1 - j$



By changing the value of K, the position of poles will be change.

* **Rules of Root-Locus plot :-**

i) The root locus is always symmetric with respect to real axis.

ii) To identify total loci i.e, max (P, Z) in Transfer function [i.e, $P = 3$, $Z = 1$] $P > Z$

iii) Total number of asymptotes $\Rightarrow P - Z$

iv) Angle of asymptotes.

$$\theta = \frac{180^\circ (2x+1)}{P - Z}, \quad x = 0, 1, 2 \ldots \text{(No. of asymptotes)}$$

v) calculation of centroid of asymptotes.

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{No. of poles} - \text{No. of zeros}}$$

i.e, $(P - Z)$

vi) calculation of break away point

identify → char. Equation $(1 + G(s)H(s) = 0)$

→ compute K = polynominal,

→ compute $dK/ds = 0$ ⇒ s = breakaway point (roots of dK/ds)

vii) To find Angle of departure and Angle of arrival.

viii) To find the Crossing point of imaginary axis

→ Char. eqn. $(1+G(s)H(s)=0)$

→ Replace s by $j\omega$

→ find k (for marginally stable system).

* Explanation for the various steps in procedure for constructing root locus:-

Step 1: Location of poles and zeros:-

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real axis and imaginary axis.

The poles are marked by cross "x" and zeros are marked as small circles "o".

The number of roots locus branches are equal to number of poles of open loop transfer function.

** The starting point of root locus is at poles and the end is at zero

Let P = number of poles

Z = number of zeros.

~~Identifying the steps to test points by relative magnitude~~

Step 2:- Root locus on Real axis:-

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of the test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3: Angle of asymptotes and centroid :-

If n is number of poles and m is number of zeros, then n-m root locus branches will terminate at zeros at infinity.

These n-m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. Hence the number of asymptotes is equal to number of root locus branches going to infinity. The angle of asymptotes and the centroid are given by the following formulae.

$$\text{Angle of asymptotes} = \frac{\pm 180(2q+1)}{n-m} \qquad q = 0,1,2 \ldots (n-m)$$

$$\text{centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}.$$

Step 4: Breakaway and Breakin points :-

The Breakaway and breakin points either lie on real axis (or) exist as complex conjugate pairs. If there is a root locus on real axis between the 2 poles then there exist a "breakaway point." If there is a root locus on real axis between 2 zeros then there exists a "breakin point." If there is root locus on real axis between pole and zero then there may be (or) may not be breakaway (or) breakin point.

By using characteristic equation find "k"

The breakaway and breakin point is given by roots of equation $dk/ds = 0$. The roots of $dk/ds = 0$ are actual breakaway (or) breakin point., the gain k should be real and positive.

Step 5: Angle of departure and angle of arrival :-

Since there is no complex poles or zeros, we need not find the angle of departure and angle of arrival.

If it has complex poles (or) zeros, we need to find both by following way.

i)

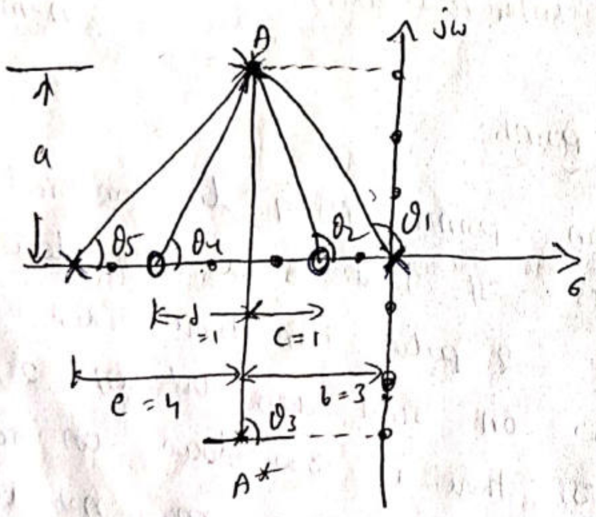
Angle of departure (from a complex pole A) = 180° − (sum of angles of vectors to the complex pole A from other poles) + (sum of angles of vectors to complex pole A from zeros)

$$\Rightarrow 180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

Example :- consider the two complex poles A and A^* shown in fig.

Angle of departure at pole A } = $180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$

Angle of departure at pole A^* } = − (Angle of departure at pole A)



$\theta_1 = 180^\circ - \tan^{-1}(a/b)$

$\theta_2 = 180^\circ - \tan^{-1}(a/c)$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}(a/d)$

$\theta_5 = \tan^{-1}(a/e)$

calculation of angle of departure.

ii)

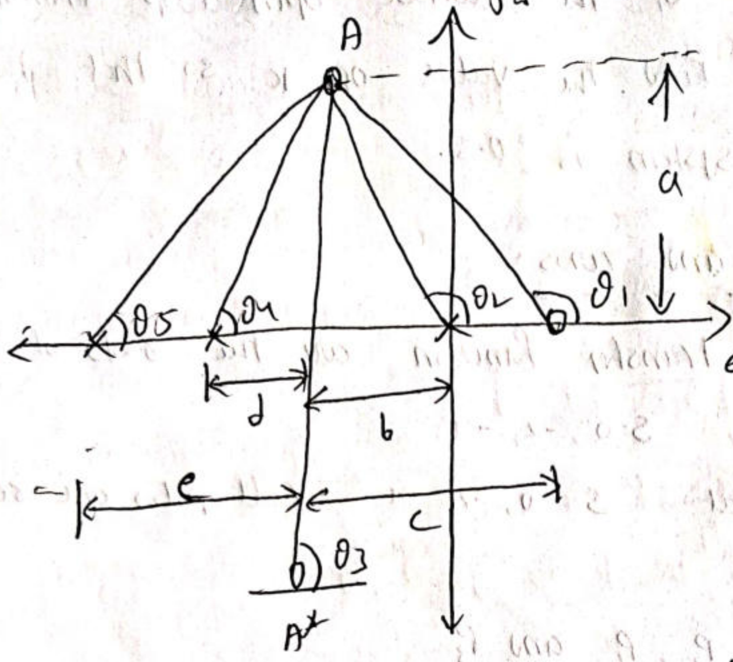
Angle of arrival at a complex zero A } = 180° − (sum of angles of vectors to the complex zero A from other zeros) + (sum of angles of vectors to the complex zero A from poles)

$$\Rightarrow 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

Example: consider the two complex ~~poles~~ zeros A and A^* shown in fig.

Angle of arrival at zero A } = $180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$

Angle of arrival at zero A^* } = − (Angle of arrival at zero A).



$\theta_1 = 180^\circ - \tan^{-1}(a/c)$

$\theta_2 = 180^\circ - \tan^{-1}(a/b)$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}(a/d)$

$\theta_5 = \tan^{-1}(a/e)$.

Calculation of Angle of arrival.

Step 6: point of intersection of root locus with imaginary axis:

letting $s = j\omega$ in the characteristic Equation and seperate the real part and imaginary part. Two Equations are obtained. One by Equating real part to zero and the other by Equating imaginary part to zero. Solve the two Equations for ω and K.

The values of ω gives the points where the root locus crosses imaginary axis. The value of K gives the value of gain K at the crossing points. Also this value of K is the limiting value K for the stability of the system.

Ex: Sketch the root locus of the whose open loop transfer function is $G(s) = \frac{K}{s(s+2)(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

Sol: Step 1: To Locate poles and zeros :-

The poles of open loop transfer function are the roots of equation $s(s+2)(s+4) = 0$. i.e, $s = 0, -2, -4$

∴ The poles are lying at $s = 0, -2, -4$ (all poles are real)

There is no zeros

Let us denote the poles P_1, P_2 and P_3

Here $P_1 = 0$ $P_2 = -2$, $P_3 = -4$ as shown in fig.

The poles are marked by cross (X)

Step 2: To find the root locus on real axis :-

There are three poles on the real axis.

- Choose a test point on real axis between $s = 0$ and $s = -2$ (consider $s = -1$). To the right of this point the total number of real poles and zero is "one" between $s = 0$ and $s = -2$ which is an odd number. Hence the real axis will be a part of root locus.
- Choose a test point on real axis between $s = -2$ and $s = -4$ (consider $s = -3$). To the right of this point the total number of real poles are zeros are "two" between $s = -2$, and $s = -4$ which is an even number. Hence the real axis will be not part of root locus.
- Choose a test point on real axis to be left of $s = -4$ ($s = -5, -6 ---$). To the right of this point, the total number of poles and zeros are "3" which is an odd number. Hence the entire negative axis from $s = -4$ to $-\infty$ will be part of root locus.

The root locus on the real axis as shown in graph.

Step 3: To find Angle of asymptotes and centroid :-

The number of asymptotes required are three

$$\text{Angle of Asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \qquad q = 0, 1, 2 \ldots n-m.$$

$q = 0, 1, 2, 3.$

$n-m \Rightarrow$ poles - zeros
$\Rightarrow 3-0 \Rightarrow 3$

$q = 0 \Rightarrow \frac{\pm 180^\circ (1)}{3} \Rightarrow \pm 60^\circ$

$q = 1 \Rightarrow \frac{\pm 180^\circ (2+1)}{3} \Rightarrow \pm 180^\circ$

$q = 2 \Rightarrow \pm \frac{180^\circ (5)}{3} \Rightarrow \pm 300^\circ = \mp 60^\circ$

$q = 3 \Rightarrow \pm \frac{180^\circ (7)}{3} \Rightarrow \pm 420^\circ = \pm 60^\circ$

Note: It is Enough if you calculate the required number of Angles. Here it is given by first three values of angles. The remaining values will be repetition of the previous values.

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} \Rightarrow \frac{0-2-4-0}{3} \Rightarrow \frac{-6}{3} \Rightarrow -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using protractor. The asymptotes are drawn by dotted lines as shown in fig.

Step 4: To find breakaway and break in point :-

The closed loop Transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

The char. Equ. is $1 + G(s)H(s) = 0$

$1 + \frac{K}{s(s+2)(s+4)} = 0 \Rightarrow s(s+2)(s+4) + K = 0$

$\Rightarrow s(s^2+6s+8) + K = 0$

$\Rightarrow s^3 + 6s^2 + 8s + K = 0$

$\therefore K = -(s^3 + 6s^2 + 8s)$

On differentiating the value of K with respect to s we get

$\frac{dK}{ds} = -(3s^2 + 12s + 8)$ Put $\frac{dK}{ds} = 0$

$$-(3s^2+12s+8)=0$$

$$3s^2+12s+8=0$$

$$s = \frac{-12 \pm \sqrt{144-96}}{6} \Rightarrow \frac{-12 \pm \sqrt{48}}{6} \Rightarrow \frac{-12 \pm 6.92}{6}$$

$$s = -0.846 \quad \text{(or)} \quad -3.153$$

$$[s=-0.846] \quad [s=-3.153]$$

Check for K: (should be positive and real).

When $s=-0.846$, the value of K is given by.

$$K = -((-0.846)^3+6(-0.846)^2+8(-0.846)) \Rightarrow -(-0.605+4.294-6.768)$$

$$\Rightarrow 3.079 \quad (+ve \text{ and real}). \checkmark$$

When $s=-3.153$, the value of K is given by

$$K = -((-3.153)^3+6(-3.153)^2+8(-3.153)) \Rightarrow -(-31.345+59.648-25.224)$$

$$\Rightarrow -3.079 \quad (-ve \text{ and real}) \times$$

Since K is negative for $s=3.153$ this is not a actual breakaway point.
The breakaway point is marked on the negative real axis as shown in graph.

Step 5: To find angle of departure and angle of arrival:

Since there are no complex pole (or) zero, no need to find angle of departure (or) Angle of arrival.

Step 6: To find point of intersection (crossing point) of root locus on imaginary axis:

The char. Equation is given by

$$s^3+6s^2+8s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+8(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+j8\omega+K=0$$

Equating imaginary parts

$$-j\omega^3+j8\omega=0$$

Equating real parts

$$-6\omega^2+K=0$$

$-\cancel{6}\omega^3 = -\cancel{6}8\omega$ $\quad\quad$ $+6\omega^2 = +K$

$\omega^2 = 8$ $\quad\quad$ $K = 6\omega^2$

$\omega = \pm\sqrt{8}$ $\quad\quad$ $K = 6(8)$

$\omega = \pm 2.8$ $\quad\quad$ $K = 48$.

$\omega = +j2.8, \ -j2.8$.

The crossing point (intersection point) of root locus is $\pm j2.8$. The value of K corresponding to this point is $K = 48$ (This is the limiting value of K for the stability of the system).

Step 7: To find the value of K corresponding to $\xi = 0.5$:-

given that $\xi = 0.5$

let $\alpha = \cos^{-1}\xi = \cos^{-1}(0.5) = 60°$

Draw a line OP, such that the angle between OP and negative real axis is 60° ($\alpha = 60°$). Shown in graph. The meeting point of OP and root locus gives the dominant pole s_d.

let K_{sd} be value of K corresponding to the point $s = s_d$.

$$K_{sd} = \frac{\text{product of length of vector from all poles to the point } s = s_d}{\text{product of length of vector from all zeros to the point } s = s_d}$$

(∵ Three poles and no zeros)

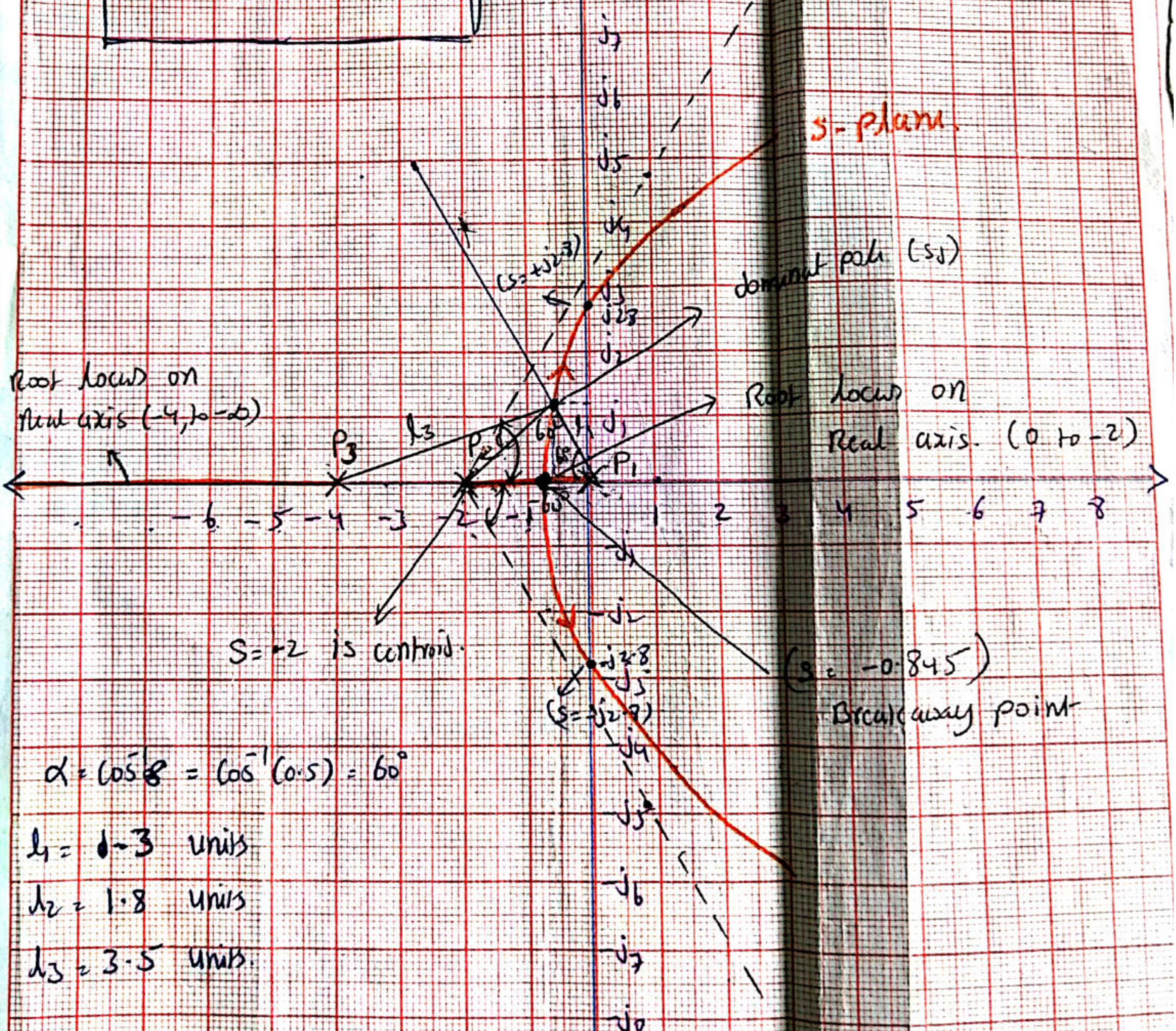
$$\Rightarrow \frac{l_1 \times l_2 \times l_3}{1}$$

$$\Rightarrow 1.3 \times 1.8 \times 3.5$$

$$\simeq 8.1$$

Scale : 1 cm = 1 unit

$$\boxed{G(s) = \frac{k}{s(s+2)(s+4)}}$$



$\alpha = \cos^{-1}\xi = \cos^{-1}(0.5) = 60^\circ$

$l_1 = 1\cdot3$ units

$l_2 = 1\cdot8$ units

$l_3 = 3\cdot5$ units

$K = l_1 \times l_2 \times l_3$

$\Rightarrow 1\cdot3 \times 1\cdot8 \times 3\cdot5 = 8$

$$G(s) = \frac{k}{s(s+2)(s+5)}$$

2) A unity feedback system has open loop transfer function $G(s) = \frac{k}{s(s^2+4s+13)}$. sketch the root locus.

Sol) 1) To locate poles and zeros :-

The poles of open loop Transfer function are the roots of

$s(s^2+4s+13) = 0$

The poles are $s_1 = 0$, $s_2 = s_3 \Rightarrow (s^2+4s+13)$

$s_2, s_3 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \frac{-4 \pm \sqrt{16 - 4 \times 13}}{2} \Rightarrow \frac{-4 \pm j6}{2}$

$\Rightarrow -2 \pm j3$

∴ The poles are $P_1 = 0$ $P_2 = -2+j3$ $P_3 = -2-j3$

The zeros are $= 0$

The poles are marked by "x" as shown in graph.

2) To find root locus on real axis :-

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then the right side of that point the total number of poles and zeros are one, which is an odd number. Hence the entire negative real axis will be a part of root locus. The root locus on real axis is shown in graph by bold line.

3) To find Angle of Asymptotes and centroid :-

$$\text{Angle of Asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m}$$

here n = No. of poles $= 3$ $q = 0, 1, 2 \ldots n-m$

m = No. of zeros. $= 0$ $\therefore q = 0, 1, 2, 3.$

when $q=0$ Angles $= \frac{\pm 180^\circ(0+1)}{3} = \pm 60^\circ$

when $q=1$ Angles $= \pm \frac{180^\circ(3)}{3} = \pm 180^\circ$

when $q=2$ Angles $= \pm \frac{180° \times 5}{3} = \pm 300° = \mp 60°$

when $q=3$ Angles $= \pm \frac{180° \times 7}{3} = \pm 420° = \pm 60°$

Note: It is enough if you calculate the required number of angles. Here it is given by first three values of Angles. The remaining values will be repetitions of the previous values.

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$\Rightarrow \frac{0 - 2 + \cancel{j3} - 2 - \cancel{j3} - 0}{3} \Rightarrow \frac{-4}{3} \Rightarrow -1.33$$

∴ The centroid point is -1.33 ~~marked~~ pointed on root locus and from this point draw Angle of Asymptotes by dotted lines as shown in graph.

3) To find breakaway and break in point:-

The closed loop T.F of $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s^2+4s+13)+K}$.

The characteristic Equation is: $s(s^2+4s+13)+K=0$

$$s^3+4s^2+13s+K=0$$

$$K = -(s^3+4s^2+13s) \rightarrow (1)$$

differentiate Equation (1) with respect to s. and Equating to zero

$$\frac{dK}{ds} = -(3s^2+8s+13) = 0$$

$$3s^2+8s+13=0$$

$$\therefore s = \frac{-8 \pm \sqrt{8^2-4 \times 13 \times 3}}{6} = -1.33 \pm j1.6$$

$$s_1 = -1.33+j1.6, \quad s_2 = -1.33-j1.6$$

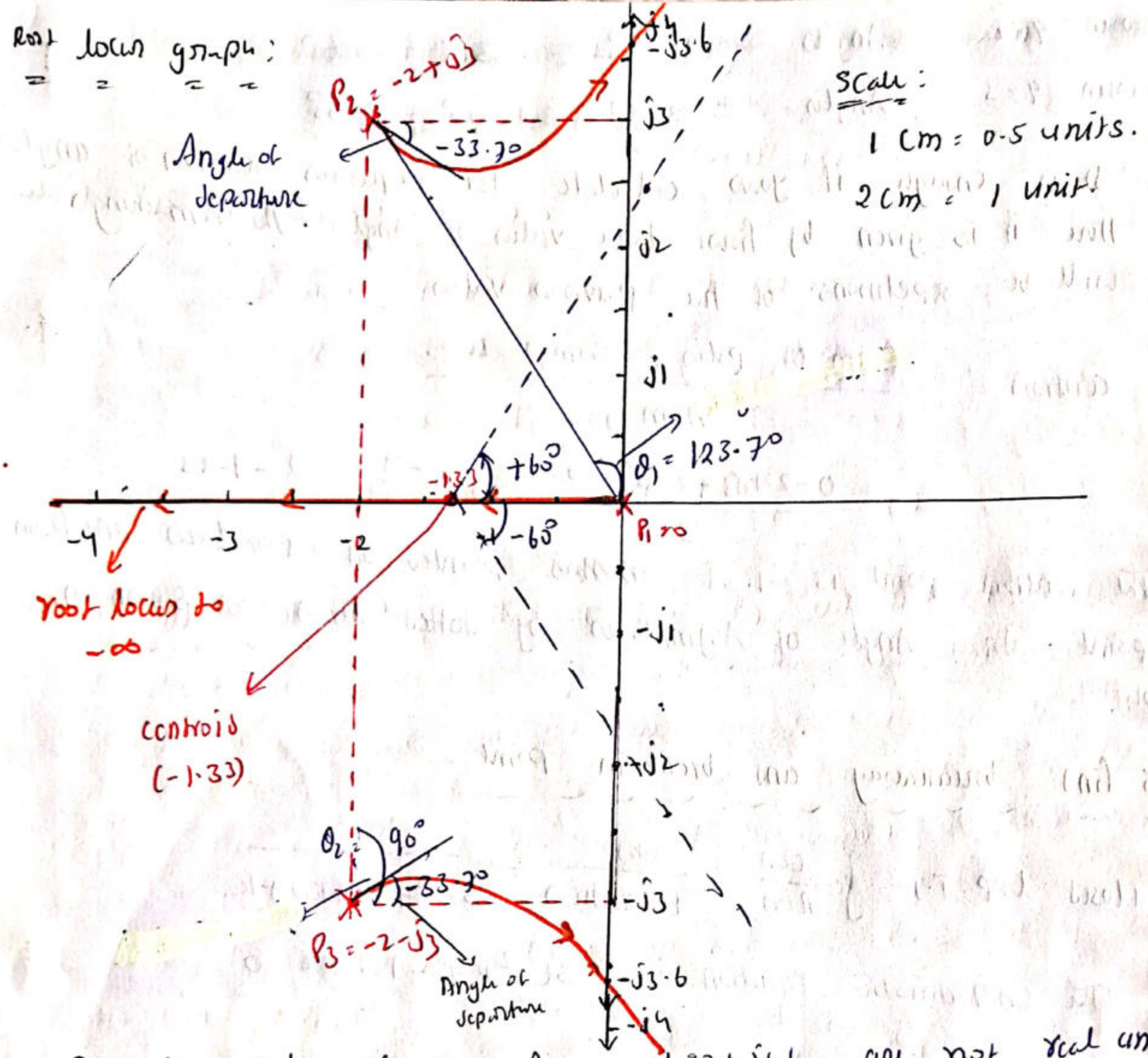
Check for K:- when $s=-1.33+j1.6$ sub. in Eqn. (1)

$K = (-(s^3+4s^2+13s) \Rightarrow -((-1.33+j1.6)^3+4(-1.33+j1.6)^2+13(-1.33+j1.6))$

$\neq$ real and positive.

Also it can be shown that when $s=-1.33-j1.6$ the value of K is not real and positive.

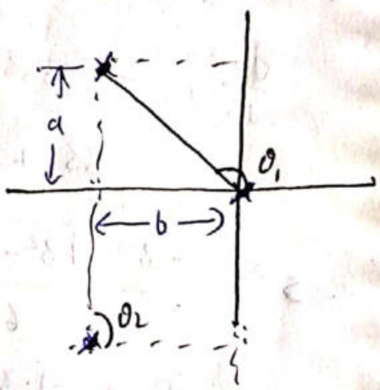
Root locus graph:



Since the values of K for $s = -1.33 \pm j1.6$ are not real and positive are not actual break away and breakin point. The root locus has neither breakaway nor breakin point.

(4) ~~Angle/at~~ To find Angle of departure:-

Since there is no Angle of arrival because no pole and No zero located on ~~left~~ right half of real axis of S-plane.



Now we have to find Angle of departure because the poles are complex conjugate.

$a = 3, \ b = 2$

$\theta_1 = 180^\circ - \tan^{-1}(a/b)$

$= 180^\circ - \tan^{-1}(3/2) \Rightarrow 123.7^\circ$

$\theta_2 = 90^\circ$

Angle of departure (from a complex pole P_2) $= 180^\circ - \begin{bmatrix}\text{sum of Angles of vectors of} \\ \text{the complex pole A from} \\ \text{other poles}\end{bmatrix} + \begin{bmatrix}\text{sum of Angles of vectors} \\ \text{of complex pole A from} \\ \text{all zeros}\end{bmatrix}$

$$\Rightarrow 180^\circ - (\theta_1 + \theta_2) \Rightarrow 180^\circ - (123.7^\circ + 90^\circ)$$

$$\Rightarrow -33.7^\circ$$

Angle of departure (from a complex pole P_3) $=$ $-$ (Angle of departure from complex poles P_2)

$$\Rightarrow -(-33.7^\circ)$$

$$\Rightarrow 33.7^\circ.$$

Mark the Angles of departure at complex poles using protractor.

5) To find the crossing point on imaginary axis

The char. Equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0$$

$$-j\omega^3 - 4\omega^2 + j13\omega + K = 0$$

separating real terms and imaginary terms

$$-j\omega^3 + j13\omega = 0$$

$$-\cancel{j}\omega^{\cancel{3}} = -\cancel{j}13\cancel{\omega}$$

$$\omega^2 = 13$$

$$\omega = \pm\sqrt{13} \Rightarrow \pm 3.6$$

$$-4\omega^2 + K = 0$$

$$+4\omega^2 = +K$$

$$K = 4 \times 13$$

$$K = 52$$

The crossing point of root locus is $\pm j3.6$. The value of K at this corresponding point is $K = 52$. (This is the limiting value of K for the stability of the system)

The root locus has three branches one branch starts at the pole at origin and travels through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles, ~~crosses~~ crosses the imaginary axis at $\pm j3.6$ and travels parallel to asymptotes to meet at zero at infinity.

3) Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$.

Sol: 1) To locate poles and zeros :-

It has 4 poles. and no zeros.

$n=4 \quad m=0 \quad n-m=4$

$P_1 = 0 \quad P_2 = -4 \quad , \quad P_3 = -2+j4 \quad P_4 = -2-j4.$

$P_3, P_4 = \frac{-4 \pm \sqrt{16-80}}{2} \Rightarrow \frac{-4 \pm j8}{2} \Rightarrow -2 \pm j4$

2) To find root locus :- There are two poles on real axis. Choose a test point between 0 and -4. To right side of this test point, the total no. of poles and zeros is one which is odd number. Hence the real axis between 0 and -4 will be a part of root locus. Choose a test point to the left side of -4, now to the right side of this test point the number of poles and zeros are two, it is even number. Hence the real axis between -4 to -∞ will not be a part of root locus. The root locus on real axis is shown by bold line as shown in graph.

3) To find Angle of Asymptotes and centroid :-

$$\text{Angle of Asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m} \qquad q = 0, 2 \cdots n-m$$
$$q = 0, 1, 2, 3, 4,$$

$n-m=4$

$q=0, \quad \frac{\pm 180^\circ(1)}{4} \Rightarrow \pm 45^\circ, \quad q=2, \quad \pm \frac{180^\circ(5)}{4} = \pm 225^\circ$

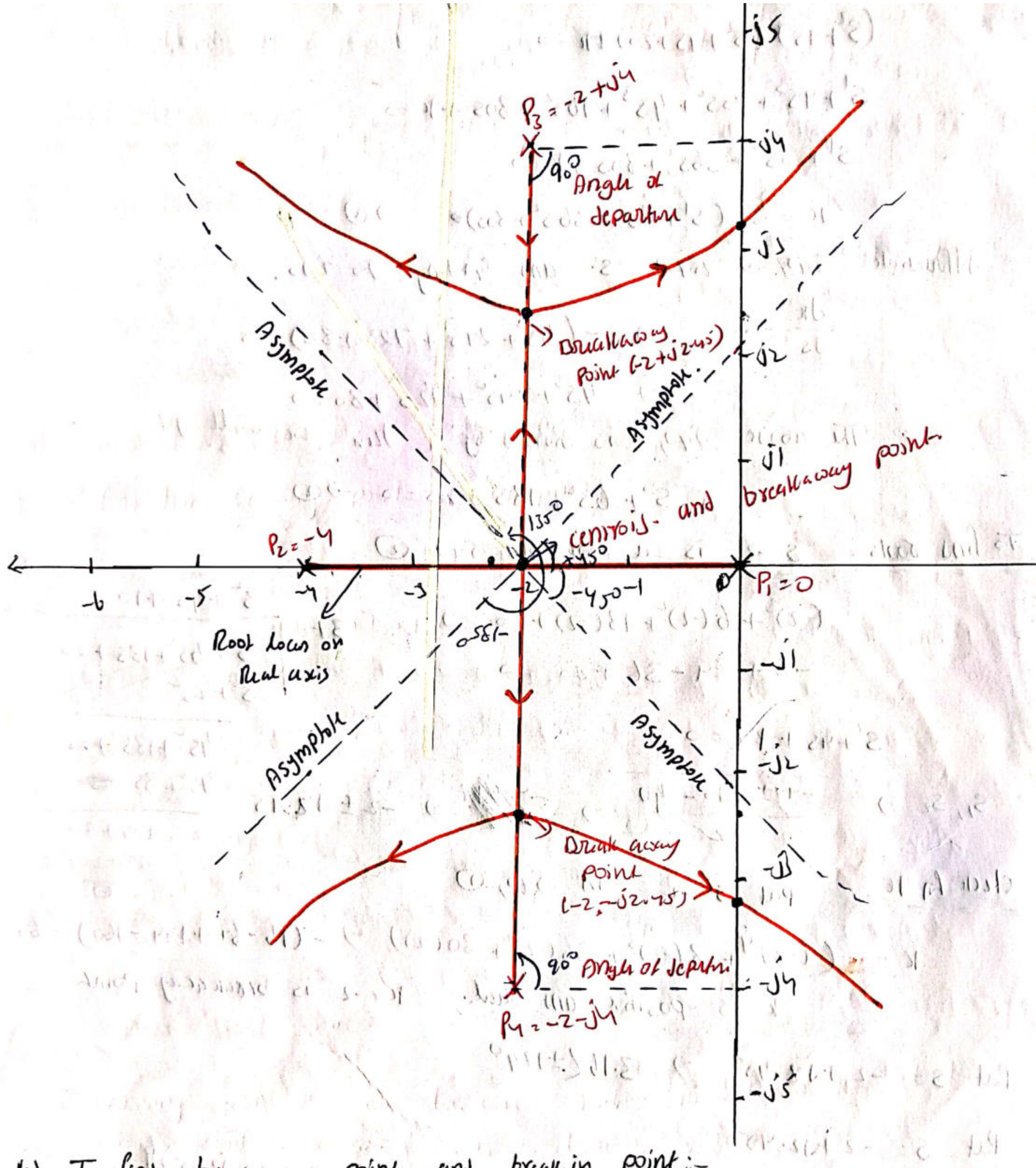
$q=1, \quad \pm \frac{180^\circ(3)}{4} \Rightarrow \pm 135^\circ,$

first 4 Angles are enough.

