ANNAMACHARYA UNIVERSITY
EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
ESTD, UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)
Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Lecture Notes
on
Theory and Analysis of Plates

Written by
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ANNAMACHARYA UNIVERSITY

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Course Structure for M. Tech-Structural Engineering

Title of the Course:	Theory and Analysis of Plates
Category:	Program Elective-II
Couse Code:	24DSTE1FT
Branch/es:	Structural Engineering
Semester:	I Semester

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	-	-	3

Course Description:
This course focuses on the theoretical foundations and practical applications of plate analysis in structural engineering. Students will explore the derivation of plate equations for various loading conditions and boundary conditions, including rectangular and circular plates

Course Objectives:
1. Describe the fundamental theories related to plate behavior, including classical plate theory, sandwich theory, and other relevant models.
2. Utilize differential equations and numerical methods to analyze the bending, stability, and vibration of various plate types under different loading conditions.
3. Examine the effects of different boundary conditions on the performance of plates and how they influence stress distribution and deflection.
4. Evaluate experimental results related to plate behavior, including load testing and material characterization, and compare them with theoretical predictions

Course Outcomes:
At the end of the course, the student will be able to
1. Derive and understand the governing equations for rectangular plates under various loading conditions.
2. Analyze circular plates, including symmetrically loaded and annular configurations.
3. Apply the principles of bending and stretching to derive governing equations and solve practical problems.
4. Explore orthotropic plate behavior and apply these concepts to grillage problems.
5. Utilize numerical methods, including finite element and variational approaches, to analyze complex plate problems and large deflections.

Unit 1 10

Derivation of Plate Equations For Rectangular Plates –In plane bending and transverse bending effects. Plates under various loading conditions like concentrated, U.D.L and hydrostatic pressure Naiver and Levy"s type of solutions for various boundary conditions.

Unit 2 10

Circular Plates: Symmetrically loaded, circular plates under various loading conditions, annular plates.

Unit 3 10

Plates Under Simultaneous Bending And Stretching: Derivation of the governing equation and application to simple cases.

10

Unit 4

Orthotropic Plates: Derivation of the governing equation, applications to grillage problems as equivalent orthotropic plates.

Unit 5 10

Numerical And Approximate Methods: Energy solutions by variational methods, finite difference and finite element methods of analysis for plate problems. Study of few simple cases for large deflection theory of plates.

Prescribed Text books:

1.Timoshenko, S., and Krieger, S.W., Theory of plates and shells, McGraw Hill Book company.
2. Theory of plates by Chandra Shekhara, K, Universities Press ltd
3. Szilard, R., Theory and Analysis of Plates, Prentice Hall Inc. 4. N.K.Bairagi, Plate analysis, Khanna Publishers, Delhi, 1986.

CO-PO Mapping

Course Outcomes	PO1	PO2	PO3	PO4	PO5	PO6
24BCIV11T.1	3	-	3	-	-	3
24BCIV11T.2	3	-	3	-	-	3
24BCIV11T.3	3	-	3	-	-	3
24BCIV11T.4	3	-	3	-	-	3
24BCIV11T.5	3	-	3	-	-	3

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Theory and Analysis of Plates

UNIT-1

Theory of Plates

UNIT- I, Derivation of plate equation for Rectangular plates

Plate - Plates are plane structural elements with a small thickness compared to the planar dimension's. Typical thickness to width ratio is less than 0.1. $t/l \ll 1$

Basically plates are classified in to Two types, they are

1) Thin plates
2) Thick plates

Again

Thin plates - Plates that are considered in to two dimensional

Again thin plates are classified in to two types

→ Thin plates with small deflection's

→ Thin plates with large deflection's

Thin plates with small deflection's - If deflection's w of a plate are small in comparison with its thickness h, a very ~~s~~ Bending of the plate by lateral loads can be developed.

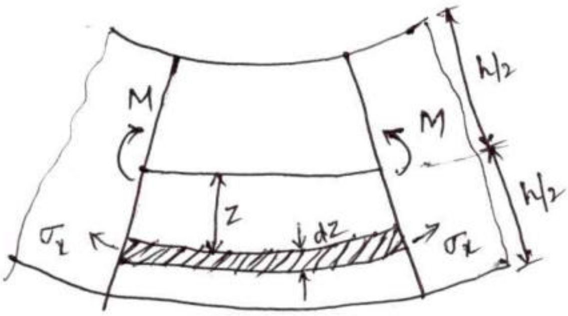
Thin plates with large deflection's - If deflection's w of a plate are large in comparison with its thickness, Bending of a plate is accompained by strain in the middle plane due to this supplimentary stresses are developed which makes the solution complicated.

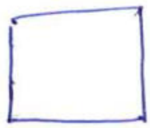
Thick plates -

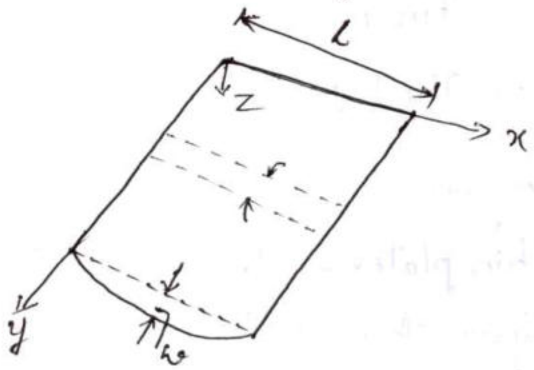
The theory of thin plates become unreliable in the case of plates of considerable thickness. This theory considers the problem of plates are three - dimensional problem of elasticity.

Differential Equation for cylindrical bending of plates -

Consider a plate of uniform thickness of h. let the xy plane as the middle plane of the plate before loading i.e., as the plane midway between the faces of the plate. let y axis coincide with one of the longitudinal edges of the plate and positive direction of the z axis be downward. l be the width of the plate. as shown in fig(1)







Consider an elemental strip of rectangular cross section which has a length l and a depth h. In calculating the bending stresses in such a bar we assume, as in the ordinary theory of beams, that c/s of the bar remain plane during bending, so that they undergo only a rotation WRT N.A. If no normal forces are applied to the end sections of the bar, the neutral surface of the bar coincides with the middle surface of the plate, and the unit elongation of a fibre parallel to the x-axis is proportional to its distance z from the middle surface.

The curvature of bar deflection curve can be taken equal to $-\frac{\partial^2 w}{\partial x^2}$, where w, is the deflection of the bar in the z direction is assumed to be small compared with the length of bar l.

$\therefore$ Unit elongation ϵ_x of a fibre at a distance z from the middle surface is then

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \longrightarrow$$

By using Hooke's law,

Unit elongation's ϵ_x and ϵ_y in terms of normal stresses σ_x and σ_y acting on the element shown shaded

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu \sigma_y}{E} \longrightarrow 1(a)$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu \sigma_x}{E} \longrightarrow 1(b)$$

where E and μ are modulus of elasticity and poisson's ratio. Lateral strain in the y-direction must be zero in order to maintain continuity in the plate during bending

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E} = 0$$

$$\frac{\sigma_y}{\cancel{E}} = \frac{\mu \sigma_x}{\cancel{E}} \Rightarrow \sigma_y = \mu\sigma_x \longrightarrow (a)$$

substituting a in 1(a) equation:

$\epsilon_x =$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu(\mu\sigma_x)}{E}$$

$$\epsilon_x = \frac{(1-\mu^2)\sigma_x}{E}$$

$$\sigma_x = \frac{E\epsilon_x}{(1-\mu^2)}$$

substituting ϵ_x in the above equation

$$\sigma_x = \frac{E}{(1-\mu^2)}\left[-z\frac{\partial^2 w}{\partial x^2}\right]$$

$$= \frac{-Ez}{(1-\mu^2)}\frac{\partial^2 w}{\partial x^2}$$

If the plate is submitted to the action of tensile and compressive forces acting along x direction and uniformly distributed along the longitudinal sides of the plate corresponding direct stress must be added to stress due to bending.

In order to have expression for bending stress σ_x, we obtain by integration the bending moment of the strip.

$$M = \int_{-h/2}^{h/2} \sigma_x \, z \, dz \longrightarrow 3$$

substituting σ_x in the above equation

$$= -\int_{-h/2}^{h/2} \frac{E(z^2)}{(1-\mu^2)} \frac{\partial^2 w}{\partial x^2} dz = \left(\frac{E}{1-\mu^2}\right)\int_{-h/2}^{h/2} z^2 \frac{\partial^2 w}{\partial x^2} dz$$

$$\frac{2E}{1-\mu^2}\int^{h/2} z^2 \, dz \frac{\partial^2 w}{\partial x^2} = \left(-\frac{E z^3}{3(1-\mu^2)} \frac{\partial^2 w}{\partial x^2}\right)_{-h/2}^{h/2} \quad \text{where } z = -h/2$$

or

let $\dfrac{E h^3}{12(1-\mu^2)} = D$

$\frac{z^3}{3}, \frac{(h/2)^3}{3}$

There fore substituting D in ③ equation, we get the equation for the deflection curve of the elemental strip.

$$D \times \frac{\partial^2 w}{\partial x^2} = -M \longrightarrow ④$$

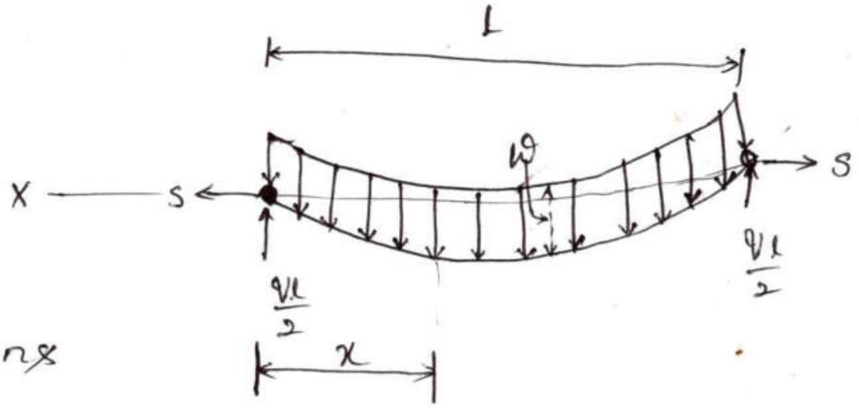
In which the quantity D, taking the place of the quantity EI in the beam is called flexural rigidity of the plate and

Considering $\mu = 0$

2 Cylindrical Bending of uniformly loaded rectangular plates with Simply Supported edges;

Consider a uniformly distributed long rectangular plate with longitudinal edges which are free to rotate but cannot move toward each other during bending. An elemental strip cut out from plate as shown in fig (1) is in the condition of a uniformly loaded bar submitted to the action of an axial force **S**. The magnitude of S is such as to prevent the ends of bar from moving along the x-axis. Denoted by Q the intensity of the uniform load, the bending moment at any section of the strip is

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw$$



we know

$$D\frac{\partial^2 w}{\partial x^2} = -M$$

Equating above two equations

$$-D\frac{\partial^2 w}{\partial x^2} = \frac{ql}{2}x - \frac{qx^2}{2} - Sw$$

$$-\frac{\partial^2 w}{\partial x^2} = \frac{ql}{2D}x - \frac{qx^2}{2D} - \frac{Sw}{D}$$

$$-\frac{qlx}{2D} + \frac{qx^2}{2D} = \frac{\partial^2 w}{\partial x^2} - \frac{Sw}{D} \longrightarrow$$

Introducing the notation $u^2 = \frac{Sl^2}{4D}$

General solution of above equation can be written in the following form

$$w = C_1 \sinh\frac{2ux}{l} + C_2\cosh\frac{2ux}{l} + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D} - \frac{ql^4}{16u^4D} \quad \rightarrow (b)$$

The constants of integration C_1 and C_2 will be determined from the conditions at the ends. Since the deflections of the strip at the ends are zero, we have

$w = 0$ for $x = 0$ and $x = l$;

Substituting for w its expression b, we obtain from these two conditions

$$C_1 = \frac{q l^4}{16 u^4 D} \times \frac{1 - \cosh 2u}{\sinh 2u}$$

$$C_2 = \frac{q l^4}{16 U^4 D}$$

Substituting C_1, C_2 in the above expression w becomes

$$w = \frac{q l^4}{16 u^4 D}\left(\frac{1-\cosh 2u}{\sinh 2u} \cdot \sinh\frac{2ux}{l} + \cosh\frac{2ux}{l} - 1\right] + \frac{q l^3 x}{8u^2 D} - \left.\frac{q l^2 x^2}{8u^2 D}\right]$$

This is the differential equation for the elementary strip. The general equation Complementary

$$w = C.F + P.I$$

Complementary function

$$D \cdot \frac{\partial^2 w}{\partial x^2} - S w = 0$$

$$\left(D_1^2 - \frac{S}{D}\right) w = 0$$

$$D_1^2 - \frac{S}{D} = 0 \qquad D_1 = \pm\sqrt{\frac{S}{D}} \rightarrow \text{roots}$$

General equation

$$w = C_1 \sinh\sqrt{\frac{S}{D}}\,x + C_2 \cosh\sqrt{\frac{S}{D}}\,x$$

Particular Integral

$$\left(D_1^2 - \frac{S}{D}\right) w = \frac{q x^2}{2D} - \frac{q l x}{2D}$$

$$PI \quad w = \frac{1}{\left(D_1^2 - \frac{S}{D}\right)}\left[\cancel{\frac{q x^2}{2D}} - \frac{q l x}{2D}\right]$$

$$\frac{1}{\frac{S}{D}\left[D_1^2\left[\frac{D}{S}\right]-1\right]}\left[\frac{vx^2}{2D}-\frac{vlx}{2D}\right]$$

$$= -\frac{D}{S}\left[1-D_1^2\left[\frac{D}{S}\right]\right]^{-1}\left[\frac{vx^2}{2D}-\frac{vlx}{2D}\right]$$

$$(1-D)^{-1} = 1+D+D^2+D^3+\dots$$

$$P.I = -\frac{D}{S}\left[1+D_1^2\left(\frac{D}{S}\right)+D_1^4\left(\frac{D}{S}\right)\right]\left[\frac{vx^2}{2D}-\frac{vlx}{2D}\right]$$

$$= -\frac{D}{S}\left[\frac{vx^2}{2D}-\frac{vlx}{2D}+\frac{vD}{DS}\right]$$

$$v = \sqrt{\frac{S}{D}}\times\frac{l}{2}$$

$$\sqrt{\frac{S}{D}} = \frac{2u}{l}$$

Introducing a term $u^2 = \frac{Sl^2}{4D}$, we get a complete G.S of a term

$$w = C.F + P.I$$

$$= C_1 \sinh\left(\sqrt{\frac{S}{D}}\right)x + C_2\cosh\left(\sqrt{\frac{S}{D}}\right)x + \left[-\frac{vx^2l^2}{8u^2D}+\frac{vl^3x}{u^2 8D}-\frac{vl^4}{16u^4D^2}\right]$$