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$\Rightarrow \frac{0-4-2+\cancel{j4}-2-\cancel{j4}-0}{4} \Rightarrow \frac{-8}{4} \Rightarrow -2$$

centroid = -2



4) To find breakaway point and break in point :-

The closed loop Transfer function $= \frac{G(s)}{1+G(s)H(s)}$

The char. Equation is $1+G(s)H(s) = 0$

$$1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

$$s(s+4)(s^2+4s+20) + K = 0$$

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$s(s+4)(s^2+4s+20)+K=0$

$(s^2+4s)(s^2+4s+20)+K=0$

$s^4+4s^3+20s^2+4s^3+16s^2+80s+K=0$

$s^4+8s^3+36s^2+80s+K=0$

$K=-(s^4+8s^3+36s^2+80s) \rightarrow ①$

differentiate eqn. ① w.r.t s and equating to zero.

$\frac{dK}{ds}=0 \Rightarrow -(4s^3+24s^2+72s+80)=0$

$\Rightarrow 4s^3+24s^2+72s+80=0$

The above eqn. is divided by '4' then we will get

$s^3+6s^2+18s+20=0 \rightarrow ②$

To find roots : $s=-2$ is a root of eqn. ②

$(-2)^3+6(-2)^2+18(-2)+20=0$

$-8+24-36+20=0$

$$
\begin{array}{r|l}
 & s^2+4s+10 \\
\hline
s+2 & s^3+6s^2+18s+20 \\
 & s^3+2s^2 \\
\hline
 & 4s^2+18s+20 \\
 & 4s^2+8s \\
\hline
 & 10s+20 \\
 & 10s+20 \\
\hline
 & 0
\end{array}
$$

$s^2+4s+10=0$

$s_1, s_2 \Rightarrow \frac{-4 \pm \sqrt{16-40}}{2} \Rightarrow \Rightarrow -2 \pm j2.45$

Check for K :- Put $s=-2$ in eqn. ①

$K=-((-2)^4+8(-2)^3+36(-2)^2+80(-2)) \Rightarrow -(16-64+144-160)=64$

K is positive and real. $K=-2$ is breakaway point

Put $s=-2 \pm j2.45 \Rightarrow 3.16 \angle \pm 129^\circ$

Put $s=-2+j2.45$:-

$K=-\{(-2+j2.45)^4+8(-2+j2.45)^3+36(-2+j2.45)^2+80(-2+j2.45)\}$

$\Rightarrow -\{-92+224.12+72.09-160\} \Rightarrow 100$ $K=100$ +ve real.

Put $s=-2-j2.45$:-

$K=-\{(-2-j2.45)^4+8(-2-j2.45)^3+36(-2-j2.45)^2+80(-2-j2.45)\}$

$\Rightarrow -\{-92.03+224.12-72.09-160\} \Rightarrow -\{-100\} \Rightarrow 100$, +ve real

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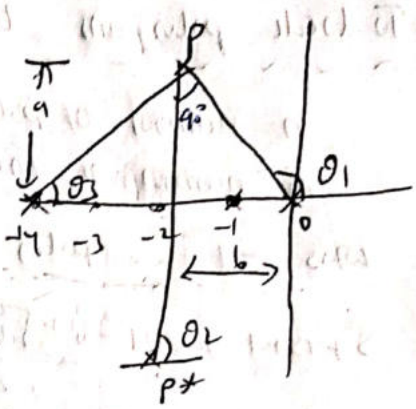
All values of S when subs in K then we get +ve real values. Hence

-2, $-2+j2.45$, $-2-j2.45$ are breakaway points.

5) Angle of departure :-



$\theta_1 = 180° - \tan^{-1}(4/6)$

$\Rightarrow 180° - \tan^{-1}(4/2) \Rightarrow 116.565°$

$\theta_2 = 90°$

$\theta_3 = \cancel{180° - \tan^{-1}}(2) \Rightarrow \tan^{-1}(4/2) \Rightarrow 63.435°$

Angle of departure (from point P) $= 180° - (116.565° + 90° + 63.435°)$

$\Rightarrow \cancel{90°} = -90°$

Angle of departure (from point P*) $= -(-90°) \Rightarrow 90°$.

6) To find crossing point on imaginary axis :-

The char. Equ. is $s^4 + 8s^3 + 36s^2 + 80s + K = 0$

Put $s = j\omega \Rightarrow (j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + K = 0$

$\omega^4 - j8\omega^3 - 36\omega^2 + j80\omega + K = 0$

On equating imaginary parts to zero

$-j8\omega^3 + j80\omega = 0$

$-j8\omega^3 = -j80\omega$

$8\omega^2 = 80$

$\omega^2 = 10$

$\omega = \sqrt{10} \Rightarrow \pm 3.2$

On equating real parts to zero

$\omega^4 - 36\omega^2 + K = 0$

$\cancel{K = -\omega^4 + 36\omega^2}$

$\cancel{K = -(\omega^2)^2 + 36\omega^2}$

$100 - 360 + K = 0$

$K = 260$

The crossing point of root locus is $\pm j3.2$. The value of K at this crossing point is $K = 260$ (This is the limiting value of K for stability)

Q) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$. Sketch the root locus of the system.

1) To locate poles and zeros :-

n = number of poles

m = number of zeros

zeros = -9, poles = $0, (s^2+4s+1)$

$s^2+4s+1 \Rightarrow \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \frac{-4 \pm \sqrt{16-4(11)}}{2} \Rightarrow \cancel{\frac{-4 \pm \sqrt{12}}{2}}$

$\Rightarrow \frac{-4 \pm \sqrt{16-44}}{2} \Rightarrow -2 \pm j2.64$

zeros $\Rightarrow Z_1 = -9$

poles $\Rightarrow P_1 = 0, \quad P_2 = -2+j2.64, \quad P_3 = -2-j2.64$

2) To find Root locus :-

Choose a Test point between 0 and -9 and when we observe Right side of the test point it has only one pole, which is odd number. Hence the real axis between 0 and -9 is a part of Root locus.

Choose a Test point left to point -9 and when we observe Right side of the test point it has one pole and one zero total two, which is even number. Hence the real axis from -9 to $-\infty$ is not part of root locus.

3) To find Angle of Asymptotes and Centroid :-

Angle of Asymptotes $= \frac{\pm 180°(2q+1)}{n-m}$ $\quad q = 0,1,2 \ldots n-m$

$q = 0,1,2$

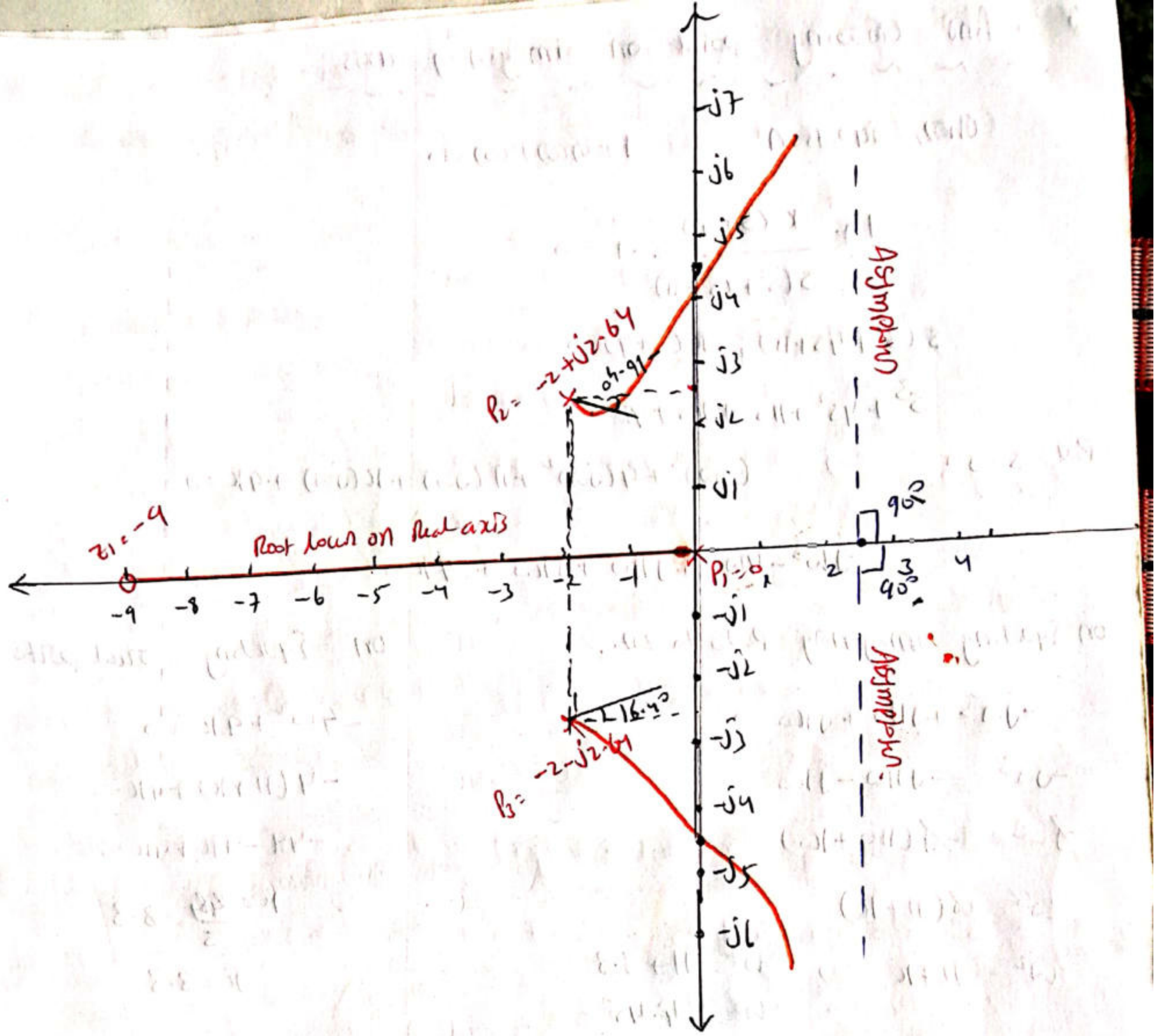
where $n-m \Rightarrow 3-1 \Rightarrow 2$

$q=0 \Rightarrow \frac{\pm 180°}{2} \Rightarrow \pm 90°$

$q=1 \Rightarrow \frac{\pm 180°(3)}{2} \Rightarrow \pm 270° = \mp 90°$ gets Change

Centroid $\Rightarrow \frac{\text{Sum of poles} - \text{sum of zeros}}{n-m} \Rightarrow \frac{0-2+\cancel{j2.64}-2-\cancel{j2.64}-(-9)}{2}$

$\Rightarrow \frac{-4+9}{2} \Rightarrow 5/2 \Rightarrow 2.5$

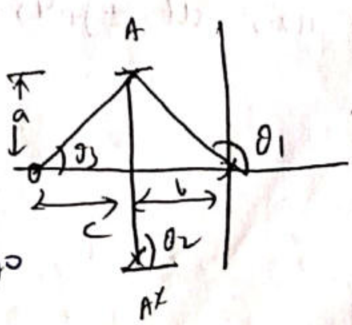


4) To find the breakaway and breakin point:-
From the location of poles and zeros and from the knowledge of typical sketches of root locus. It can be concluded that is no possibility of breakaway (or) breakin point.

5) Angle of arrival and Angle departure:-

$\theta_1 = 180° - \tan^{-1}(a/b) \Rightarrow 180° - \tan^{-1}\left(\frac{2.64}{2}\right)$

$\Rightarrow 127.1°$



$\theta_2 = 90°$, $\theta_3 = \tan^{-1}(a/c) \Rightarrow \tan^{-1}\left(\frac{2.64}{7}\right) = 20.7°$

Angle of departure from point A $= 180° - (\theta_1 + \theta_2) + \theta_3 \Rightarrow 180° - (127.1° + 90°) + 20.7° \Rightarrow -16.4°$

Angle of departure from point A* $= 16.4°$

6) To find crossing point on imaginary axis :-

Charr. Equation $\quad 1+G(s)H(s)=0$

$$1+\frac{K(s+9)}{s(s^2+4s+11)}\cdot 1=0$$

$$s(s^2+4s+11)+K(s+9)=0$$

$$s^3+4s^2+11s+Ks+9K=0$$

Put $s=j\omega$ $\Rightarrow$ $(j\omega)^3+4(j\omega)^2+11(j\omega)+K(j\omega)+9K=0$

$$-j\omega^3-4\omega^2+j11\omega+jK\omega+9K=0$$

on Equating imaginary parts to zero.

$$-j\omega^3+j11\omega+jK\omega=0$$

$$-j\omega^3=-j11\omega-jK\omega$$

$$\cancel{-j}\omega^3=\cancel{-j}(11\omega+K\omega)$$

$$\omega^{\cancel{3}}=\cancel{\omega}(11+K)$$

$$\omega^2=11+K \Rightarrow \omega^2=11+8.8$$

$$\omega=\pm 4.45$$

on Equating real parts to zero.

$$-4\omega^2+9K=0$$

$$-4(11+K)+9K=0$$

$$-44-4K+9K=0$$

$$K=\frac{44}{5}=8.8$$

$$K=8.8$$

The crossing point of root locus is $\pm j4.45$. The value of K at this crossing point is $K=8.8$ (This is limiting value of K for the stability of the system)

The root locus has three branches. one branch is start from origin and travel through negative real axis and meet at $s=-9$

The other two root locus branches starts at complex poles crosses the imaginary axis at $\pm j4.45$. and travels parallel to asymptotes to meet zero at infinity.

Ex: Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s)H(s) = \dfrac{K(s+1.5)}{s(s+1)(s+5)}$.

Sol) 1) To locate poles and zeros:

number of poles (n) = 3 $\qquad n-m = 3-1 = 2$.

number of zeros (m) = 1

poles => $P_1 = 0$, $P_2 = -1$, $P_3 = -5$

zeros => $Z_1 = -1.5$

2) To find root locus on real axis:-

Choose test point between 0 and -1 and we observe right side to the test point the number of poles and zeros is one which is odd number. Hence the real axis between 0 and -1 is a part of root locus.

Choose the test point between -1 and -1.5 and observe right side to the test point the number of poles and zeros are two which is even number. Hence the real axis between -1 and -1.5 is not part of root locus.

Choose the test point between -1.5 and -5 and observe right side to the test point the number of poles and zeros are three which is odd number. Hence the real axis between -1.5 and -5 is part of root locus.

3) To find Angle of Asymptotes and centroid:-

Angle of Asymptotes => $\dfrac{\pm 180^\circ(2q+1)}{n-m}$ $\qquad q = 0, 1, 2$

$q=0 \Rightarrow \dfrac{\pm 180^\circ}{2} \Rightarrow \pm 90^\circ$

$q=1 \Rightarrow \pm \dfrac{\cancel{180}^{90}{}^\circ(3)}{\cancel{2}} \Rightarrow \pm 270^\circ \Rightarrow \mp 90^\circ$

centroid => $\dfrac{0-1-5+1.5}{2} \Rightarrow -2.25$

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4) To find Breakaway and Breakin points

Chas. Equation => $1+G(s)H(s)=0$

$$1+\frac{K(s+1.5)}{s(s+1)(s+5)}\cdot 1=0$$

$$s(s+1)(s+5)+K(s+1.5)=0$$

$$(s^2+s)(s+5)+K(s+1.5)=0$$

$$s^3+5s^2+s^2+5s+K(s+1.5)=0$$

$$K(s+1.5)=-(s^3+6s^2+5s) \Rightarrow K=\frac{-(s^3+6s^2+5s)}{(s+1.5)}=\frac{u}{v} \rightarrow ①$$

differentiate equ. ① with respect to s and equating to zero

$$\frac{dK}{ds}=\frac{-(3s^2+12s+5)(s+1.5)-[-(s^3+6s^2+5s)(1)]}{(s+1.5)^2}=0 \qquad \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}$$

$$-(3s^3+12s^2+5s+4.5s^2+18s+7.5)+s^3+6s^2+5s=0$$

$$\Rightarrow -3s^3-12s^2-5s-4.5s^2-18s-7.5+s^3+6s^2+5s=0$$

$$\Rightarrow -2s^3-10.5s^2-18s-7.5=0$$

$$\Rightarrow -2(s^3+5.25s^2+9s+3.75)=0$$

$$\Rightarrow s^3+5.25s^2+9s+3.75=0 \rightarrow ②$$

The above Equation has three roots.

· To find the real roots of $s^3+5.25s^2+9s+3.75=0$ by Lin's method.

— Last two terms of the polynomial are choosen as 1st trial divisor

Trail - I :

$9s+3.75=0 \Rightarrow s+\frac{3.75}{9}=0$

$\Rightarrow s+0.42=0$

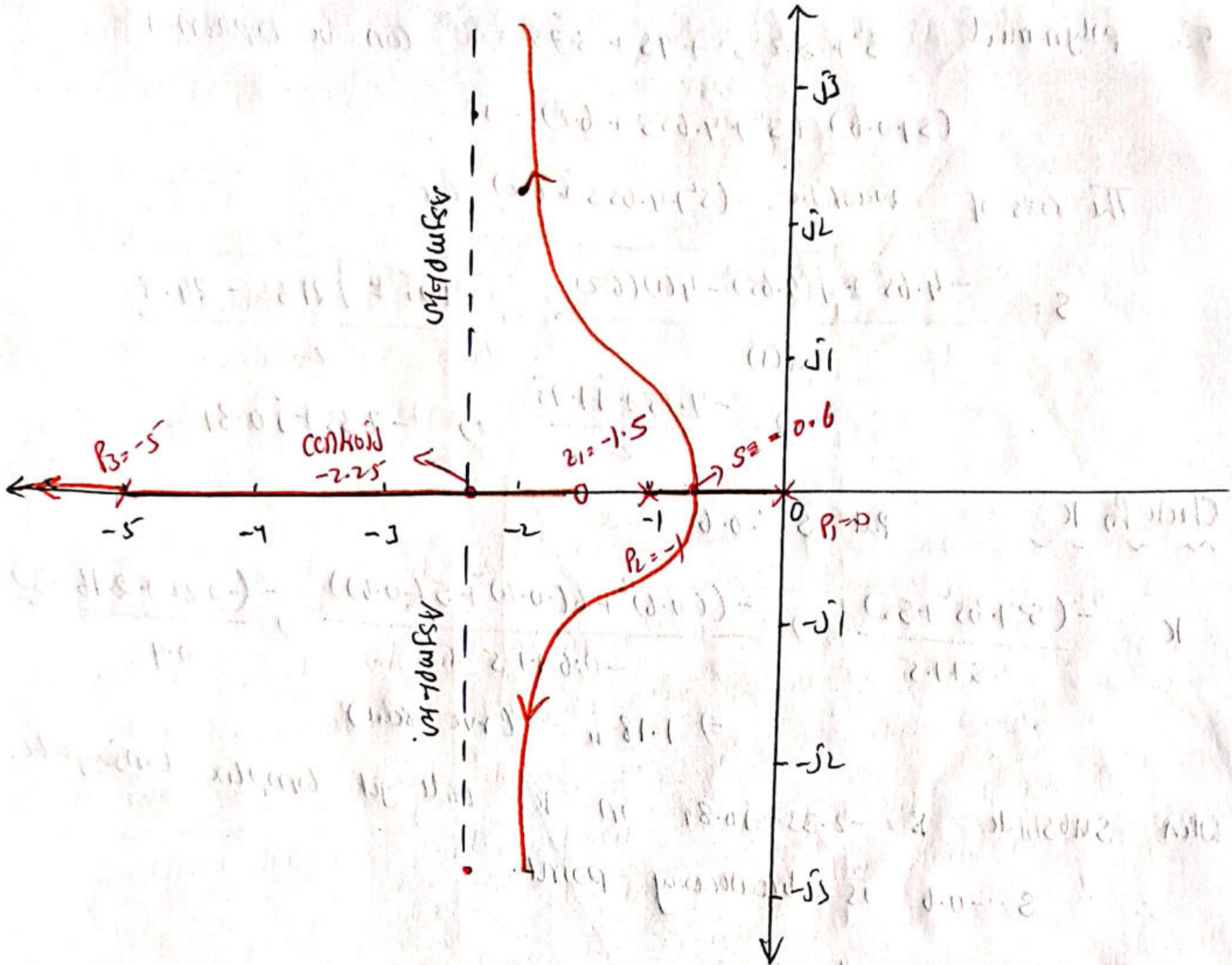
$$
\begin{array}{r|l}
 & s^2+4.83s+6.97 \\
s+0.42 & s^3+5.25s^2+9s+3.75 \\
 & s^3+0.42s^2 \\
\hline
 & 4.83s^2+9s+3.75 \\
 & 4.83s^2+2.03s \\
\hline
 & 6.97s+3.75 \\
 & (-)\ 6.97s+2.93 \\
\hline
 & 0.82
\end{array}
$$

Trail - II

$6.97s+3.75=0$

$s+\frac{3.75}{6.97}=0$, $s+0.54=0$

$$
\begin{array}{r|l}
 & s^2+4.71s+6.46 \\
s+0.54 & s^3+5.25s^2+9s+3.75 \\
 & s^3+0.54s^2 \\
\hline
 & 4.71s^2+9s+3.75 \\
 & 4.71s^2+2.54s \\
\hline
 & 6.46s+3.75 \\
 & 6.46s+3.49 \\
\hline
 & 0.26
\end{array}
$$



Trail - III

$6.46S + 3.75 = 0$

$S + \frac{3.75}{6.46} = 0$

$S + 0.58 = 0$

$$
\begin{array}{r|l}
 & S^2 + 4.67S + 6.29 \\
S+0.58 & S^3 + 5.25S^2 + 9S + 3.75 \\
 & S^3 + 0.58S^2 \\
\hline
 & 4.67S^2 + 9S + 3.75 \\
 & 4.67S^2 + 2.71S + 3.75 \\
\hline
 & 6.29S + 3.75 \\
 & 6.29S + 3.65 \\
\hline
 & 0.05
\end{array}
$$

Trail - IV

$6.29S + 3.75 = 0$

$S + \frac{3.75}{6.29} = 0$

$S + 0.6 = 0$

$$
\begin{array}{r|l}
 & S^2 + 4.65 + 6.21 \\
S+0.6 & S^3 + 5.25S^2 + 9S + 3.75 \\
 & S^3 + 0.6S^2 \\
\hline
 & 4.65S^2 + 9S \\
 & 4.65S^2 + 2.79S \\
\hline
 & 6.21S + 3.75 \\
 & 6.21S + 3.73 \\
\hline
 & 0.02
\end{array}
$$

on neglecting the small value of 0.02, one of the root of the polynomial is $S = -0.6$

The polynomial $s^3 + 5.25s^2 + 9s + 3.75 = 0$ can be expressed

$$(s+0.6)\ (s^2 + 4.65s + 6.2) = 0$$

The roots of quadratic, $(s^2+4.65s+6.2)$ are

$$s = \frac{-4.65 \pm \sqrt{(4.65)^2 - 4(1)(6.2)}}{2(1)} \Rightarrow \frac{-4.65 \pm \sqrt{21.62 - 24.8}}{2}$$

$$\Rightarrow \frac{-4.65 \pm j1.78}{2} \Rightarrow -2.33 \pm j0.89.$$

Check for K :- Put $s = -0.6$

$$K = \frac{-(s^3 + 6s^2 + 5s)}{s+1.5} \Rightarrow \frac{-((-0.6)^3 + 6(-0.6)^2 + 5(-0.6))}{-0.6+1.5} \Rightarrow \frac{-(-0.22 + 2.16 - 3)}{0.9}$$

$$\Rightarrow 1.18 \text{ (+ve real)}$$

When substitute $s = -2.33 \pm j0.89$ in K will get complex conjugate.

$\therefore$ $s = -0.6$ is breakaway point.

5) To find angle of departure :-

since there are no complex pole (or) zero we need not find angle of departure (or) arrival.

6) To find crossing point on imaginary axis :-

The characteristic equation is $s^3 + 6s^2 + 5s + Ks + 1.5K = 0$

Put $s = j\omega$ in above equ.

$$(j\omega)^3 + 6(j\omega)^2 + 5(j\omega) + K(j\omega) + 1.5K = 0$$

$$-j\omega^3 - 6\omega^2 + j5\omega + jK\omega + 1.5K = 0$$

on equating imaginary part to zero

$$-j\omega^3 + j5\omega + jK\omega = 0$$

$$\cancel{j}\omega^3 = \cancel{j}(5\omega + K\omega)$$

$$\omega^2 = 5 + K$$

on equating real part to zero

$$-6\omega^2 + 1.5K = 0$$

$$-6(5+K) + 1.5K = 0$$

$$-30 - 6K + 1.5K = 0$$

$$-30 = 4.5K$$

$$K = -6.67$$

since the value of k is negative, there is no crossing point on imaginary axis ~~(or) for any positive~~ so the root locus will not cross imaginary axis.

Ex. Sketch root locus for the unity feedback system whose open loop transfer function is $G(s) = \dfrac{K(s^2+6s+25)}{s(s+1)(s+2)}$

1) To locate poles and zeros :-

no. of poles $(n) = 3$

no. of zeros $(m) = 2$

poles $\Rightarrow P_1 = 0, \quad P_2 = -1, \quad P_3 = -2$

zeros $\Rightarrow s^2+6s+25 \quad \Rightarrow \dfrac{-6 \pm \sqrt{36-4(25)}}{2}$

$\Rightarrow \dfrac{-6 \pm \sqrt{-64}}{2} \Rightarrow -3 \pm j4 \qquad z_1 = -3+j4$

$z_2 = -3-j4$

2) To find root locus on real axis :-

The segment of real axis between 0 and -1, and the entire negative real axis from -2 will be part of root locus.

3) To find angle of asymptotes and centroid :-

$q = 0, 1,$

$n-m \Rightarrow 3-2 \Rightarrow 1$

Angle of Asymptotes $\Rightarrow \dfrac{\pm 180^\circ(2q+1)}{n-m}$

$q=0 \Rightarrow \dfrac{\pm 180^\circ}{1} \Rightarrow \pm 180^\circ$

$q=1 \Rightarrow \pm 180^\circ(3) \Rightarrow \pm 540^\circ = \mp 180^\circ$

4) To find breakaway point and break in point :-

Char. Equation $\Rightarrow 1+G(s)H(s)=0$

$1 + \dfrac{K(s^2+6s+25)}{s(s+1)(s+2)} \cdot 1 = 0$

$s(s+1)(s+2)$

$\Rightarrow s(s^2+3s+2)$

$\Rightarrow s^3+3s^2+2s$

$s(s+1)(s+2) + K(s^2+6s+25) = 0$

$K = \dfrac{-(s(s+1)(s+2))}{s^2+6s+25}$

$K = \dfrac{-(s^3+3s^2+2s)}{s^2+6s+25} \rightarrow (1)$

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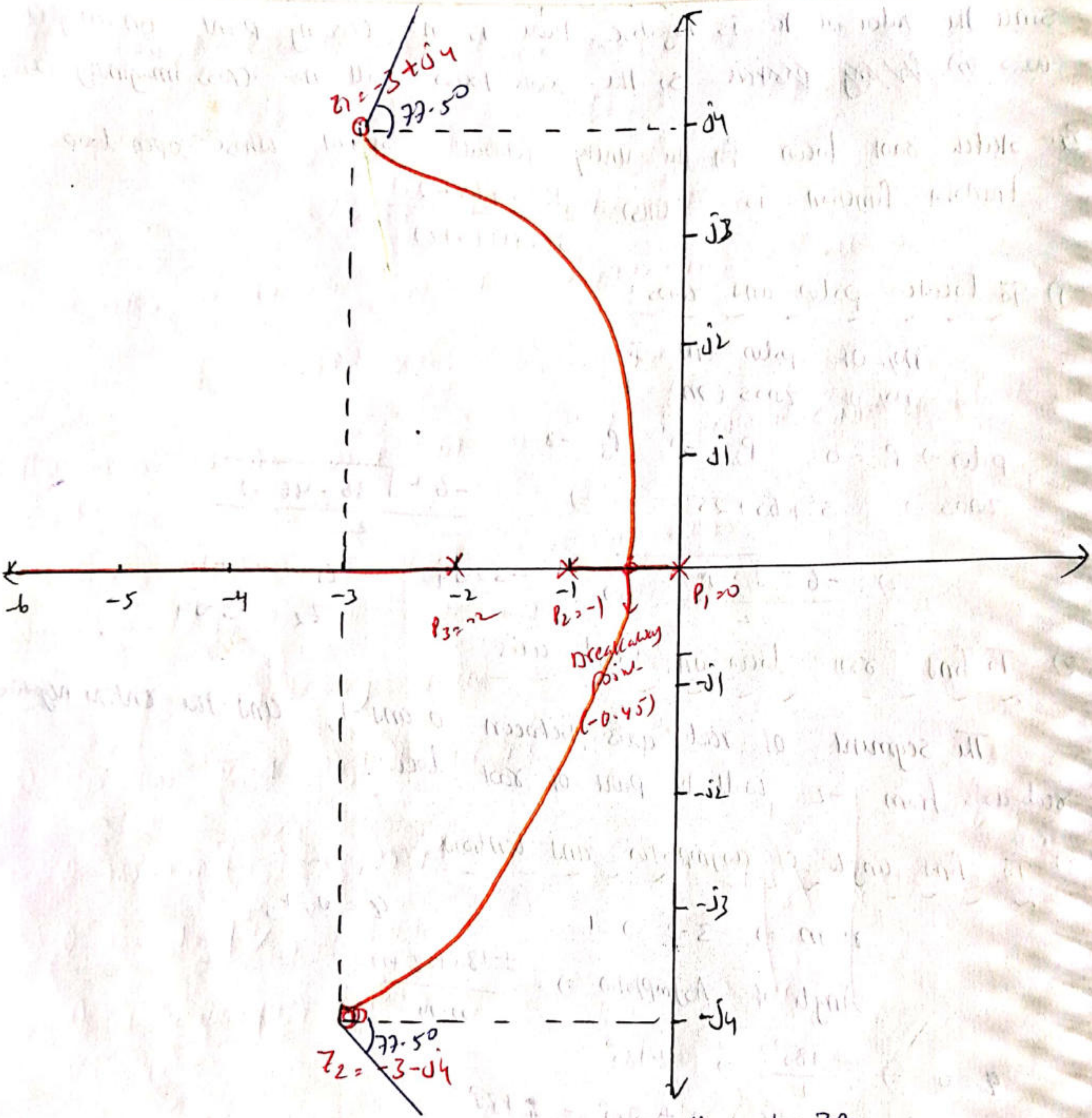


differentiating Equ. ① w.r.t s and equating to zero

$$\frac{dk}{ds} = \frac{(s^2+6s+25)\times-(3s^2+6s+2) - (-(s^3+3s^2+2s)\times(2s+6))}{(s^2+6s+25)^2} = 0$$

$$\Rightarrow -(s^2+6s+25)(3s^2+6s+2) + (s^3+3s^2+2s)(2s+6) = 0$$

$$\Rightarrow -(3s^4+6s^3+2s^2+18s^3+36s^2+12s+75s^2+150s+50) + (2s^4+6s^3+6s^3+18s^2+4s^2+12s)$$

$$\Rightarrow \cancel{-3s^4+30s^3+80s^2}$$

$$\Rightarrow -3s^4-24s^3-113s^2-162s-50+2s^4+6s^3+6s^3+18s^2+4s^2+24s=0$$

$$\Rightarrow -3s^4-24s^3-113s^2-162s-50=0$$

$$-s^4-12s^3-91s^2-150s-50=0$$

$$\Rightarrow s^4+12s^3+91s^2+150s+50=0$$

$\Rightarrow$

The fourth order polynomial equation can be split up into two quadratic equations. The two quadratic factors can be obtained from by Lin's method.

To find quadratic factors by Lin's method :-

Char. eq. $\Rightarrow$ $s^4 + 12s^3 + 91s^2 + 150s + 50 = 0$

consider last three terms.