P.I is divided and multiplied by v^2

$$P.I = -\frac{D}{S}\left[\frac{vx^2u^2}{2u^2}-\frac{vlx}{2u^2}\times u^2-\frac{Dv}{S}\times\frac{u^4}{u^4}\right]$$

$$\frac{vx^2}{2Du^2}\times\frac{Sl^2}{4D} = \frac{vx^2l^2S}{8u^2D^2}$$

$$\frac{vlx}{2D}\times\frac{v^2}{v^2} = \frac{vlx}{2Du^2}\times\frac{Sl^2}{4D} = \frac{Svl^3x}{8u^2D^2}$$

$$u^4 = \frac{S^2l^4}{16D^2}$$

$$\frac{v}{S}\frac{u^4}{u^4} \Rightarrow \frac{v}{Su^4}\times\frac{S^2l^4}{16D^2} = \frac{vSl^4}{16u^4D^2}$$

$$P.I = -\frac{D}{S}\left[\frac{vx^2l^2S}{8u^2D^2}-\frac{Svl^3x}{8u^2D^2}+\frac{vSl^4}{u^4 16D^2}\right]$$

$$P.I = \left(\frac{vx^2l^2}{8u^2D}+\frac{vl^3x}{u^2 8D}-\frac{vl^4}{16u^4D^2}\right)$$

$$w = C_1 \sinh\frac{2ux}{l} + C_2 \cosh\frac{2ux}{l} - \frac{qx^2l^2}{8u^2D} + \frac{ql^3x}{8u^2D} - \frac{ql^4}{16u^4D^2}$$

~~From the boun~~

So let us substitute

$$1-\cosh 2u = (\cosh^2 u - \sinh^2 u) - (\cosh^2 u + \sinh^2 u)$$

$$= -2\sinh^2 u$$

~~sinh2u~~ $\sinh 2u = 2\sinh u \cosh u$

$$w = \frac{-ql^4}{16u^4D}\left[\frac{-2\sinh^2 u}{2\sinh u \cosh u}\right]\sinh\frac{2u}{l}x + \frac{ql^4}{16u^4D}\cosh\frac{2ux}{l} + \frac{ql^3}{8u^2D}$$

$$-\frac{ql^2x^2}{8u^2D} - \frac{ql^4}{16u^4D}$$

$$= \frac{ql^4}{16u^4D}\left[-\frac{\sinh u}{\cosh u}\times\frac{\sinh 2u}{l}x + \cosh\frac{2u}{l}x\right] + \frac{ql^3x}{8u^2D} -$$

$$\frac{ql^2x^2}{8u^2D} - \frac{ql^4}{16u^4D}$$

finally

$$w = \frac{ql^4}{16u^4D}\left[\frac{\cosh u\left(1-\frac{2x}{l}\right)}{\cosh hu} - 1\right] + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D}$$

Thus the deflection curve or equation is mainly depend upon 'u'. which is the function of axial force 'S'. The magnitude of the axial force enables that the ends of simply supported plate does not move along x-direc

Hence the extension of strip produced by axial force S is equal to the difference b/n the length of the arc along the deflection curve and chord length 'l' is represented by λ

$$\lambda = \frac{1}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx.$$

$\frac{d^2w}{dx^2}$ = curvature

$\frac{dw}{dx}$ = slope

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} \qquad \epsilon_x = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

$\epsilon_y = 0 \quad \sigma_y = \mu \sigma_x$

$\epsilon_x = \frac{\sigma_x}{E}\left[1-\mu^2\right)$ also $\epsilon_x = \frac{\delta l}{l}$

$\frac{\delta l}{L} = \frac{\sigma_x}{E}(1-\mu^2) \qquad \sigma_x = \frac{S}{1\times h} = \frac{S}{h}$

$\delta l = \frac{S l}{hE}(1-\mu^2)$

$\frac{1}{2}\int_0^l \left(\frac{d^2 w}{dx^2}\right) dx = \frac{S l}{hE}(1-\mu^2)$

λ = Extension produced in the elemental strip due to axial force 'S'.

$\frac{Sl}{Eh}(1-\mu^2) = \frac{1}{2}\int_0^l \left[\frac{\partial}{\partial x}\frac{q l^4}{16 U^4 D}\frac{\cosh u(1-\frac{2x}{l})}{\cosh u} + \frac{q l^3 x}{8u^2 D} - \frac{q l^2 x^2}{8u^2}\right)$

$\frac{SL}{Eh}(1-\mu^2) = \frac{q^2 l^7}{D^2}\left[\frac{5}{256}\frac{\tanh u}{u^7} + \frac{1}{256}\frac{\tanh^2 u}{U^6} - \frac{5}{256}\frac{1}{u^6} + \frac{1}{384}\times\frac{1}{u^4}\right]$

$D = \frac{Eh^3}{12(1-\mu^2)} \Rightarrow 12(1-\mu^2) = \frac{Eh^3}{D} \qquad S = \frac{4u^2 D}{l^2}$

$\Rightarrow \frac{1-\mu^2}{Eh} = \frac{h^2}{12D}$ In the above equation we get

$\frac{E^2 h^8}{(1-\mu^2) q^2 l^8} = \frac{135}{16}\frac{\tanh u}{u^9} + \frac{27}{16}\frac{\tanh^2 u}{u^8} - \frac{135}{16 u^8} + \frac{9}{8 u^6}$

Max BM can be calculated from Differential Equation

$D \times \frac{\partial^2 w}{\partial x^2} = -M$

$M_{max} = -D \times \frac{\partial^2 w}{\partial x^2}$

$w = \frac{q l^4}{16 U^4 D}\left[\frac{\cosh u(1-\frac{2x}{l})}{\cosh u} - 1\right] + \frac{q l^3 x}{8u^2 D} - \frac{q l^2 x^2}{8u^2 D}$

$\frac{\partial w}{\partial x} = \frac{q l^4}{16 U^4 D}\left[\frac{\sinh u(1-\frac{2x}{l})}{\cosh u}\left[\frac{-2u}{l}\right]\right] + \frac{q l^3}{8u^2 D} - \frac{2 q l^2 x}{8u^2 D}$

$$\frac{\partial^2 w}{\partial x^2} = \frac{q l^2}{4 u^2 D}\left[-1 + \frac{\cosh u\left(1-\frac{2x}{l}\right)}{\cosh u}\right]$$

At $x = l/2$ we get

$$= \frac{q l^2}{4 u^2 D}\left[\frac{\cosh u\,\cancel{(0)}}{\cosh u} - 1\right]$$

$$= \frac{q l^2}{4 u^2 D}\left[\frac{1-\cosh u}{\cosh u}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{q l^2}{4 u^2 D}\left[\operatorname{sech} u - 1\right] \times \frac{2}{2}$$

$$= \frac{q l^2}{8 u^2 D}\left(2\operatorname{sech} u - 1\right)$$

Max B.M

$$M_{max} = -D\left[\frac{\partial^2 w}{\partial x^2}\right]_{x=l/2}$$

$$= \frac{-q l^2}{8 u^2}\left[2(\operatorname{sech} u - 1)\right]$$

$$M_{max} = \frac{q l^2}{8 u^2}\left[2(1-\operatorname{sech} u)\right]$$

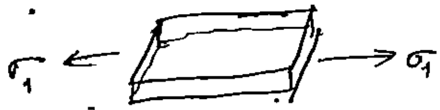
$$\times \div\ = \frac{q l^2}{8}\left[\frac{2}{u^2}(1-\operatorname{sech} u)\right]$$

$$= \frac{q l^2}{8}\left[(1-\operatorname{sech} u) \Big/ \left(\frac{u^2}{2}\right)\right]$$

$$\psi_0(u) = \left[1-\operatorname{sech} u / (u^2/2)\right]$$

$$M_{max} = \frac{q l^2}{8}\psi_0(u)$$

Maximum stress in plate $\sigma_{max} = \sigma_1 + \sigma_2$



$$\sigma_1 = \frac{S}{h \times 1}$$

$$\sigma_2 = \frac{M_{max}}{h^3/12} \times h/2$$

$$\sigma_2 = \frac{6 M_{max}}{h^2} \qquad \because U^2 = \frac{S l^2}{4D}$$

$$\sigma_{max} = \frac{S}{h} + \frac{6}{h^2}\frac{q l^2}{8}\psi_0(u) \qquad S = \frac{4Du^2}{l^2}$$

$$= \frac{4Du^2}{h l^2} + \frac{6}{h^2}\frac{q l^2}{8}\psi_0(u) \qquad D = \frac{Eh^3}{12(1-\mu^2)}$$

$$\sigma_{max} = \frac{1}{3}\frac{E h^2 u^2}{l^2(1-\mu^2)} + \frac{3}{4}\frac{q l^2}{h^2}\psi(u)$$

$$= \frac{1}{3}\frac{E u^2}{(1-\mu^2)}\left(\frac{h}{l}\right)^2 + \frac{3}{4} q \psi_0(u)\left(\frac{l}{h}\right)^2$$

Max deflection of the long rectangular plate will be

$$w_{max} = w_{at\, x = l/2}$$

$$w = \frac{q l^4}{16 u^4 D}\left[\frac{\cosh u\left(1 - \frac{2x}{l}\right)}{\cosh u} - 1\right] + \frac{q l^3 x}{8u^2 D} - \frac{q l^2 x^2}{8u^2 D} \quad \text{when } x = l/2$$

we get

$$= \frac{q l^4}{16 u^2 D}\left[\frac{1}{u^2}(\operatorname{sech} u - 1) + \frac{1}{2}\right]$$

$$w = \frac{q l^4}{16 u^2 D} \times \frac{5 \times 24}{5 \times 24}\left(\frac{1}{u^2}(\operatorname{sech} u - 1) + \frac{1}{2}\right)$$

$$= \frac{5 q l^4}{384 u^2 D} \times \frac{24}{5}\left[\frac{1}{u^2}(\operatorname{sech} u - 1) + \frac{1}{2}\right]$$

$$= \frac{5 q l^4}{384 D}\left[\frac{24(\operatorname{sech} u - 1)}{5u^4} + \frac{24}{5u^2} \times \frac{1}{2}\right]$$

$$= \frac{5 q l^4}{384 D}\left[\frac{24(\operatorname{sech} u - 1)}{5u^4} + \frac{12}{5u^2}\right]$$

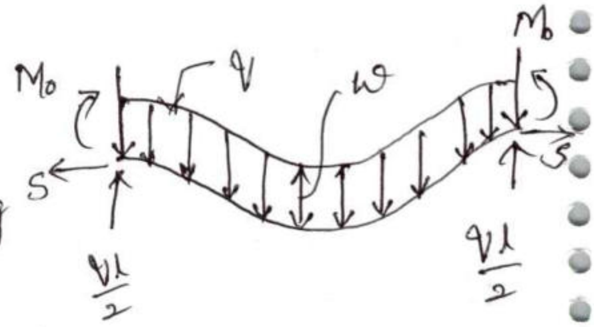
$$w_{max} = \frac{5 q l^4}{384 D}\left(\frac{(\operatorname{sech} u - 1)}{\left(\frac{5u^4}{24}\right)} + \frac{1}{\left(\frac{5u^2}{12}\right)}\right)$$

$$w_{max} = \frac{5ql^4}{384D} f_0(u)$$

If there were no tensile forces acting along the edges of the strip the max deflection be $\frac{5ql^4}{384D}$

Cylindrical bending of rectangular plate subjected to UDL with built in edges;

Let us consider a long rectangular plate having thickness 'h' subjected to UDL. The longitudinal edges are built in. edges. i.e., means the edges cannot rotate. An elemental strip cut out from the plate and the forces acting as shown in figure.



q = instensity of load

M_0 = Bending moment per unit weight along the longitudinal edges

S = Axial force

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw + M_0$$

$$M = -D\frac{\partial^2 w}{\partial x^2} \qquad D \times \frac{\partial^2 w}{\partial x^2} = -M$$

$$D\frac{\partial^2 w}{\partial x^2} = \frac{qlx}{2} - \frac{qx^2}{2} - Sw + M_0$$

$$v^2 = \frac{Sl^2}{4D}$$

$$D \times \frac{\partial^2 w}{\partial x^2} = -\frac{qlx}{2} + \frac{qx^2}{2} + Sw - M_0$$

$$\frac{S}{D} = \frac{4u^2}{l^2}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{-qlx}{2D} + \frac{qx^2}{2D} + \frac{Sw}{D} - \frac{M_0}{D}$$

C.F $\quad w = C_1 \sinh\frac{2ux}{l} + C_2 \cosh\frac{2ux}{l}$

P.I $\quad w = \frac{qlx}{2D}\left[\frac{D}{S}\right] - \frac{qx^2}{2D}\left[\frac{D}{S}\right] - q\left[\frac{D}{S}\right]^{*} + \frac{M_0}{D}\left[\frac{D}{S}\right]$

q/S

$$= \frac{vlx}{2D}\left[\frac{l^2}{4U^2}\right] - \frac{vx^2}{2D}\left[\frac{l^2}{4U^2}\right] - v\left(\frac{l^2}{4U^2}\right)^2 + \frac{M_0}{D}\left[\frac{l^2}{4U^2}\right]$$

$$w = \frac{vl^3x}{8U^2D} - \frac{vx^2l^2}{8U^2D} - \frac{vl^4}{16U^4} + \frac{M_0l^2}{4DU^2}$$

$$= C_1\sinh\left[\frac{2ux}{l}\right] + C_2\cosh\left[\frac{2ux}{l}\right] + \frac{vl^3x}{8U^2D} - \frac{vx^2l^2}{8U^2D} - \frac{vl^4}{16U^4} + \frac{M_0l^2}{4U^2D}$$

Boundary conditions

At $x=0$, $w=0$, $\frac{\partial w}{\partial x}=0$

$x=l/2$, $w=0$, $\frac{\partial w}{\partial x}$ = max deflection

$x=0$, $w=0$,

$S = \frac{Sl^2}{4D}$

$S/D = 4U^2/l^2$

D/S

Case(i) $x=0$, $w=0$

$$\frac{\partial w}{\partial x} = C_1\cosh\frac{2ux}{l}\times\frac{2u}{l} + C_2\sinh\frac{2ux}{l}\times\frac{2u}{l} + \frac{vl^3}{8U^2D} - \frac{2vxl^2}{8U^2D}$$

$$= C_1\cosh\left(\frac{2ux}{l}\right)\left(\frac{2u}{l}\right) + C_2\sinh\left(\frac{2ux}{l}\right)\frac{2u}{l} + \frac{vl^3}{8U^2D} - \frac{2vxl^2}{8U^2D}$$