Trail - I :-

$91s^2 + 150s + 50 = 0$

$s^2 + \frac{150}{91}s + \frac{50}{91} = 0$

$s^2 + 1.65s + 0.55$

$$
\begin{array}{r|l}
 & s^2 + 10.35s + 73.37 \\
\hline
s^2 + 1.65s + 0.55 & s^4 + 12s^3 + 91s^2 + 150s + 50 \\
 & \cancel{s^4} + 1.65s^3 + 0.55s^2 \\
\hline
 & 10.35s^3 + 90.45s^2 + 150s + 50 \\
 & \cancel{10.35s^3} + 17.08s^2 + 5.7s \\
\hline
 & 73.37s^2 + 144.25s + 50 \\
 & \cancel{73.37s^2} + 121.1s + 40.35 \\
\hline
 & 23.15s + 9.65
\end{array}
$$

$\rightarrow$ bigger value go for Trail - II

Trail - II $\quad 73.37s^2 + 144.25s + 50 \quad \Rightarrow s^2 + 2s + 0.7 = 0$

$$
\begin{array}{r|l}
 & s^2 + 10s + 70.3 \\
\hline
s^2 + 2s + 0.7 & s^4 + 12s^3 + 91s^2 + 150s + 50 \\
 & s^4 + 2s^3 + 0.7s^2 \\
\hline
 & 10s^3 + 90.3s^2 + 150s \\
 & 10s^3 + 20s^2 + 7s \\
\hline
 & 70.3s^2 + 143s + 50 \\
 & 70.3s^2 + 140.6s + 49.2 \\
\hline
 & 2.4s + 0.8
\end{array}
$$

on neglecting the small remainder we can write.

$$s^4 + 12s^3 + 91s^2 + 150s + 50 = (s^2 + 2s + 0.7)(s^2 + 10s + 70.3)$$

The roots of Quadratic Equation $s^2+2s+0.7=0$ we can write

$$\frac{-2\pm\sqrt{4-2.8}}{2} \Rightarrow \frac{-2\pm\sqrt{1.2}}{2} \Rightarrow \frac{-2\pm 1.01}{2}$$

$s_1, s_2 \Rightarrow$ ~~$-1.45/0.545$~~ $\Rightarrow -0.45, -1.55$

The roots of Quadratic Equation $s^2+10s+70.3=0$ are

$$s = \frac{-10\pm\sqrt{10^2-4\times70.3}}{2} = -5\pm j6.73.$$

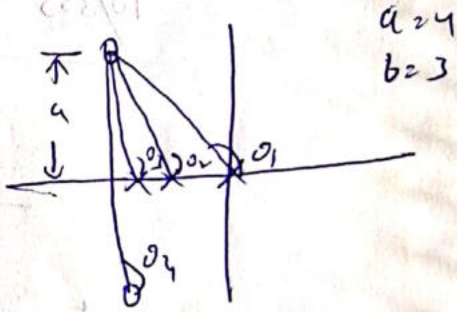
Here -1.55 is not point on root locus, hence it cannot be a breakaway point.

When $s=-0.45$ $\quad K = 0.07$ (+ve and real)

When $s=-5\pm j6.73$ $\quad K =$ Not positive and real.

$\therefore$ Breakaway point $= -0.45$

5) To find ~~angle~~ Angle of arrival :-



$\theta_1 = 180^\circ - \tan^{-1}(4/3) \Rightarrow 126.9^\circ$

$\theta_2 = 180^\circ - \tan^{-1}(4/2) \Rightarrow 116.6^\circ$

$\theta_3 = 180^\circ - \tan^{-1}(4/1) \Rightarrow 104^\circ$

$\theta_4 = 90^\circ$

Angle of arrival from A $= 180^\circ - (\theta_4) + (\theta_1+\theta_2+\theta_3)$

$\Rightarrow 180^\circ - 90^\circ + 126.9^\circ + 116.6^\circ + 104^\circ$

$\Rightarrow 437.5^\circ \Rightarrow 77.5^\circ$

Angle of arrival from A^* $= -(A) \Rightarrow -77.5^\circ$

b) To find the crossing point on imaginary axis :-

The char. Equation is

$$s(s+1)(s+2) + K(s^2+6s+25) = 0$$

$$s^3 + (3+K)s^2 + (2+6K)s + 25K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + (3+K)(j\omega)^2 + j(2+6K)\omega + 25K = 0$$

$$-j\omega^3 - (3+K)\omega^2 + j(2+6K)\omega + 25K = 0$$

On Equating imaginary part to zero

$$-j\omega^3 + j(2+6K)\omega = 0$$

$$\cancel{\omega^2 = (2+6K)}$$

$$-\cancel{j}\omega^{\cancel{3}} = -\cancel{j}(2+6K)\cancel{\omega}$$

$$\omega^2 = (2+6K)$$

On Equating real part to zero

$$-(3+K)\omega^2 + 25K = 0$$

$$-(3+K)(2+6K) + 25K = 0$$

$$-(6 + 18K + 2K + 6K^2) + 25K = 0$$

$$-6 + 20K - 6K^2 + 25K = 0$$

$$-6K^2 + 5K - 6 = 0$$

$$K = 0.4 \pm j0.9$$

Since the value of K is not real and positive, there is no crossing point on imaginary axis, (or) for any positive value of K the root locus will not cross imaginary axis.

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Unit-IV

Fruquancy ~~domain~~ response Analysis

Introduction :-

Sinusoidal Transfer function and frequency response :-

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response to sinusoidal input is called sinusoidal Transfer function. of the system and it is denoted by $T(j\omega)$.

The sinusoidal Transfer function $T(j\omega)$ can be obtained as follows.

→ construct a physical model of system using basic elements / parameters.

→ determine the differential equations, governing of the system from the physical model of the system.

→ Take the Laplace transform of differential equations in order to convert them to s-domain equation.

→ determine s-domain transfer function, $T(s)$ which is ratio of s-domain o/p & i/p.

→ determine the frequency domain transfer function, $T(j\omega)$ by replacing "s" by "$j\omega$" in the s-domain transfer function $T(s)$.

i.e, $T(s) \xrightarrow{s=j\omega} T(j\omega)$.

Frequency Response :-

The frequency domain transfer function $T(j\omega)$ is a complex function of "ω". Hence it can be separated into magnitude function and phase function. Now the magnitude and phase functions are real functions of ω. and they are called frequency Response.

Open Loop Transfer function → $G(s) \longrightarrow G(j\omega) = |G(j\omega)| \angle G(j\omega)$ → ①

Loop transfer function → $G(s)H(s) \longrightarrow G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle (G(j\omega)(H(j\omega))$ → ②

closed loop Transfer function → $M(s) \rightarrow M(j\omega) = |M(j\omega)| \angle M(j\omega)$ → ③

* Fruquency domain specifications :-

The performance and characteristics of a system in frequency domain are measured interms of frequency domain specifications.

The frequency domain specifications are,

i) Resonant peak, M_r
ii) Resonant frequency, ω_r
iii) Bandwidth, ω_b
iv) Cut-off rate
v) Gain margin, K_g
vi) Phase margin γ

i) Resonant peak :-

The maximum value of the magnitude of closed loop Transfer function is called "resonant peak", M_r

$$\text{Resonant peak, } M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

ii) Resonant frequency, ω_r :-

The frequency at which resonant peak occurs is called resonant frequency. It is denoted by ω_r

$$\text{Resonant frequency } \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

iii) Band width, ω_b :-

The bandwidth is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain -3db is called "cut-off frequency". Bandwidth is a measure of the ability of a feedback system to reproduce the i/p signal, noise rejection and rise time. A large bandwidth corresponds fast response.

$$\text{Band width } (\omega_b) = \omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{1/2}$$

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iv) Cut-off rate :-

The slope of the log magnitude curve near the cut off frequency is called cut-off rate. The cut-off rate indicates the ability of the system ~~the~~ to distinguish the signal from noise.

v) Gain Margin, K_g :-

The gain margin, K_g is defined as the value of the gain, to be added to system, in order to bring the system in stability condition.

The gain margin K_g is given by reciprocal of magnitude of open loop transfer function at phase cross over frequency.

The frequency at which the phase of open loop transfer function is -180° is called phase cross over frequency: ω_{pc}.

units dB

$$\text{Gain Margin } K_g = \frac{1}{|G(j\omega)|}$$

The gain margin in db can be expressed as

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in db is given by the negative value of magnitude of $G(j\omega)$ at phase cross-over frequency. The gain margin indicates that the additional gain that can be provided to system without affecting the stability of the system.

vi) Phase Margin (γ) :- The phase margin γ is obtained by adding 180° to the phase angle (ϕ) of the open loop transfer function at the gain cross over frequency.

units - degree

$$\text{Phase Margin } \gamma = 180^\circ + \phi_{gc} \qquad \phi_{gc} = \angle G(j\omega_{gc})$$

* Rules for construction of Bode plots :-

Follow the rules while constructing a bode plot.

i) Represent the open loop Transfer function in the standard time constant

Ex: Consider open loop Transfer function $\Rightarrow \dfrac{10s}{(s+2)(s+5)}$

Convert into time constant form i.e., $(1+sT)$

$$\frac{10s}{2(1+s/2)\,5(1+s/5)} \Rightarrow \frac{10s}{10(1+0.5s)(1+0.2s)} \Rightarrow \frac{s}{(1+0.5s)(1+0.2s)}$$

ii) Substitute $s = j\omega$ in the above Equation

$$\frac{j\omega}{(1+j0.5\omega)(1+j0.2\omega)}$$

iii) Find the corner frequencies and arrange them in ascending order.

iv) Consider the starting corner frequency of bode plot as $1/10^{th}$ of the minimum corner frequency and higher corner frequency which more than the maximum higher corner frequency.

v) Draw the magnitude plots for each term and ~~combined~~ combine these plots properly.

vi) Draw the phase plots for each term and combine these properly.

* Stability Analysis with bode plots :-

From the bode plot we can say whether the control system is stable, marginally stable (or) unstable based on the values of these parameters.

i) Gain cross over frequency and phase cross over frequency.

ii) Gain margin and phase margin.

* Gain cross over frequency :-

The frequency at which the magnitude plot is having the magnitude of 0dB is known as "Gain cross over frequency". It is denoted by "ω_{gc}"

The unit of gain cross over frequency is rad/sec.

* Phase cross over frequency :-

The frequency at which the phase plot is having phase of $-180°$ is known as "phase cross over frequency". It is denoted by ω_{pc}. The unit of phase cross over frequency is rad/sec.

The stability of the control system based on the relation between the gain cross over frequency and phase cross over frequency.

→ If the phase cross over frequency (ω_{pc}) is greater than the gain cross over frequency (ω_{gc}), then the control system is <u>stable.</u>

→ If the phase cross over frequency (ω_{pc}) is equal to the gain cross over frequency (ω_{gc}), then the control system is <u>marginally stable.</u>

→ If the phase cross over frequency (ω_{pc}) is less than the gain cross over frequency (ω_{gc}), then the control system is <u>unstable.</u>

The stability of the control system based on the relation between GM and PM.

→ If both GM and PM are positive then control system is stable.

→ If both GM and PM are equal to zero, then control system is marginally stable.

→ If both GM (or) PM are/is negative, then system is unstable.

NOTE:

$S \rightarrow +20$ db/dec

$S^2 \rightarrow +40$ db/dec

$1/S \rightarrow -20$ db/dec

$1/S^2 \rightarrow -40$ db/dec.

Ex: Sketch the bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$ [(1+sT) form]

Sol :- The given transfer function in s-domain. The sinusoidal transfer function $G(j\omega)$ can be obtained by replacing "s" by "$j\omega$".

$$\therefore G(j\omega) = \frac{K(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

Magnitude plot :- The corner frequencies are

$\omega_{c_1} = \frac{1}{0.2} \Rightarrow 5$ rad/sec $\quad \omega_{c_2} = \frac{1}{0.02} \Rightarrow 50$ rad/sec.

The various terms of $G(j\omega)$ are listed in Table-1 in the increasing order of corner frequency. Also the table shows the slope contributed by each term and change in slope at the corner frequency.

Table-I :-

Term	Corner frequency rad/sec	Slope db/dec	change in slope db/dec.
$(j\omega)^2$	—	$+40^\circ$	
$\frac{1}{(1+j0.2\omega)}$	$\omega_{c_1} = \frac{1}{0.2} = 5$	-20°	$40 - 20 \Rightarrow +20^\circ$
$\frac{1}{(1+j0.02\omega)}$	$\omega_{c_2} = \frac{1}{0.02} = 50$	-20°	$20^\circ - 20^\circ = 0^\circ$

Choose a lower frequency ω_l ~~which~~ is such that $\omega_l < \omega_{c_1}$ and choose a higher frequency ω_h such that $\omega_h > \omega_{c_2}$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 100$ rad/sec.

Let $A = |G(j\omega)|$ in dB.

Let us calculate A at ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_l \Rightarrow \quad A = 20 \log |(j\omega)^2| \Rightarrow 20 \log \omega_l^2 = 20 \log (0.5)^2 \Rightarrow -12$ db.

At $\omega = \omega_{c_1} \Rightarrow \quad A = 20 \log |(j\omega)^2| \Rightarrow 20 \log \omega_{c_1}^2 \Rightarrow 20 \log (5)^2 \Rightarrow 28$ db.

At $\omega = \omega_{c_2} \Rightarrow \quad A = \left[\text{Slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}} \right] + A_{(\text{at } \omega = \omega_{c_1})}$

$$\Rightarrow 20 \times \log \frac{50}{5} + 28 \Rightarrow 48 \text{ db.}$$

At $\omega = \omega_h \Rightarrow \quad A = \left[\text{Slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}} \right] + A_{(\text{at } \omega = \omega_{c_2})}$

$$\Rightarrow 0 \times \log \left(\frac{100}{50}\right) + 48 = 48 \text{ db.}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c_1}, ω_{c_2}, ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 10 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scale on x-axis.

Phase plot:- The phase angle of $G(j\omega)$ as a function of ω is given by -

$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}(0.2\omega) + \tan^{-1}(0.02)\omega$$

The phase angle of $G(j\omega)$ are calculated for various values of ω are listed in Table-2.

Table-2:

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg.
0.5	5.7	0.6	173.7 ≃ 174
1	11.3	1.1	167.6 ≃ 168
5	45	5.7	124.3 ≃ 130
10	63.4	11.3	105.3 ≃ 106
50	84.3	45	50.7 ≃ 51
100	87.1	63.4	29.5 ≃ 30

on the Same semilog sheet choose a scale of 1 unit = 20° on the y-axis on the right side of semi log graph. Mark the calculated points on the graph sheet. Join the points by smooth curve.

Calculation of K :-

At $\omega = 5$ rad/sec the gain value is 28 db. If the gain cross over frequency is 5 rad/sec then at the frequency the db gain should be zero. Hence every point of magnitude plot a db gain of -28 db should be added. The additional -28 db shifts the plot downwards.

The value of K is calculated by equating 20 log K to -28 db.

$$20 \log K = -28$$

$$\log K = \frac{-28}{20} \therefore K = 10^{-(28/20)} \Rightarrow K = 10^{-1.4}$$

$$K = 0.0398$$

The magnitude plot with $K = 1$ and $K = 0.0398$ and phase plot are shown in graph.

Note :

The freq. $\omega = 5$ rad/sec is corner freq. Hence in the exact plot the db gain at $\omega = 5$ rad/sec will be 3 db less than approximate graph.

$\therefore$ Therefore for exact plot the 20 log K will contribute a gain of -25 db

$$20 \log K = -25$$

$$K = 10^{-(25/20)} = 0.0562.$$

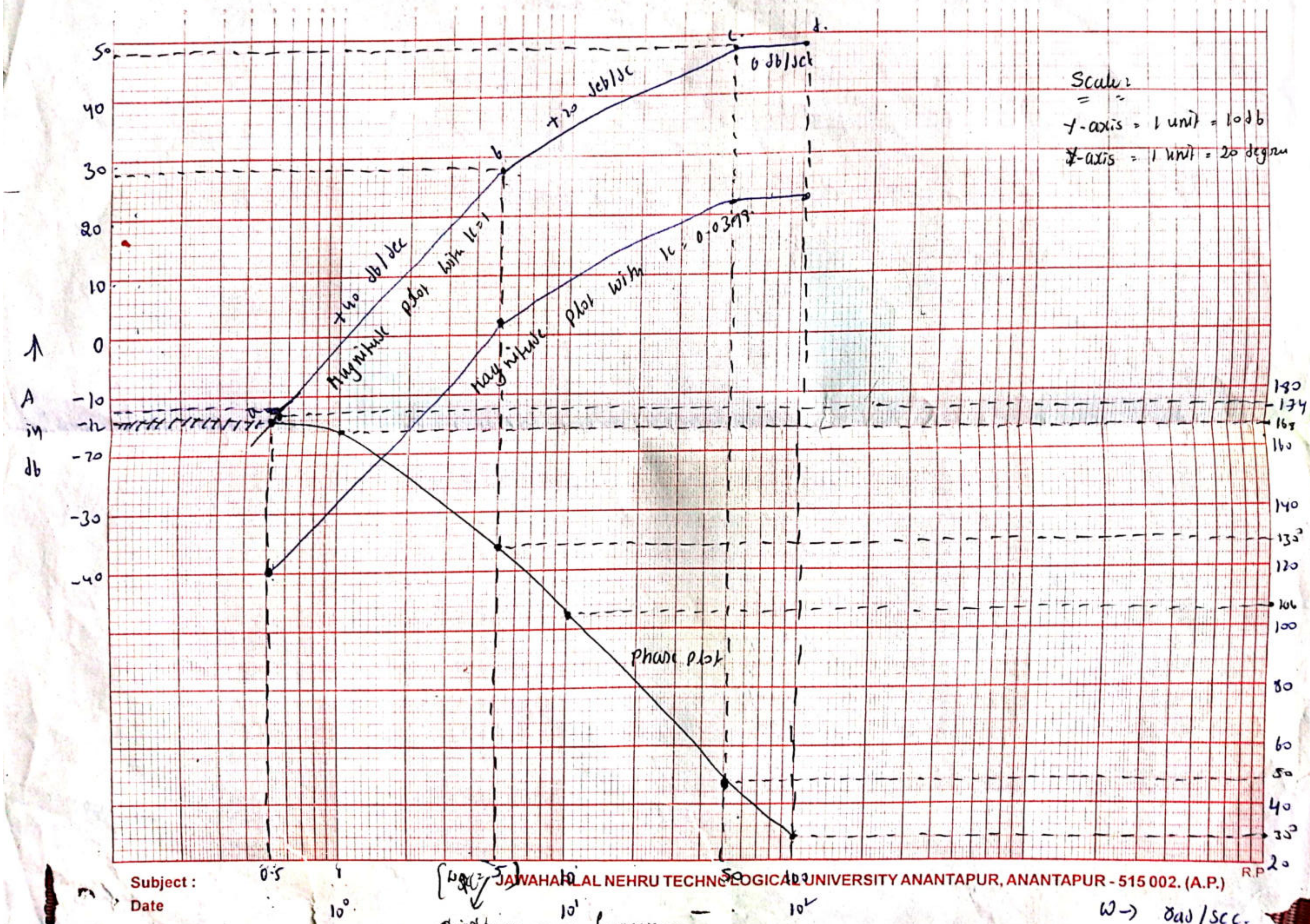
$$G(s) = \frac{ks^2}{(1+0.2s)(1+0.02s)}$$

SEMI-LOG PAPER (5 CYCLES X 1/10")

Scale:

Y-axis = 1 unit = 10 db

X-axis = 1 unit = 20 degree



Subject :
Date

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR, ANANTAPUR - 515 002. (A.P.)

R.P.

[ω_{gc} = 5] gain cross over frequency.

Ex: 2 Plot the bode diagram for the following open loop Transfer function and Obtain the gain and phase cross over frequencies.

$$G(s) = \frac{10}{S(1+0.4S)(1+0.1S)}$$

Sol: The sinusoidal transfer function of $G(j\omega)$ is obtained by replacing S by $j\omega$ in the given transfer function.

$$\therefore G(j\omega) = \frac{10}{j\omega(1+j0.4\omega)(1+j0.1\omega)}$$

Magnitude plot:-

The corner frequencies are

$$\omega_{c_1} = \frac{1}{0.4} = 2.5 \text{ rad/sec} \qquad \omega_{c_2} = \frac{1}{0.1} = 10 \text{ rad/sec.}$$

The various terms of $G(j\omega)$ are listed in Table-1 in the increasing order of their corner frequencies. And also table shows the slope contributed by each term and the change in slope.

Table-1:

terms	corner frequencies rad/sec	Slope db/dec	change in slope db/dec.
$\frac{10}{j\omega}$	—	-20	
$\frac{1}{(1+j0.4\omega)}$	$\omega_{c_1} = \frac{1}{0.4} = 2.5$	-20	$-20-20 = -40$
$\frac{1}{(1+j0.1\omega)}$	$\omega_{c_2} = \frac{1}{0.1} = 10$	-20	$-40-20 = -60$

Choose the low corner frequency ω_l which is less than ω_{c_1} and choose high corner frequency which is greater than ω_{c_2}

$$\omega_l < \omega_{c_1} \qquad \omega_h > \omega_{c_2}$$

$$\omega_l = \frac{1}{10}(\omega_{c_1}) \Rightarrow 0.25 \text{ rad/sec} \qquad \omega_h = 50 \text{ rad/sec.}$$

Let $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_l \Rightarrow 20\log\left|\frac{10}{j\omega}\right| \Rightarrow 20\log\left|\frac{10}{\omega_l}\right| \Rightarrow 20\log\left|\frac{10}{0.25}\right| \Rightarrow 32\text{db}$

At $\omega = \omega_{c_1} \Rightarrow 20\log\left|\frac{10}{j\omega}\right| \Rightarrow 20\log\left|\frac{10}{\omega_{c_1}}\right| \Rightarrow 20\log\left|\frac{10}{2.5}\right| \Rightarrow 12\text{db}$

At $\omega = \omega_{c2}$ $A = \left\{ \text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right) \right\} + A_{(\omega=\omega_{c1})}$

$$A = \left\{ -40 \times \log\left(\frac{10}{2.5}\right) \right\} + 12$$

$\Rightarrow -12$ db.

At $\omega = \omega_h$ $A = \left\{ \text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\left(\frac{\omega_h}{\omega_{c2}}\right) \right\} + A_{(\omega=\omega_{c2})}$

$$\Rightarrow \left\{ -60 \times \log\left(\frac{50}{10}\right) \right\} - 12 \Rightarrow -54 \text{ db.}$$

Let the points a, b, c and d be the points corresponding to $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 10db on y-axis. The frequencies are marked in decades from 0.1 to 100 on x-axis.

Phase plot:-

The phase angle of $G(j\omega)$ is given by.

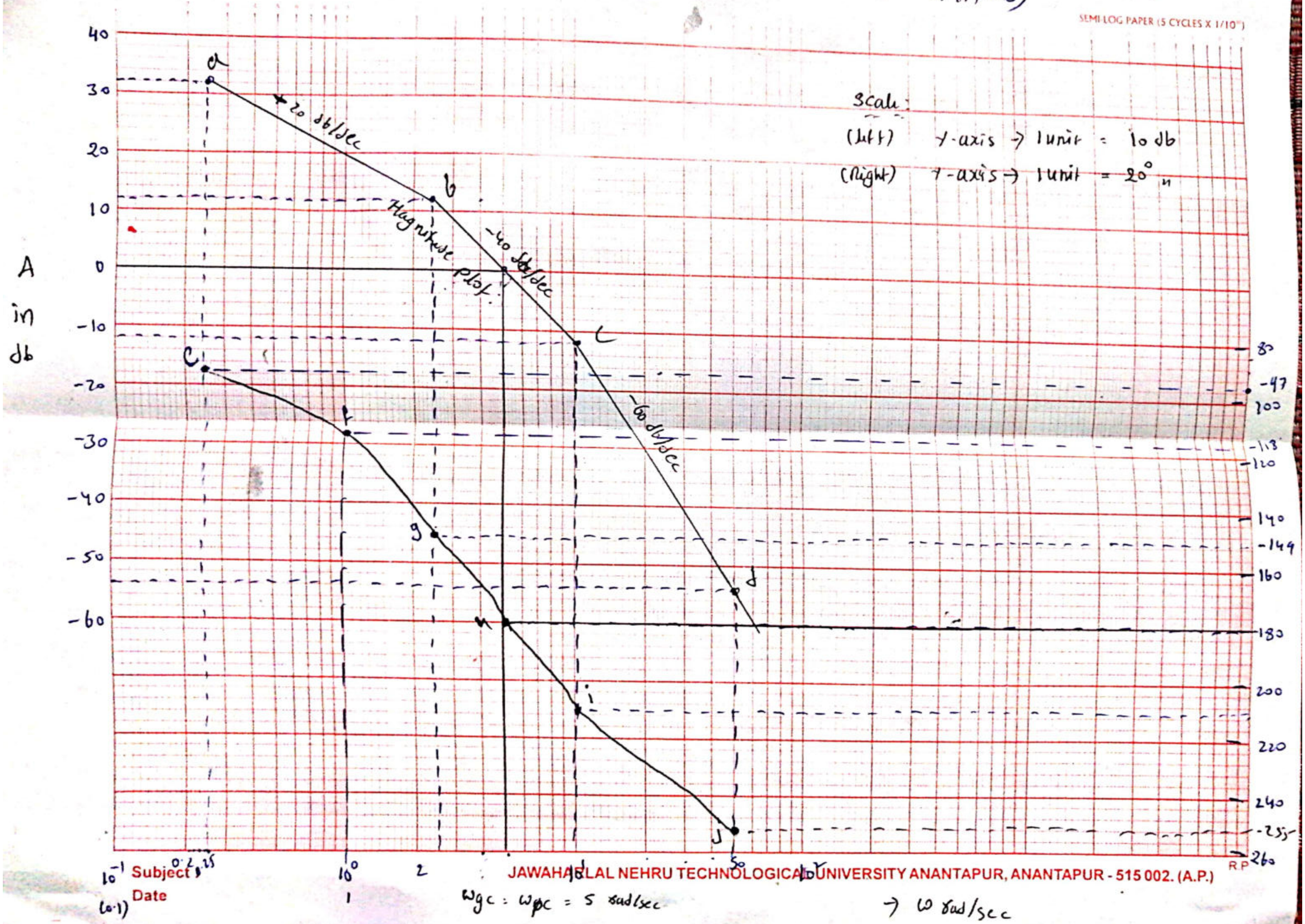
$$\phi = \angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}(0.4\omega) - \tan^{-1}(0.1\omega)$$

The phase angle of $G(j\omega)$ are calculated for various values of ω listed in Table-2.

Table-2:

ω	$\tan^{-1}(0.4\omega)$	$\tan^{-1}(0.1\omega)$	ϕ	points
0.25	5.71	1.43	-97	e
1	21.80	5.71	-118	f
2.5	45	14	-149	g
5	63.43	26.56	-180	h
10	75.96	45	-210	i
50	87.13	78.69	-255	j

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$



Ex:3: For the following transfer function draw bode plot and obtain gain cross over frequency.

$$G(s) = \frac{30}{s(1+3s)(1+4s)}.$$

Sol: The sinusoidal transfer function $G(j\omega)$ is obtained by replacing "s" by $j\omega$ in the given transfer function.

$$G(j\omega) = \frac{30}{j\omega(1+j3\omega)(1+j4\omega)}$$

Magnitude plot:-

The corner frequencies are

$\omega_{c_1} = \frac{1}{4} = 0.25$ rad/sec. $\omega_{c_2} = \frac{1}{3} = 0.33$ rad/sec

The various terms of $G(j\omega)$ are listed in Table-1. in the increasing order of their frequencies. And also table shows slope contributed by each term and change in slope.

terms	corner frequencies rad/sec	slope db/dec	change in slope db/dec.
$\frac{30}{j\omega}$	—	-20	
$\frac{1}{1+j4\omega}$	$\omega_{c_1} = \frac{1}{4} = 0.25$	-20	$-20-20 = -40$
$\frac{1}{1+j3\omega}$	$\omega_{c_2} = 1/3 = 0.33$	-20	$-40-20 = -60$

Choose lower corner frequency such that $\omega_l < \omega_{c_1}$ and choose higher corner frequency such that $\omega_h > \omega_{c_2}$

let $\omega_l = 0.1$ rad/sec $\omega_h = 1$ rad/sec.

let $A = |G(j\omega)|$ in db.

let us calculate A for ω_l, ω_{c_1}, ω_{c_2} and ω_h.

At $\omega = \omega_l$ $\Rightarrow A = 20\log\left|\frac{30}{j\omega}\right| \Rightarrow 20\log\left|\frac{30}{\omega_l}\right| \Rightarrow 20\log\left(\frac{30}{0.1}\right) \Rightarrow$ ~~49.5~~ 46 db

At $\omega = \omega_{c_1}$ $\Rightarrow A = 20\log\left(\frac{30}{0.25}\right) \Rightarrow 38$ db.

At $\omega = \omega_{c_2}$ $\Rightarrow A = \left[\text{Slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\left(\frac{\omega_{c_2}}{\omega_{c_1}}\right)\right] + A_{(\omega = \omega_{c_1})}$

$\Rightarrow A = \left[-40 \times \log\left(\frac{0.33}{0.25}\right)\right\} + 38 \Rightarrow 33$ db.

At $\omega = \omega_h$ $\Rightarrow A = \left[-60 \times \log\left(\frac{1}{0.33}\right)\right\} + 33 \Rightarrow 4$ db.

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c_1}, ω_{c_2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 10 db on y-axis. The frequencies are marked in decades from 0.1 to 10 rad/sec on x-axis.

Phase plot:

The phase angle of $G(j\omega) \Rightarrow \phi = -90^\circ - \tan^{-1}3\omega - \tan^{-1}4\omega$.

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in Table-2

Table-2:

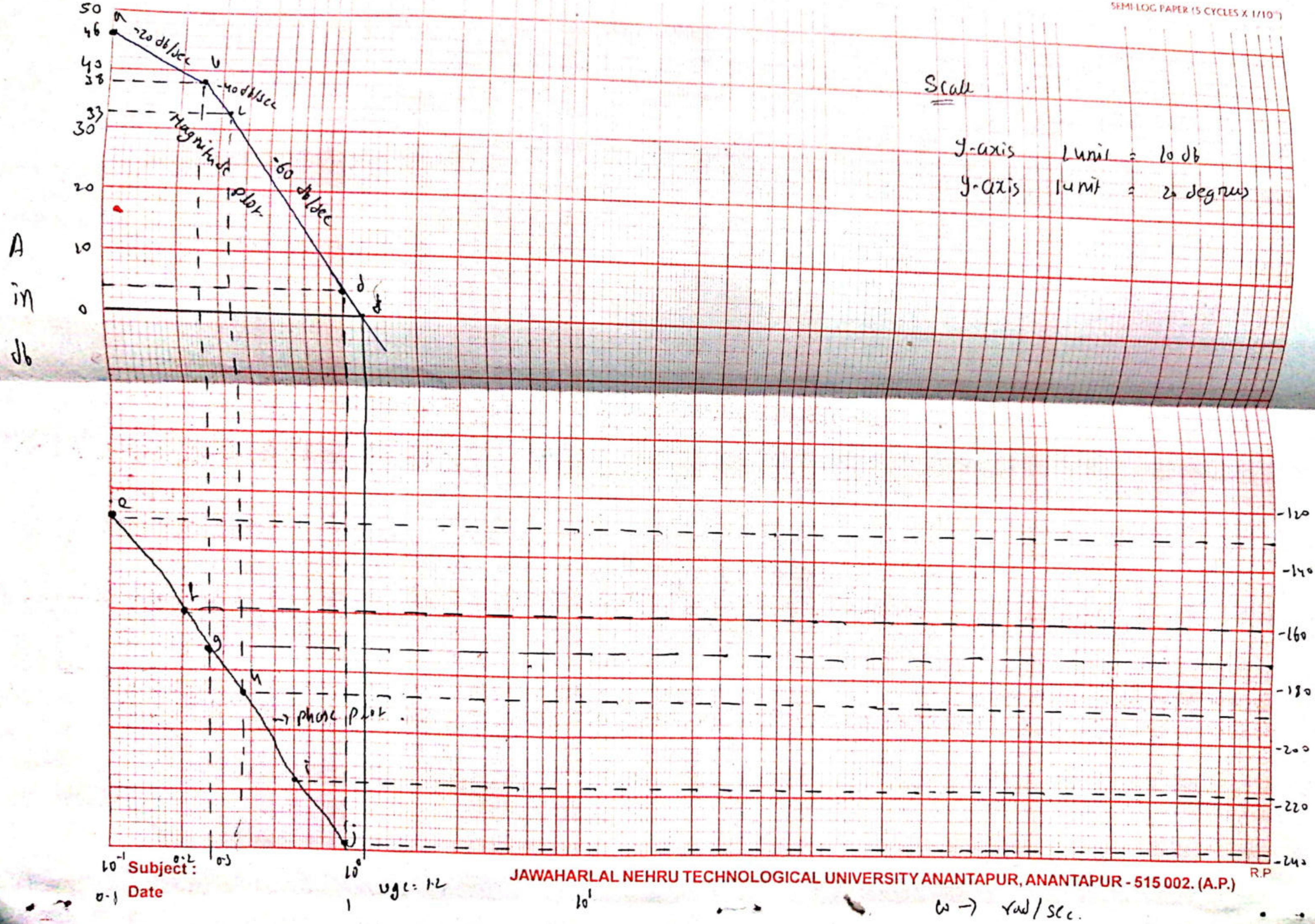
ω	$\tan^{-1}3\omega$ degree	$\tan^{-1}4\omega$ degree	ϕ degree	Points of phase plot
0.1	16.7	21.8	$-128.5 \simeq -129$	e
0.2	31	38.7	-160	f
0.25	36.9	45	-172	g
0.33	44.7	52.9	-188	h
0.6	60.9	67.4	-218	i
1	71.6	76	-238	j

On the same semilog graph sheet choose a scale of 1 unit = 20° on the y-axis on the right side of the semilog graph sheet.