$x=0$, $w=0$ $\frac{\partial w}{\partial x}=0$

$$0 = C_1\cosh^{(0)}\left[\frac{2u}{l}\right] + C_2\sinh^{(0)}\left(\frac{2u}{l}\right) + \frac{vl^3}{8U^2D}$$

$$C_1\left[\frac{2u}{l}\right] = \frac{-vl^3}{8U^2D} \quad \therefore C_1 = \frac{-vl^4}{16U^3D}$$

Case ii $x=l/2$ $\frac{\partial w}{\partial x}=0$

$$0 = \frac{-vl^4}{16U^3D}\times\cosh\left[\frac{2u(l/2)}{l}\right]\times\frac{2u}{l} + C_2\sinh\left(\frac{2u\times l}{2l}\right)\times\frac{2u}{l} + \frac{vl^3}{8U^2D} - \frac{2v(l/2)l^2}{8U^2D}$$

$$= \frac{-\cancel{2}vl^4u}{\underset{8}{\cancel{16}}U^3D\cancel{l}}\cosh u + \frac{2u}{l}C_2\sinh u + \cancel{\frac{vl^3}{8U^2D}} - \cancel{\frac{vl^3}{8U^2D}}$$

$$= \frac{vl^3}{8U^2D}\cosh u + \frac{2u}{l}C_2\sinh u$$

$$\frac{2u}{l} C_2 \sinh u = \frac{q l^3}{8U^2 D} \cosh u$$

$$C_2 = \frac{q l^3}{8U^2 D} \times \frac{l}{2u} \frac{\cosh u}{\sinh u}$$

$$= \frac{q l^4}{16 U^3 D} \coth u$$

Case (iii) $x = 0 \quad w = 0$;

$$w = C_1 \sinh \frac{2ux}{l} + C_2 \cosh\left[\frac{2ux}{l}\right] + \frac{q l^3 x}{8U^2 D} - \frac{q x^2 l^2}{8U^2 D} - \frac{q l^4}{16 U^4 D} + \frac{M_0 l^2}{4U^2 D}$$

$$0 = 0 + C_2 \cosh\left[\frac{2u0}{l}\right] + 0 - 0 - \frac{q l^4}{16 U^4 D} + \frac{M_0 l^2}{4U^2 D}$$

$$C_2 - \frac{q l^4}{16 U^4 D} + \frac{M_0 l^2}{4U^2 D} = 0$$

$$\frac{q l^4}{16 U^3 D} \coth u - \frac{q l^4}{16 U^4 D} + \frac{M_0 l^2}{4 U^2 D} = 0$$

$$\frac{q l^4}{16 U^3 D}\left[\coth u - \frac{1}{u}\right] = -\frac{M_0 l^2}{4U^2 D}$$

$$-M_0 = \frac{q l^2}{4u}\left[\coth u - \frac{1}{u}\right]$$

$$-M_0 = \frac{q l^2}{4u} \coth u - \frac{q l^2}{4U^2}$$

$$w = C_1 \sinh\left[\frac{2ux}{l}\right] + C_2 \cosh\left[\frac{2ux}{l}\right] + \frac{q l^3 x}{8U^2 D} - \frac{q x^2 l^2}{8U^2 D} - \frac{q l^4}{16 U^4 D} + \frac{M_0 l^2}{4U^2 D}$$

$$C_1 = \frac{-q l^4}{16 U^3 D}, \quad C_2 = \frac{q l^4}{16 U^3 D} \coth u$$

$$M_0 = \frac{-q l^2}{4u} \coth u + \frac{q l^2}{4U^2}$$

$$w = \frac{q l^4}{16 U^3 D \tanh u}\left[\frac{\cosh u\left(1-\frac{2x}{l}\right)}{\cosh u} - 1\right] + \frac{q l^3 x}{8 U^2 D} - \frac{q l^2 x^2}{8 U^2 D}$$

The deflection expression w depends on U^2 which is a function of axial force 'S'.

$$\lambda = \frac{1}{2}\int_0^l \frac{d^2 w}{\partial x^2}\,dx$$

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} = 0 \qquad \sigma_y = \mu\sigma_x$$

$$\epsilon_x = \frac{(1-\mu^2)\,\sigma_x}{E} = \frac{\delta l}{l}$$

$$\delta l = \frac{(1-\mu^2)\,\sigma_x \times l}{E}$$

$$\delta l = \frac{(1-\mu^2)\times S \times l}{hE} = \frac{Sl}{hE}(1-\mu^2)$$

$$\frac{1}{2}\int_0^l \frac{\partial^2 w}{\partial x^2}\,dx = \frac{Sl}{hE}(1-\mu^2)$$

Substituting w in the above equation and performing integration we get

$$\frac{Sl}{hE}(1-\mu^2) = \frac{q^2 l^2}{D^2}\left[-\frac{3}{256\,U^5\tanh u} \div \frac{1}{256\,U^4\sinh^2 u} + \frac{1}{64\,U^6} + \frac{1}{384\,U^4}\right]$$

Substituting $D = \frac{Eh^3}{12(1-\mu^2)}$ $S = \frac{4U^2 D}{l^2}$ in the above equation we get

$$\frac{E^2 h^8}{q^2 l^8 (1-\mu^2)^2} = \frac{-81}{16\,U^7\tanh u} - \frac{27}{16\,U^6\sinh^2 u} + \frac{27}{4U^8} + \frac{9}{8U^6} \longrightarrow$$

Maximum B.M

$$M_{max} = M_0$$

$$\frac{ql^2}{4U^2} - \frac{ql}{4u}\coth u$$

$$\frac{ql^2}{4u}\left[\frac{1}{u} - \frac{1}{\tanh u}\right] \times \frac{3}{3}$$

$$-\frac{3ql^2}{12u}\left[\frac{\tanh u - u}{u\tanh u}\right]$$

$$= \frac{-ql^2}{12} \times \frac{3}{u}\left[\frac{u-\tanh u}{u\tanh u}\right]$$

$$M_{max} = \frac{-ql^2}{12} \times \psi_1(u)$$

Max deflection at $x = l/2$,

$$w = \frac{ql^4}{16u^3 \tanh u D}\left[\frac{\cosh u\left(1-\frac{2x}{l}\right)}{\cosh u} - 1\right] + \frac{ql^3x}{8u^2D} - \frac{ql^2x^2}{8u^2D}$$

$$\frac{ql^4}{16u^3 \tanh u D}\left[\frac{\cosh u\left(1-\frac{2l}{2l}\right)}{\cosh u} - 1\right] + \frac{ql^3 \times l}{16u^2D} - \frac{ql^2 \times l^2}{32u^2D}$$

$$= \frac{ql^4}{16u^3 \tanh u}\left[\frac{1-\cosh u}{\cosh u}\right] + \frac{ql^4}{16u^2D}\left[1-\frac{1}{2}\right]$$

$$= \cancel{\frac{ql^4}{16 u^3 D}} \cdot \cancel{\frac{1}{u^3 \tanh u D}} \left[\cancel{\frac{\text{sech}\, u - 1}{\tanh u}}\right] + \cancel{x}$$

$$\frac{ql^4}{16D} \times \frac{24}{24} f_0(u) \times 24$$

$$\frac{ql^4}{384D} f_0(u)$$

~~...ted Rectangular plates under Hydrostatic press~~

~~Assume a long rectangular plate under a hydrostatic pressure~~

Navier solution for simply supported rectangular plates;

Consider a long rectangular plates having thicknes simply supported having thickness 'h' by considering the coordinates axis by the equation

$$q = f(x,y)$$

For this purpose, we represent the function $f(x,y)$ in the form of a double trigonometric series.

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

a_{mn} = coefficient of series

To calculate any particular coefficient a_{mn} of this series we multiply both sides $\sin\left(\frac{n'\pi y}{b}\right)$

$$\cancel{\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy = 0 \quad \text{when } n \neq n'}$$

$$\therefore \int_0^b \left(\sin\frac{n'\pi y}{b}\right) dy \,(f(xy)) = \int_0^b \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \times \sin\frac{n'\pi y}{b}\, dy$$

$$= \sum_{m=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \int_0^b \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy$$

where $n = n'$ we get

$$\therefore \int_0^b \sin\left(\frac{n'\pi y}{b}\right) dy = \sum_{m=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} = 0$$

$$\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy$$

$$= \frac{1}{2}\int_0^b 2\sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,dy$$

$$= \frac{1}{2}\int_0^b \cos\left[\frac{n\pi y}{b}-\frac{n'\pi y}{b}\right]-\cos\left[\frac{n\pi y}{b}+\frac{n'\pi y}{b}\right]dy$$

$n = n'$

$$= \frac{1}{2}\int_0^b \left(\cos 0 - \cos\frac{2n\pi y}{b}\right)dy$$

$$= \frac{1}{2}\int_0^b \left(1-\cos\frac{2n\pi y}{b}\right)dy$$

$$= \frac{1}{2}\left[y+\sin\frac{2n\pi y}{b}\times\frac{2n\pi}{b}\right]_0^b$$

$$\frac{1}{2}\left[b+\sin 2\pi \times \frac{b}{2\pi}\right]$$

$$\therefore \int_0^b f(x,y)\sin\left(\frac{n'\pi y}{b}\right)dy = b/2\, a_{mn}\sin\frac{m\pi x}{a}$$

Again multiplying with the above equation $\sin\frac{m'\pi x}{a}dx$ on both sides and integrating from 0 to a

$$\int_0^b\int_0^a f(x,y)\sin\left[\frac{n'\pi y}{b}\right]\sin\left[\frac{m'\pi x}{a}\right]dy\,dx$$

$$= b/2\, a_{m'n'}\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dx$$

$$= \int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}dx = \qquad m=m',\ m\neq m' = 0$$

$$\int_0^a\int_0^b f(x,y)\sin\frac{n'\pi y}{b}\sin\frac{m'\pi x}{a}dy\,dx = \frac{b}{2}\times\frac{a}{2}\times a_{m'n'}$$

$$a_{m'n'} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)dy\,dx \longrightarrow (1)$$

In this manner we can find out any type of coefficient equation.

Summation of such terms for the given loading f(x,y) are as same as sinusoidal load then deflection equation becomes

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{a_{mn} \left[\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\right]}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \right] \longrightarrow (2)$$

Boundary conditions are

$w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$ @ $x = 0$, $x = a$

$w = 0$, $\frac{\partial^2 w}{\partial y^2} = 0$ @ $y = 0$ $y = a$

If $f(x,y) = q_0$ in eq q_0 = intensity of UDL from formula (1)

~~where a_{mn}~~ $$a_{mn} = \frac{4q_0}{ab} \int_0^a \int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) dx\, dy \longrightarrow (2)$$

$= 16q_0/\pi^2 mn$

where m and n are odd integers. If m or n or both of them are even $a_{mn} = 0$ substituting in eq (2) we get

$$w = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn \left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2} \longrightarrow (3)$$

where $m = 1, 3, 5$ and $n = 1, 3, 5$

In case of uniform load we have deflection surface symmetrical with respect to axis $x = a/2$ $y = b/2$ and quite naturally all terms with even numbers for m or n in series (3) since they are unsymmetrical with respect to above mentioned axis. The max deflection of the plate is at its center and is found by substituting $x = a/2$, $y = b/2$

$$w_{max} = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n}{2} - 1}}{mn\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2} \longrightarrow$$

This a xapi for example Square plate $a=b$ and $m=n=1$

$$w_{max} = \frac{16q_0}{\pi^6 D} \times \frac{a^4}{4}$$

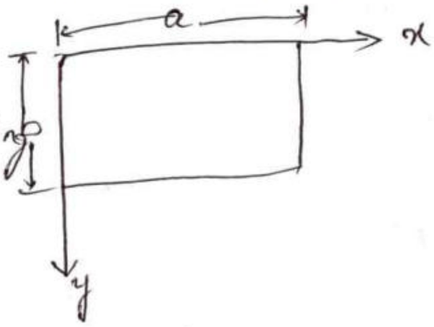
$$= \frac{4q_0 a^4}{\pi^6 D} = 4.16\times10^{-3}\frac{q_0 a^4}{D}$$

If $D = \frac{Eh^3}{12(1-\mu^2)}$ $D = 0.0915\,Eh^3$ for $\mu = 0.3$

Then $w_{max} = 4.16\times10^{-3} \times \frac{q_0 a^4}{0.0915\,Eh^3}$

$$w_{max} = 0.045\frac{q_0 a^4}{Eh^3}$$

Simply supported rectangular plate under sinusoidal loads;



Consider a ▭le plate of $a\times b$ which is subjected to sinusoidal loading the load distributed over the surface of the plate is given by

$$q = q_0 \sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b} \longrightarrow (1)$$

q_0 represents intensity of load at center of the plate is given as

$$\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 = q/D$$

$$\frac{\partial^4 w}{\partial x^4} + 2\times\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_0}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \longrightarrow 2$$

boundary conditions for simply supported edges are

1 $w=0$ 2 $\frac{\partial^2 w}{\partial x^2}=0$ for $x=0$ to a

3 $w=0$ 4 $\frac{\partial^2 w}{\partial y^2}=0$ for $y=0$ to b

let us consider $w = c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \longrightarrow ③$