From the graph the gain cross over frequency is found to be

$\omega_{gc} =$

$$G(s) = \frac{\ \ }{s(1+3s)(1+4s)}$$



Ex:4 For the function $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$, draw the bode plot

Sol The sinusoidal Transfer function $G(j\omega)$ is obtained by replacing the s by $j\omega$ in $G(s)$.

$$\therefore G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}.$$

Magnitude plot:-

The corner frequencies are $\omega_{c_1} = \frac{1}{4} = 0.25$ rad/sec, $\omega_{c_2} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c_3} = \frac{1}{0.25} = 4$ rad sec.

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by the each term and the change in slope at the corner frequency.

Terms	Corner frequencies rad/sec	Slope db/dec	Change in slope db/dec.
5	–	0	
$\frac{1}{1+j4\omega}$	$\omega_{c_1} = \frac{1}{4} = 0.25$	−20	$0-20 = -20$
$1+j2\omega$	$\omega_{c_2} = \frac{1}{2} = 0.5$	20	$-20+20 = 0$
$\frac{1}{1+j0.25\omega}$	$\omega_{c_3} = \frac{1}{0.25} = 4$	−20	$0-20 = -20$

Choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c_3}$.

Let $\omega_l = 0.025$ rad/sec and $\omega_h = 10$ rad/sec.

Let $A = |G(j\omega)|$ in db and let us calculate. A at ω_l, ω_{c_1}, ω_{c_2}, ω_{c_3} and ω_h.

At $\omega = \omega_l \Rightarrow A = 20\log 5 \Rightarrow +14$ db.

At $\omega = \omega_{c_1} \Rightarrow A = 20\log 5 \Rightarrow +14$ db.

At $\omega = \omega_{c_2} \Rightarrow A = \left[\text{Change slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] + A(\omega=\omega_{c_1})$

$\Rightarrow \left[-20 \times \log\left(\frac{0.5}{0.25}\right)\right] + 14 \Rightarrow 8$ db

At $\omega = \omega_{c_3} \Rightarrow A = \left[\text{Slope from } \omega_{c_2} \text{ to } \omega_{c_3} \times \log \frac{\omega_{c_3}}{\omega_{c_2}}\right] + A(\omega=\omega_{c_2})$

$\Rightarrow 0 + 8\text{db} \Rightarrow 8$ db.

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At $\omega = \omega_h \Rightarrow A = \left\{ \text{slope from } \omega_{c3} \text{ to } \omega_h \times \log\left(\frac{\omega_h}{\omega_{c3}}\right) \right\} + A_{(\omega = \omega_{c3})}$.

$$A = \left\{ -20 \times \log\left(\frac{40}{4}\right) \right\} + 8 \Rightarrow 0\,db$$

Let the points a, b, c, d and e be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 5 db on y-axis. The frequencies are marked in decades from 0.01 to 100 rad sec on logarithmic scale on x-axis.

Phase plot:

The phase angle of $G(j\omega)$ is ϕ

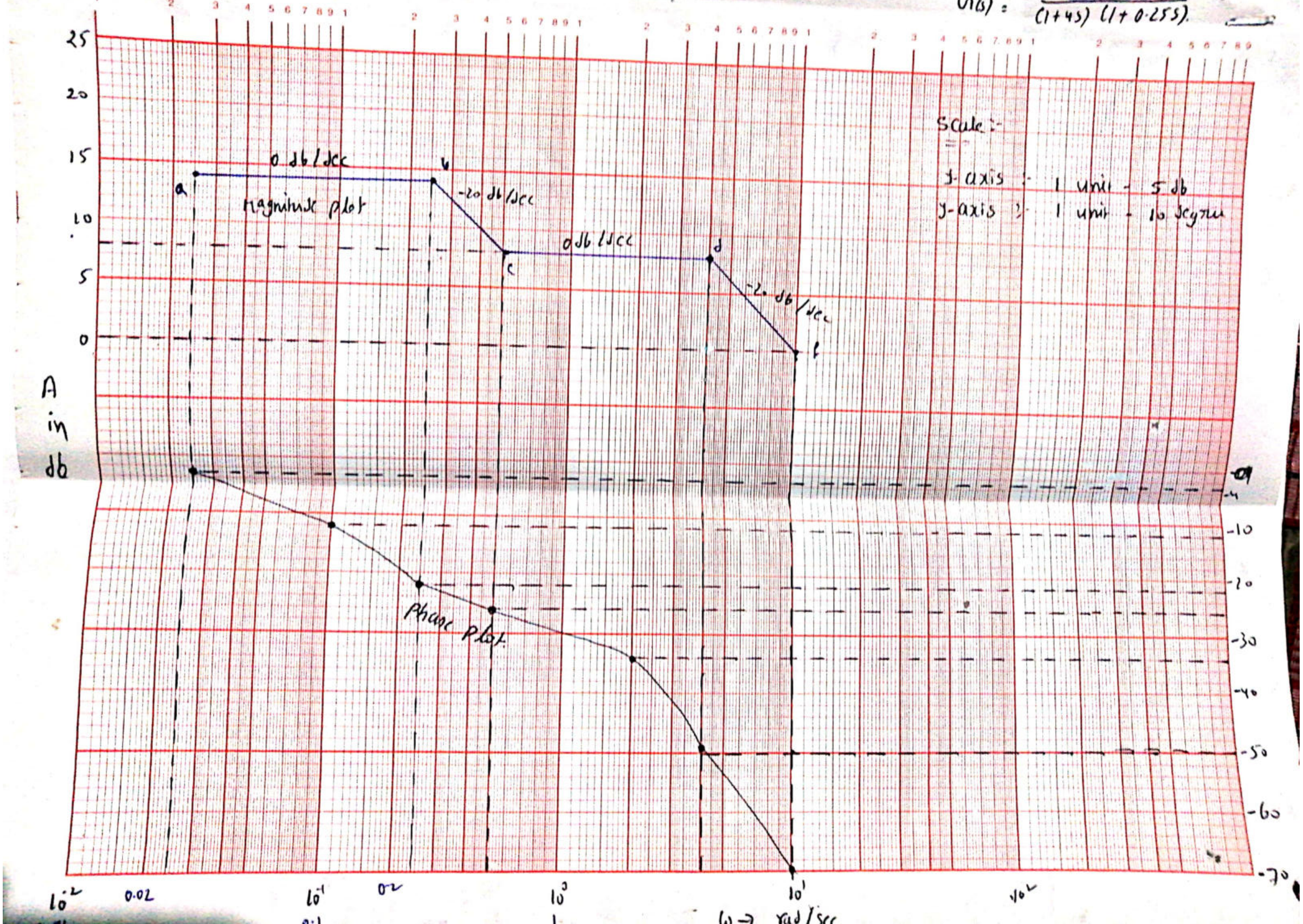
$$\phi = \angle G(j\omega) = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega).$$

The phase angle of $G(j\omega)$ are calculated for various values of ω are listed in Table-2

ω	$\tan^{-1}(2\omega)$	$\tan^{-1}(4\omega)$	$\tan^{-1}(0.25\omega)$	ϕ
0.025	2.9	5.7	0.4	-3.2
0.1	11.3	21.8	1.4	$-11.9 \simeq -12$
0.25	26.6	45	3.6	-22
0.5	45	63.4	7.1	-26
2	76	82.9	26.6	-34
4	82.9	86.4	45	-49
10	87.1	88.6	68.2	-70

On the same semilog graph sheet choose a scale of 1 unit = 10° on y-axis on the right side of the graph sheet.

$$G(s) = \frac{s(1+2s)}{(1+4s)(1+0.25s)}$$



Ex: 5.2 Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}.$$

Sol: on comparing the quadratic factor in the denominator of $G(s)$ with standard form of quadratic factor we can estimate ζ and ω_n.

$$\therefore s^2+16s+100 = s^2+2\zeta\omega_n s+\omega_n^2$$

$2\zeta\omega_n = 16$ $\qquad$ $\omega_n^2 = 100$

$\qquad\qquad\qquad\qquad\quad$ $\omega_n = 10$

$$\zeta = \frac{16}{20} = 0.8$$

Let us convert the given s-domain transfer function into time constant form.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} \Rightarrow \frac{75(1+0.2s)}{s\times 100\left(1+\frac{16}{100}s+\frac{s^2}{100}\right)} \Rightarrow \frac{0.75(1+0.2s)}{s(1+0.01s^2+0.16s)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in $G(s)$.

$$\therefore G(j\omega) = \frac{0.75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16s)}$$

Magnitude plot:

The corner frequencies are $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec, $\omega_{c2} = \omega_n = 10$ rad/sec

Note: For quadratic factor the corner frequency is ω_n.

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope of corner frequency.

Terms	Corner frequency rad/sec	Slope db/sec	Change in slope db/sec.
$\frac{0.75}{j\omega}$	–	-20	
$(1+j0.2\omega)$	$\omega_{c1} = 5$	20	$-20+20=0$
$\frac{1}{(1-0.01\omega^2+j0.16)}$	$\omega_{c2} = \omega_n = 10$	-40	$0-40=-40$

Choose a lower frequency value such that $\omega_l < \omega_{c1}$ and choose higher frequency value such that $\omega_h > \omega_{c2}$

let $\omega_l = 0.5$ rad/sec $\quad \omega_h = 50$ rad/sec

$\omega_{c1} = 5$ rad/sec $\quad \omega_{c2} = 10$ rad/sec.

Let $A = |G(j\omega)|$ in db

Let us calculate A at ω_l, ω_{c1}, ω_{c2}, and ω_h.

At $\omega = \omega_l \Rightarrow A = 20 \log\left(\frac{0.75}{0.5}\right) \Rightarrow 3.5$ db

At $\omega = \omega_{c1} \Rightarrow A = 20 \log\left(\frac{0.75}{5}\right) \Rightarrow -16.5$ db

At $\omega = \omega_{c2} \Rightarrow A = \left\{\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right)\right\} + A(\omega = \omega_{c1})$

$\Rightarrow -16.5$ db.

At $\omega = \omega_h \Rightarrow A = \left\{\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\left(\frac{\omega_h}{\omega_{c2}}\right)\right\} + A(\omega = \omega_{c2})$

$\Rightarrow \left\{-40 \times \log\left(\frac{50}{10}\right)\right\} - 16.5 \Rightarrow -44.5$ db.

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 5 db on y-axis. The frequencies are on logarithmic scale on x-axis marked in decades from 0.1 to 100 rad/sec

Phase plot :-

The phase angle of $G(j\omega)$ as a function of "ω" is given by,

$$\phi = \angle G(j\omega) = -90^\circ + \tan^{-1}(0.2\omega) - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) \quad \text{for } \omega \le \omega_n.$$

$$\phi = \angle G(j\omega) = -90^\circ + \tan^{-1}(0.2\omega) - \left[\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) + 180^\circ\right] \quad \text{for } \omega > \omega_n.$$

Note: In Quadratic factors the phase varies from 0° to 180°. But calculator calculates only between 0° to 90°. Hence a correction of 180° should be added to phase after ω_n.

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in Table-2

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$	ϕ
0.5	5.7	4.6	$-88.9 \simeq -89$
1	11.3	9.2	$-87.9 \simeq -88$
5	45	46.8	$-91.8 \simeq 92$
		90	$-116.6 \simeq 117$
10	63.4	$-46.8 + 180 = 133.2$	$-147.3 = -148$
20	75.4	$-18.4 + 180 = 161.6$.	$-167.3 \simeq -168$.
50	84.3		

On the same semilog graph sheet choose a scale of 1 unit $= 20^\circ$ on the y-axis on the right side of semilog graph sheet. Magnitude plot

Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross over frequency, ω_{gc}.

From the we get $\phi_{gc} = -88^\circ$

$\therefore$ phase margin $= 180^\circ + \phi_{gc}$

$\Rightarrow 92^\circ$.

The phase plot crosses -180° only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ db.

$\therefore$ Hence gain margin is "$-\infty$"

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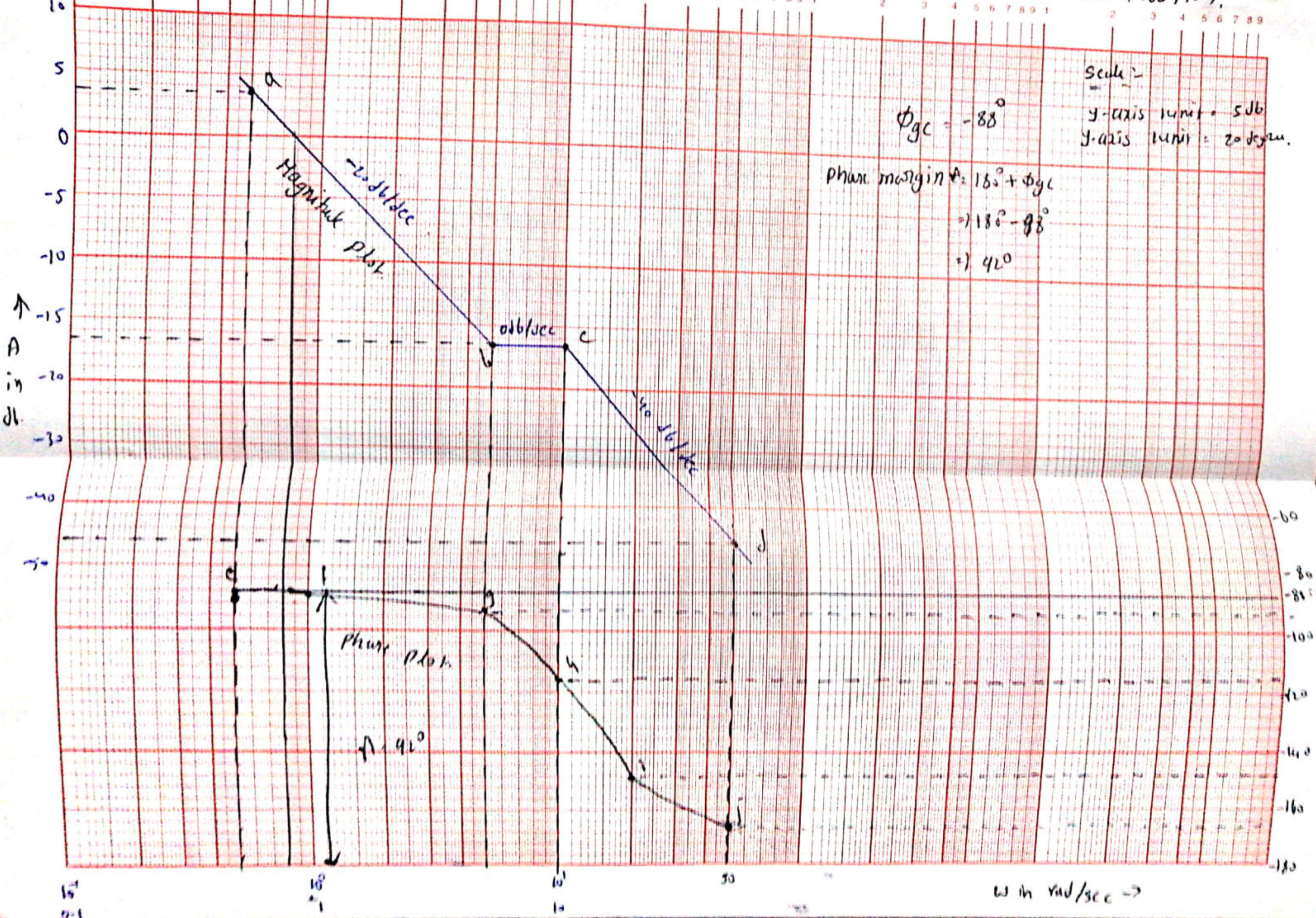
$$G(s) = \frac{75(1+0.2S)}{S(S^2+16S+100)}$$



Ex: Given $G(s) = \frac{K e^{-0.2s}}{s(s+2)(s+8)}$, Find K so that the system is stable with a) gain margin equal to 2db b) phase margin equal to 45°.

Sol: Let us take K = 1 and convert the transfer function to time constant form (or) bode form.

$$\therefore G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} \Rightarrow \frac{e^{-0.2s}}{s \times 2(1+s/2) \times 8(1+s/8)}$$

$$\Rightarrow \frac{0.0625\, e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$

$$\therefore G(j\omega) = \frac{0.0625\, e^{-j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$$

Note: $|0.0625 e^{-j0.2\omega}| = 0.0625$ and $\angle 0.0625 e^{-j0.2\omega} = -0.2\omega$ radians

Magnitude plot:

The corner frequencies are $\omega_{c1} = \frac{1}{0.5} = 2$ rad/sec $\omega_{c2} = \frac{1}{0.125} = 8$ rad/sec.

The various terms of $G(j\omega)$ are listed in Table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope.

Table-1:

Terms	Corner frequencies rad/sec	Slope db/dec	Change in slope db/dec.
$\frac{0.0625}{j\omega}$	–	-20	
$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = 2$ rad/sec	-20	-40
$\frac{1}{1+j0.125\omega}$	$\omega_{c2} = 8$ rad/sec	-20	-60

Choose a low frequency value such that $\omega_l < \omega_{c1}$ and choose high frequency value such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.2$ rad/sec $\omega_h = 50$ rad/sec

Let $A = |G(j\omega)|$ in db

Let us calculate A at $\omega_l, \omega_{c_1}, \omega_{c_2}, \omega_h$.

At $\omega = \omega_l \Rightarrow A = 20\log\left(\frac{0.0625}{0.2}\right) = -10db$

At $\omega = \omega_{c_1} \Rightarrow A = 20\log\left(\frac{0.0625}{2}\right) = -30db$

At $\omega = \omega_{c_2} \Rightarrow A = \{\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\left(\frac{\omega_{c_2}}{\omega_{c_1}}\right)\} + A_{(\omega = \omega_{c_1})}$

$A = \{-40 \times \log\left(\frac{8}{2}\right)\} - 30 \Rightarrow -54db.$

At $\omega = \omega_h \Rightarrow A = \{-60 \times \log\left(\frac{50}{8}\right)\} - 54 \Rightarrow -102db.$

Let the point a, b, c, d be the points corresponding to $\omega_l, \omega_{c_1}, \omega_{c_2}$ and ω_h respectively in the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 10 db on y-axis. The frequencies are choosen in decades from 10^{-1} to 100 on x-axis.

phase plot:-

The phase angle of $G(j\omega)$ as a function of ω is given by.

$$\phi = -0.2\omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(0.125\omega)$$

values of ω and The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table - 2

Table-2 :

ω	$-0.2\omega \times \frac{180}{\pi}$	$\tan^{-1} 0.5\omega$	$\tan^{-1} 0.125\omega$	ϕ
0.2	-2.3	5.7	1.4	$-99.4 \simeq -100$
1	-11.5	26.6	7.1	-135
2	-22.9	45	14	-172
5	-57.3	68.2	32	-247
8	-91.7	76	45	-302 ×
10	-114.6	78.7	51.3	-335 ×
50	-573	87.7	80.9	-831.6 ×

Calculation of K :-

Phase margin $\gamma = 180^\circ + \phi_{gc}$, Where ϕ_{gc} is phase of $G(j\omega)$ at $\omega = \omega_{gc}$

When $\gamma = 45^\circ$ $\quad \phi_{gc} = \gamma - 180^\circ \Rightarrow 45^\circ - 180^\circ = -135^\circ$

When K=1, the gain at $\phi = -135^\circ$ is -24 db. The gain should be made zero to have to phase margin of 45°. Hence every point of magnitude plot a db gain of 24 db should be added. The corrected magnitude plot is obtained by shifting the plot with K=1 by 24 db upwards.

$\therefore 20 \log K = 24$

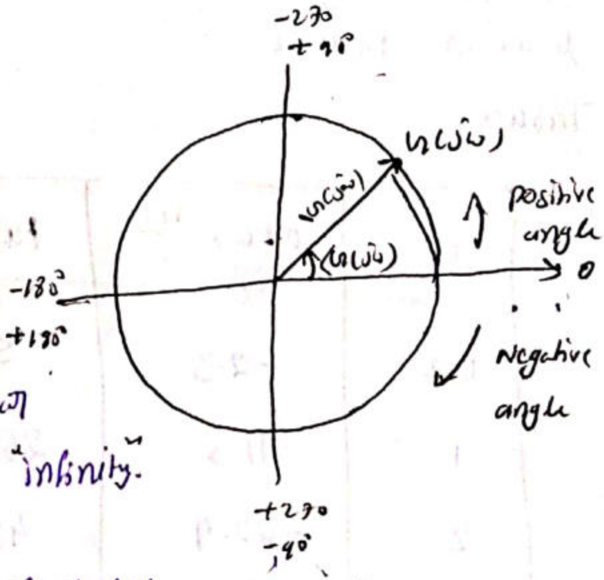
$K = 10^{(24/20)} \Rightarrow 15.8$ "

With K=1, the gain margin $= -(-32) = 32$ db. But the required gain margin is 2 db. Hence every point of magnitude plot a db gain of 30 db should be added.

$\therefore 20 \log K = 30$

$K = 10^{(30/20)} \Rightarrow K = 31.62$ "

* POLAR PLOT :-



The polar plot for a sinusoidal Transfer function $G(j\omega)$ is a plot drawn between magnitude of $G(j\omega)$ i.e, "$|G(j\omega)|$" and the phase angle of $G(j\omega)$ i.e, "$\angle G(j\omega)$" on polar co-ordinates as ω is varied from "zero" to "infinity".

Thus the polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

The polar plot is also called "Nyquist plot."

The polar plot is usually plotted on polar graph sheet. The polar graph sheet consists of concentric circles and radial lines. The circles represents the magnitude and ~~phase angle~~ ~~when~~ the radial lines represents the phase angles. Each point on a polar graph has a magnitude and phase-angle.

In a polar graph sheet a positive phase angle is measured by in anticlock wise direction from the reference axis (0°) and a negative angle is measured clockwise from the reference axis.

In order to plot the polar plot, magnitude and phase angle of $G(j\omega)$ are computed for various values of 'ω' and tabulated. usually the choice of frequencies (ω) are corner frequencies and frequencies around corner frequencies. Choose the proper scale for the magnitude circles. Fix all the points on polar graph sheet and join the points by smooth curve.

Alternatively, if $G(j\omega)$ can be expressed in rectangular coordinates as

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

where $G_R(j\omega)$ = Real part of $G(j\omega)$

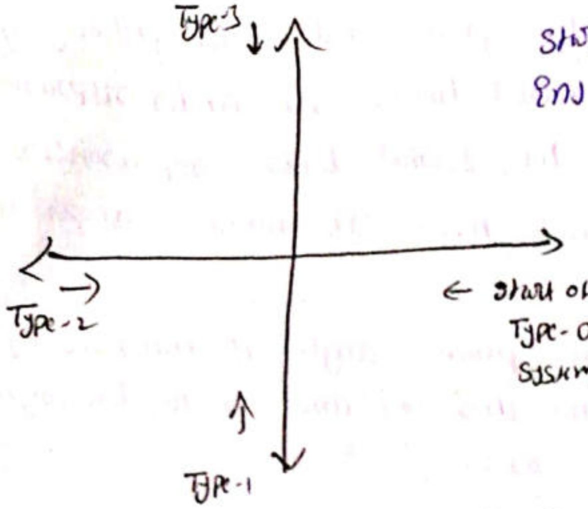
$G_I(j\omega)$ = Imaginary part of $G(j\omega)$.

then the polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞. In order to plot the polar plot on ordinary graph sheet, the magnitude and phase of $G(j\omega)$ are computed for various values of ω. Then convert the polar coordinates to rectangular coordinates using P→R conversion in the calculator. Sketch the polar plot using ordinary graph sheet.

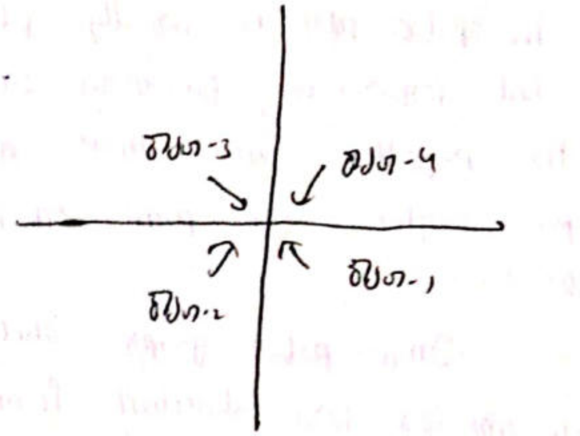
The change in shape of polar plot can be predicted due to addition of a pole (or) zero.

i) When a pole is added to a system, the polar plot end point will shift by $-90°$

ii) when a zero is added to a system, the polar plot end point will shift by $+90°$.



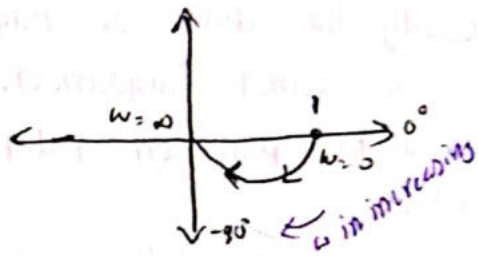
Start of polar plot of all pole minimum phase system.



End of polar plot of all pole minimum phase system.

* Typical Sketches of polar plots :-

i) Type-0, Order-1 :-



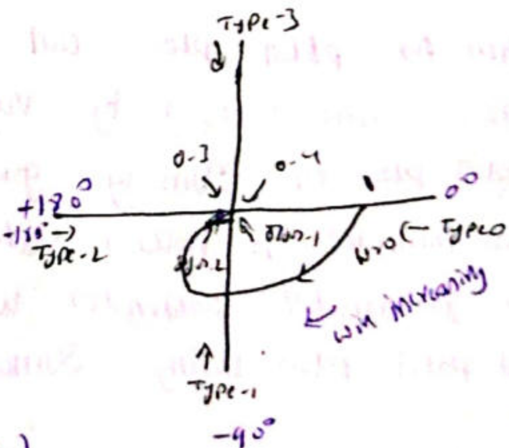
$$G(s) = \frac{1}{1+sT}$$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\ \angle \tan^{-1}\omega T} \Rightarrow \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

At $\omega \to 0$ $\quad G(j\omega) = 1\angle 0^\circ$

At $\omega \to \infty$ $\quad G(j\omega) = 0\angle -90^\circ$

ii) Type-0 Order-2 :- $\quad G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$



$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

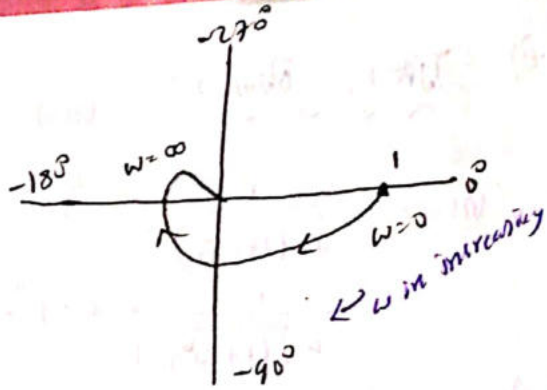
$$\Rightarrow \frac{1}{\sqrt{(1+\omega^2T_1^2)}\ \angle \tan^{-1}(\omega T_1) \cdot \sqrt{(1+\omega^2T_2^2)}\ \angle \tan^{-1}(\omega T_2)}$$

$$\Rightarrow \frac{1}{\sqrt{(1+\omega^2T_1^2)}\sqrt{1+\omega^2T_2^2}} \angle(-\tan^{-1}\omega T_1) - \tan^{-1}\omega T_2)$$

$\omega \to 0 \Rightarrow G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty \Rightarrow G(j\omega) = 0\angle -180^\circ$

iii) Type-0, Order-3 :-



$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

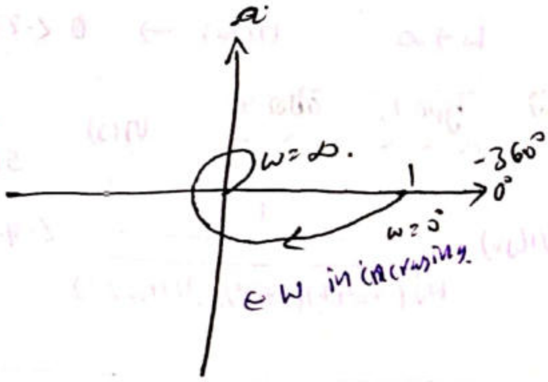
$$\Rightarrow \frac{1}{\sqrt{(1+\omega^2T_1^2)}\angle\tan^{-1}\omega T_1 \cdot \sqrt{(1+\omega^2T_2^2)}\angle\tan^{-1}\omega T_2 \cdot \sqrt{(1+\omega^2T_3^2)}\angle\tan^{-1}\omega T_3}$$

$$\Rightarrow \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}} \cdot \angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

At $\omega \to 0$, $G(j\omega) \to 1\angle 0^\circ$

At $\omega \to \infty$, $G(j\omega) \to 0\angle -270^\circ$

iv) Type-0 Order-4 :-



$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)(1+sT_4)}$$

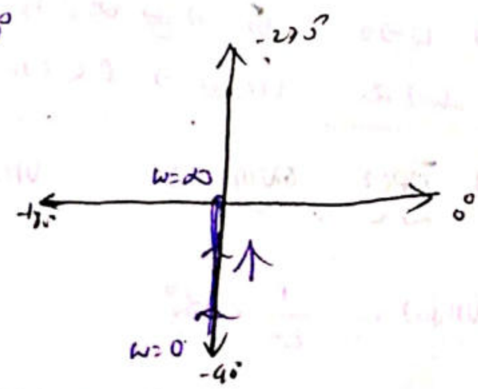
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)(1+j\omega T_4)}$$

$$\Rightarrow \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)(1+\omega^2T_4^2)}} \angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3 - \tan^{-1}\omega T_4)$$

At $\omega \to 0$, $G(j\omega) \to 1\angle 0^\circ$

$\omega \to \infty$ $G(j\omega) \to 0\angle -360^\circ$ (or) $0\angle 0^\circ$

v) Type-1, Order-1 :-



$$G(s) = \frac{1}{s}$$

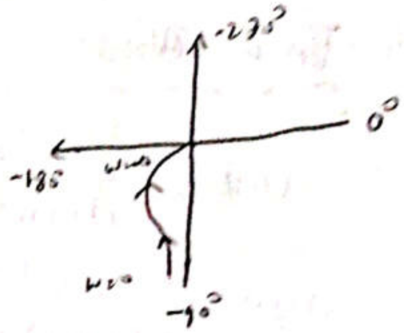
$$G(j\omega) = \frac{1}{j\omega}$$

$$\Rightarrow \frac{1}{\omega\angle 90^\circ} \Rightarrow \frac{1}{\omega}\angle -90^\circ$$

At $\omega \to 0$ $G(j\omega) \to \infty\angle -90^\circ$

At $\omega \to \infty$ $G(j\omega) \to 0\angle -90^\circ$

vi) Type-1, Order-2 :- $G(s) = \dfrac{1}{s(1+sT_1)}$



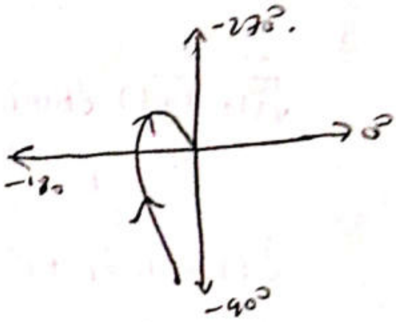
$G(j\omega) = \dfrac{1}{j\omega(1+j\omega T_1)}$

$\Rightarrow \dfrac{1}{\omega\sqrt{(1+\omega^2 T_1^2)}} \angle -90° - \tan^{-1}\omega T_1$

At $\omega \to 0$ $\quad G(j\omega) \to \infty \angle -90°$

$\omega \to \infty$ $\quad G(j\omega) \to 0 \angle -180°$

vii) Type-1, Order-3 :- $G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)}$

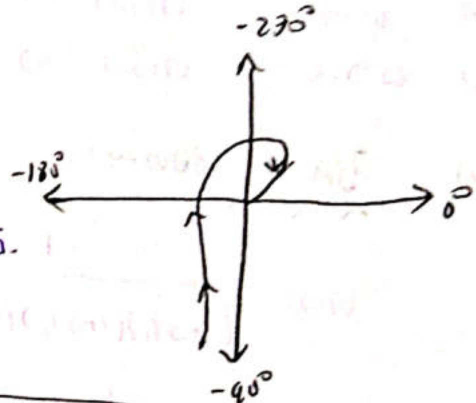


$G(j\omega) = \dfrac{1}{\omega\sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle -90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$

At $\omega \to 0$, $\quad G(j\omega) \to \infty \angle -90°$

$\omega \to \infty$ $\quad G(j\omega) \to 0 \angle -270°$

viii) Type-1, Order-4 :- $G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)(1+sT_3)}$



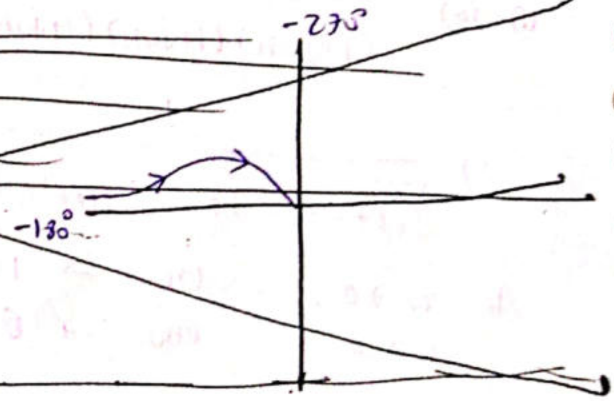
$G(j\omega) = \dfrac{1}{\omega\sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}} \angle -90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$.

ix) Type-2, Order-1 :- $G(s) = \dfrac{1}{s^2(1+sT)}$

$G(j\omega) = \dfrac{1}{\omega^2\sqrt{(1+\omega^2 T_1^2)}} \angle -180° - \tan^{-1}\omega T_1$

At $\omega \to 0$, $\quad G(j\omega) \to \infty \angle -180°$

$\omega \to \infty$ $\quad G(j\omega) \to 0 \angle -270°$

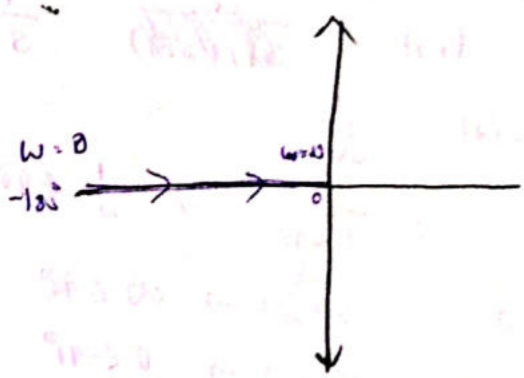


x) Type-2, Order-2 :- $G(s) = \dfrac{1}{s^2}$

$G(j\omega) = \dfrac{1}{\omega^2} \angle -180°$

$\omega \to 0$ $\quad G(j\omega) \to \infty \angle -180°$

$\omega \to \infty$ $\quad G(j\omega) \to 0° \angle -180°$



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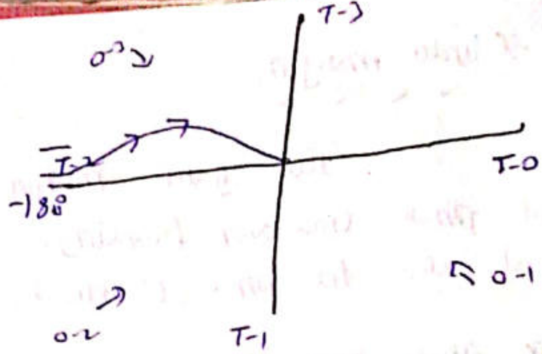
xi) Type-2, Order-3

$G(s) = \dfrac{1}{s^2(1+sT_1)}$



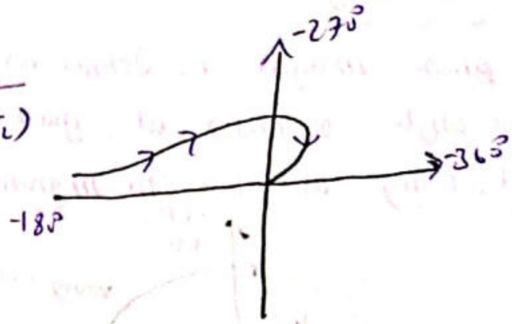
$$G(j\omega) = \frac{1}{\omega^2\sqrt{(1+\omega^2 T_1^2)}} \angle -180^\circ - \tan^{-1}\omega T_1$$

At $\omega \to 0 \qquad G(j\omega) \to \infty \angle -180^\circ$

$\omega \to \infty \qquad G(j\omega) \to 0 \angle -270^\circ$

xii) Type-2, Order-4

$G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$



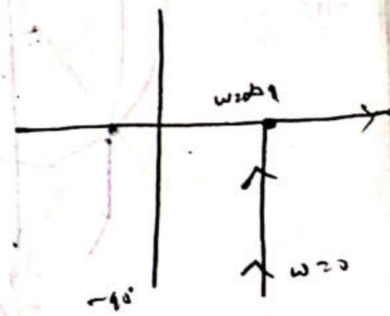
$$G(j\omega) = \frac{1}{\omega^2\sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

At $\omega \to 0 \qquad G(j\omega) \to \infty \angle -180^\circ$

$\omega \to \infty \qquad G(j\omega) \to 0 \angle -360^\circ$

xiii) $G(s) = \dfrac{1+sT}{sT}$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T} \Rightarrow \frac{1}{j\omega T} + 1 \Rightarrow \frac{1}{\omega T}\angle -90^\circ + 1$$



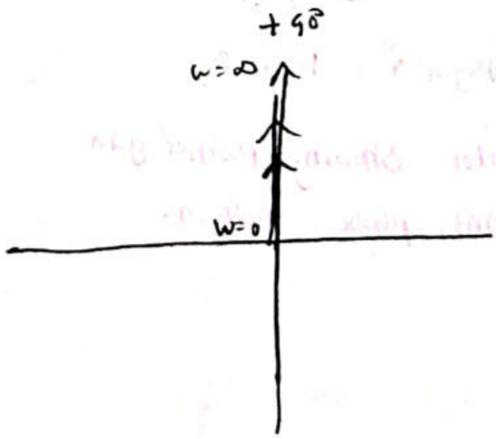
As $\omega \to 0 \qquad G(j\omega) \to \infty \angle -90^\circ + 1$

$\omega \to \infty \qquad G(j\omega) \to 0 \angle -90^\circ + 1$

xiv) $G(s) = s$

$G(j\omega) = j\omega$

$\Rightarrow \omega \angle 90^\circ$



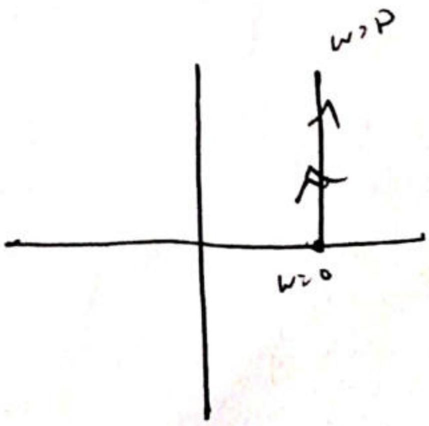
As $\omega \to 0, \qquad G(j\omega) \to 0 \angle 90^\circ$

$\omega \to \infty \qquad G(j\omega) \to \infty \angle 90^\circ$

xv) $G(s) = 1 + sT$

$G(j\omega) = 1 + j\omega T$

$\Rightarrow 1 + \omega T \angle 90^\circ$



As $\omega \to 0 \qquad G(j\omega) \to 1 + 0\angle 90^\circ$

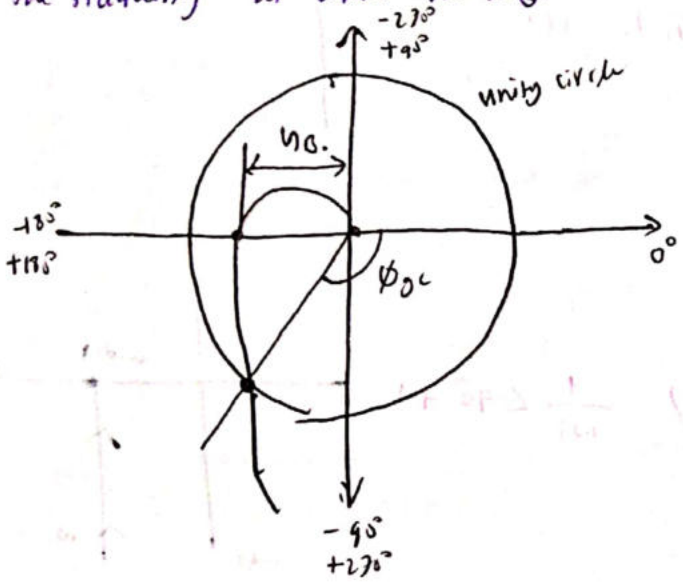
$\omega \to \infty \qquad G(j\omega) \to 1 + \infty \angle 90^\circ$

* Gain margin:-

The gain margin is defined as the inverse of magnitude of $G(j\omega)$ at phase cross over frequency. The phase cross over frequency is the frequency at which the phase of $G(j\omega)$ is 180°

* Phase margin:-

The phase margin is defined as, the phase margin $\gamma = 180^\circ + \phi_{gc}$ where ϕ_{gc} is phase angle of $G(j\omega)$ at gain cross over frequency. The gain crossover frequency is the frequency at which the magnitude of $G(j\omega)$ is unity.



Gain margin $K_g = \frac{1}{|G_{B}|}$

Phase margin $\gamma = 180^\circ + \phi_{gc}$

Polar plot showing positive gain margin and phase margin.

Ex. The open loop Transfer function of a unity feedback. System is given by $G(s) = \frac{1}{s(1+s)(1+2s)}$. Sketch the polar plot and determine the gain margin and phase margin.

Sol. given that $G(s) = \frac{1}{s(1+s)(1+2s)}$

The sinusoidal Transfer function $G(j\omega)$ is obtained by replacing "s" by $j\omega$ then

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)}.$$

The corner frequencies. $\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec $\omega_{c_2} = \frac{1}{1} = 1$ rad/sec.

The magnitude & phase angle of $G(j\omega)$ are calculated for corner frequencies and ~~___~~ frequencies around corner frequencies. and tabulated in Table-1. using polar to rectangular conversion, the values for real part and imaginary part for different values of "ω" are tabulated in Table-2. The polar plot as shown in fig-1 using rectangular coordinates is sketched in ordinary graph

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)} = \frac{1}{\omega\angle 90^\circ \cdot \sqrt{1+\omega^2}\angle\tan^{-1}\omega \cdot \sqrt{1+4\omega^2}\angle\tan^{-1}2\omega}$$

$$\Rightarrow \frac{1}{\omega\sqrt{1+\omega^2}\cdot\sqrt{1+4\omega^2}} \cdot \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega \Rightarrow \phi$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} \Rightarrow \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} \Rightarrow \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

Table-1:

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	0.8	1
$\lvert G(j\omega)\rvert$	2.2	1.7	1.4	1.3	0.9	0.7	0.5	0.3
$\angle G(j\omega)$ deg	-144°	-153	-156°	-162°	-171°	-180°	-187°	-198°

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At $\omega = 0.35$:-

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}} \Rightarrow \frac{1}{0.35\sqrt{1+0.61+0.06}} \Rightarrow \frac{1}{0.45} \Rightarrow 2.22$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega \Rightarrow -90^\circ - \tan^{-1}(0.35) - \tan^{-1}(0.7) \Rightarrow -144^\circ.$$

At $\omega = 0.4$:-

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}} \Rightarrow \frac{1}{0.4\sqrt{1+0.8+0.1}} \Rightarrow \frac{1}{0.6} \Rightarrow 1.7$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.4) - \tan^{-1}(0.8) \Rightarrow -150^\circ.$$

At $\omega = 0.45$:-

$$|G(j\omega)| = \frac{1}{0.45\sqrt{1+1+0.2}} \Rightarrow \frac{1}{0.7} \Rightarrow 1.4$$

$$\angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}(0.45) - \tan^{-1}(0.9) \Rightarrow -156^\circ$$

At $\omega = 0.5$:-

$$|G(j\omega)| = \frac{1}{0.5\sqrt{1+1.3+0.3}} \Rightarrow \frac{1}{0.8} \Rightarrow 1.3$$

$$\angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}(0.5) - \tan^{-1}(1) = -162^\circ$$

At $\omega = 0.6$:-

$$|G(j\omega)| = \frac{1}{0.6\sqrt{1+1.8+0.5}} \Rightarrow \frac{1}{1.1} \Rightarrow 0.9$$

$$\angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}(0.6) - \tan^{-1}(1.2) \Rightarrow -171^\circ$$

At $\omega = 0.7$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.7\sqrt{1+2.5+1}} \Rightarrow \frac{1}{1.5} \Rightarrow 0.7$$

$$\angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}(0.7) - \tan^{-1}(1.4) \Rightarrow -180^\circ$$

At $\omega = 0.8$:-

$$|G(j\omega)| = \frac{1}{0.8\sqrt{1+3.2+1.6}} \Rightarrow \frac{1}{1.9} \Rightarrow 0.5$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.8) - \tan^{-1}(1.6) \Rightarrow -187^\circ$$

At $\omega = 1$:-

$$|G(j\omega)| \Rightarrow \frac{1}{1\sqrt{1+5+4}} \Rightarrow 0.31$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(1) - \tan^{-1}(2) \Rightarrow -198^\circ$$

Table-2:-

ω	0.35	0.4	0.45	0.5	0.6	0.7	0.8	1
$G_R(\omega)$	-1.79	-1.48	-1.27	-1.24	-0.89	-0.7	-0.5	-0.29
$G_I(\omega)$	-1.3	-0.9	-0.6	-0.4	-0.14	0	0.06	0.09.

$$G(j\omega) = \frac{\angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

At $\omega = 0.35$:-

$G(j\omega) \Rightarrow 2.22\angle -144^\circ \Rightarrow -1.79 - j1.3$

At $\omega = 0.4$:-

$G(j\omega) \Rightarrow 1.7\angle -150^\circ \Rightarrow -1.48 + j0.9$

At $\omega = 0.45$:-

$G(j\omega) = 1.4\angle -156^\circ \Rightarrow -1.27 - j0.6$

At $\omega = 0.5$:-

$G(j\omega) = 1.3\angle -162^\circ \Rightarrow -1.24 - j0.4$

At $\omega = 0.6$:

$G(j\omega) = 0.9\angle -171^\circ \Rightarrow -0.89 + j0.14$

At $\omega = 0.7$:-

$G(j\omega) = 0.7\angle -180^\circ \Rightarrow -0.7 + j0$

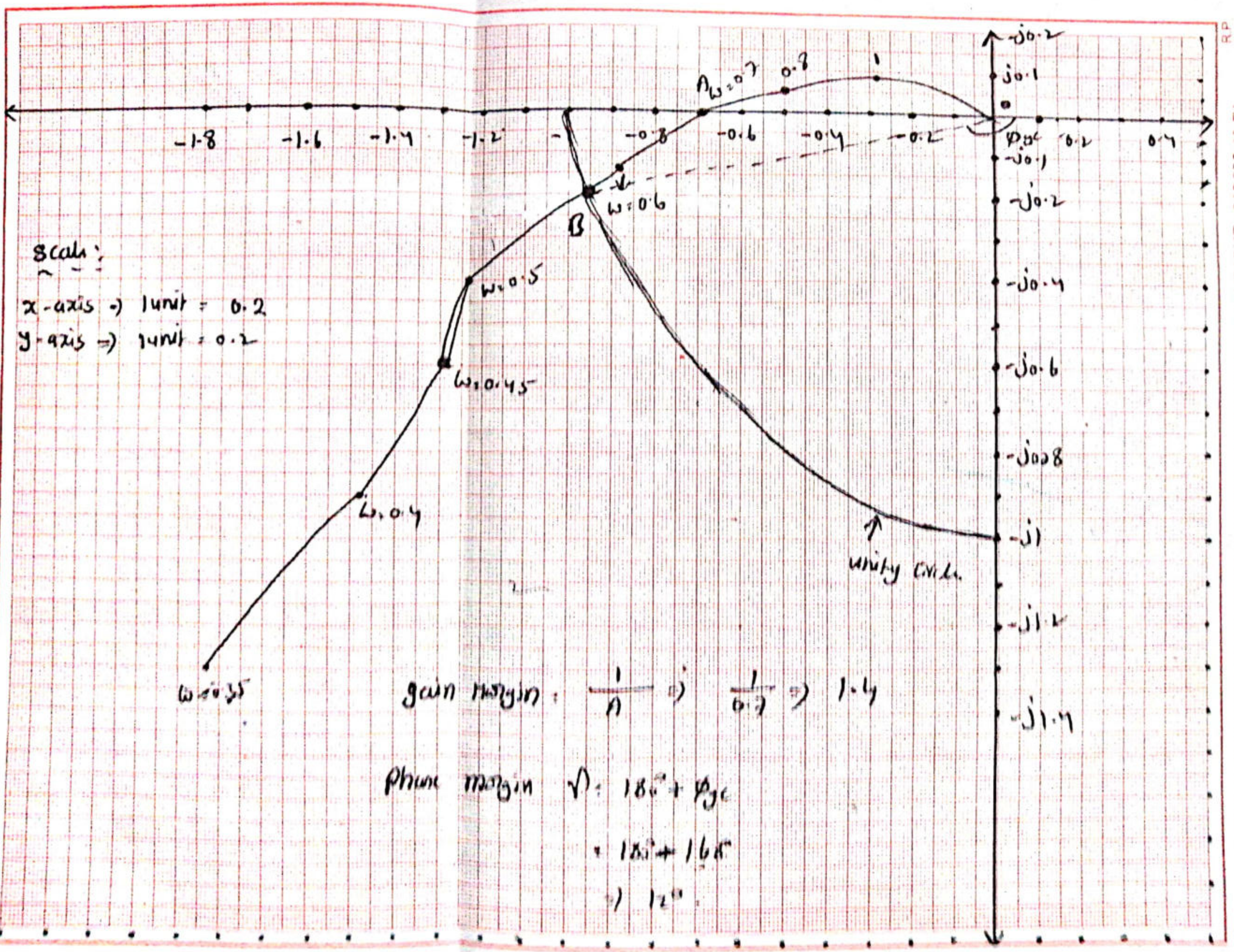
At $\omega = 0.8$:-

$G(j\omega) = 0.5\angle -187^\circ \Rightarrow -0.5 + j0.06$

At $\omega = 1$:-

$G(j\omega) = 0.3\angle -198^\circ \Rightarrow -0.29 + j0.09$

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$ (using rectangular coordinates).



Scale:

x-axis ⇒ 1 unit = 0.2

y-axis ⇒ 1 unit = 0.2

gain margin : $\frac{1}{A} \Rightarrow \frac{1}{0.7} \Rightarrow 1.4$

Phase margin γ : $180° + \phi_{gc}$

$= 180° + 168°$

$\Rightarrow 12°$

2x32

The open loop Transfer function of a unity feedback system is given by

$G(s) = \frac{1}{s^2(1+s)(1+2s)}$. Sketch the polar plot and determine the G.M & P.M.

Sol, given that $G(s) = \frac{1}{s^2(1+s)(1+2s)}$

The sinusoidal Transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$

$$\therefore G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

The corner frequencies are $\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c_2} = 1$ rad/sec

The polar plot can be obtained by determination of magnitude of $G(j\omega)$ and phase angle of $G(j\omega)$ for different values of "ω". But choose values of ω as a corner frequencies and frequencies around corner frequencies and values of magnitude of $G(j\omega)$ and phase angle of $G(j\omega)$ value are Tabulated in Table-1. and by using these values polar plot can be drawn in polar graph sheet. Another method to draw polar plot by using polar to rectangular coordinates conversion, then seperate the real and imaginary parts after that find these values for different values of "ω" and tabulated in Table-2 and by using these points draw the polar plot in normal graph sheet.

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

$$\Rightarrow \frac{1}{\omega^2\angle 180^\circ \cdot \sqrt{1+\omega^2}\angle\tan^{-1}\omega \cdot \sqrt{1+4\omega^2} \cdot \angle\tan^{-1}2\omega}$$

$$\Rightarrow \frac{1}{\omega^2\sqrt{(1+\omega^2)(1+4\omega^2)}} \angle -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

$$|G(j\omega)| \Rightarrow \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) \Rightarrow \angle -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

Table-1:-

ω rad/sec	0.3	0.4	0.5	0.6	0.7	0.8	1
$\lvert G(j\omega)\rvert$	9.1	5.6	2.5	1.5	1	0.65	0.3
$\angle G(j\omega)$	-229	-240	-252	-261	-269	-277	-288°

At $\omega = 0.3$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.09\sqrt{1+0.45+0.03}} \Rightarrow \frac{1}{0.11} \Rightarrow 9.09 \simeq 9.1$$

$$\angle G(j\omega) \Rightarrow \angle -180^\circ - 16.70 - 30.96 \Rightarrow -229$$

At $\omega = 0.4$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.16\sqrt{1+0.13+0.1}} \Rightarrow \frac{1}{0.18} \Rightarrow 5.6$$

$$\angle G(j\omega) = -180^\circ - 21.80 - 38.66^\circ \Rightarrow -240$$

At $\omega = 0.5$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.25\sqrt{1+1.25+0.25}} \Rightarrow \frac{1}{0.4} \Rightarrow 2.5$$

$$\angle G(j\omega) \Rightarrow -252$$

At $\omega = 0.6$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.36\sqrt{1+1.8+0.52}} \Rightarrow \frac{1}{0.66} \Rightarrow 1.5$$

$$\angle G(j\omega) \Rightarrow -261$$

At $\omega = 0.7$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.49\sqrt{1+2.45+0.46}} \Rightarrow \frac{1}{0.9} \Rightarrow 1.11$$

$$\angle G(j\omega) \Rightarrow -269^\circ$$

At $\omega = 0.8$:-

$$|G(j\omega)| \Rightarrow \frac{1}{0.64\sqrt{1+3.2+1.64}} \Rightarrow \frac{1}{1.55} \Rightarrow 0.65 \simeq 0.7$$

$$\angle G(j\omega) \Rightarrow -277^\circ$$

At $\omega = 1$:

$$|G(j\omega)| \Rightarrow \frac{1}{1\sqrt{1+5+4}} \Rightarrow \frac{1}{\sqrt{10}} \Rightarrow 0.32 \simeq 0.3$$

$$\angle G(j\omega) \Rightarrow -288^\circ$$

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

Subject :-

Date

Scale:

x-axis ⇒ 1cm = 0.2 units

y-axis ⇒ 1cm = 0.2 units.

Gain margin = $\frac{1}{\infty} = 0$

Phase margin = $180^\circ - 270^\circ$

⇒ -95

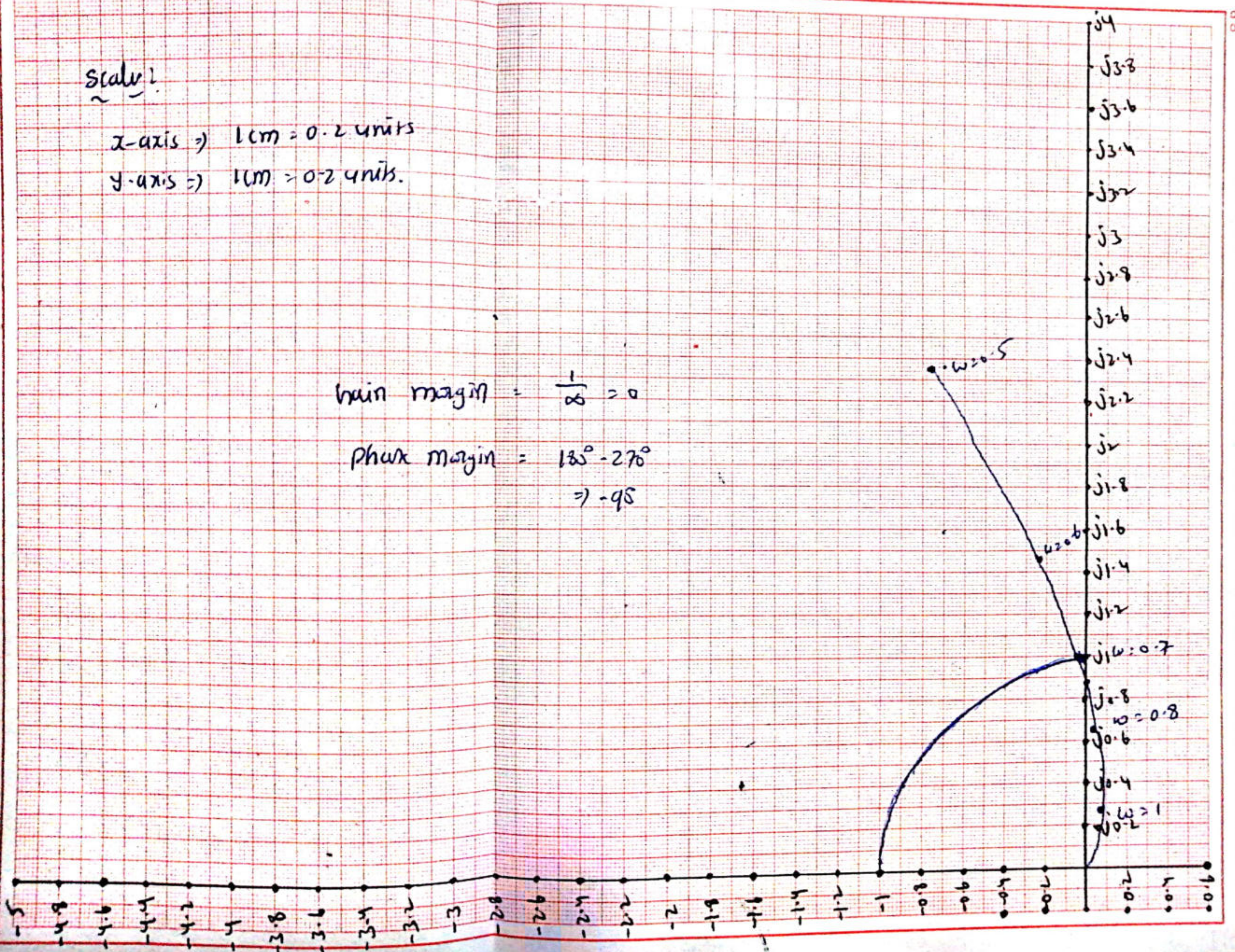


Table-2:

ω	0.3	0.4	0.5	0.6	0.7	0.8	1
$G_R(\omega)$	-6.08	-2.8	-0.77	-0.23	-0.08	0.08	0.09
$G_I(\omega)$	6.76	4.84	2.37	1.48	1	0.64	0.29

At ω = 0.3:

$|G(j\omega)| \angle G(j\omega) \Rightarrow 9.1 \angle -228^\circ \Rightarrow -6.08 + j6.76$

At ω = 0.4:

$|G(j\omega)| \angle G(j\omega) \Rightarrow 5.6 \angle -240 \Rightarrow -2.8 + j4.84$

At ω = 0.5:

$|G(j\omega)| \angle G(j\omega) \Rightarrow 2.5 \angle -252 \Rightarrow -0.77 + j2.37$

At ω = 0.6: $|G(j\omega)| \angle G(j\omega) \Rightarrow 1.5 \angle -261 \Rightarrow -0.23 + j1.48$.

At ω = 0.7: $|G(j\omega)| \angle G(j\omega) = 1 \angle -264 \Rightarrow 0 + j1.0$

At ω = 0.8: $|G(j\omega)| \angle G(j\omega) = 0.65 \angle -277^\circ \Rightarrow 0.08 + j0.64$

At ω = 1: $|G(j\omega)| \angle G(j\omega) = 0.3 \angle -288^\circ \Rightarrow 0.09 + j0.29$

Ex: 3: The open loop Transfer function of a unity feedback system is given by $G(s) = \dfrac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$. Sketch the polar plot and determine the phase margin.

Sol:
$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$

The sinusoidal Transfer function is obtained by ~~using~~ replacing "s" by "jω"

$$G(j\omega) = \frac{(1+j0.2\omega)(1+j0.025\omega)}{(j\omega)^3(1+j0.005\omega)(1+j0.001\omega)}$$

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$$G(j\omega) = \frac{\sqrt{1+(0.2\omega)^2}\angle \tan^{-1}0.2\omega \cdot \sqrt{1+(0.025\omega)^2}\angle \tan^{-1}0.025\omega}{\omega^3\angle 270^\circ \cdot \sqrt{1+(0.005\omega)^2}\angle \tan^{-1}0.005\omega \cdot \sqrt{1+(0.001\omega)^2}\angle \tan^{-1}0.001\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+(0.2\omega)^2}\cdot\sqrt{(1+(0.025\omega)^2}}{\omega^3\sqrt{(1+0.005\omega)^2}\cdot\sqrt{1+(0.001\omega)^2}}$$

$$\angle G(j\omega) = \tan^{-1}0.2\omega + \tan^{-1}0.025\omega - 270^\circ - \tan^{-1}0.005\omega - \tan^{-1}0.001\omega.$$

The magnitude and phase angle of $G(j\omega)$ are calculated for various frequencies and listed in Table-1. Using polar to rectangular conversion, the polar coordinates listed in Table-1 are converted to rectangular co-ordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet.

Table-1:-

ω rad/sec	0.5	0.6	0.7	0.8	1	1.1	1.2	1.4	1.6
$\lvert G(j\omega)\rvert$	8	6	3	1.8	1	0.8	0.6	0.4	0.3
$\angle G(j\omega)$	-264	-263°	-261	-260	-258°	-256°	-255°	-253°	-251°

At $\omega = 0.5$:-

$$|G(j\omega)| = \frac{\sqrt{(1+(0.2\omega)^2}\cdot\sqrt{(1+(0.025\omega)^2}}{\omega^3\sqrt{(1+(0.005\omega)^2}\sqrt{(1+(0.001\omega)^2}} \Rightarrow \frac{1\times1}{0.13\times1\times1} \Rightarrow 8$$

$\angle G(j\omega) = -264^\circ$

At $\omega = 0.6$:- $|G(j\omega)| \Rightarrow \frac{1.01\times1}{0.22\times1\times1} \Rightarrow 6$

$\angle G(j\omega) \Rightarrow -263^\circ$

At $\omega = 0.7$: $|G(j\omega)| \Rightarrow \frac{1.01\times1}{0.34\times1\times1} \Rightarrow 3$

$\angle G(j\omega) \Rightarrow -261$

At $\omega = 0.8$:-

$$|G(j\omega)| \Rightarrow \frac{1.01 \times 1}{0.51 \times 1 \times 1} \Rightarrow 1.8$$

$$\angle G(j\omega) \Rightarrow -260^\circ$$

At $\omega = 1$:-

$$|G(j\omega)| \Rightarrow \frac{1.02 \times 1}{1 \times 1 \times 1} \Rightarrow 1$$

$$\angle G(j\omega) \Rightarrow -258^\circ$$

At $\omega = 1.1$:-

$$|G(j\omega)| \Rightarrow \frac{1.02 \times 1}{1.21 \times 1 \times 1} \Rightarrow 0.8$$

$$\angle G(j\omega) \Rightarrow -256^\circ$$

At $\omega = 1.2$:-

$$|G(j\omega)| \Rightarrow \frac{1.03 \times 1}{1.73 \times 1 \times 1} \Rightarrow 0.6$$

$$\angle G(j\omega) \Rightarrow -255^\circ$$

At $\omega = 1.4$:-

$$|G(j\omega)| \Rightarrow \frac{1.04 \times 1}{2.74 \times 1 \times 1} \Rightarrow 0.4$$

$$\angle G(j\omega) \Rightarrow -253^\circ$$

At $\omega = 1.6$:-

$$|G(j\omega)| \Rightarrow \frac{1.05 \times 1}{4.1 \times 1 \times 1} \Rightarrow 0.3$$

$$\angle G(j\omega) = \angle -251^\circ$$

Table-2:-

ω	0.5	0.6	0.7	0.8	1	1.1	1.2	1.4	1.6
	-0.8	-0.7	-0.5	-0.3	-0.2	-0.19	-0.15	-0.11	-0.09
$G_R(j\omega)$	8	6	3	2	0.97	0.77	0.58	0.38	0.28.
$G_I(j\omega)$									

phase angle $\gamma = 185 + \phi_{gc}$

$\Rightarrow 180^\circ - 257^\circ$

$\Rightarrow -77^\circ$

Scale:

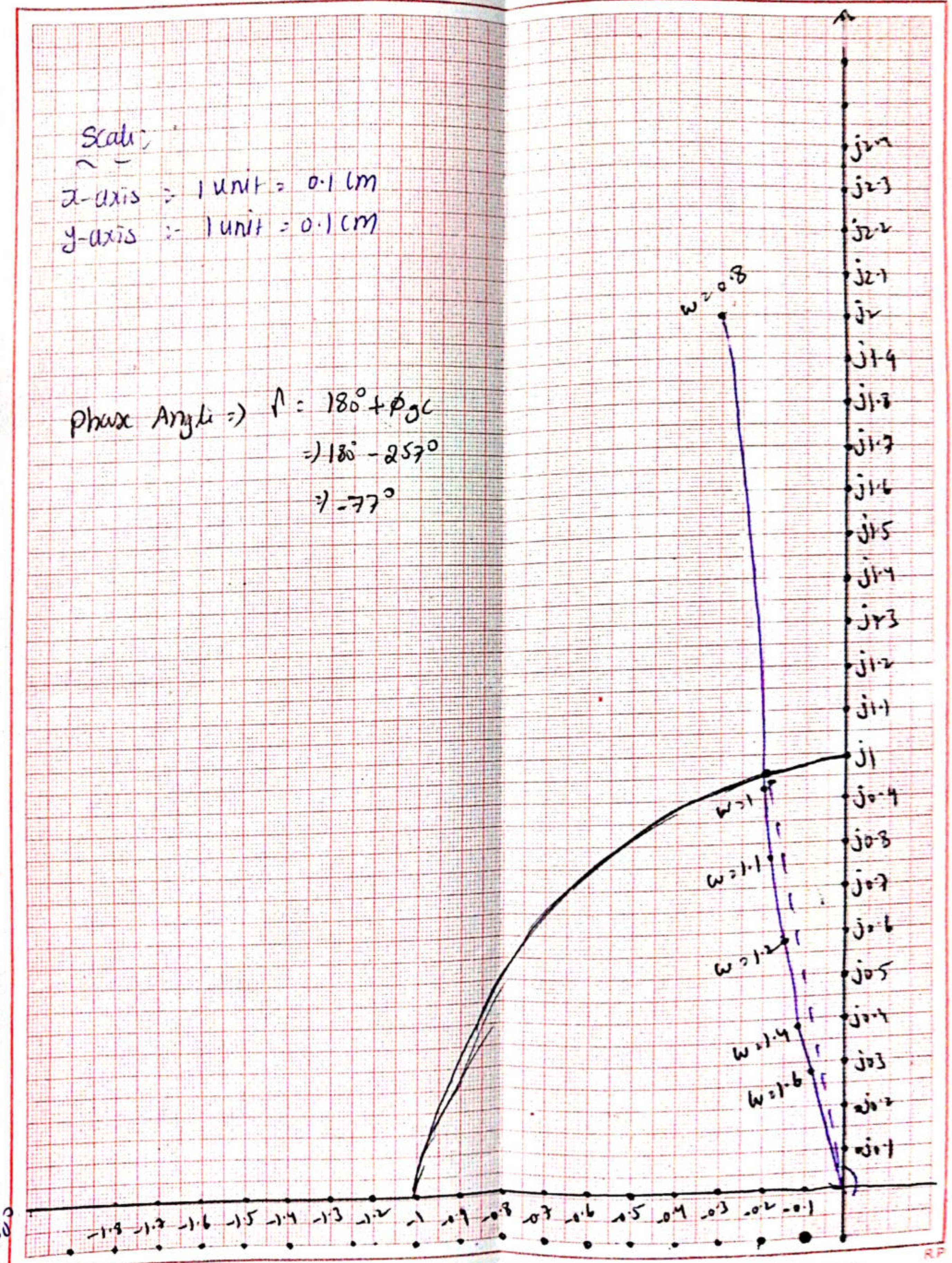
x-axis : 1 unit = 0.1 cm

y-axis : 1 unit = 0.1 cm

Phase Angle => $\gamma = 180^\circ + \phi_{gc}$

$\Rightarrow 180^\circ - 257^\circ$

$\Rightarrow -77^\circ$

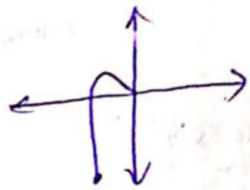


$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$

Ex 24 The open loop T.F of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)^2}$. Sketch the polar plot and determine the gain and phase margin.