Constant C to ~~satisfy~~ Eq ② substituting expression

$$\pi^4 \left[\frac{1}{a^2} + \frac{1}{b^2}\right]^2 C = \frac{q_0}{D}$$

$$\frac{\partial w}{\partial x} = c \sin\frac{n\pi y}{b} \overset{\cos}{\sin}\frac{m\pi x}{a} \times \frac{m\pi}{a}$$

$$\frac{\partial^2 w}{\partial x^2} = -c \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a} \times \frac{m^2\pi^2}{a^2}$$

$$\frac{\partial^3 w}{\partial x^3} = c \sin\frac{n\pi y}{b} \overset{\cos}{\sin}\frac{m\pi x}{a} \times \frac{m^3\pi^3}{a^3}$$

$$\frac{\partial^4 w}{\partial x^4} = +c \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a} \times \frac{m^4\pi^4}{a^4}$$

$$\frac{\partial w}{\partial y} = c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \times \frac{n\pi}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \times \frac{n^2\pi^2}{b^2}$$

$$\frac{\partial^3 w}{\partial y^3} = c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \times \frac{n^3\pi^3}{b^3}$$

$$\frac{\partial^4 w}{\partial y^4} = c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \times \frac{n^4\pi^4}{b^4}$$

$$\left[-\cos\frac{m\pi x}{a}\right]_0^a \cdot \frac{m\pi}{a}\Bigg]_0^a$$

$$\cos 0 \quad 1+1$$

$$\left[1 - \cos\frac{m\pi a}{a}\right] \times \frac{m\pi}{a}$$

$$\left((1-(-1))\frac{m\pi}{a}\right) \quad (1-(-1)) \frac{n\pi}{b}$$

$$4 \frac{m^2\pi^2 n^2}{ab}$$

$$4q_0 \times \frac{4 m^2 n^2 \pi^2}{ab} \times \frac{1}{a} \cdot \frac{1}{b}$$

$$1-(-1) \cos\frac{m\pi x}{a}$$

$$\cos\pi \; (m \times n) \times \frac{m\pi}{a} \times \frac{n\pi}{b}$$

$$\frac{16 q_0}{ab} \quad \frac{4 m^2\pi^2 n^2}{ab}$$

if $a=n$, $b=n$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} =$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} = -c \overset{\cos}{\sin}\frac{n\pi y}{b} \times \frac{n\pi}{b} \sin\frac{m\pi x}{a} \times \frac{m^2\pi^2}{a^2}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = +c \sin\frac{n\pi y}{b} \times \frac{n^2\pi^2}{b^2} \sin\frac{m\pi x}{a} \times \frac{m^2\pi^2}{a^2}$$

$$\frac{\partial^4 w}{\partial x^4} + 2 \times \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

$$= \frac{q_0}{D} \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b}$$

$$c \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a} \times \frac{m^4\pi^4}{a^4}$$

$$\downarrow$$

$$+ 2 \times c \sin\frac{m\pi x}{a} \times \frac{m^2\pi^2}{a^2} \times \sin\frac{n\pi y}{b} \times \frac{n^2\pi^2}{b^2} + c \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \times \frac{n^4\pi^4}{b^4}$$

$$c\, \pi^4 \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b} \left[\frac{m^4}{a^4} + 2\left[\frac{mn}{ab}\right]^2 + \frac{n^4}{b^4}\right]$$

$$C\pi^4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2 = \frac{q_0}{D}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$C\pi^4\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2 = \frac{q_0}{D}$$

$$C = \frac{q_0}{D\pi^4}\times\frac{1}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}$$

$$w = C\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}$$

$$w = \frac{q_0}{D\pi^4}\times\frac{1}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}$$

Navier solution for simply supported rectangular plates;

Deflection produced in a simply supported rectangular plate by any kind of loading is given by

$$q = f(x,y) \longrightarrow (1)$$

To represent the function $f(x,y)$ in the form of double trigonometric series

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \longrightarrow (2)$$

To evaluate any particular coefficient a_{mn} of this series we multiply both sides of Eq by $\sin \frac{n'\pi y}{b} dy$ and integrate from 0 to b

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b} dy = 0 \quad \text{where } n \neq n'$$

$$\int_0^b \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b} dy = b/2 \quad \text{when } n = n'$$

$$\int_0^b f(x,y) \sin \frac{n'\pi y}{b} dy = b/2 \sum_{m=1}^{\infty} a_{mn'} \sin \frac{m\pi x}{a} \longrightarrow 3$$

Mul'

$$\int_0^b f(x,y) \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b} dy = b/2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

multiplying both sides of equation 3 by $\sin\left(\frac{m'\pi x}{a}\right) dx$ and integrating from 0 to a

$$\int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} dx\, dy = \frac{ab}{4} a_{mn}$$

$$a_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} dx\, dy \longrightarrow 4$$

Performing the integration on the above equation of given load distribution i.e., for a given $f(x,y)$. In order to find coefficients of series and represent in this way the given load as a sum of partial sinusoidal loadings

deflection is given by

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b}$$

Consider the case of UDL over the entire surface of the plate as an example

If $f(x,y) = q_0$

where q_0 is the intensity of UDL. From formula

$$a_{mn} = \frac{4q_0}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b}\, dx\, dy$$

$$a_{mn} = \frac{16 q_0}{\pi^2 mn}$$

where m and n are odd integers. If m or n or both of them are even $a_{mn} = 0$

$$w = \frac{a_{mn}}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)mn}$$

$$w = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \times \sin\frac{n\pi y}{b}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)mn}$$

where $m = 1, 3, 5$ and $n = 1, 3, 5$

In case of uniform load we have deflection surface symmetrical WRT to the axis $x = a/2$, $y = b/2$ — max deflection

$$w = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^2}$$

This is rapidly converging series, a satisfactory approximation is obtained by taking only the first term of series which for example in case of square plate

$$w = \frac{4 q_0 a^4}{\pi^6 D}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$w = 0.0454 \frac{q_0 a^4}{Eh^2}$$

Levy's solution for a Simply Supported $\square^{le}$ plate

For a Simply Supported rectangular plate subjected to UDL, M. levy's suggested taking the solution in the form of a series.

$$w = \sum_{m=1}^{\infty} y_{mn} \sin \frac{m\pi x}{a}$$

y_m is pure form 'y' value

Differential equation for deflection surface is

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

In this case at $x=0$ to $x=a$; $w=0$, $\frac{\partial^2 w}{\partial y^2} = 0$

General solution $w = w_1 + w_2$

w_1 = function of x w_2 = function of y

let $w_1 = \frac{q}{24D}\left[x^4 - 2ax^3 + a^3 x\right]$

$$\frac{\partial w_1}{\partial x} = \frac{q}{24D}\left[4x^3 - 6ax^2 + a^3\right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{q}{24D}\left[12x^2 - 12ax\right]$$

$$\frac{\partial^3 w}{\partial x^3} = \frac{q}{24D}\left[24x - 12a\right]$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{q}{24D}\left[24\right] = q/D$$

$$w_2 = \sum_{m=1}^{\infty} y_m \sin\frac{m\pi x}{a}$$

$$\frac{\partial w_2}{\partial x} = \sum_{m=1}^{\infty} y_m^{*}\left(-\cos\frac{m\pi x}{a} \times \frac{m\pi}{a}\right.$$

$$\frac{\partial^2 w_2}{\partial x^2} = \sum_{m=1}^{\infty} y_m^{**} \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^2$$

$$\frac{\partial^3 w_2}{\partial y^3} = \sum_{m=1}^{\infty} y_m^{***} + \cos\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^3$$

$$\frac{\partial^4 w}{\partial y^4} = \sum_{m=1}^{\infty} y_m^{**} \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^4 \qquad \frac{\partial^4 w}{\partial y^4} = \sum_{m=1}^{\infty} y_m^{IV} \sin\frac{m\pi x}{a}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{\partial^2 w_2}{\partial x^2} = -\sum_{m=1}^{\infty}\left[-\sin\frac{m\pi x}{a}\right]\left[\frac{m\pi}{a}\right]^2 y_m$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} = -\cancel{\sum_{m=1}^{\infty}\left(-\cos\frac{m\pi x}{a}\right]\left[\frac{m\pi}{a}\right]^3 y_m'}$$

$$= \sum_{m=1}^{\infty} \cdot \left[\sin\frac{m\pi x}{a}\right]\left(\frac{m\pi}{a}\right)^2 y_m'$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = -\left[\frac{m\pi}{a}\right]^2 \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a}\, y_m''$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \left[\left(\frac{m\pi}{a}\right)^4 y_m + 2\left(\frac{m\pi}{a}\right)^2 y_m'' + y_m^{IV}\right] \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a}$$

let us consider

$$\left[y_m^{IV} + 2\left(\frac{m\pi}{a}\right)^2 y_m'' + \left(\frac{m\pi}{a}\right)^4 y_m\right] \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} = 0$$

Above equation is a D.E. of fourth degree solution general solution is

$$y_m = \frac{qa^4}{D}\left[A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m \sinh\frac{m\pi y}{a} + D_m \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)$$

The deflection surface is symmetrical WRT x-axis and in the expression only even function of y and the let the integration constant C_m, $D_m = 0$

$$y_m = \frac{qa^4}{D}\left[A_m \cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

$w = w_1 + w_2$

$$w = \frac{q}{24D}\left[x^4 + 2ax^3 + a^3x\right] + \sum_{m=1}^{\infty} A_m \cosh\frac{m\pi y}{a} + B_m \sin\frac{m\pi y}{a} \cdot \frac{m\pi y}{a}\Big] \sum_{m=1}^{\alpha} \sin\frac{m\pi x}{a}$$

we develop the expression (3) in trigonometric series

$$w = \frac{q}{24D}\left[x^4 + 2ax^3 + a^3x\right]$$

$$= \frac{4qa^4}{\pi^5 D}\sum_{m=1}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a}$$

The $$w = \frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a} + \Big[\frac{qa^4}{D}(A_m \cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} \times \sum_{m=1}^{\infty}\sin\frac{m\pi x}{a}\Big]$$

$$w = \frac{qa^4}{D}\Big[\Big[\frac{4}{\pi^5}\sum_{m=1,3,5}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a}\Big] + \Big[A_m \cosh\frac{m\pi y}{a} + B_m \times \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} \times \sum_{m=1,3,5}^{\infty}\sin\frac{m\pi x}{a}\Big]$$

$$w = \frac{qa^4}{D}\sum_{m=1,3,5}^{\infty}\sin\frac{m\pi x}{a}\left[\frac{4}{\pi^5 m^5} + A_m\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

$$w = \frac{4}{\pi^5 m^5} + A_m \cosh\alpha_m + \alpha_m B_m \sinh\alpha_m = 0$$

$$\alpha_m = \frac{m\pi b}{2a}$$

$$= (A_m + 2B_m)\cosh\alpha_m + \alpha_m B_m \sinh\alpha_m$$

By solving abore equation

$$A_m = \frac{-2(\alpha_m \tanh\alpha_m + 2)}{\pi^5 m^5 \cosh\alpha_m} \qquad B_m = \frac{2}{\pi^5 m^5 \cosh\alpha_m}$$

Substitute the above expression in the deflection equation of deflection surface and apply boundary conditions along the edge $y = \pm b/2$

i.e., $w = 0$, $\frac{\partial^2 w}{\partial y^2} = 0$ then weget

At $x = a/2$ $y = 0$ we can get maximum deflection at the centre of the plate

$$w_{max} = \frac{q a^4}{D} \sum_{m=1}^{\infty} \left[\frac{4}{\pi^5 m^5} + \frac{(-2(\alpha_m \tanh\alpha_m + 2)}{\pi^5 m^5 \cosh\alpha_m} \right] \cosh\frac{m\pi y}{a}$$

$$+ \frac{m\pi y}{a} \frac{2}{\pi^5 m^5 \cosh\alpha_m} \sinh\frac{m\pi y}{a} \Bigg] \sin\frac{m\pi x}{a}$$

$$\alpha_m = \frac{m\pi b}{2a}$$

Substitute $x = a/2$ $y = 0$

$$w_{max} = \frac{q a^4}{D} \sum_{m=1,3,5}^{\infty} \left(\frac{4}{\pi^5 m^5} - \frac{2\left[\frac{m\pi b}{2a} \tanh\frac{m\pi b}{2a} + 2\right]}{\pi^5 m^5 \cosh\frac{m\pi b}{2a}} \right) \times \sin\frac{m\pi a}{2a}$$

$$= \frac{4 q a^4}{D \pi^5 m^5} \sum_{m=1,3,5}^{\infty} (-1)^{\frac{(m-1)}{2}} \left[1 - \frac{(\alpha_m \tanh\alpha_m + 2)}{\cosh\alpha_m} \right]$$

$$\alpha_m = \frac{m\pi b}{2a}$$

when we ignore 2nd term in the paranthesis. It represents the deflection of the middle plane of a uniformly loaded strip.

Then we can represent the above expression

$$W_{max} = \frac{5qa^4}{384D} - \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \left[\frac{\alpha_m \tanh \alpha_m + 2}{2\cosh \alpha_m} \right.$$

If $a = b$ for square plate

$$\alpha_m = \frac{m\pi}{2} \times \frac{b}{a} \qquad \alpha_m = m\pi/2$$

$$\alpha_1 = \pi/2 \ , \ \alpha_3 = 3\pi/2 \quad \alpha_5 = 5\pi/2 \ . \ . \ . \ . \text{ for odd}$$

Integers, when we expand the expression (6).

$$W_{max} = \frac{5qa^4}{384D} - \frac{4qa^4}{\pi^5 D} \left[0.68562 - 0.00025 + \ldots \right]$$

$$\frac{5qa^4}{384D} - \frac{4qa^4}{\pi^5 D} (0.68562)$$

where α is a numerical factor which depend upon b/a

$$a_{mn} = \frac{4q_0}{ab} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx\, dy = \frac{16 q_0}{\pi^2 mn}$$

Levy's Solution

SS and UDL

Bending of rectangular plates that have two opposite edges simply supported. M. Levy suggested taking the solution

$$w = \sum_{m=1}^{\infty} Y_m \sin \frac{m\pi x}{a}$$

Y_m function of y only $x = 0$ to $x = a$ $w = 0$ $\partial^2 w / \partial x^2 = 0$.