Sol

$$G(s) = \frac{1}{s(1+s)(1+s)}$$



$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

$$\Rightarrow \frac{1}{\omega\angle 90^\circ \cdot \sqrt{(1+\omega^2)}\angle \tan^{-1}\omega \cdot \sqrt{(1+\omega^2)} \cdot \tan^{-1}\omega}$$

$$|G(j\omega)| \Rightarrow \frac{1}{\omega\sqrt{(1+\omega^2)(1+\omega^2)}} \Rightarrow \frac{1}{\omega\sqrt{(1+\omega^2)^4}} \Rightarrow \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) \Rightarrow -90^\circ - 2\tan^{-1}\omega.$$

Table-1:

ω	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
$\lvert G(j\omega)\rvert$	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4	0.3	0.24
$\angle G(j\omega)$	-134°	-143	-152°	-160°	-167°	-174°	-180°	-185°	-19°	-195°

At $\omega = 0.4$:

$$|G(j\omega)| \Rightarrow \frac{1}{0.4 + 0.06} \Rightarrow 2.2$$

$$\angle G(j\omega) \Rightarrow -134^\circ$$

At $\omega = 0.5$:

$$|G(j\omega)| \Rightarrow \frac{1}{0.5 + 0.13} \Rightarrow 1.6$$

$$\angle G(j\omega) \Rightarrow -143$$

At $\omega = 0.6$:

$$|G(j\omega)| \Rightarrow \frac{1}{0.6 + 0.22} \Rightarrow 1.2$$

$$\angle G(j\omega) = -152$$

At $\omega = 0.7$:

$$|G(j\omega)| = \frac{1}{0.7 + 0.34} \Rightarrow 1$$

$$\angle G(j\omega) \Rightarrow -160^\circ$$

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At $\omega = 0.8$:-

$$|G(j\omega)| = \frac{1}{0.8 + 0.5} \Rightarrow 0.8$$

$$\angle G(j\omega) \Rightarrow -167^\circ$$

At $\omega = 0.9$:-

$$|G(j\omega)| = \frac{1}{0.9 + 0.73} \Rightarrow 0.6$$

$$\angle G(j\omega) \Rightarrow -174^\circ$$

At $\omega = 1$:-

$$|G(j\omega)| = 0.5$$

$$\angle G(j\omega) \Rightarrow -180^\circ$$

At $\omega = 1.1$:-

$$|G(j\omega)| = \frac{1}{1.1 + 1.33} \Rightarrow 0.4$$

$$\angle G(j\omega) \Rightarrow -185^\circ$$

At $\omega = 1.2$:-

$$|G(j\omega)| \Rightarrow \frac{1}{1.2 + 1.73} \Rightarrow 0.3$$

$$\angle G(j\omega) \Rightarrow -190^\circ$$

At $\omega = 1.3$:-

$$|G(j\omega)| \Rightarrow \frac{1}{1.3 + 2.2} \Rightarrow 0.29$$

$$\angle G(j\omega) \Rightarrow -195^\circ$$

Table 2 :-

ω	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
$G_R(j\omega)$	-1.53	-1.28	-1.05	-0.93	-0.78	-0.6	-0.5	-0.4	-0.29	-0.28
$G_I(j\omega)$	-1.58	-0.96	-0.58	-0.36	-0.18	-0.06	0	0.03	0.05	0.07

$$\text{Gain Margin} = \frac{1}{0.5} = 2$$

$$\text{Phase margin} = 180^\circ - 159^\circ$$

$$\Rightarrow 21^\circ$$

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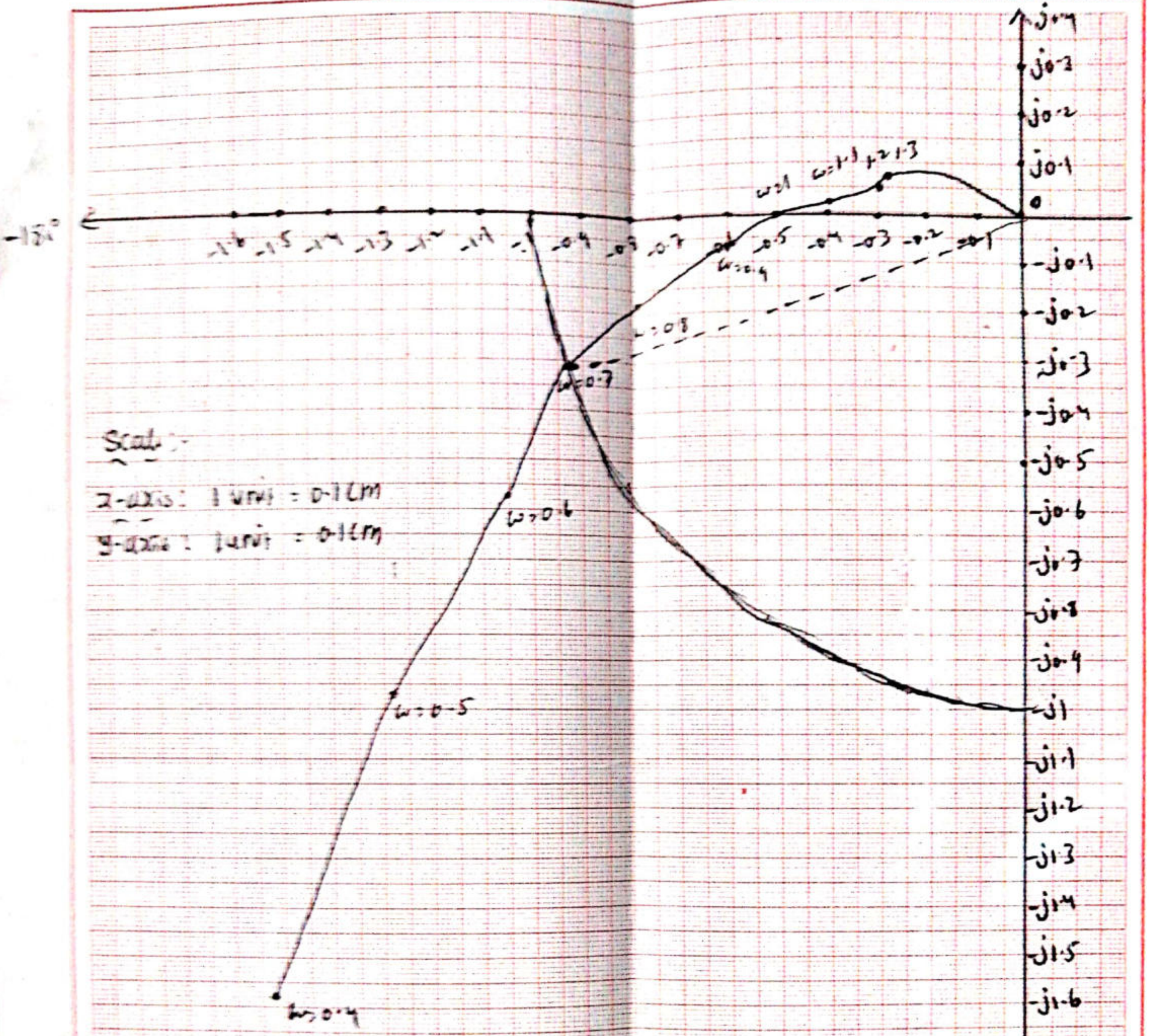

Scale :-

x-axis : 1 unit = 0.1 cm

y-axis : 1 unit = 0.1 cm

Gain margin = $\frac{1}{|G(j\omega_{pc})|} \Rightarrow \frac{1}{0.5} \Rightarrow 2$

Phase margin γ = $180° - 154°$

$\Rightarrow 21°$

-j3

$$G(s) = \frac{1}{s(1+s)^2}$$

Ex 25: Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db (ii) phase margin is 60°.

Sol

$$G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$$. The polar plot sketched by taking K = 1

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

The corner frequencies are $\omega_{c1} = 1/0.2 = 5$ and $\omega_{c2} = 1/0.05 = 20$ rad/sec. The magnitude and phase angle of $G(j\omega)$ are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular co-ordinates and tabulated in table-2.

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$$\Rightarrow \frac{1}{\omega\angle 90^\circ \cdot \sqrt{1+(0.2\omega)^2}\,\angle\tan^{-1}0.2\omega \cdot \sqrt{1+(0.05\omega)^2}\,\angle\tan^{-1}0.05\omega}$$

$$\therefore |G(j\omega)| \Rightarrow \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) \Rightarrow -90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

Table-1: Magnitude and phase of $G(j\omega)$ at various frequencies.

ω	0.6	0.8	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert G(j\omega)\rvert$	1.65	1.24	1	0.5	0.3	0.2	0.14	0.1	0.08	0.06	0.05	0.04	0.03	0.03
$\angle G(j\omega)$	−99°	−101°	−104°	−113°	−129°	−140°	−149°	−157°	−164°	−170°	−175°	−180°	[illegible]	[illegible]

At $\omega = 0.6$: $|G(j\omega)| \Rightarrow \frac{1}{0.6\times1.01\times1} \Rightarrow 1.65$, $\angle G(j\omega) \Rightarrow -99^\circ$

At $\omega = 0.8$: $|G(j\omega)| \Rightarrow \frac{1}{0.8\times1.01\times1} \Rightarrow 1.24$, $\angle G(j\omega) \Rightarrow -101^\circ$

At $\omega = 1$: $|G(j\omega)| \Rightarrow \cancel{1\times1.02\times0.65} \Rightarrow \frac{1}{1\times1.02\times1} \Rightarrow 1$, $\angle G(j\omega) = -104^\circ$

At $\omega = 2$:- $|G(\omega j)| = \dfrac{1}{2 \times 1.08 \times 1} \Rightarrow 0.5$, $\angle G(j\omega) = 118^\circ$

At $\omega = 3$:- $|G(j\omega)| \Rightarrow \dfrac{1}{3 \times 1.17 \times 1.01} \Rightarrow 0.3$, $\angle G(j\omega) \Rightarrow -129^\circ$

At $\omega = 4$:- $|G(\omega j)| \Rightarrow \dfrac{1}{4 \times 1.28 \times 1.02} \Rightarrow 0.2$, $\angle G(j\omega) \Rightarrow -140^\circ$

At $\omega = 5$:- $|G(\omega j)| \Rightarrow \dfrac{1}{5 \times 1.41 \times 1.03} \Rightarrow 0.14$, $\angle G(j\omega) \Rightarrow -149^\circ$

At $\omega = 6$:- $|G(\omega j)| \Rightarrow \dfrac{1}{6 \times 1.56 \times 1.04} \Rightarrow 0.1$, $\angle G(j\omega) \Rightarrow -157^\circ$

At $\omega = 7$:- $|G(\omega j)| \Rightarrow \dfrac{1}{7 \times 1.72 \times 1.06} \Rightarrow 0.08$, $\angle G(j\omega) \Rightarrow -164^\circ$

At $\omega = 8$:- $|G(j\omega)| \Rightarrow \dfrac{1}{8 \times 1.89 \times 1.08} \Rightarrow 0.06$, $\angle G(j\omega) \Rightarrow -170^\circ$

At $\omega = 9$:- $|G(\omega j)| \Rightarrow \dfrac{1}{9 \times 2.06 \times 1.10} \Rightarrow 0.05$, $\angle G(j\omega) \Rightarrow -175^\circ$

At $\omega = 10$:- $|G(\omega j)| \Rightarrow \dfrac{1}{10 \times 2.24 \times 1.12} \Rightarrow 0.04$, $\angle G(j\omega) \Rightarrow -180^\circ$

At $\omega = 11$:- $|G(\omega j)| \Rightarrow \dfrac{1}{11 \times 2.42 \times 1.14} \Rightarrow 0.03$, $\angle G(j\omega) \Rightarrow -184^\circ$

At $\omega = 12$:- $|G(\omega j)| \Rightarrow \dfrac{1}{12 \times 2.6 \times 1.17} \Rightarrow 0.03$, $\angle G(j\omega) \Rightarrow -188^\circ$

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Table-2: Real and imaginary part of $G(j\omega)$ at various frequencies.

ω	0.6	0.8	1	2	3	4	5	6	7	8	9	10	11	12
$G_R(j\omega)$	-0.25	-0.23	-0.24	-0.23	-0.19	-0.15	-0.12	-0.09	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02
$G_I(j\omega)$	-1.63	-1.21	-0.97	-0.44	-0.23	-0.13	-0.07	-0.04	-0.02	-0.01	-0.004	0	0.007	0.005

There are two plots marked as Curve-I and Curve-II. These two loci are sketched with different scales to clearly determine the gain margin and phase margin.

From the polar plot with $K = 1$

$$\text{Gain margin } K_g = 1/0.04 \Rightarrow 25$$

$$\text{Gain margin in db} \Rightarrow 20 \log 25 \Rightarrow 28\ db.$$

$$\text{Phase margin } \gamma = 180^\circ + \phi_{gc} \Rightarrow 76^\circ$$

Case 1: with $K = 1$, let $G(j\omega)$ cut the -180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$.

The gain margin of 28 db with $K = 1$ has to be reduced to 18 db so K has to be increased to a value greater than 1.

Let G_A be the gain at -180° for a gain margin of 18 db.

$$20 \log \frac{1}{G_A} = 18 \qquad \log \frac{1}{G_A} = \frac{18}{20} \Rightarrow \frac{1}{G_A} = 10^{18/20}$$

$$G_A = \frac{1}{10^{18/20}} \Rightarrow 0.125.$$

$$\therefore \text{The value of } K \text{ is given by } \frac{G_A}{G_B} = \frac{0.125}{0.04} = 3.125$$

Case 2: with $K = 1$, the phase margin $= 76^\circ$. This has to be reduced to 60°

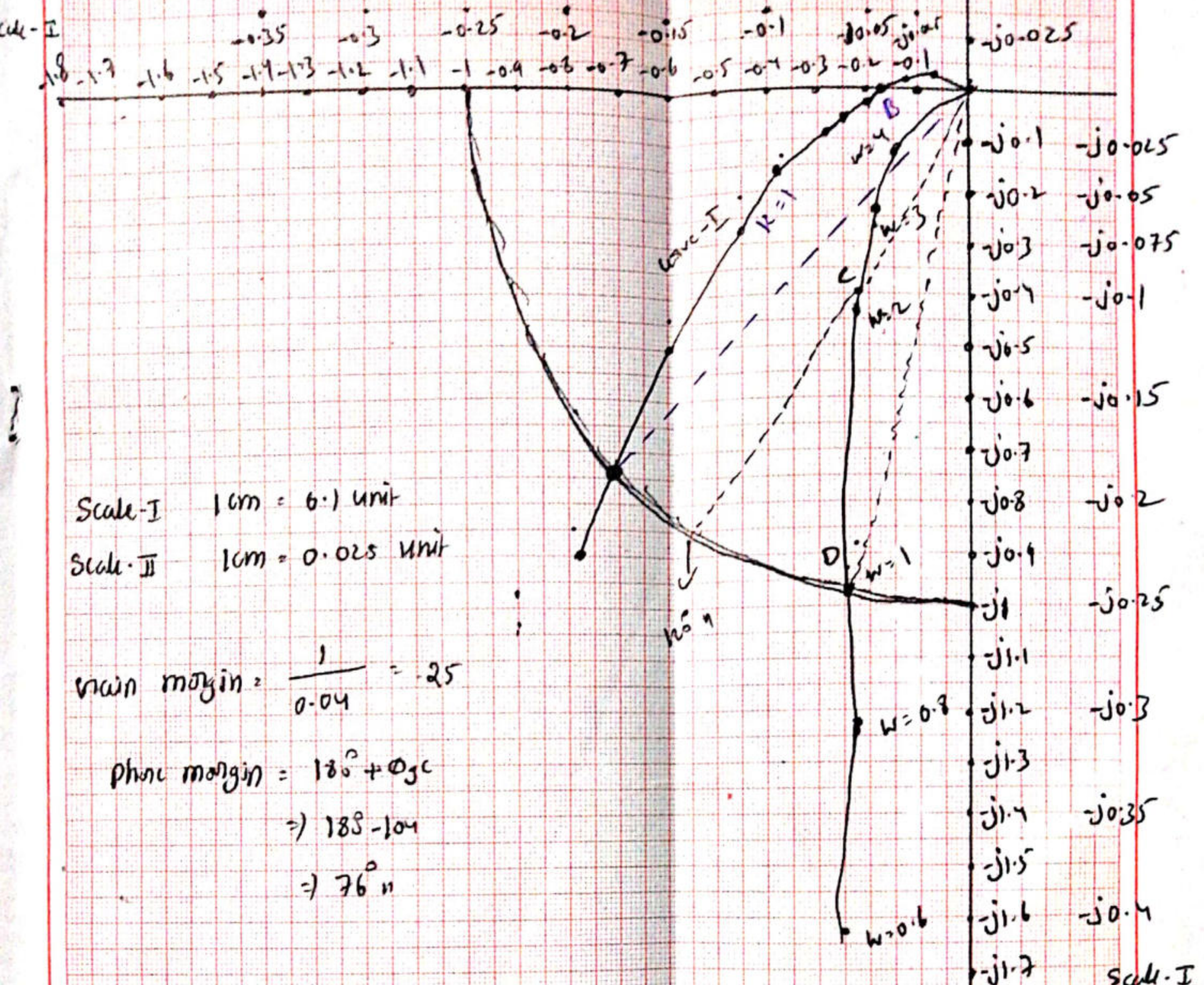
$$\text{For phase margin} = 60^\circ$$

$$60^\circ = 180^\circ + \phi_{gc} \Rightarrow \phi_{gc} = 120^\circ$$

Let the polar plot the -120° line at the locus of $G(j\omega)$ at point C and cut the unity ~~at~~ circle at point D. $\quad G_C = 0.4 \quad G_D = 1$

$$K = \frac{G_D}{G_C} = \frac{1}{0.4} \Rightarrow 2.5$$

For gain margin of 18 db, $K = 3.125$, For phase margin of 60°, $K = 2.5$



Scale-I 1cm = 0.1 unit

Scale-II 1cm = 0.025 unit

Gain margin = $\frac{1}{0.04} = 25$

Phase margin = $180° + \phi_{gc}$

$\Rightarrow 180 - 104$

$\Rightarrow 76°$

$$G(w) = \frac{K}{S(1+0.2S)(1+0.05S)}$$

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Unit-V

STATE SPACE ANALYSIS

(1)

* Introduction :-

The state variable approach is a powerful technique for analysis and design of control systems. The analysis and design of the following system can be carried using state space method.

→ Linear system
→ Non-linear system
→ Time invariant system
→ Time varying system.
→ Multiple input and multiple output system.

The state space Analysis is a modern approach and also easier for analysis using digital computers. The old methods of analysis employs the transfer function of the system. The drawbacks in the transfer function model and Analysis are,

* Transfer function is defined under zero initial conditions.
* Transfer function is applicable to linear time invariant systems.
* Transfer function analysis is restricted to single input & single output
* Transfer function analysis is ... state of the system.
* Does not provides information regarding the internal state of the system.

* State space formulation :- (State variables).

The state of a dynamic system is a minimal set of variables (known as state variables). A set of variables which describes the system at any time instant are called state variables.

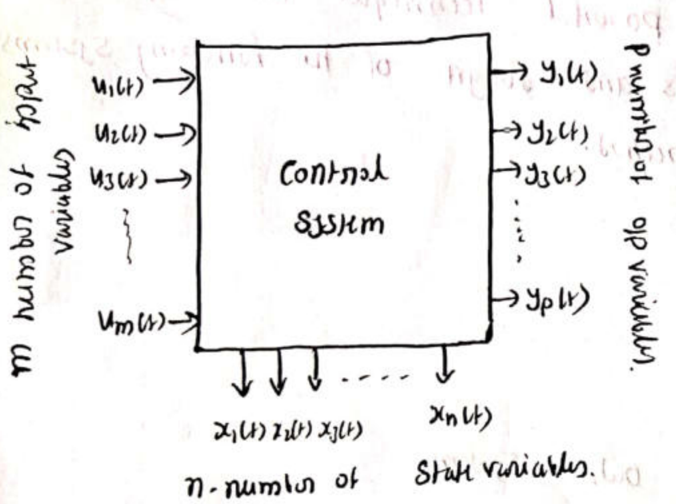
In general a system consists of m-inputs, p-outputs and n-state variables. The state space representation of the system is shown in fig.

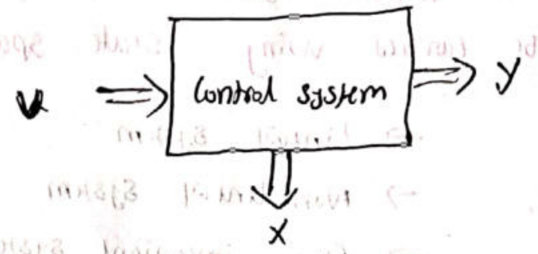
Let

input variables = $u_1(t), u_2(t), u_3(t) \ldots u_m(t)$

output variables = $y_1(t), y_2(t), y_3(t) \ldots y_p(t)$ ②

State variables = $x_1(t), x_2(t), x_3(t) \ldots x_n(t)$.





State space representation of system.

The different variables may be represented by vectors (column matrix) as shown below.

input vector $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$ output vector $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$ State variable vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

State Equations :-

The state variable representation can be arranged in the form of "n" number of first order differential equations as shown below.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2 \ldots x_n ; u_1, u_2 \ldots u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2 \ldots x_n ; u_1, u_2 \ldots u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2 \ldots x_n ; u_1, u_2 \ldots u_m)$$

The number of differential equations can be represented in vector notation as

$$\dot{X}(t) = f[X(t), U(t)] \rightarrow ①$$

The Equ. ① is called "State Equation"

* State model of Linear System :-

The state model of a system consists of state equation and O/P equation. The state equation of a system is a function of state variable and inputs as defined by Eqn. ①.

The first derivative of state variables can be expressed as a linear combination of state variable and inputs.

$$\left.\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m \\
&\vdots \\
\dot{x}_n &= a_{n1}x_n + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m
\end{aligned}\right\} \rightarrow ②$$

where the co-efficients a_{ij} and b_{ij} are constants; for $i = 1,2 \dots n$, $j = 1,2 \dots n$.

The above equation can be represented in the form of matrix as shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \rightarrow ③$$

The equation (3) can also be written as.

$$\boxed{\dot{X}(t) = AX(t) + BU(t)} \rightarrow (4) \quad \text{(state Equation)}.$$

where $X(t)$ = state vector of order $(n\times1)$

$U(t)$ = input vector of order $(m\times1)$

A = system matrix of order $(n\times n)$.

B = input matrix of order $(n\times m)$.

The equation $\dot{X}(t) = AX(t) + BU(t)$ is called state equation of linear time invariant (LTI) system.

The output equation is a function of state variable and inputs.

$\therefore$ o/p vector $Y(t) = f(X(t), U(t)) \rightarrow (5)$

The output variable can be expressed as a linear combination of state variable and input variables. (4)

$$\left.\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m \\ y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m \\ &\vdots \\ y_p &= c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m \end{aligned}\right\} \rightarrow (6)$$

Where the co-efficients c_{ij}, d_{ij} are constants.

The Equation (6) can be represented in the form of matrix as shown below.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \\ u_m \end{bmatrix} \rightarrow (7)$$

The above Equation also written as $Y(t) = CX(t) + DU(t)$. $\rightarrow (8)$

Where $X(t)$ = State vector of order $(n \times 1)$

$U(t)$ = input vector of order $(m \times 1)$

C = O/P matrix of order $(p \times n)$

D = Transmission matrix $(p \times m)$

The Equation $Y(t) = CX(t) + DU(t)$ is also called O/P Equation of Linear time invariant (LTI) System.

The State model of Linear System consists of State Equation and Output Equation.

$\dot{X}(t) = AX(t) + BU(t)$ — State Equation

$Y(t) = CX(t) + DU(t)$ — O/P Equation.

* State Diagram:

⑤

The pictorial representation of state model of the system is called "State diagram". The state diagram of a state model is constructed by using three elements are scalar, Adder and integrators.

The time domain, s-domain elements of block diagram are shown below.

Element	S-domain	T-domain
Scalar.	$X(s) \rightarrow \boxed{a} \rightarrow a(X(s))$	$x(t) \rightarrow \boxed{a} \rightarrow a\,x(t)$
Adder.	$X_1(s) \xrightarrow{+} \bigotimes \rightarrow X_1(s)+X_2(s)$, $X_2(s) \xrightarrow{+} \bigotimes$	$x_1(t) \xrightarrow{+} \bigotimes \rightarrow x_1(t)+x_2(t)$, $x_2(t) \xrightarrow{+} \bigotimes$
integrator.	~~$\dot{x}(s) \rightarrow \boxed{\int} \rightarrow \bigotimes$~~ $X(s) \rightarrow \boxed{1/s} \rightarrow \bigotimes \rightarrow \frac{X(s)}{s}+\frac{X(0)}{s}$, $X(0)/s \xrightarrow{+} \bigotimes$	$\dot{x}(t) \rightarrow \boxed{\int} \xrightarrow{+} \bigotimes \rightarrow x(t)+X(0)$, $X(0) \xrightarrow{+} \bigotimes$

The time domain and s-domain elements of signal flowgraph are shown below.

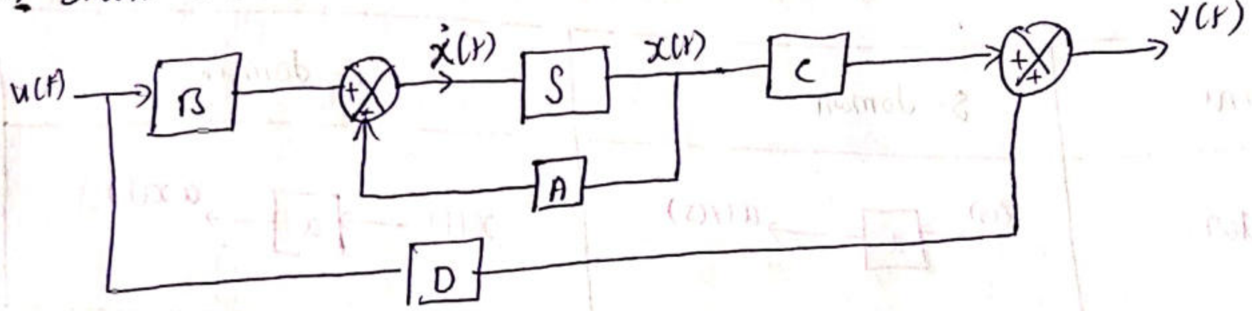
Element	S-domain	T-domain
Scalar	$X_0(s) \xrightarrow{a} aX(s)$	$x(t) \xrightarrow{a} ax(t)$
Adder.	$X_1(s) \xrightarrow{1} X_1(s)+X_2(s)$, $X_2(s) \xrightarrow{1} X_1(s)+X_2(s)$	$x_1(t) \xrightarrow{1} x_1(t)+x_2(t)$, $x_2(t) \xrightarrow{1} x_1(t)+x_2(t)$
integrator.	$\dot{X}(s) \xrightarrow{1/s} \frac{X(s)}{S}+\frac{X(0)}{S}$, $X(0) \xrightarrow{1/s} \frac{X(s)}{S}+\frac{X(0)}{S}$	$\dot{x}(t) \xrightarrow{\int} x(t)+x(0)$, $X(0) \xrightarrow{1} x(t)+x(0)$

The state model of linear time invariant system is given by the Equations.

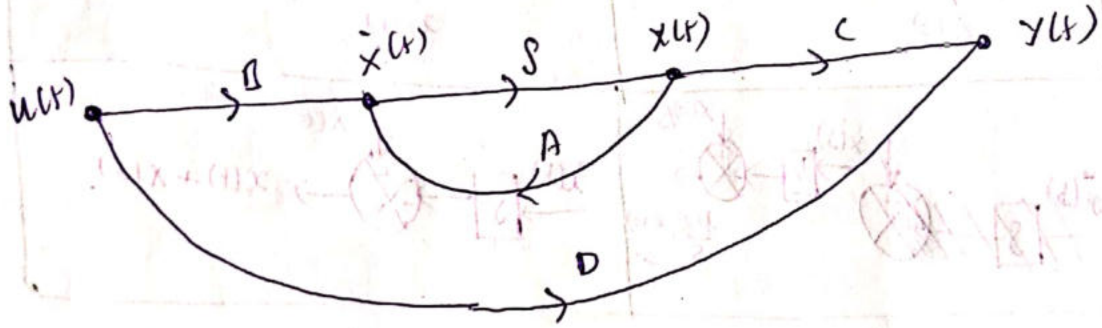
$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{--- State Eqn. } (1)$$

$$y(t) = Cx(t) + Du(t) \quad \text{--- O/P Eqn.}$$

The block diagram and signal flow graph representation in time-domain as shown below.



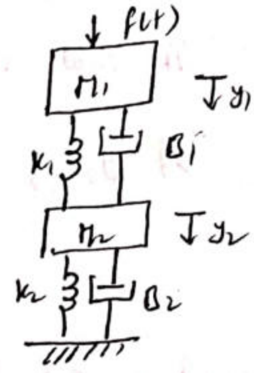
Block diagram of State model



Signal flow graph of State model.

* Problems based on State models from differential Equations

Ex: Construct state model from mechanical system shown in fig. (7)

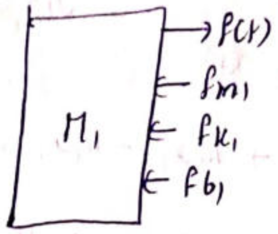


Sol: free body diagram of M_1 is

$$f_{m1} = M_1 \frac{d^2 y_1}{dt^2}$$

$$f_{b1} = B_1 \frac{d(y_1 - y_2)}{dt}$$

$$f_{k1} = k_1 (y_1 - y_2)$$



By newton's second's law, the force balance equation at node M_1 is

$$f(t) = f_{m_1} + f_{b_1} + f_{k_1}$$

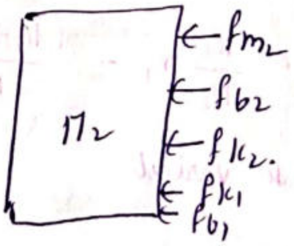
$$f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + k_1 (y_1 - y_2) \rightarrow \cancel{(1)}$$

$$\int \left[f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{dy_1}{dt} - B_1 \frac{dy_2}{dt} + k_1 y_1 - k_2 y_2 \right] \rightarrow (1)$$

free body diagram of M_2 is

$$f_{m2} = M_2 \frac{d^2 y_2}{dt^2} \qquad \cancel{f_{b2} = B_2 \frac{d}{dt}(y_2 - y_1)}$$

$$\cancel{f_{k2} = k_2 (y_2 - y_1)}$$



By newton's second law, the force balance equation at node m_2 is

$$f_{m2} + f_{b2} + f_{k2} + f_{b1} + f_{k1} = 0$$

$$\cancel{M_1 \frac{d^2 y_2}{dt^2} + B_2 \frac{d}{dt}(y_2 - y_1) + k_2 (y_2 - y_1) = 0}$$

$$\cancel{M_1 \frac{d^2 y_2}{dt^2} + B_2 \frac{dy_2}{dt} - B_2 \frac{dy_1}{dt} + k_2 y_2 - k_2 y_1 = 0 \rightarrow (2)}$$

$$M_2 \frac{d^2 y_2}{dt^2} + B_1 \frac{d}{dt}(y_2 - y_1) + B_2 \frac{dy_2}{dt} + k_1 (y_2 - y_1) + k_2 y_2 = 0$$

$$\int \left[M_2 \frac{d^2 y_2}{dt^2} + B_1 \frac{dy_2}{dt} - B_1 \frac{dy_1}{dt} + B_2 \frac{dy_2}{dt} + k_1 y_2 - k_1 y_1 + k_2 y_2 = 0 \right] \rightarrow (2)$$

Let us choose four state variables x_1, x_2, x_3 and x_4. Let the input $f(t) = u$. The state variables are related to physical variables as follows

$$\left.\begin{aligned} x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \frac{dy_1}{dt}, \quad x_4 = \frac{dy_2}{dt} \\ \dot{x}_3 = \frac{d^2y_1}{dt^2} \quad \dot{x}_4 = \frac{d^2y_2}{dt^2} \end{aligned}\right\} \rightarrow (3)$$

On substituting (3) in Eqn. (1)

(8)

$$u = M_1\dot{x}_3 + B_1x_3 - B_1x_4 + K_1x_1 - K_1x_2$$

$$M_1\dot{x}_3 = -B_1x_3 + B_1x_4 - K_1x_1 + K_1x_2 + u.$$

$$\dot{x}_3 = -\frac{K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{1}{M_1}u \rightarrow (4)$$

On substituting (3) in Equation (2)

$$M_2\dot{x}_4 + B_2x_4 + B_1x_4 - B_1x_3 + K_2x_2 + K_1x_2 - K_1x_1 = 0$$

$$M_2\dot{x}_4 = -B_2x_4 - B_1x_4 + B_1x_3 - K_2x_2 - K_1x_2 + K_1x_1$$

$$\dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4 \rightarrow (5)$$

The state variable $x_1 = y_1$

$$\frac{dx_1}{dt} = \frac{dy_1}{dt} \Rightarrow \dot{x}_1 = x_3 \rightarrow (6)$$

The state variable $x_2 = y_2$

$$\frac{dx_2}{dt} = \frac{dy_2}{dt} \Rightarrow \dot{x}_2 = x_4 \rightarrow (7)$$

The Equations 4, 5, 6, 7 are state Equations of mechanical system.
Hence the state Equations of mechanical system are

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{1}{M_1}u$$

$$\dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4$$

On arranging state equations in matrix form as shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1}{M_1} & \frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{B_1}{M_1} \\ \frac{K_1}{M_2} & -\frac{(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & -\frac{(B_1+B_2)}{M_2} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix}}_{B} [u] \rightarrow (8)$$

Let the displacements y_1 and y_2 be the o/p of the system.

(9)

$$\therefore \quad y_1 = x_1 \qquad y_2 = x_2$$

The o/p equation in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \rightarrow \langle 9 \rangle$$

The eqn. (8) & (9) together called state model of the system.

Ex 2: construct a state model for a system characterized by the differential equation $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$. Give the block diagram representation of the state model.

Sol: Let us choose y and their derivatives as state variables. The system is governed by third order differential equation. and so the number of state variables are three.

The state variables x_1 x_2 x_3 are related to phase variables as follows.

$$x_1 = y \qquad x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1 \qquad \dot{x}_3 = \frac{d^3y}{dt^3}$$

Put $y = x_1$, $dy/dt = x_2$ and $d^2y/dt^2 = x_3$ and $\frac{d^3y}{dt^3} = \dot{x}_3$ in the given eqn

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u \rightarrow (1)$$

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$$\dot{x}_1 = \frac{dy}{dt} = x_2 \rightarrow (2)$$

$$\dot{x}_2 = \frac{d^2y}{dt^2} = x_3 \rightarrow (3)$$

The Equations (1) (2), (3) are represented in the form of matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u] \quad (10)$$

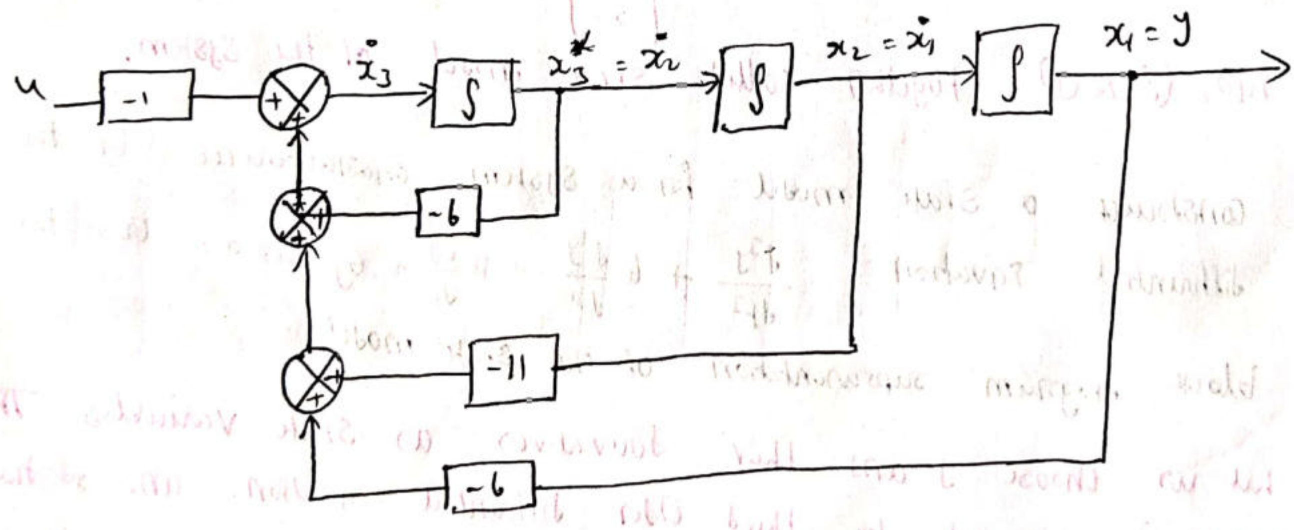
Here y = output

$y = x_1$

$\therefore$ The o/p Equation $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The State Equation and output Equation, constitutes the state model of the system.

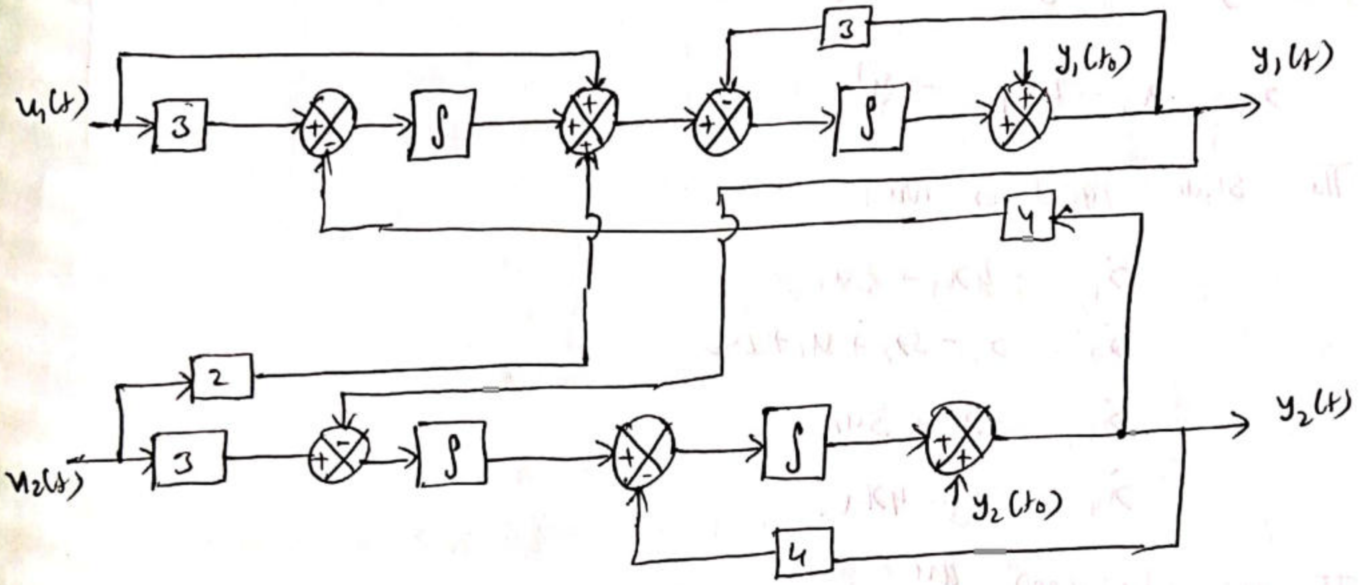
The block diagram representation of state model as shown below.



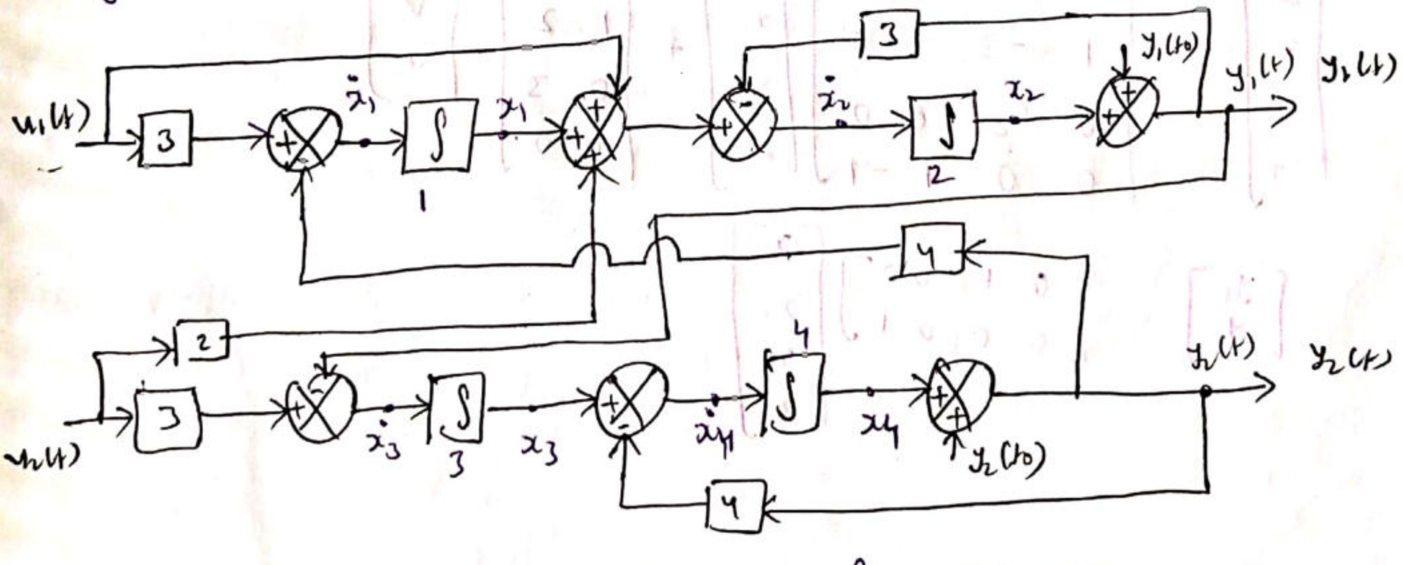
block diagram form of State model.

* Problem based on State model from block diagram :-

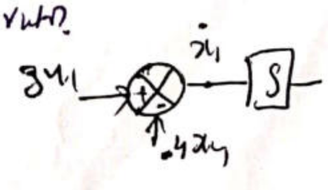
Ex. The state diagram of a system is shown in fig. Assign state variables and obtain the state model of the system. (11)



sol, Since there are four integrators in the state diagram we can assign 4 state variables. The state variables can be assigned at the o/p of the integrators as shown in fig. Hence at the input of the integrator, the first derivative of the state variable will be available.

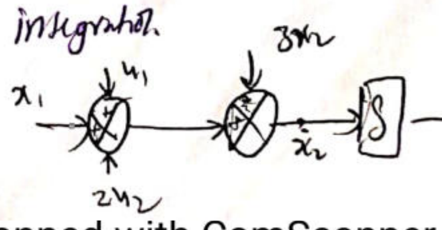


On adding the signals coming to the first integrator.



$$\dot{x}_1 = -4x_4 + 3u_1 \rightarrow (1)$$

On adding the signal coming to the second integrator.



$$\dot{x}_2 = x_1 - 3x_2 + u_1 + 2u_2 \rightarrow (2)$$

On adding the signals coming to the integrator 3.

$\dot{x}_3 = -x_2 + 3u_2 \rightarrow (3)$

On adding the signals coming to the integrator-4

(12)

$\dot{x}_4 = x_3 - 4x_4 \rightarrow (4)$.

The state equations are

$$\dot{x}_1 = -4x_4 + 3u_1$$
$$\dot{x}_2 = x_1 - 3x_2 + u_1 + 2u_2$$
$$\dot{x}_3 = -x_2 + 3u_2.$$
$$\dot{x}_4 = x_3 - 4x_4.$$

The o/p equations are

$y_1 = x_2$ and $y_2 = x_4$

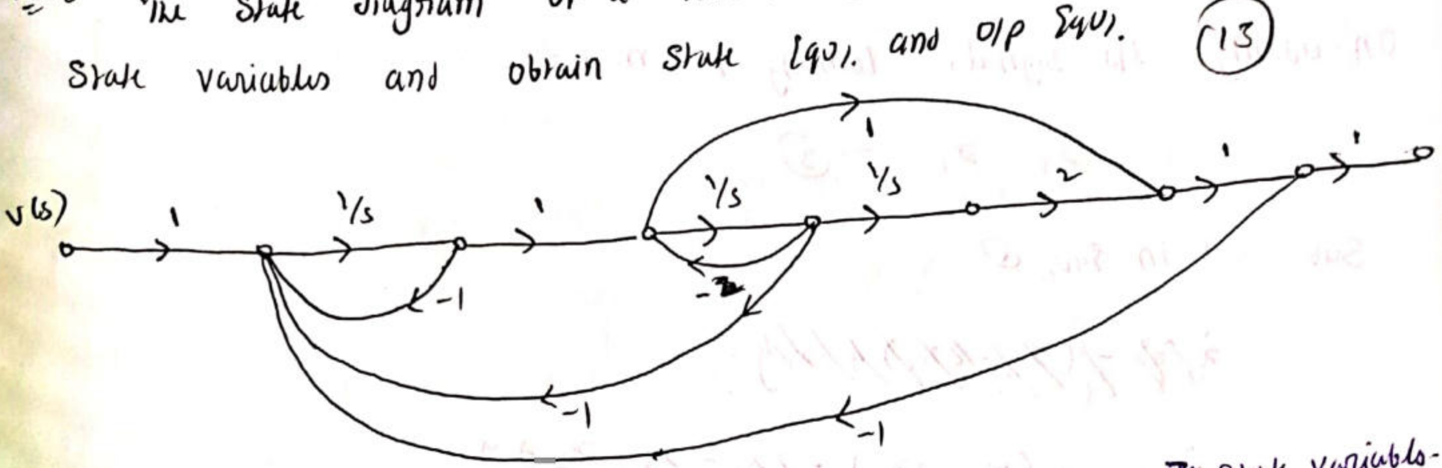
The state equ/, and o/p equations are arranged in matrix form as shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & -3 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

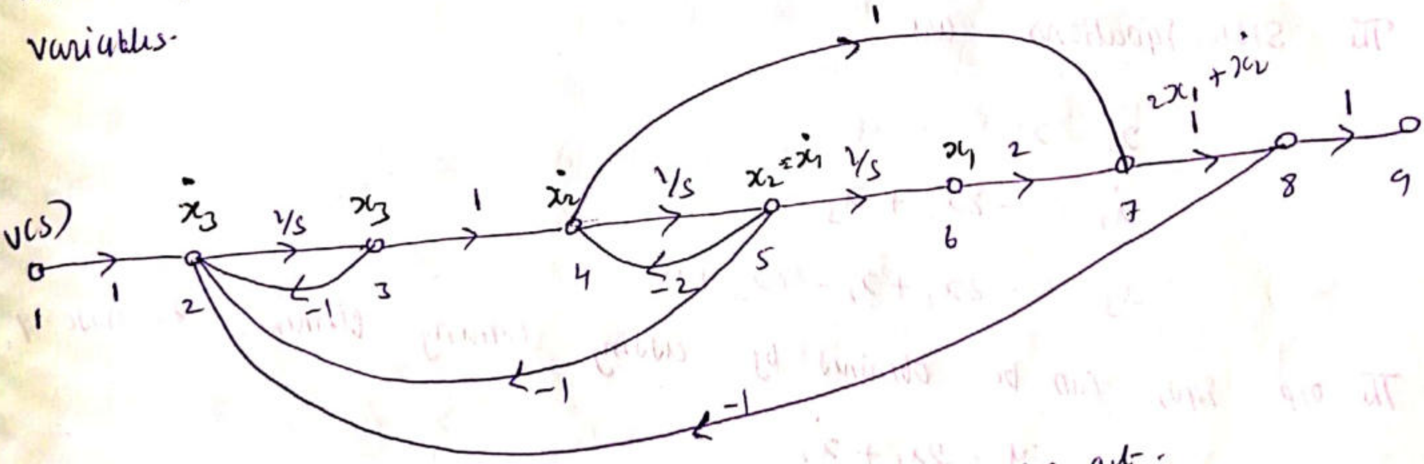
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Problems based on state model from signal flow graph:-

Ex:- The state diagram of a linear system is given below. Assign state variables and obtain state eqn. and o/p eqn. (13)



Sol: Since there are three integrators (1/s) we can assign 3 state variables. The state variables are assigned at the o/p of integrator as shown in fig. At the i/p of the integrator we have to assign first derivative of state variables.



On adding the signals coming to node-2 we get.

$$\dot{x}_3 = u - x_3 - x_2 - (2x_1 + \dot{x}_2)$$

$$\dot{x}_3 = u - x_3 - x_2 - 2x_1 - \dot{x}_2$$

$$\dot{x}_3 = -\dot{x}_2 - 2x_1 - x_2 - x_3 - u \rightarrow (1)$$

On adding the signals coming to node-4.

$$\dot{x}_2 = x_3 - 2x_2 \rightarrow ②$$

On adding the signals coming to node-5

$$\dot{x}_1 = x_2 \rightarrow ③$$

(14)

Sub. ② in equ. ①

$$\dot{x}_3 = -(x_3 - 2x_2) - 2x_1 - x_2 - x_3 + u.$$

$$\dot{x}_3 = -x_3 + 2x_2 - 2x_1 - x_2 - x_3 + u.$$

$$\dot{x}_3 = -2x_1 + x_2 - 2x_3 + u \rightarrow ④$$

The state equations are :

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 + x_3$$

$$\dot{x}_3 = -2x_1 + x_2 - 2x_3 + u$$

The o/p equ. can be obtained by adding coming elements to node-9.

$$y = 2x_1 + \dot{x}_2$$

$$y = 2x_1 - 2x_2 + x_3$$

The state equation and o/p equation are arranged in matrix form as shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problems based on state model from Transfer functions :-

Ex:1 Obtain the state model of a system whose transfer function is given as $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$ (15)

Sol, given that $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$ → (1)

on cross multiplying the Equ. (1) we get.

$$Y(s)[s^3+4s^2+2s+1] = 10\,U(s)$$

$$s^3Y(s) + 4s^2Y(s) + 2sY(s) + Y(s) = 10\,U(s) \quad \rightarrow (2)$$

on taking inverse Laplace transform to (2) we get

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u \quad \rightarrow (3)$$

Let us define state variables as follows.

$$x_1 = y \qquad x_2 = \frac{dy}{dt} = \dot{y}, \qquad x_3 = \frac{d^2y}{dt^2} = \ddot{y}$$

Put $\dddot{y} = \dot{x}_3$; $\ddot{y} = x_3$, $\dot{y} = x_2$ $y = x_1$ on Equ. (3)

$$\dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$$

$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u \quad \rightarrow (4)$$

$$\dot{x}_2 = \ddot{y} = x_3 \quad \rightarrow (5)$$

$$\dot{x}_1 = \dot{y} = x_2 \quad \rightarrow (6)$$

The O/P Equ. is $y = x_1$ → (7)

The matrix form representation of 4, 5, 6, 7 Equ. as shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}[u]$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A feedback system has a closed loop transfer function

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)} \quad (1c)$$

Construct state model for this system and give block diagram representation

Sol

given that $\dfrac{Y(s)}{U(s)} = \dfrac{10(s+4)}{s(s+1)(s+3)}$

$$\frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$$

Let $\dfrac{X_1(s)}{U(s)} = \dfrac{1}{s(s+1)(s+3)}$ and $\dfrac{Y(s)}{X_1(s)} = 10(s+4)$

consider, $\dfrac{X_1(s)}{U(s)} = \dfrac{1}{(s^2+s)(s+3)} \Rightarrow \dfrac{1}{s^3+3s^2+s^2+3s} \Rightarrow \dfrac{1}{s^3+4s^2+3s} \rightarrow (1)$

on cross multiplying the Eqn. (1) we get

$$X_1(s)\left[s^3+4s^2+3s\right] = U(s)$$

$$\therefore s^3X_1(s) + 4s^2X_1(s) + 3sX_1(s) = U(s) \rightarrow (2)$$

Applying inverse Laplace transform to above Eqn. we get

$$\dddot{x}_1 + 4\ddot{x}_1 + 3\dot{x}_1 = u \rightarrow (3)$$

Let the state variables, x_1, x_2, x_3 ; where $x_2 = \dot{x}_1$ and $x_3 = \ddot{x}_1$

$(\dot{x}_3 = \dddot{x}_1$

sub. in Eq. (3)

$$\dot{x}_3 + 4x_3 + 3x_2 = u$$

$$\dot{x}_3 = -3x_2 - 4x_3 + u \rightarrow (4)$$

$$\dot{x}_2 = \ddot{x}_1 = x_3 \rightarrow (5)$$

$$\dot{x}_1 = x_2 \rightarrow (6)$$

consider second part of transfer function.

$$\frac{Y(s)}{X_1(s)} = 10s+40 \Rightarrow Y(s) = 10sX_1(s) + 40X_1(s) \rightarrow (7)$$

on taking inverse L.P.T to Eqn. (7) we get

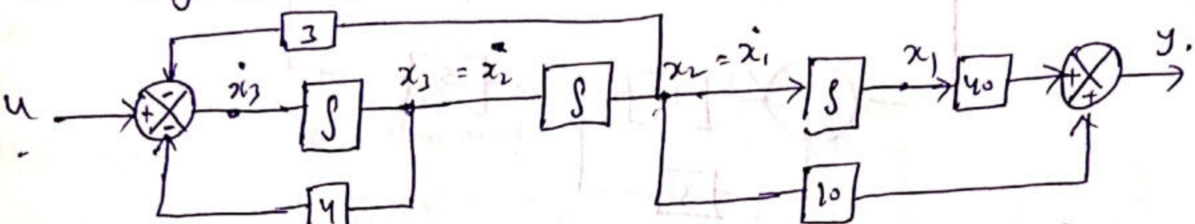
$y = 10\dot{x}_1 + 40x_1$

$y = 40x_1 + 10x_2 \rightarrow (8)$

The state model in matrix form is shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u], \quad y = \begin{bmatrix} 40 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The block diagram representation of this state model is shown below



model-II: Representation of block diagram in canonical form

given that $\dfrac{Y(s)}{U(s)} = \dfrac{10(s+4)}{s(s+1)(s+3)}$

By partial fraction expansion $Y(s)/U(s)$ can be expressed as

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} \rightarrow (1)$$

$$10(s+4) = A(s+1)(s+3) + Bs(s+3) + Cs(s+1)$$

Put $s=0$ $\quad 40 = 3A \Rightarrow \boxed{A = 40/3}$

Put $s=-1$ $\quad 30 = B(-1)(2) \Rightarrow \boxed{B = -15}$

Put $s=-3$ $\quad 10 = C(-3)(-2) \Rightarrow 10 = 6C \Rightarrow \boxed{C = 5/3}$

Substitute A, B, C values in Equ. (1)

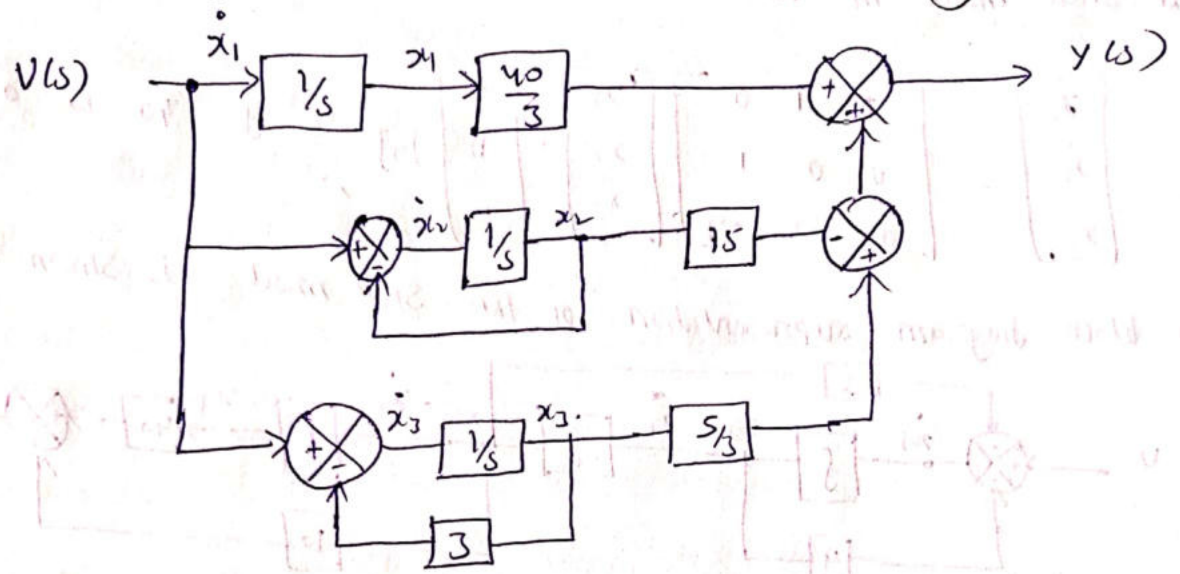
$$\frac{Y(s)}{U(s)} = \frac{40/3}{s} - \frac{15}{s+1} + \frac{5/3}{s+3} \rightarrow (2)$$

The Equ. (2) can be rearranged as shown below.

$$\frac{Y(s)}{U(s)} = \frac{40/3}{s} - \frac{15}{s(1+1/s)} + \frac{5/3}{s(1+3/s)}$$

$$Y(s) = \left[\frac{1}{s} \times \frac{40}{3}\right] V(s) - \left[\frac{1/s}{1+1/s} \times 15\right] V(s) + \left[\frac{1/s}{1+3/s} \times \frac{5}{3}\right] V(s) \rightarrow (3)$$

The block diagram of Equ. (3) shown below (18)



Ex: 3 Determine the canonical State model of the system, whose transfer function is $T(s) = \dfrac{2(s+5)}{(s+2)(s+3)(s+4)}$

Sol given that $T(s) = \dfrac{Y(s)}{V(s)} = \dfrac{2(s+5)}{(s+2)(s+3)(s+4)}$

By partial fraction Expansion

$$\frac{Y(s)}{V(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} \rightarrow (1)$$

$$2(s+5) = A(s+3)(s+4) + B(s+2)(s+4) + C(s+2)(s+3)$$

Put $s = -2$

$6 = A(1)(2) \Rightarrow [A = 3]$

Put $s = -3$ $\quad 4 = B(-1)(1) \Rightarrow [B = -4]$

Put $s = -4$ $\quad 2 = C(-2)(-1) \Rightarrow [C = 1]$

Substitute A, B, C in Equ. (1) we get

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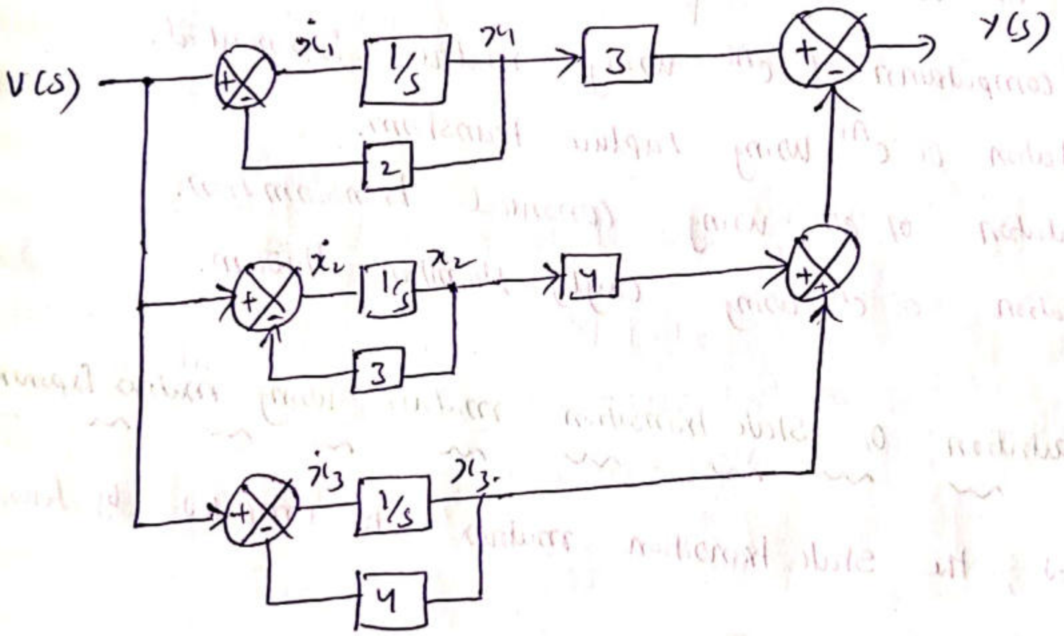
$$\frac{Y(s)}{U(s)} = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4}.$$

$$\frac{Y(s)}{U(s)} = \frac{3}{s(1+2/s)} - \frac{4}{s(1+3/s)} + \frac{1}{s(1+4/s)}.$$ (19)

$$Y(s) = \left[\frac{\frac{1}{s}}{1+\frac{1}{s}\times 2}\times 3\right]U(s) - \left[\frac{1/s}{1+1/s\times 3}\times 4\right]U(s) + \left[\frac{1/s}{1+1/s\times 4}\right]U(s).$$

The block diagram of above equation is shown below.



State transition matrix:-

A matrix which satisfies Linear homogenous state equation then it is called "State transition matrix".