It remains to determine Y_m in such a form as to satisfy the boundary conditions on the sides $y = \pm b/2$ and also the equation of the deflection surface

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Applying for further simplification

$$w = w_1 + w_2$$

where

$$w_1 = \frac{v}{24D}(x^4 - 2ax^2 + a^3x)$$

w_1 represents the deflection of a uniformly loaded strip parallel to x-axis

$x=0$ to $x=a$

The expression for w_2 evidently has to satisfy the equation

$$\frac{\partial^4 w_2}{\partial x^4} + \frac{2\partial^4 w_2}{\partial x^2 \partial y^2} + \frac{\partial^4 w_2}{\partial y^4} = 0$$

$$w = \sum_{m=1}^{\infty} y_m \sin\frac{m\pi x}{a}$$

$$\frac{\partial^2 w}{\partial x^2} = \sum_{m=1}^{\infty} y_m \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^2$$

$$\frac{\partial^4 w}{\partial x^4} = \sum_{m=1}^{\infty} y_m \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^4$$

$$\frac{\partial w}{\partial y} = \sum_{m=1}^{\infty} y_m' \sin\frac{m\pi x}{a} \qquad \frac{\partial^3 w}{\partial x^2 \partial y} = \sum_{m=1}^{\infty} y_m' \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^2$$

$$\frac{\partial^2 w}{\partial y^2} = \sum_{m=1}^{\infty} y_m'' \sin\frac{m\pi x}{a} \qquad \frac{\partial^4 w}{\partial x^2 \partial y^2} = \sum_{m=1}^{\infty} y_m'' \sin\frac{m\pi x}{a} \times \left(\frac{m\pi}{a}\right)^2$$

$$\frac{\partial^3 w}{\partial y^3} \quad \sum_{m=1}^{\infty} y_m''' \sin\frac{m\pi x}{a}$$

$$\frac{\partial^4 w}{\partial y^4} = \sum_{m=1}^{\infty} y_m^{IV} \sin\frac{m\pi x}{a}$$

substituting in above equation

$$\sum_{m=1}^{\infty} y_m \sin\frac{m\pi x}{a}\left(\frac{m\pi}{a}\right)^4 + 2 \times \sum_{m=1}^{\infty} y_m'' \sin\frac{m\pi x}{a}\left(\frac{m\pi}{a}\right)^2 +$$

$$\sum_{m=1}^{\infty} y_m^{IV} \sin\frac{m\pi x}{a}$$

$$\sum_{m=1}^{\infty}\left[y_m^{IV} - 2\frac{m^2\pi^2}{a^2}y_m'' + \frac{m^4\pi^4}{a^4}\right]\sin\frac{m\pi x}{a} = 0$$

Equation should satisfy for all values of x only if the function y_m satisfies the equation

$$y_m^{IV} - 2\frac{m^2\pi^2}{a^2}y_m'' + \frac{m^4\pi^4}{a^4}y_m = 0$$

General Integral of this equation is

$$y_m = \frac{qa^4}{D}\left[A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m \sinh\frac{m\pi y}{a} + D_m \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]$$

observing that the deflection surface of the plate is symmetrical with respect to x-axis we keep above expression only even function of y and let the integration constants $C_m = D_m = 0$

Deflection surface is represented by

$$w = \frac{q}{24D}\left(x^4 - 2ax^3 + a^3x\right) + \frac{qa^4}{D}\sum_{m=1}^{\infty}\left[A_m \cosh\frac{m\pi y}{a} + B_n \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

Boundary condition $x=0$ to $x=a$ $w=0$ $\frac{\partial^2 w}{\partial y^2}=0$ on the sides $y = \pm b/2$

we begin by developing expression (c) in a Trigonometric series

$$w = \frac{q}{24D}\left[x^4 - 2ax^3 + a^3x\right] \text{ in trigonometric series}$$

$$= \frac{4qa^4}{\pi^5 D}\sum_{m=1}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a}$$

where $m = 1, 3, 5 \ldots$ the deflection surface (e) will be Represented

$$w = \frac{qa^4}{D}\sum_{m=1}^{\infty}\left[\frac{4}{\pi^5 m^5} + A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

where $m = 1, 3, 5, \ldots$ Substituting this expression in the boundary conditions and using the notation

$$\frac{m\pi b}{2a} = \alpha_m$$

$$w = w_1 + w_2$$

$$\frac{4qa^4}{\pi^5 D}\sum_{m=1}^{\infty}\frac{1}{m^5}\sin\frac{m\pi x}{a} + \frac{qa^4}{D}\sum_{m=1}^{\infty} A_m \cosh\frac{m\pi y}{a} + B_n \frac{m\pi y}{a} \times \sinh\frac{m\pi y}{a}\Big]$$

where $m = 1, 3, 5, \ldots$ substituting this expression in the boundary conding by using

$$\alpha_m = \frac{m\pi b}{2a}$$

we obtain following equation for determining coefficients A_m and B_n

$$\frac{4}{\pi^5 m^5} + A_m \cosh\alpha_m + \alpha_m B_m \sinh\alpha_m = 0$$

$$(A_m + 2B_m \cosh\alpha_m) + \alpha_m B_m \sinh\alpha_m = 0$$

Solving we get

$$\frac{4}{\pi^5 m^5} + A_m \cosh\alpha_m - (A_m + 2B_m) \cosh\alpha_m = 0$$

$$\frac{4}{\pi^5 m^5} + (\cancel{A_m} - \cancel{A_m} + 2B_m) \cosh\alpha_m$$

$$\frac{4}{\pi^5 m^5} + 2B_m \cosh\alpha_m$$

$$\frac{\cancel{4}\,2}{\cancel{2} \times \pi^5 m^5 \times \cosh\alpha_m} = B_m$$

$$B_m = \frac{2}{\pi^5 m^5 \cosh\alpha_m}$$

$$\frac{4}{\pi^5 m^5} + A_m \cosh\alpha_m + \alpha_m B_m \sinh\alpha_m = 0$$

$$\frac{4}{\pi^5 m^5} + A_m \cosh\alpha_m + \alpha_m \times \frac{2}{\pi^5 m^5 \cosh\alpha_m} \times \sinh\alpha_m = 0$$

$$\frac{4}{\pi^5 m^5} + \frac{2\alpha_m}{\pi^5 m^5 \cancel{\cosh\alpha_m}} \tanh\alpha_m = -A_m \cosh\alpha_m$$

$$A_m = \frac{-2}{\pi^5 m^5} \left[\frac{\alpha_m \tanh\alpha_m + 2}{\cosh\alpha_m} \right]$$

substituting these values of Constants in $w = w_1 + w_2$ we obtain the deflection surface of the plate -

$$w = \frac{4qa^4}{\pi^5 D} \sum_{m=1}^{\infty} \frac{1}{m^5}\left[A_m \cosh\frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \times \sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi y}{a}$$

~~and bou~~

$$w = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5}^{\alpha} \left[\frac{-2}{\pi^5 m^5}\left[\frac{\alpha_m \tanh\alpha_m + 2}{\cosh\alpha_m}\right]\cosh\frac{m\pi y}{a} + \right.$$

$$\left.\frac{2}{\pi^5 m^5 \cosh\alpha_m} \times \frac{m\pi y}{a} \times \sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi y}{a}$$

Deflection at any point can be calculated by using tables of hyperbolic functions The max deflection is obtained at middle of plate

$x = a/2$ $y=0$ where

$$w_{max} = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5}\left[1 - \frac{\alpha_m \tanh\alpha_m + 2}{2\cosh\alpha_m}\right)$$

This series represents the deflection of middle of a uniformly loaded strip. Hence we can represent expression in the following form

$$w_{max} = \frac{5}{384}\frac{qa^4}{D} - \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5}^{\infty} \frac{(-1)^{(m-1)/2}}{m^5} \frac{\alpha_m \tanh\alpha_m + 2}{2\cosh\alpha_m}\Big]$$

To converge very rapidly and sufficient accuracy is obtained by taking. $\alpha_1 = \frac{\pi}{2}$ $\alpha_2 = \frac{3\pi}{2}$ Square plate

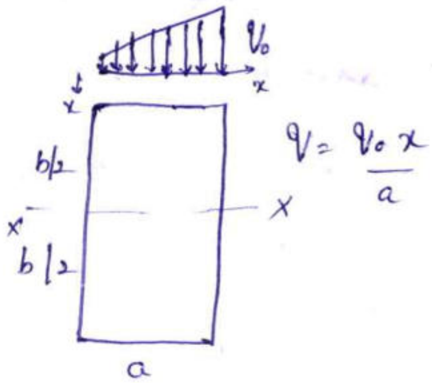
which gives

$$w_{max} = \frac{5}{384}\frac{qa^4}{D} - \frac{4qa^4}{\pi^5 D}(0.68562 - 0.00025 + \ldots)$$

$$= 0.00406\,\frac{qa^4}{D}$$

Simply Supported Rectangular plate under Hydrostatic pressure

Simply supported plate is loaded as shown



$q = \frac{q_0 x}{a}$

$$w = w_1 + w_2 \longrightarrow a$$

$$w_1 = \frac{q_0}{360D}\left[\frac{3x^5}{a} - 10ax^3 + 7a^3x\right) \longrightarrow b$$

Represents the deflection of a strip under the Triangular loading writting above in Trigonometric series

$$w_1 = \frac{2q_0 a^4}{D\pi^5}\sum_{m=1,2,3}^{\infty}\frac{(-1)^{m+1}}{m^5}\sin\frac{m\pi x}{a} \longrightarrow b(1)$$

This expression should satisfy the D.E

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} = \frac{q_0 x}{aD} \longrightarrow \cancel{b(1)}\ C$$

$w = 0$ $\frac{\partial^2 w}{\partial x^2}$ at $x = 0$ to a

Part w_2 is taken as $w_2 = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a} \longrightarrow d$

where Y_m have the same form substituting w_1 and w_2

$$W = W_1 + W_2$$

$$= \frac{2q_0 a^4}{D\pi^5}\sum_{m=1,2,3}^{\infty}\left[\frac{(-1)^{m+1}}{m^5}\sin\frac{m\pi x}{a}\right] + \frac{q_0 a^4}{D}\sum_{m=1}^{\infty}\left(A_m\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

$$= \frac{q_0 a^4}{D}\sum_{m=1}^{\infty}\left[\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m\cosh\frac{m\pi y}{a} + B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right]$$

$$\sin\frac{m\pi x}{a}$$

where the constants A_m and B_m from boundary condition

$$w=0 \quad \frac{\partial^2 w}{\partial y^2}=0 \quad \text{for } y=\pm b/2$$

from this condition

$$\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \cosh\alpha_m + B_m \alpha_m \sinh\alpha_m = 0$$

$$(2B_m + A_m)\cosh\alpha_m + B_m \alpha_m \sinh\alpha_m = 0$$

Solving we find

$$\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \cosh\alpha_m - (2B_m + A_m)\cosh\alpha_m = 0$$

$$(\cancel{A_m} - 2B_m - \cancel{A_m})\cosh\alpha_m = \frac{2(-1)^{m+1}}{\pi^5 m^5}$$

$$-2B_m \cosh\alpha_m = \frac{2(-1)^{m+1}}{\pi^5 m^5}$$

$$B_m = \frac{\cancel{2}(-1)^{m+1} + 2\cosh\alpha_m}{\pi^5 m^5 \cancel{2}\cosh\alpha_m}$$

$$B_m = \frac{(-1)^{m+1}}{\pi^5 m^5 \cosh\alpha_m}$$

$$\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \cosh\alpha_m + \frac{(-1)^{m+1}}{\pi^5 m^5 \cosh\alpha_m}\alpha_m \sinh\alpha_m$$

$$\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \cosh\alpha_m + \frac{(-1)^{m+1}\alpha_m \tanh\alpha_m}{\pi^5 m^5 \alpha_1}$$

$$A_m \cosh \alpha_m = \frac{(-1)^{m+1} \alpha_m \tanh \alpha_m + 2(-1)^{m+1}}{\pi^5 m^5}$$

$$A_m = -\frac{(2+\alpha_m \tanh \alpha_m)(-1)^{m+1}}{\pi^5 m^5 \cosh \alpha_m}$$

Deflection of the plate along x-axis is

$$(w)_{y=0} = \frac{q_0 a^4}{D} \sum_{m=1}^{\infty} \left(\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

for a square plate $a=b$

$$w_{y=0} = \frac{q_0 a^4}{D} \sum_{m=1}^{\infty} \left[\frac{2(-1)^{m+1}}{\pi^5 m^5} + A_m \right] \sin \frac{m\pi x}{a}$$

$$w_y = \frac{q_0 a^4}{D} \left[0.002055 \sin \frac{\pi x}{a} - 0.000177 \sin \frac{2\pi x}{a} + 0.000025 \sin \frac{3\pi x}{a} - \cdots \right)$$

6/4/15 Two hours All present

Simplifying we get at $x = a/2$

$$w_{max\, x=a/2,\, y=0} = 0.00203 \frac{q_0 a^4}{D}$$

CIVIL ENGINEERING

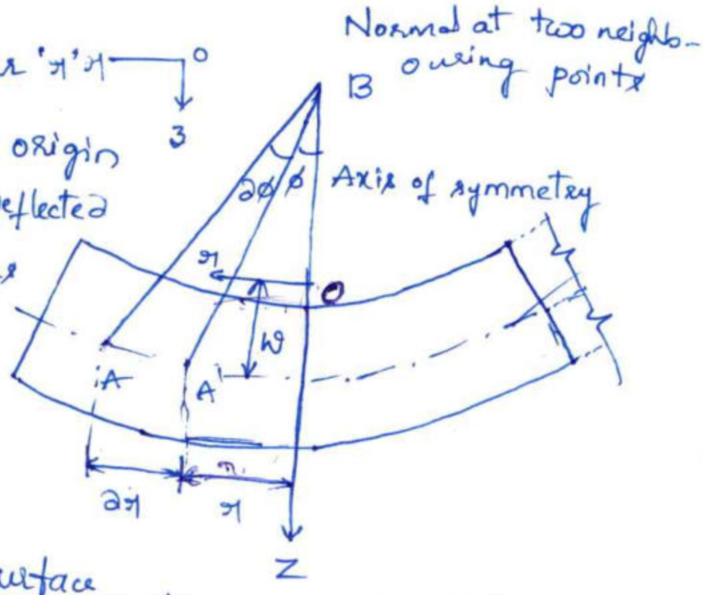
Theory and Analysis of Plates

UNIT-2

CIRCULAR PLATES

Symmetrical Bending of circular plate;

Consider a circular plate of radius 'r'. Let us take a small c/s of a plate, its origin of coordinates o at the centre of the undeflected plate denoted by 'r', the radial distances of points in the middle plane of the plate and w be the deflections in the down ward direction.



Section of circular plate

Maximum slope of the deflection surface at any point A is then equal to $-\partial w/\partial r$, and the curvature of the middle surface of plate in the diametral section rz for small deflection is

$$\frac{1}{r_n} = \frac{-\partial^2 w}{\partial r^2} = \frac{\partial \phi}{\partial r}$$

ϕ is the small angle between the normal to deflection surface at A and the axis of symmetry OB.