Generally the state equation is given by $\dot{x}(t) = Ax(t) + Bu(t)$

The state equation without input vector is called as "Linear-homogeneous state equation." and is given by

$$\left[\dot{x}(t) = Ax(t)\right]$$

It is denoted by $\phi(t)$ (or) e^{At}

Properties of State transition matrix:- (20)

1) At $t=0$, $\phi(t) = e^{At}$ $\Rightarrow$ $\phi(0) = e^{A\times 0} = I$ (unit matrix)

2) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$ (or) $\phi^{-1}(t) = \phi(-t)$

3) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1) \cdot \phi(t_2)$ "

* Computation of State transition matrix:-

The State transition matrix (e^{At}) can be computed by any one of the following methods.

Method-1:- computation of e^{At} using matrix Exponential.

Method-2:- Computation of e^{At} using Laplace transform.

Method-3:- Computation of e^{At} using Canonical transformation.

Method-4:- Computation of e^{At} using Cayley-Hamilton theorem.

Method-1:- Computation of State transition matrix using matrix Exponential:-

In this method, the State transition matrix is computed by following Equation.

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \frac{1}{n!}A^nt^n$$

where,

e^{At} = State transition matrix of order $n\times n$

A = System matrix of order $n\times n$

I = unit matrix of order $n\times n$.

The disadvantage in this method is that Each term of e^{At} will be an infinite series and the convergence of the infinite series are obtained by trail and Error.

Method-2: Computation of e^{At} by using Laplace transform.

Consider a state equation without unit vector. 21

$$\dot{x}(t) = A\,x(t) \rightarrow (1)$$

The above equation is called as Linear homogeneous state equation.

~~on taking~~ on taking Laplace transform at zero initial condition.

$$L\{\dot{x}(t)\} = L\left\{\frac{dx}{dt}\right\} = L\{Ax(t)\} + L\{x(0)\}$$

$$sX(s) = AX(s) + X(0)$$

here 's' should be in matrix form and when it is multiplied with unit matrix (I) we will get same matrix

i.e, $SI = S$

$$ISX(s) = AX(s) + X(0)$$

$$X(s)[SI - A] = X(0)$$

$$X(s) = [SI-A]^{-1}X(0) \rightarrow (2)$$

on taking inverse Laplace transform.

$$L^{-1}\{X(s)\} = L^{-1}\{[SI-A]^{-1}X(0)\}$$ $X(0) \rightarrow$ is constant.

$$x(t) = L^{-1}[SI-A]^{-1}x(0)$$

The above equ. can be written as

$$x(t) = e^{At}x(0) \quad (or) \quad x(t) = \phi(t)\,x(0)$$

$$\therefore e^{At} = \phi(t) = L^{-1}\{[SI-A]^{-1}\} \rightarrow \text{State transition matrix}$$

Ex:1) Determine the state transition matrix from the given Equations.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \quad ; \quad y(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

(22)

Sol: given that

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}, \quad y(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad y(t) = C\,x(t) + D(0)$$

State transition matrix is given by

$$\phi(t) = L^{-1}[SI - A]^{-1} \rightarrow ①$$

here $A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ $\quad I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\quad SI = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix}$

$B = \begin{bmatrix}1\\ 1\end{bmatrix}$ $\quad C = \begin{bmatrix}1 & 1\end{bmatrix}$.

SI-A :- $\begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}(S-1) & 0\\ -1 & (S-1)\end{bmatrix}$

$$(SI-A)^{-1} \Rightarrow \frac{Adj\,(SI-A)}{|SI-A|} \rightarrow ②$$

Note: If $x = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then $adj\,x = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$|x| = ad - bc$

$|SI-A|$:- $\begin{vmatrix}S-1 & 0\\ -1 & S-1\end{vmatrix} \Rightarrow (S-1)(S-1) \Rightarrow (S-1)^2 \rightarrow ③$

Adj (SI-A) :- $adj\begin{vmatrix}(S-1) & 0\\ -1 & (S-1)\end{vmatrix} \Rightarrow \begin{bmatrix}(S-1) & 0\\ 1 & (S-1)\end{bmatrix} \rightarrow ④$

Substitute (3) and (4) in Eqn. (2)

$$(SI-A)^{-1} \Rightarrow \frac{1}{(s-1)^2}\begin{bmatrix}(s-1) & 0\\ 1 & (s-1)\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{1}{s-1} & 0\\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)}\end{bmatrix}$$

Now Apply inverse Laplace transform to the above Eqn, we will get.

(23)

$$\phi(t) = L^{-1}[SI-A]^{-1}$$

$$\Rightarrow L^{-1}\begin{bmatrix}\frac{1}{s-1} & 0\\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)}\end{bmatrix} \Rightarrow \begin{bmatrix}e^t & 0\\ te^t & e^t\end{bmatrix}$$

$$\therefore \quad \phi(t) = \begin{bmatrix}e^t & 0\\ te^t & e^t\end{bmatrix}$$ → State transition matrix for given state model.

Ex 2) Determine the state transition matrix from the given Equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u(t), \qquad y(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

Sol: The state transition matrix is given by

$$\phi(t) = L^{-1}[SI-A]^{-1}$$

from the give model $A = \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}$ $\qquad SI = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix}$.

$$(SI-A) \Rightarrow \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}S+1 & -1\\ 0 & S+2\end{bmatrix}$$

$$(SI-A)^{-1} \Rightarrow \frac{Adj\,(SI-A)}{|SI-A|}$$

$$Adj\,(SI-A) \Rightarrow \begin{bmatrix}S+2 & 1\\ 0 & S+1\end{bmatrix}$$

$$|SI-A| \Rightarrow (s+1)(s+2) \Rightarrow s^2+3s+2 \quad (or) \quad (s+1)(s+2)$$

$$\therefore (SI-A)^{-1} \Rightarrow \frac{1}{(s+1)(s+2)}\begin{bmatrix}S+2 & 1\\ 0 & S+1\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{1}{S+1} & \frac{1}{(s+1)(s+2)}\\ 0 & \frac{1}{S+2}\end{bmatrix}$$

On taking Lap the transform to the above Eqn,, will get

$$L^{-1}[SI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{S+1} & \frac{1}{(S+1)(S+2)} \\ 0 & \frac{1}{S+2} \end{bmatrix} = \phi(t) \quad (24)$$

Consider $\frac{1}{(S+1)(S+2)}$ and it can be Expressed in Partial fraction

$$\frac{1}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$1 = A(S+2) + B(S+1)$$

Put $S=-1$, $[A=1]$ $\therefore \frac{1}{(S+1)(S+2)} = \frac{1}{S+1} - \frac{1}{(S+2)}$

Put $S=-2$ $[B=-1]$

$$\therefore \phi(t) = L^{-1}[SI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{S+1} & \frac{1}{S+1} - \frac{1}{S+2} \\ 0 & \frac{1}{S+2} \end{bmatrix} \Rightarrow \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$\left\{ \therefore \phi(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \right\}$$

Ex:3 Consider ~~Compute~~ the matrix A. Compute the state transition matrix by two methods (M-1, M-2) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol Method-1 :- $e^{At} = \left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots\right]$.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$$

$$A^3 = A^2 \cdot A \Rightarrow \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}$$

$$A^4 = A^3 \cdot A \Rightarrow \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}$$

25

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots$$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}t + \frac{1}{2}\begin{bmatrix}-2 & -3\\6 & 7\end{bmatrix}t^2 + \frac{1}{6}\begin{bmatrix}6 & 7\\-14 & -15\end{bmatrix}t^3 + \frac{1}{24}\begin{bmatrix}-14 & -15\\30 & 31\end{bmatrix}t^4 + \cdots$$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}0 & t\\-2t & -3t\end{bmatrix} + \begin{bmatrix}-t^2 & -\frac{3}{2}t^2\\3t^2 & \frac{7}{2}t^2\end{bmatrix} + \begin{bmatrix}t^3 & \frac{7}{6}t^3\\-\frac{7}{3}t^3 & -\frac{5}{2}t^3\end{bmatrix} + \begin{bmatrix}-\frac{7}{12}t^4 & -\frac{5}{8}t^4\\\frac{5}{4}t^4 & \frac{31}{24}t^4\end{bmatrix} + \cdots$$

$$\Rightarrow \begin{bmatrix}1 - t^2 + t^3 - \frac{7}{12}t^4 + \cdots & t - \frac{3}{2}t^2 + \frac{7}{6}t^3 - \frac{5}{8}t^4 + \cdots\\ -2t + 3t^2 - \frac{7}{3}t^3 + \frac{5}{4}t^4 + \cdots & 1 - 3t + \frac{7}{2}t^2 - \frac{5}{2}t^3 + \frac{31}{24}t^4 + \cdots\end{bmatrix}$$

The each term in the matrix is an expansion of e^{at}. The convergence of series is obtained by trail and error. Consider the expansions e^{-t} and e^{-2t}

$\cancel{e^{-t} = 1 + At + \frac{t^2}{2!}}$

$$e^{-t} = 1 - t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 \cdots = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots$$

$$e^{-2t} = 1 - 2t + \frac{1}{2!}2^2t^2 - \frac{1}{3!}2^3t^3 + \frac{1}{4!}2^4t^4 \cdots \Rightarrow 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 + \cdots$$

$$2e^{-t} - e^{-2t} \Rightarrow 2 - 2t + t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 + \cdots \; -1 + 2t - 2t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 + \cdots$$

$$\Rightarrow (2-1) + (-2t + 2t) + (t^2 - 2t^2) + \left(-\frac{t^3}{3} + \frac{4}{3}t^3\right) + \left(\frac{1}{12}t^4 - \frac{2}{3}t^4\right) + \cdots$$

$$\underline{2e^{-t} - e^{-2t}} \Rightarrow 1 - t^2 + t^3 - \frac{7}{12}t^4 + \cdots \rightarrow (1)$$

$$\underline{e^{-t} - e^{-2t}} \Rightarrow 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots \; -1 + 2t - 2t^2 + \frac{4}{3}t^3 - \frac{2}{3}t^4 + \cdots$$

$$\Rightarrow (1-1) + (-t + 2t) + \left(\frac{1}{2}t^2 - 2t^2\right) + \left(-\frac{1}{6}t^3 + \frac{4}{3}t^3\right) + \left(\frac{1}{24}t^4 - \frac{2}{3}t^4\right) \cdots$$

$$\Rightarrow t - \frac{3}{2}t^2 + \frac{7}{6}t^3 - \frac{5}{8}t^4 + \cdots \rightarrow (2)$$

$$\underline{-2e^{-t} + 2e^{-2t}} \Rightarrow -2 + 2t - t^2 + \frac{1}{3}t^3 - \frac{1}{12}t^4 + \cdots + 2 - 4t + 4t^2 - \frac{8}{3}t^3 + \frac{4}{3}t^4 \cdots$$

$$\Rightarrow (-2+2) + (2t - 4t) + (-t^2 + 4t^2) + \left(\frac{1}{3}t^3 - \frac{8}{3}t^3\right) + \left(\frac{-1}{12}t^4 + \frac{4}{3}t^4\right) \cdots$$

$$\Rightarrow -2t + 3t^2 - \frac{7}{3}t^3 + \frac{5}{4}t^4 + \cdots \rightarrow (3)$$

$-e^{-t}+2e^{-2t}$:- $-1+t-\frac{1}{2}t^2+\frac{1}{6}t^3-\frac{1}{24}t^4 \dots + 2-4t+4t^2-\frac{8}{3}t^3+\frac{4}{3}t^4+\dots$

$\Rightarrow (-1+2)+(t-4t)+(-\frac{1}{2}t^2+4t^2)+(\frac{1}{6}t^3-\frac{8}{3}t^3)+(-\frac{1}{24}t^4+\frac{4}{3}t^4)\dots$

$\Rightarrow 1-3t+\frac{7}{2}t^2-\frac{5}{2}t^3+\frac{31}{24}t^4+\dots \rightarrow (4)$

Now ~~sub~~ replace Equation (1), (2), (3) & (4) in matrix.

$$e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

(26)

Method: 2 :-

The Equation of State transition matrix

$\phi(t) = L^{-1}[SI-A]^{-1}$ here $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

$SI-A$: $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$(SI-A)^{-1} = \frac{Adj(SI-A)}{|SI-A|}$

$Adj(SI-A) \Rightarrow \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

$|SI-A| \Rightarrow s(s+3)+2 \Rightarrow s^2+3s+2 \Rightarrow (s+1)(s+2)$

$(SI-A)^{-1} \Rightarrow \frac{1}{(s+1)(s+2)} \begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \rightarrow (1)$$

Consider $\frac{s+3}{(s+1)(s+2)}$, in partial fraction expansion

(27)

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$s+3 = A(s+2) + B(s+1)$$

Put $s=-2$, $-B = 1$ $\Rightarrow$ $[B=-1]$

Put $s=-1$ $[A=2]$

$$\frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} - \frac{1}{(s+2)} \rightarrow \langle 2 \rangle$$

Consider $\frac{1}{(s+1)(s+2)}$ in partial fraction expansion.

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$1 = A(s+2) + B(s+1)$$

Put $s=-1$ $[A=1]$

Put $s=-2$ $[B=-1]$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)} \rightarrow \langle 3 \rangle$$

Consider $\frac{-2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

$$-2 = A(s+2) + B(s+1)$$

Put $s=-1$ $\Rightarrow$ $[A=-2]$

Put $s=-2$ $\Rightarrow$ $-B=-2$ $\Rightarrow$ $[B=2]$

$$\frac{-2}{(s+1)(s+2)} = \frac{-2}{(s+1)} + \frac{2}{(s+2)} \rightarrow (4)$$

Consider $\frac{s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

$$s = A(s+2) + B(s+1)$$

Put $s=-1$ $\Rightarrow$ $-1 = A$ $\Rightarrow$ $[A=-1]$

Put $s=-2$ $\Rightarrow$ $-2 = -B$ $\Rightarrow$ $[B=2]$

$$\frac{s}{(s+1)(s+2)} = \frac{-1}{(s+1)} + \frac{2}{(s+2)} \rightarrow (5)$$

Sub. (2) (3) (4) (5) in Eq. ①

$$(SI-A)^{-1} = \begin{bmatrix} \frac{2}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{(s+1)} - \frac{1}{(s+2)} \\ \frac{-2}{(s+1)} + \frac{2}{(s+2)} & \frac{-1}{(s+1)} + \frac{2}{(s+2)} \end{bmatrix}$$

Applying inverse L.P.T to above matrix

$$L^{-1}[SI-A]^{-1} \Rightarrow \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \phi(t)$$

Ex: A linear time-invariant system is characterized by homogeneous state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute the solution of the homogeneous equation, assuming the initial state vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Sol: Here $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $(SI-A) \Rightarrow \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} (s-1) & 0 \\ -1 & (s-1) \end{bmatrix}$

$$|SI-A| \Rightarrow (s-1)^2$$

$$(SI-A)^{-1} \Rightarrow \frac{1}{(s-1)^2} \begin{bmatrix} (s-1) & 0 \\ 1 & (s-1) \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{(s-1)} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix} \rightarrow ①$$

The state transition matrix $\Rightarrow \phi(t) = L^{-1}(SI-A)^{-1}$

Now Apply I.L.T to Matrix-①

$$L^{-1}(SI-A)^{-1} \Rightarrow L^{-1} \begin{bmatrix} \frac{1}{(s-1)} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix} \Rightarrow \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

∴ The solution of state equation is

$$x(t) = \phi(t)\, x(0) \Rightarrow \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

Ex: Determine the state vector $x(t)$ for the state model $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -12 & 2/3 \\ -36 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} u$ and the initial conditions are $x_1(0) = 2$, $x_2(0) = 1$

Sol: The state vector $x(t)$ is given by (24)

$$x(t) = \phi(t)\, x(0)$$

where $\phi(t) \rightarrow$ state transition matrix $= L^{-1}(SI-A)^{-1}$

$x(0) \Rightarrow$ initial conditions $= \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

To find $\phi(t)$: here $A = \begin{bmatrix} -12 & 2/3 \\ -36 & -1 \end{bmatrix}$

$$(SI-A) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -12 & 2/3 \\ -36 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} (S+12) & -2/3 \\ +36 & (S+1) \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{Adj\,(SI-A)}{|SI-A|}$$

$$|SI-A| \Rightarrow (s+12)(s+1) + 24 \Rightarrow s^2 + s + 12s + 12 + 24$$

$$\Rightarrow s^2 + 13s + 36 \Rightarrow$$

$$\Rightarrow (s+9)(s+4)$$

$$(SI-A)^{-1} \Rightarrow \frac{1}{(s+9)(s+4)} \begin{bmatrix} (s+1) & 2/3 \\ 36 & (s+12) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dfrac{(s+1)}{(s+9)(s+4)} & \dfrac{2/3}{(s+9)(s+4)} \\ \dfrac{36}{(s+9)(s+4)} & \dfrac{(s+12)}{(s+9)(s+4)} \end{bmatrix} \rightarrow (1)$$

Apply I.L.T to above matrix we get ~~befo~~ before that we have express each term in $L^{-1}(SI-A)^{-1}$ partial fraction.

Consider $\dfrac{(s+1)}{(s+4)(s+9)} = \dfrac{A}{(s+4)} + \dfrac{B}{(s+9)}$

$(s+1) = A(s+9) + B(s+4)$

Put $s=-4$ $\quad (-4+1) = A(-4+9) \Rightarrow 5A = -3 \Rightarrow \left[A = -3/5\right]$

Put $s=-9$ $\quad (-9+1) = B(-9+4) \Rightarrow -8 = -5B \Rightarrow \left[B = 8/5\right]$

$$\frac{(s+1)}{(s+4)(s+9)} = \frac{-3/5}{(s+4)} + \frac{8/5}{(s+9)} \rightarrow (2)$$

consider $\dfrac{2/3}{(s+4)(s+9)} = \dfrac{A}{(s+4)} + \dfrac{B}{(s+9)}$ (3)

$2/3 = A(s+9) + B(s+4)$

Put $s=-4 \Rightarrow$ $\quad 2/3 = A(-4+9) \Rightarrow 2/3 = 5A \Rightarrow \left[A = 2/15\right]$

Put $s=-9 \Rightarrow$ $\quad 2/3 = B(-9+4) \Rightarrow 2/3 = -5B \Rightarrow \left[B = -2/15\right]$

$$\frac{2/3}{(s+4)(s+9)} = \frac{2/15}{(s+4)} + \frac{-2/15}{(s+9)} \rightarrow (3)$$

consider $\dfrac{36}{(s+4)(s+9)} = \dfrac{A}{(s+4)} + \dfrac{B}{(s+9)}$

$36 = A(s+9) + B(s+4)$

Put $s=-4$, $\quad 36 = 5A \Rightarrow \left[A = 36/5\right]$

Put $s=-9$ $\quad 36 = -5B \Rightarrow \left[B = -36/5\right]$

$$\frac{36}{(s+4)(s+9)} = \frac{36/5}{(s+4)} - \frac{36/5}{(s+9)} \rightarrow (4)$$

consider $\dfrac{(s+12)}{(s+4)(s+9)} = \dfrac{A}{(s+4)} + \dfrac{B}{(s+9)}$

$(s+12) = A(s+9) + B(s+4)$

put $s=-4$, $\quad 8 = 5A \Rightarrow \left[A = 8/5\right]$

Put $s=-9$, $\quad 3 = -5B \Rightarrow \left[B = -3/5\right]$

$$\frac{(s+12)}{(s+4)(s+9)} = \frac{8/5}{(s+4)} - \frac{3/5}{(s+9)} \rightarrow (5)$$

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Substitute (2), (3), (4), (5) in Eq (1)

$$(SI-A)^{-1} = \begin{bmatrix} \frac{-3/5}{(s+4)} + \frac{8/5}{(s+9)} & \frac{2/15}{(s+4)} - \frac{2/15}{(s+9)} \\ \frac{36/5}{(s+4)} - \frac{36/5}{(s+9)} & \frac{8/5}{(s+4)} - \frac{3/5}{(s+9)} \end{bmatrix} \quad (31)$$

$$L^{-1}(SI-A)^{-1} = \begin{bmatrix} -3/5\, e^{-4t} + 8/5\, e^{-9t} & 2/15\, e^{-4t} - 2/15\, e^{-9t} \\ \frac{36}{5} e^{-4t} - \frac{36}{5} e^{-9t} & 8/5\, e^{-4t} - 3/5\, e^{-9t} \end{bmatrix}$$

The solution of State Equation is

$$x(t) = \phi(t)x(0)$$

$$x(t) = \begin{bmatrix} -3/5\, e^{-4t} + 8/5\, e^{-9t} & 2/15\, e^{-4t} - 2/15\, e^{-9t} \\ \frac{36}{5} e^{-4t} - \frac{36}{5} e^{-9t} & 8/5\, e^{-4t} - 3/5\, e^{-9t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6/5\, e^{-4t} + \frac{16}{5} e^{-9t} - 3/5\, e^{-4t} + 8/5\, e^{-9t}, & 4/15\, e^{-4t} - \frac{4}{15} e^{-9t} + \frac{2}{15} e^{-4t} - 2/15\, e^{-9t} \\ \frac{72}{5} e^{-4t} - \frac{72}{5} e^{-9t} + \frac{36}{5} e^{-4t} - \frac{36}{5} e^{-9t} & \frac{16}{5} e^{-4t} - 6/5\, e^{-9t} + 8/5\, e^{-4t} - 3/5\, e^{-9t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} -9/5\, e^{-4t} + \frac{24}{5} e^{-9t} & 6/15\, e^{-4t} - 6/15\, e^{-9t} \\ \frac{108}{5} e^{-4t} - \frac{108}{5} e^{-9t} & \frac{24}{5} e^{-4t} - 9/5\, e^{-9t} \end{bmatrix}$$

* CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY :-

* Controllability :- (32)

The controllability verifies the usefulness of state variable. In the controllability test we can find, whether the state variable can be controlled to achieve the desired output. The choice of state variables is arbitrary while forming the state model. After determining the state model, the controllability of the state variable is verified. If the state variable is not controllable then we have to go for another choice of state variable.

Defination :- A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_d)$ in specified infinite time by a control vector $U(t)$.

The controllability of a state model can be tested by "Kalman's test" (or) "Gilbert's test."

* Observability :-

In observability test we can find whether the state variable is observable. The concept of observability is useful in solving the problems of reconstructing unmeasurable state variables from measurable ones in the min. possible length of time. In state feedback control the estimation of unmeasurable state variables is essential in order to construct the control signals.

Defination :- A system is said to be completely observable if every state $x(t)$ can be completely identified by measurements of the o/p $y(t)$ over a finite time interval.

The observability of a system can be tested by either "Kalman's Test" (or) "Gilbert's test."

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* Method - II :- Kalman's method :-

Kalman's method of testing controllability :-

Consider a system with state equation, $\dot{X} = AX + BU$. For this system, a composite matrix, Q_c can be formed such that,

$$Q_c = [B \quad AB \quad A^2B \; --- \; A^{n-1}B]$$

Condition for controllability :-

$$|Q_c| \neq 0$$

Kalman's method of testing observability :-

Consider a system with state model $\dot{X} = AX + BU$, $Y = CX + DU$ such that, for the system a composite matrix, Q_o can be formed such that

$$Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T \; --- \; (A^T)^{n-1}C^T]$$

Condition for observability :-

$$|Q_o| \neq 0.$$

Ex: State whether the system is controllable and observable for the linear time invariant system characterized by state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u \quad ; \quad y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Sol: Kalman's test for controllability :-

Here $A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

here $n = 3$

$n - 1 = 2$

Q_c is given by

$$Q_c = [B \quad AB \quad A^2B]$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

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$$A^2B = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

The composite matrix for controllability $Q_c = [B \ \ AB \ \ A^2B]$.

$$Q_c = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$$

Determinant of $Q_c \Rightarrow |Q_c| = 32$

since $|Q_c| \neq 0$, the rank of $Q_c = 3$

Hence the system is completely state "controllable"

Kalman's test for observability:-

$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$C^T = [1 \ 0 \ 0]^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A^T)^2 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & -2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & -2 \\ 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix}$$

$$A^TC^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A^T)^2C^T = \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

The composite matrix for observability,

$$Q_0 = [C^T \;\; A^TC^T \;\; (A^T)^2C^T] \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\text{Determinant of } Q_0 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} = 1 \times -2 = -2$$

Since $|Q_0| \neq 0$, the rank of $Q_0 = 3$

Hence the system is completely observable. 35

* Method :- 2 :- Gilbert's Test :-

Case-1 :- When the system matrix has distinct eigen values.

In this case the system matrix can be "diagonalized" and the state model can be converted to canonical form.

Consider the state model of the system

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The state model can be converted to canonical form by a transformation. The transformed state model is given by

$$\dot{Z} = \Lambda Z + \tilde{B}U$$
$$Y = \tilde{C}Z + DU$$

Where $\Lambda = M^{-1}AM$

$\tilde{B} = M^{-1}B$

$\tilde{C} = CM$.

In this case the necessary and sufficient condition for complete controllability is that, the matrix $\tilde{B}$ must have no rows with all zeros. and sufficient condition for complete observability is that, the matrix $\tilde{C}$ must have no columns with all zeros.

Case 2 :- When the system matrix has repeated eigen values.

In this case, the system matrix cannot be diagonalized but can be transformed to Jordan canonical form.

Consider the state model of the system

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The state model can be transformed to Jordan canonical form by a transformation,

The transformed state model is given by,

$$\dot{Z} = JZ + \tilde{B}U \qquad J = M^{-1}AM$$
$$y = \tilde{C}Z + DU \qquad \tilde{B} = M^{-1}B$$
$$\tilde{C} = CM$$

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In this case, the system is completely controllable if the elements of any row of $\tilde{B}$ that corresponding to the last row of each Jordan block are not all zero and the columns of matrix $\tilde{C}$ are not equal to zero.

Ex: State whether system is controllable and observable for the given state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u \; ; \; y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Sol: To find Eigen values :-

Hence the system matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$

The char. Equation is $|\lambda I - A| = 0$

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} \Rightarrow \lambda(\lambda+3)^2 - (-4) \Rightarrow \lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0$$

$\therefore$ The Eigen values are $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = -4$

To find Eigenvectors :-

$$(\lambda_1 I - A) = \begin{bmatrix} \lambda_1 & 0 & -1 \\ 2 & \lambda_1+3 & 0 \\ 0 & -2 & \lambda_1+3 \end{bmatrix}$$

Let C_{11}, C_{12} and C_{13} be cofactors along 1^{st} row of the matrix $(\lambda_1 I - A)$

$$C_{11} = (+1)\begin{vmatrix} \lambda_1+3 & 0 \\ -2 & \lambda_1+3 \end{vmatrix} = (\lambda_1+3)^2 \Rightarrow \lambda_1^2+6\lambda+9$$

$$C_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 0 & \lambda_1+3 \end{vmatrix} = -2\lambda_1-6$$

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$$C_{13} = (+1)\begin{vmatrix} 2 & \lambda_1+3 \\ 0 & -2 \end{vmatrix} = -4$$

Sub, $\lambda_1 = -1$

$$m_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} \lambda_1^2+6\lambda+9 \\ -2\lambda_1-6 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$$

* When repeated Eigen values, we have to do like this

$$m_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} \frac{d}{d\lambda_1} C_{11} \\ \frac{d}{d\lambda} C_{12} \\ \frac{d}{d\lambda_1} C_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_1+6 \\ -2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \quad \lambda_1 = -1$$

$$(\lambda_3 I - A) = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

$$C_{11} = (+1)\begin{vmatrix} -1 & 0 \\ -2 & -1 \end{vmatrix} = 1, \quad C_{12} = (-1)\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = 2, \quad C_{13} = (+1)\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = -4$$

$$m_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}.$$

To find canonical form of state model :-

The modal matrix, M is given by.

$$M = [m_1 \; m_2 \; m_3] = \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}.$$

$$M^{-1} = \frac{\text{Adj } M}{|A|} = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}.$$

$$J = M^{-1}AM = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

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$$\tilde{B} = M^{-1}B = \frac{1}{18}\begin{bmatrix} -2 & -4 & -2.5 \\ 6 & 3 & 3 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}.$$

The Jordan canonical form of state model is shown below

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -4/9 \\ 3/9 \\ 4/9 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

(Here DU is not defined).

Conclusion:

It is observed that the elements of the rows of $\tilde{B}$ are not all zeros. Hence the system is completely controllable.

It is observed that the elements of the columns of $\tilde{C}$ are not all zeros. Hence the system is completely observable.

1) What are the advantages and disadvantages of phase variables.

Sol: Advantages:- The state space model can be directly formed by inspection from the differential equations governing the system. The phase variable provides a link b/w a transfer function design approach and time domain design approach.

Disadvantages:- The phase variables are not physical variables of the system and therefore are not available for measurement and control purposes.

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2) What are the advantages of state space Analysis?

Sol: i) The state space Analysis is applicable to any type of systems. They can be used for modelling and analysis of linear & non-linear systems, time invariant & time variant systems and multiple i/ps & multiple o/ps system

ii) The state space analysis can be performed with initial conditions.

iii) Using this Analysis the internal state of the system at any time instant can be predicted.

3) What are the drawbacks in transfer function model Analysis?

Sol: i) Transfer function is defined under zero initial conditions.

ii) Transfer function is applicable to linear time invariant system.

iii) Transfer function restricted to single i/p and single o/p.

iv) Does not provides information regarding the internal state of the system.

4) What is state and state variables?

The state is the condition of system at any time instant "t". A set of variables which describes the state of the system at any time instant are called state variables.

5) What is state vector.?

The state vector is a $(n \times 1)$ column matrix whose elements are state variables of the system. It is denoted by $X(t)$

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

n is order of system

6) Write the State model of n^{th} order system?

The State model of a System consists of State eq/. and o/p eq/.

The State model of a n^{th} order system with m- inputs and P- outputs are

$\dot{X}(t) = AX(t) + BU(t)$ → State eq/.

$Y(t) = CX(t) + DU(t)$ → o/p eq/.

Where $X(t)$ = State vector of order $(n \times 1)$

$U(t)$ = input vector of order $(m \times 1)$

$Y(t)$ = output vector of order $(P \times 1)$

A = System matrix of order $(n \times n)$

B = input matrix of order $(n \times m)$

C = output matrix of order $(P \times n)$

D = transmission matrix of order $(P \times m)$

7) The state model of a linear time invariant system is given by,

$\dot{X}(t) = AX(t) + BU(t)$
$Y(t) = CX(t) + DU(t)$
. obtain the Expression for transfer function of the system.

so/ given that $\dot{X}(t) = AX(t) + BU(t)$ → ①

$Y(t) = CX(t) + DU(t)$ → ②

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on taking L.P.T to eq/. ①

$$SX(S) = AX(S) + BU(S)$$

$$SX(S) - AX(S) = BU(S)$$

$$SIX(S) - AX(S) = BU(S)$$

$$X(S)[SI - A] = BU(S)$$

$$X(S) = (SI - A)^{-1} B U(S) \rightarrow ③$$

on taking L.P.T to eq/. ②

$$Y(S) = CX(S) + DU(S) \rightarrow ④$$

sub/. ③ in ④

$$Y(S) = C(SI - A)^{-1} BU(S) + DU(S)$$

$$Y(S) = U(S)[C(SI - A)^{-1} B + D]$$

$$\boxed{\frac{Y(S)}{U(S)} = C(SI - A)^{-1} B + D}$$

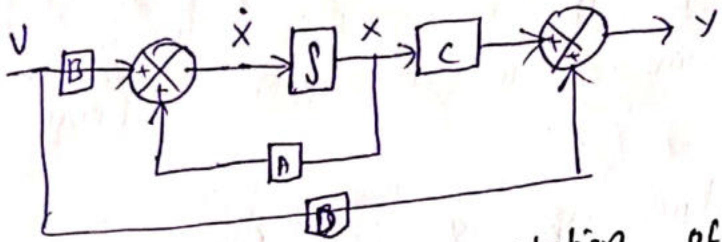
8) What is state diagram?

The pictorial representation of the state model of the system is called state diagram. The state diagram of the system can be either in block diagram (or) in signal flow graph form.

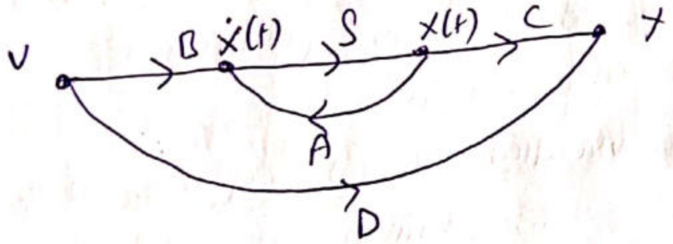
9) Draw the block diagram representation of state model?

$\dot{X} = AX + BU$

$Y = CX + DU.$



10) Draw the signal flow graph representation of state model.



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