Since the circular plate is symmetrical, if $1/r_n$ is the one of the principal curvature at the deflection surface at A. The second principle curvature is found to be equal to A'B' in length. This found to geometrically circular plate. Second principal curvature will be such that it will be normal to 'A B' and ⊥r to rz plane

from Δ^{le} ABC Angle = $\frac{\text{arc}}{\text{radius}}$ $\quad \phi = r/r_t$

$$r_t = r/\phi$$

$$\frac{1}{r_t} = \frac{1}{r}\frac{\partial w}{\partial r} = \frac{\phi}{r}$$

$$\frac{1}{r_t} = \frac{\phi}{r} = \frac{-1}{r}\frac{\partial w}{\partial r}$$

Moment curvature relation

$$M_{r} = -D\left[\frac{\partial^2 \omega}{\partial r^2} + \frac{\nu}{r}\frac{\partial \omega}{\partial r}\right] = D\left[\frac{\partial \phi}{\partial r} + \frac{\nu}{r}\phi\right] \longrightarrow (1)$$

$$M_t = -D\left[\frac{1}{r}\frac{\partial \omega}{\partial r} + \nu\frac{\partial^2 \omega}{\partial r^2}\right] = D\left[\frac{\phi}{r} + \nu\frac{\partial \phi}{\partial r}\right] \longrightarrow (2)$$

$$M_x = D\left[\frac{1}{R_x} + \nu\frac{1}{R_y}\right] \qquad \frac{1}{R_x} = \frac{1}{R_n}$$

$$M_y = D\left[\frac{1}{R_y} + \nu\frac{1}{R_x}\right] \qquad \frac{1}{R_y} = \frac{1}{R_t}$$

Hence above two equations can be moulded for circular as follows

$$M_{r} = D\left[\frac{1}{R_n} + \nu\frac{1}{R_t}\right]$$

$$M_t = D\left[\frac{1}{R_t} + \nu\frac{1}{R_n}\right.$$

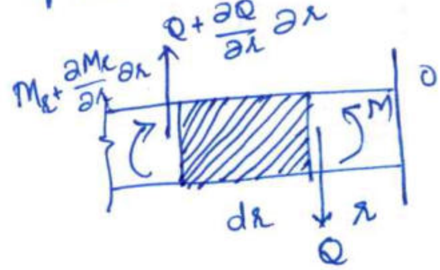
where M_r and M_t B.M per unit length along circular section and diametral section of the plate. The direction of actions will be as shown in follows. Substituting in the value of $\frac{1}{R_n}$ & $\frac{1}{R_t}$ both terms of ϕ and w

$$M_r = D\left[\frac{\partial \phi}{\partial r} + \nu\frac{\phi}{r}\right] = -D\left[\frac{\partial^2 \omega}{\partial r^2} + \frac{\nu}{r}\frac{\partial \omega}{\partial r}\right]$$

$$M_t = D\left[\frac{\phi}{r} + \nu\frac{\partial \phi}{\partial r}\right] = -D\left[\frac{1}{r}\frac{\partial \omega}{\partial r} + \nu\frac{\partial^2 \omega}{\partial r^2}\right]$$

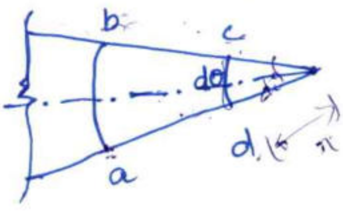
Considering small section ab and cd by two diametral sections ad and bc. The couple acting on the side of the element

$$M_r\, r\, \partial\theta \longrightarrow c$$



corresponding couple on the side ab is

$$\left[M_r + \frac{\partial M_r}{\partial r}\partial r\right]\left[r + \partial r\right]\partial\theta \longrightarrow d$$



Couple on the sides of ab an bc of the element are each $M_t \partial r$ and they give a resultant couple in the plane rOz equal to

$$M_t \, \partial r \, \partial\theta$$

Let Q be the shear forces that may act on the element. Shearing force per unit length of the cylindrical section of radius r, the total shearing force acting on the side cd of the element is $Q_r \partial\theta$, and the corresponding forces on the side ab is

$$\left[Q + \left(\frac{\partial Q}{\partial r}\right)\partial r\right](r+\partial r)\,\partial\theta$$

Small difference between the shearing forces on the two opposite sides of the element, we can state that these forces give a couple in the rz plane $Q_r \partial\theta \, \partial r$

Couple acting on the side bc due to $M_r = M_r r \partial\theta$

Couple acting on the side ad due to $M_r = \left[M_r + \frac{\partial M_r}{\partial r}\partial\theta\right](r+\partial r)\partial\theta$

The B.M M_t acting along sides ad and bc produces equal amount of couple $(M_t \partial r)$ then resultant of couple of element along 'rOz' plane.

For equilibrium of the element (clockwise couple - anticlockwise due to M_r, M_t and S.F

$$M_r = D\left[\frac{\partial\phi}{\partial r} + \mu\frac{\phi}{r}\right] = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu \times \frac{1}{r}\frac{\partial w}{\partial r}\right]$$

$$M_t = D\left[\frac{\phi}{r} + \mu\frac{\partial\phi}{\partial r}\right] = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \mu \times \frac{\partial^2 w}{\partial r^2}\right]$$

$$\left[M_r + \frac{\partial M_r}{\partial r}\partial r\right][r+\partial r]\partial\theta - M_r r \, d\theta - M_t \partial r \, \partial\theta + Q_r \partial r \, \partial\theta = 0$$

Neglecting small quantities

$$M_r + \frac{\partial M_r}{\partial r} r - M_t + Q_r = 0$$

substituting expressions 1 and 2 for M_t and M_r

$$\phi = -\frac{\partial w}{\partial r}$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} = -\frac{Q}{D} \longrightarrow$$

$$\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} = -\frac{Q}{D} \longrightarrow$$

From Equation above Eq we can determine the slope ϕ or the deflection w of the circular plate provided Q is known. In simplified manner

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\phi)\right] = -\frac{Q}{D}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right]\right] = \frac{Q}{D}$$

Uniformly loaded ~~c~~ plates; $\rightarrow$

Consider a circular plate of radius 'a' which is subjected to UDL q over the area then the shear force Q at a distance of 'r' from the the centre 'c' is can calculated from the expression

$$2\pi r Q = \pi r^2 q$$

$$Q = qr/2$$

Differential equation of deflection surface for circular plate

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r} r \cancel{\phi}\left(\frac{\partial w}{\partial r}\right)\right] = \frac{Q}{D}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = -\frac{qr}{2D}$$

Integrating the above equation WRT r

$$\int \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right)\right] = \int \frac{qr}{2D}\partial r$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right] = \frac{qr^2}{4D} + C_1$$

multiplying with r on both sides

$$\frac{r}{r}\frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right] = \frac{qr^3}{4D} + C_1 r$$

Again Integrating the above expression WRT 'r'

$$\int \frac{\partial}{\partial r}\left[r\frac{\partial w}{\partial r}\right)\partial r = \int\left(\frac{qr^3}{4D} + C_1 r\right)\partial r$$

$$r\frac{\partial w}{\partial r} = \frac{qr^4}{16D} + C_1\frac{r^2}{2} + C_2$$

~~Differentiating~~ WRT 'r' $\frac{\partial w}{\partial r} = \frac{1}{r}\frac{qr^4}{16D} + \frac{C_1 r^2}{2r} + \frac{C_2}{r}$

$$\frac{\partial w}{\partial r} = \frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r} \longrightarrow (1)$$

Again integrating the above expression WRT r

$$\int\left(\frac{\partial w}{\partial r}\right)\partial r = \int\left[\frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}\right]\partial r$$

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2\log r + C_3 \longrightarrow (2)$$

Equation for deflection for UDL circular plates

Case i; Circular plates with clamped edge condition;

The constants C_1, C_2, C_3 are to be found from B.C of circular plate at centre and at edge of the plate

Boundary conditions are

At $r=0$ $r=a$ $-\frac{\partial w}{\partial r_1}=0$ $w=0$

slope of deflection curve and slope of deflection surface is '0' $\phi = \frac{\partial w}{\partial r} = 0$

From the first equation we write

$$\left[\frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}\right] = 0 \quad \text{if } C_2 = 0$$

$$\left(\frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}\right)_{r=a} = 0$$

$$\frac{qa^3}{16D} + \frac{C_1 a}{2} = 0$$

$$C_1 = -\frac{qa^3}{16D} \times \frac{2}{a} = \frac{-qa^2}{8D}$$

Now slope $\phi = -\frac{\partial w}{\partial r}$

$$\phi = -\left[\frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}\right]$$

substitituting the values of C_1 & C_2 in above equation

$$\phi = -\left[\frac{qr^3}{16D} + \left[\frac{-qa^2}{8D}\right]\frac{r}{2} + 0\right]$$

$$\phi = \left[-\frac{qr^3}{16D} + \frac{qa^2 r}{16D} + 0\right]$$

$$\phi = \frac{-qr^3}{16D} + \frac{qa^2 r}{16D}$$

$$\phi = \frac{qr}{16D}(a^2 - r^2)$$

The above expression represents slope of the circular plate having clamped edge subjected to UDL

iii) Boundary condition $w = 0$ @ $r = a$

From eq ②

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log e + C_3$$

substituting C_1 and C_2 values @ $r = a$

$$\frac{qa^4}{64D} + \left[\frac{-qa^2}{8D}\right]\frac{a^2}{4} + 0 + C_3 = 0$$

$$C_3 = \frac{qa^4}{32D} - \frac{qa^4}{64D}$$

$$C_3 = \frac{qa^4}{64D}$$

Then the deflection surface or deflection equation becomes

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log r + C_3$$

Substituting C_1, C_2 and C_3 values

$$w = \frac{qr^4}{64D} - \frac{qa^2r^2}{32D} + 0 + \frac{qa^4}{64D}$$

$$w = \frac{qr^4}{64D} - \frac{qa^2r^2}{32D} + \frac{qa^4}{64D}$$

$$w = \frac{q}{64D}\left[r^4 - 2a^2r^2 + a^4\right]$$

$$w = \frac{q}{64D}\left[r^2 - a^2\right]^2$$ which represents the deflection surface of clamped edge circular plate.

Then $w_{max} = \frac{q}{64D}(-a^2)^2$

$$w_{max} = \frac{qa^4}{64D}$$

The max deflection occurs at centre of the plate

$$M_r = D\left[\frac{\partial\phi}{\partial r} + \mu\phi/r\right] = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right]$$

$\frac{\partial w}{\partial r} =$

$$M_r = \frac{q}{16}\left[a^2(1+\nu) - r^2(3+\nu)\right]$$

$$M_t = \frac{q}{16}\left[a^2(1+\nu) - r^2(1+3\nu)\right]$$

Substituting $r = a$ in these expressions we find B.M at boundary

$(M_r)_{max} = \frac{qa^2}{8}$ $\qquad M_{t(max)} = -\frac{\nu q a^2}{8}$

At the centre of the plate where $r=0$

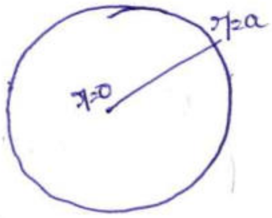
$$M_r = M_t = \frac{qa^2}{16}(1+\nu) \qquad \sigma_{rmax} = \frac{M}{I}\times y = \frac{12M}{bd^3}\times \frac{d}{2} = \frac{6M_r}{h^2}$$

$$= \frac{6^{3}}{h^2}\,\frac{qa^2}{\cancel{16}_{4}}(1+\nu) \qquad \frac{3}{4}\,\frac{qa^2}{h^2}$$

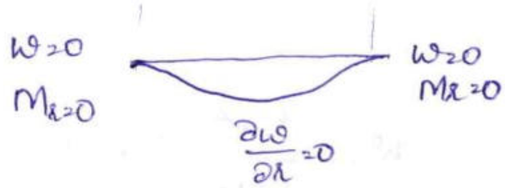
CIRCULAR PLATES WITH SIMPLY SUPPORTED EDGE CONDITION;

Deflection equation

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log r + C_3$$



Boundary conditions



1. $r=0 \quad \frac{\partial w}{\partial r} = 0$

2. $r=a \quad \cancel{\frac{\partial w}{\partial r}}\; w=0$

3. $r=a \quad M_r = 0$

Deflection equation for a circular plates

$$w = \frac{qr^4}{64D} + C_1\frac{r^2}{4} + C_2 \log r + C_3$$

First B.C

$\rightarrow r=0 \;;\; \frac{\partial w}{\partial r} = 0$ $\quad$ ~~r=a~~

$$\frac{\partial w}{\partial r} = \frac{qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$r=0,\; \frac{\partial w}{\partial r} = 0 \quad C_2 = 0$$

$\rightarrow r=a \quad w=0$

$$\frac{qa^4}{64D} + \frac{C_1 a^2}{4} + 0 + C_3 = 0$$

$$\frac{C_1 a^2}{4} + C_3 = \frac{-qa^4}{64D} \longrightarrow 1$$

$\rightarrow M_r = 0, \quad r = a$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right]$$

$$w = \frac{qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log r + C_3$$

$$\frac{\partial w}{\partial r} = \frac{4qr^3}{64D} + \frac{2C_1 r}{4} + C_2 \times \frac{1}{r} + 0 \quad \text{where } C_2 = 0$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{12qr^2}{64D} + \frac{2C_1}{4}$$

$$M_r = -D\left[\frac{12qr^2}{64D} + \frac{\cancel{2}C_1}{\cancel{4}} + \frac{\mu}{r}\left[\frac{4qr^3}{64D} + \frac{2C_1 r}{4} + \frac{\cancel{C_2}}{\cancel{r}}\right]\right]$$

$$= -D\left[\frac{3qr^2}{16D} + 0.5C_1 + \mu\left[\frac{qr^2}{16D} + \frac{\cancel{2C_1}}{\cancel{4}} 0.5C_1\right]\right]$$

$$= -D\left[\frac{3qr^2}{16D} + 0.5C_1 + \mu 0.5C_1 + \mu\frac{qr^2}{16D}\right]$$

$$= -D\left[\frac{qr^2}{16D}(3+\mu) + 0.5C_1(1+\mu)\right]$$

$$= \left[\frac{-qr^2}{16D}(3+\mu) - 0.5C_1(1+\mu)\right]D$$

$$\frac{0.5C_1(1+\mu)}{D} = \frac{-qr^2}{16D}(3+\mu)$$

$$C_1 = \frac{-qr^2}{16D \times 0.5(1+\mu)}[3+\mu]$$

$$C_1 = \frac{-qr^2}{8D}\left[\frac{3+\mu}{1+\mu}\right]$$

At $r = a$ $\quad C_1 = \frac{-qa^2}{8D}\left[\frac{3+\mu}{1+\mu}\right]$

$$\frac{C_1 a^2}{4} + C_3 = \frac{-qa^4}{64D}$$

$$\frac{-qa^2}{8D}\left[\frac{3+\mu}{1+\mu}\right] \times \frac{a^2}{4} + C_3 = \frac{-qa^4}{64D}$$

$$w = \frac{-qa^4}{32D}\left[\frac{3+\mu}{1+\mu}\right] + C_3 = \frac{-qa^4}{64D}$$

$$C_3 = \frac{-qa^4}{64D} + \frac{qa^4}{32D}\left[\frac{3+\mu}{1+\mu}\right]$$

$$= \frac{qa^4}{64D}\left[-1 + \frac{2(3+\mu)}{(1+\mu)}\right]$$

$$= \frac{qa^4}{64D}\left[\frac{-(1+\mu)+2(3+\mu)}{(1+\mu)}\right]$$

$$= \frac{qa^4}{64D}\left[\frac{-1-\mu+6+2\mu}{(1+\mu)}\right]$$

$$C_3 = \frac{qa^4}{64D}\left[\frac{5+\mu}{1+\mu}\right]$$

$$w = \frac{qr^4}{64D} - C_1\times\frac{r^2}{4} + C_2\log r + C_3$$

$C_2 = 0$

$$w = \frac{qr^4}{64D} - \frac{qa^2}{8D}\left[\frac{3+\mu}{1+\mu}\right]\times\frac{r^2}{4} + 0 + \left[\frac{5+\mu}{1+\mu}\right]\frac{qa^4}{64D}$$

$$= \frac{qr^4}{64D} - \frac{qa^2r^2}{32D}\left[\frac{3+\mu}{1+\mu}\right]\times\frac{2}{2} + \frac{qa^4}{64D}\left[\frac{5+\mu}{1+\mu}\right]$$

$$= \frac{qr^4}{64D} - \frac{qa^2r^2}{64D}\left[\frac{6+2\mu}{1+\mu}\right] + \frac{qa^4}{64D}\left[\frac{5+\mu}{1+\mu}\right]$$

$$= \frac{qr^4}{64D} - \frac{qa^2r^2}{64D}\left[\frac{5+\mu}{1+\mu}+1\right] + \frac{qa^4}{64D}\left[\frac{5+\mu}{1+\mu}\right]$$

$$= \frac{qr^4}{64D} - \frac{qa^2r^2}{64D}\left[\frac{5+\mu}{1+\mu}\right] - \frac{qa^2r^2}{64D} + \frac{qa^4}{64D}\left(\frac{5+\mu}{1+\mu}\right)$$

$$= \frac{qr^4}{64D} - \frac{qa^2}{64D}(r^2-a^2)\left[\frac{5+\mu}{1+\mu}\right] - \frac{qa^2r^2}{64D}$$

$$= \frac{q r^2}{64D}\left[r^2-a^2\right] - \frac{q a^2}{64D}\left(\frac{5+\mu}{1+\mu}\right)\left[r^2-a^2\right]$$

$$= \left[r^2-a^2\right]\left[\frac{q r^2}{64D} - \frac{q a^2}{64D}\left[\frac{5+\mu}{1+\mu}\right]\right]$$

$$w = \frac{q}{64D}(r^2-a^2)\left[(r^2-a^2)\left[\frac{5+\mu}{1+\mu}\right]\right]$$

$$= \frac{q(a^2-r^2)}{64D}\left[\left(\frac{5+\mu}{1+\mu}\right)(a^2-r^2)\right]$$

$r=0$

$r=0;\quad w_{max} = \frac{q a^2}{64D}\left[\left(\frac{5+\mu}{1+\mu}\right)a^2\right]$

$$= \frac{q a^4}{64D}\left[\frac{5+\mu}{1+\mu}\right] \qquad \text{If } \mu = 0.3$$

$$w_{max} = 0.0637\,\frac{q a^4}{D}$$

$$M_r = \frac{q}{16}(3+\nu)(a^2-r^2)$$

$$M_t = \frac{q}{16}\left[a^2(3+\nu) - r^2(1+3\mu)\right)$$

Max B.M at centre

$$M_r = M_t = \frac{3+\nu}{16} q a^2 \qquad \frac{M}{I} = \frac{\sigma}{y} \quad \sigma = \frac{M}{I}\times y \quad \frac{M \times 12}{bd^3}\times d/2 \quad \frac{6M}{h^2}$$

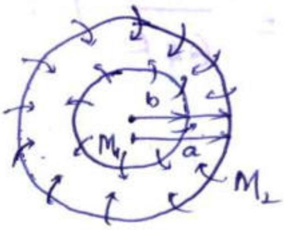
$$\sigma_{r\,max} = \sigma_{t\,max} = \frac{6M}{h^2} = \frac{3(3+\nu)qa^2}{8h^2}$$

Circular plate with a circular hole at its centre;

Consider a circular plate of radius 'a' which has a circular hole having radius 'b' subjected to pure moments M_1 and M_2

We know that

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\times\frac{\partial\omega}{\partial r}\right)\right] = \frac{Q}{D}$$



Integrating with Respect to 'r' on both sides

$$\int\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\times\frac{\partial\omega}{\partial r}\right)\right]\partial r = \int\frac{Q}{D}\,\partial r$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\times\frac{\partial\omega}{\partial r}\right] = C_1$$

$$\frac{\partial}{\partial r}\left(r\times\frac{\partial\omega}{\partial r}\right] = C_1 r$$

Again integrating WRT 'r' on both sides

$$r\times\frac{\partial\omega}{\partial r} = \frac{C_1 r^2}{2} + C_2$$

$$\frac{\partial\omega}{\partial r} = \frac{C_1 r}{2} + \frac{C_2}{r}$$

(-ve sign indicates decrease in angle)

$$-\frac{\partial\omega}{\partial r} = -\frac{C_1 r}{2} - \frac{C_2}{r}$$

Again integrating the above equation WRT 'r' on both sides

$$w = -\frac{C_1 r^2}{2} - C_2\log\frac{r}{a} + C_3$$

C_1, C_2 & C_3 Constants of Integration

If the circular plate is simply supported then the boundary conditions are

(i) $r = a \quad M_r = M_2$

(ii) $r = b \quad M_r = M_1$

(iii) $r = a \quad \omega = 0$

we know that

$$M_r = -D\left[\frac{\partial^2\omega}{\partial r^2} + \frac{\mu}{r}\frac{\partial\omega}{\partial r}\right]$$

Applying first boundary condition

(i) $r = a, \quad M_r = M_2$

$$\frac{\partial\omega}{\partial r} = \frac{-C_1(2r)}{4}^{V(r)} - \frac{C_2}{r}$$

$$\frac{\partial^2 w}{\partial r^2} = -\frac{2C_1}{4} + \frac{C_2}{r^2}$$

$$-\frac{\partial^2 w}{\partial r^2} = -\frac{2C_1}{4} + \frac{C_2}{r^2}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\mu}{r}\frac{\partial w}{\partial r}\right]$$

$$= -D\left[-\frac{2C_1}{4} + \frac{C_2}{r^2} + \frac{\mu}{r}\left[-\frac{2C_1 r}{4} + \frac{C_2}{r}\right)\right]$$

$$= D\left[\frac{2C_1}{4} - \frac{C_2}{r^2} + \frac{\mu}{r}\left[\frac{2C_1 r}{4} - \frac{C_2}{r}\right)\right]$$

$$= D\left[\frac{2C_1}{4} - \frac{C_2}{r^2} + \left[\frac{\mu}{r}\frac{2C_1 r}{4} - \frac{\mu}{r}\times\frac{C_2}{r}\right]\right]$$

$$= D\left[\frac{2C_1}{4} - \frac{C_2}{r^2} + \left[\mu\times\frac{2C_1}{4} - \frac{\mu C_2}{r^2}\right]\right)$$

$$D\left[\frac{\cancel{2}C_1}{\cancel{4}_2}(1+\mu) - \frac{C_2}{r^2}(1-\mu)\right]$$

@ $r = a$ $M_r = M_2$

$$M_2 = D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{r^2}(1-\mu)\right]$$

$$M_2 = D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{a^2}(1-\mu)\right] \longrightarrow 2a^2$$

@ $r = b$ $M_r = M_1$

$$M_1 = D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{b^2}(1-\mu)\right] \longrightarrow 2b^2$$

Inorder to find constants

$$\frac{2a^2 M_2}{D} = \left[C_1 a^2(1+\mu) - 2C_2(1-\mu)\right]$$

$$\frac{2b^2 M_1}{D} = \left[C_1 b^2(1+\mu) - 2C_2(1-\mu)\right]$$

$-$ $-$ $(+)$

$$\frac{2a^2M_2}{D} - \frac{2b^2M_1}{D} = C_1(1+\mu)(a^2-b^2)$$ 8 8 8

$$\frac{2}{D}(a^2M_2 - b^2M_1) = (1+\mu)(a^2-b^2)C_1$$

$$C_1 = \frac{2(a^2M_2 - b^2M_1)}{D(1+\mu)(a^2-b^2)}$$

ii @ $r=b$ $M_1 = D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{b^2}(1-\mu)\right]$

$r=a$ $M_2 = D\left[\frac{C_1}{2}(1+\mu) - \frac{C_2}{a^2}(1+\mu)\right]$

$$\frac{M_2-M_1}{D} = \left[\frac{C_1}{2}\cancel{(1+\mu)} - \frac{C_2}{b^2}(1-\mu)\right] - \frac{C_1}{2}\cancel{(1+\mu)} + \frac{C_2}{a^2}(1+\mu)\Big]$$

$$\frac{M_2-M_1}{D} = \frac{-C_2}{b^2}(1-\mu) + \frac{C_2}{a^2}(1-\mu)$$

$$= +\left[C_2(1-\mu)\left[\frac{1}{a^2} - \frac{1}{b^2}\right]\right]$$

$$\frac{M_2-M_1}{D} = +\left[C_2(1-\mu)\left[\frac{b^2-a^2}{a^2b^2}\right]\right.$$

$$\frac{a^2b^2(M_2-M_1)}{D(b^2-a^2)(1-\mu)} = C_2$$

From 3^{rd} boundary condition @ $r=a$; $w=0$

$$w = \frac{-C_1r^2}{4} - c_2\log\frac{r}{a} + C_3$$

$$0 = -\left[\frac{2(a^2M_2 - b^2m_1)}{D(1+\mu)(a^2-b^2)}\right]\frac{a^2}{4} - C_2\log\frac{a}{a} + C_3$$

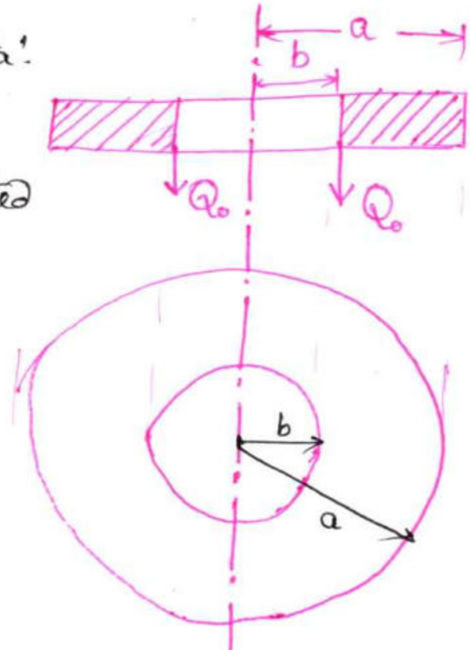
$$= \quad C_3 = \frac{C_1a^2}{4} = \left[\frac{\cancel{2}(a^2M_2 - b^2M_1)a^2}{{}_2\cancel{4}D(1+\mu)(a^2-b^2)}\right]$$

If the ends are simply supported In this case $M_2 = 0$ then

$$w = \frac{-C_1 r^2}{4} - C_2 \log\frac{r}{a} + C_3$$

$$w = \frac{-2(M_1 b^2)}{D(1+\mu)(a^2-b^2)} + \frac{M_1 a^2 b^2 \log r/a}{D(1-\mu)(a^2-b^2)} - \frac{M_1 a^2 b^2}{2D(1+\mu)(a^2-b^2)}$$

Circular plate with a circular hole at its centre subjected to shearing force Q_0 at centre-



Consider a circular plate having radius 'a'. let us drill a circular hole of radius 'b' at the centre of the plate. The whole plate subjected to shearing force Q_0 at inner edge

Magnitude of shearing Q_0 at any distance 'r' from the centre of plate is calculated

$$2\pi r Q = 2\pi b Q_0$$

$$Q = \frac{Q_0 b}{r} = \frac{P}{2\pi r} \quad \emptyset$$

where $P = 2\pi b Q_0$ denotes the total load acting on circular plate.

Differential equation of circular plates

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial \omega}{\partial r}\right)\right] = \frac{Q}{D}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \times \frac{\partial \omega}{\partial r}\right)\right] = \frac{P}{2\pi r D}$$

Integrating the above expression WRT 'r' on both sides

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \times \frac{\partial \omega}{\partial r}\right) = \frac{P}{2\pi D}\log\frac{r}{a} + C_1$$

$$\frac{\partial}{\partial r}\left(r \times \frac{\partial \omega}{\partial r}\right) = \frac{P}{2\pi D} r \log\frac{r}{a} + C_1 r$$

Again integrating the above expression

$$r \times \frac{\partial w}{\partial r} = \frac{P}{2\pi D}\left[\frac{r^2}{2}\log\frac{r}{a} - \frac{r^2}{4}\right] + \frac{C_1 r^2}{2} + C_2$$

$$\frac{\partial w}{\partial r} = \frac{P}{2\pi D}\left[\frac{r}{2}\log\frac{r}{a} - \frac{r}{4}\right] + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$\int UV = U\int V - \int\left[U'\int V\,\partial x\right]\partial x$$

Again integrating the above expression

$$= \log\frac{r}{a}\times\frac{r^2}{2} - \int \frac{\cancel{a}}{\cancel{r}}\times\frac{1}{\cancel{a}}\,\frac{r^{\cancel{2}}}{2}\,\partial x$$

$$= \frac{r^2}{2}\log\frac{r}{a} - \frac{r^2}{4}$$

$$\cancel{\int\frac{\partial w}{\partial r} = \frac{P}{2\pi D}\int\left[\frac{r}{2}\log\frac{r}{a} - \frac{r}{4}\right] + \int\frac{C_1 r}{2} + \int\frac{C_2}{r}}$$

$$\cancel{\frac{\partial w}{\partial r} = \frac{P}{2\pi D}}$$

$$\cancel{\frac{\partial w}{\partial r} = \frac{P}{}}$$

$$\frac{\partial w}{\partial r} = \frac{P}{4\pi D}\left[r\log r/a - r\right] + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$\frac{\partial w}{\partial r} = \frac{Pr}{8\pi D}\left[2\log\frac{r}{a} - 1\right] + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$\phi = -\frac{\partial w}{\partial r}$$

$$\phi = -\frac{\partial w}{\partial r} = \frac{-Pr}{8\pi D}\left[2\log\frac{r}{a} - 1\right] - \frac{C_1 r}{2} - \frac{C_2}{r}$$

To get exact solution P is taken as +ve

$$\phi = \frac{Pr}{8\pi D}\left[2\log\frac{r}{a} - 1\right] - \frac{C_1 r}{2} - \frac{C_2}{r}$$

To obtain deflection ~~Differentia~~ Integrating the above equation WRT 'r'

$$= \int\frac{Pr}{8\pi D}\left(2\log\frac{r}{a} - 1\right) - \frac{C_1 r}{2} - \frac{C_2}{r}$$

$$= \frac{P}{\cancel{8}_{4}\pi D}\left[\frac{r^2}{4}\times\cancel{2}\log\frac{r}{a} - r^2\right.$$

$$-\left(\frac{P}{4\pi D}\left[r\times\cancel{2}\log\frac{r}{a} - r\right] + \frac{C_1 r}{2} + \frac{C_2}{r}\right)$$

$$= -\frac{P}{4\pi D}\left[\frac{r^2}{2}\log\frac{r}{a} - \frac{r^2}{2}\right] - \frac{C_1 r^2}{2} - C_2\log\frac{r}{a} + C_3$$

$\times \div 2$

$$w = -\frac{P r^2}{8\pi D}\left[\log\frac{r}{a} - 1\right] - \frac{C_1 r^2}{2} - C_2\log\frac{r}{a} + C_3$$

C_1, C_2 & C_3 are constants of integration, will now be calculated from B.C. Assuming that the plate is a simply supported along the outer edge.

$$(w)_{r=a} = 0$$

$$M_r = 0 \; ; \; r = a \quad , \quad M_r = 0 \; ; \; r = b$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right]$$

$$\frac{\partial w}{\partial r} = \frac{P r}{8\pi D}\left[2\log\frac{r}{a} - 1\right] - \frac{C_1 r}{2} - \frac{C_2}{r}$$

$$\cancel{\frac{\partial^2 w}{\partial r^2} = \frac{P}{8\pi D}\left[2\log\frac{r}{a} + 2r \cdot \frac{1}{r/a}\cdot\frac{1}{a} - 1\right] - \frac{C_1 r^2}{2} - \frac{C_2}{r}}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{P}{8\pi D}\left[2\log\frac{r}{a} + 1\right] - \frac{C_1}{2} + \frac{C_2}{r^2}$$

$$(M_r) = 0 \ @ \ r = a$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right]$$

$$0 = -D\left[\left(\frac{P}{8\pi D}\left(2\log\frac{r}{a} + 1\right) - \frac{C_1}{2} + \frac{C_2}{r^2}\right) - \frac{\nu}{r}\left(\frac{P r}{8\pi D}\left(2\log\frac{r}{a} - 1\right) - \frac{C_1 r}{2} - \frac{C_2}{r}\right)\right]$$

$$= \frac{P}{8\pi D}\left[2\log\frac{a}{a} + 1\right) - \frac{C_1}{2} + \frac{C_2}{a^2}\Big) + \frac{\nu}{r}\left[\left(a\, 2\log\overset{0}{\frac{a}{a}} - 1\right) - \frac{C_1 a}{2} - \frac{C_2}{a}\right)\right]$$

$$= \frac{P}{8\pi D}\left[\left[-\frac{C_1}{2} + \frac{C_2}{a^2}\right) + \frac{\nu}{r}\left(-\frac{C_1 a}{2} - \frac{C_2}{a}\right)\right] \quad r = a$$

$$= \frac{P}{8\pi D}\left[-\frac{C_1}{2} + \frac{C_2}{a^2} - \frac{C_1 \cancel{a}\nu}{2\cancel{a}} - \frac{C_2\nu}{a \times a}\right] = \frac{P}{8\pi D}\left[-\frac{C_1}{2} + \frac{C_2}{a^2} - \frac{C_1\nu}{2} - \frac{C_2\nu}{a^2}\right]$$

$$\frac{P}{8\pi D}\left[-\frac{C_1}{2}(1+\nu)+\frac{C_2}{a^2}(1-\nu)\right] \longrightarrow (1)$$

$(M_r) = 0 \quad @ \; r = b$

$$= -D\left[\frac{P}{8\pi D}\left(2\log\frac{r}{a}+1\right)-\frac{C_1}{2}+\frac{C_2}{r^2}+\frac{\nu}{r}\left(\frac{Pr}{8\pi D}\left(2\log\frac{r}{a}-1\right)-\frac{C_1 r}{2}-\frac{C_2}{r}\right)\right]$$

$$= D\frac{P}{8\pi D}\left[\left[\left(2\log\frac{b}{a}+1\right)-\frac{C_1}{2}+\frac{C_2}{b^2}\right]+\frac{\nu}{r}\left[2b\log\frac{b}{a}-1-\frac{C_1 b}{2}-\frac{C_2}{b}\right]\right]$$

$$= \frac{P}{8\pi D}\left[2\log\frac{b}{a}(1+\nu)+(1-\nu)\right]-\frac{C_1}{2}(1+\nu)+\frac{C_2}{b^2}(1-\nu)\Big] \longrightarrow (2)$$

solving (1) and (2) equations $2-1$

$$\frac{P}{8\pi D}\left[\cancel{-\frac{C_1}{2}(1+\nu)}+\cancel{\frac{P}{8\pi D}}\left[2\log\frac{b}{a}(1+\nu)+(1-\nu)\right)\right]-\cancel{\frac{C_1}{2}(1+\nu)}+\frac{C_2}{b^2}(1-\nu)\Big]$$

$$+\left[\cancel{\frac{C_1}{2}(1+\nu)}-\frac{C_2}{a^2}(1-\nu)\right]$$

$$\frac{P}{\cancel{8}_4\pi D}\left(\cancel{2}\log\frac{b}{a}(1+\nu)+C_2(1-\nu)\left[\frac{1}{b^2}-\frac{1}{a^2}\right]\right)=0$$

$$C_2 = \frac{-P}{4\pi D}\log b/a\,\frac{(1+\nu)}{(1-\nu)}\left[\frac{a^2b^2}{a^2-b^2}\right]$$

from eq (2) substituting C_2 in (2) eq.

$$\frac{P}{8\pi D}\left(2\log\frac{b}{a}(1+\nu)-(1-\nu)-\frac{C_1}{2}(1+\nu)+\frac{C_2}{b^2}(1-\nu)\right)$$

$$C_1 = \frac{P\cdot 2}{8\pi D}\left[\frac{2\log\frac{b}{a}(1+\nu)-(1-\nu)}{(1+\nu)}+\frac{2C_2}{b^2}\frac{(1-\nu)}{1+\nu}\right]$$

$$C_1 = \frac{P}{4\pi D}\cancel{2\log}\frac{b}{a}\left[\frac{1-\mu}{1+\mu}-\frac{2b^2}{a^2-b^2}\log b/a\right]$$

Using third boundary condition

$w_{r=a} = 0$

Substituting C_1 C_2 in w equation we get

$$C_3 = \frac{Pa^2}{8\pi D}\left[1 + \frac{1}{2}\frac{1-\nu}{1+\nu} - \frac{b^2}{a^2-b^2}\log\frac{b}{a}\right)$$

Substituting all of these constants in $\partial w/\partial r$ and w equation we get

$$\left(\frac{\partial w}{\partial r}\right)_{r=b} = \frac{Pb}{8\pi D}\left[2\log\frac{b}{a} - 1 - \frac{1-\nu}{1+\nu} + \frac{2b^2}{a^2-b^2}\log\frac{b}{a}\left(1+\frac{a^2}{b^2}\frac{1+\nu}{1-\nu}\right)\right]$$

In the limiting case where b is infinitely small $b^2\log(b/a)$ approaches zero, and the constants of integration become

$$C_1 = \frac{1-\nu}{1+\nu}\frac{P}{4\pi D} \qquad C_2 = 0 \qquad C_3 = \frac{Pa^2}{8\pi D}\left[1 + \frac{1}{2}\frac{1-\nu}{1+\nu}\right]$$

Substituting C_1 C_2 & C_3 in the above we get

$$w = \frac{P}{8\pi D}\left[\frac{3+\nu}{2(1+\nu)}(a^2-r^2) + r^2\log\frac{r}{a}\right]$$

C_1 explanation
from eq ②

$$\frac{C_1}{2}(1+\mu) = \frac{P}{8\pi D}\left(2\log\frac{b}{a}(1+\mu) - (1-\mu) + \frac{C_2}{b^2}(1-\mu)\right)$$

$$= \frac{P}{4\pi D}2\log\frac{b}{a}\frac{(1+\cancel{\mu})}{(\cancel{1}+\mu)} + \frac{(1-\mu)}{(1+\mu)} + \frac{2C_2}{b^2}\frac{(1-\cancel{\mu})}{(\cancel{1+\nu})}$$

$$= \frac{P}{4\pi D}\left[2\log\frac{b}{a} + \frac{1-\mu}{1+\mu}\right) - \frac{2}{b^2}\left[\frac{P}{4\pi D}\log\frac{b}{a}\left[\cancel{\frac{1+\mu}{1-\mu}}\right]\frac{a^2b^2}{a^2-b^2}\right.$$

$$\left[\cancel{\frac{1-\mu}{1+\mu}}\right)$$

$$C_1 = \frac{P}{4\pi D}\left[2\log b/a + \frac{1-\mu}{1+\mu}\right) - \frac{2P}{4\pi D}\log\frac{b}{a}\frac{a^2}{a^2-b^2}$$

$$C_1 = \frac{P}{4\pi D}2\log\frac{b}{a}\left(\frac{a^2-b^2-a^2}{a^2-b^2} + \frac{1-\mu}{1+\mu}\right).$$

ANNAMACHARYA UNIVERSITY
EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
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Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Theory and Analysis of Plates

UNIT-3

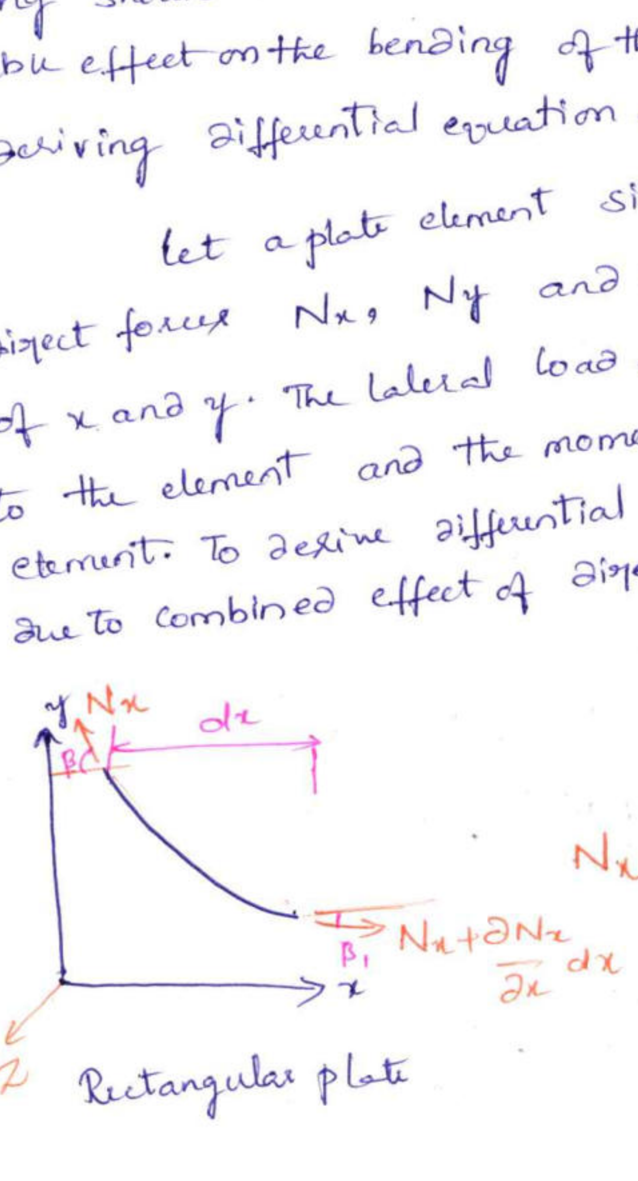
UNIT-III Plate subjected to ~~subjected~~ to Simultaneous Bending & Stretching

Plate is bent by lateral load only and deflections are so small that the plate middle surface was assumed to be unstrained.

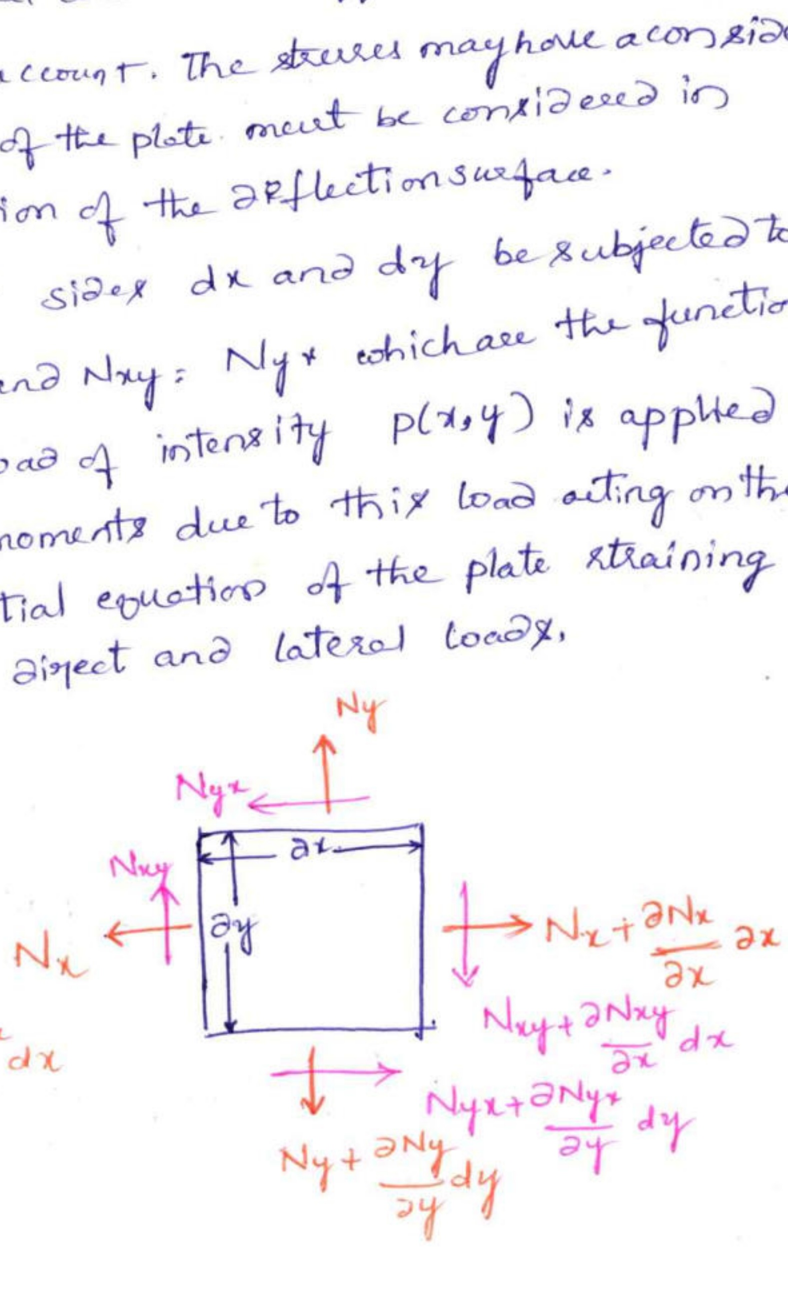
→ Stresses in the middle surface are small and there fore, their influence on the plate bending is negligible. So, that the total stress may be obtained by adding stresses caused by stretching and by bending of the plate middle surface.

→ Direct stresses are not small and their effect on the plate bending should be taken into account. The stresses may have a considerable effect on the bending of the plate must be considered in deriving differential equation of the deflection surface.

Let a plate element sides dx and dy be subjected to direct forces N_x, N_y and $N_{xy} = N_{yx}$ which are the functions of x and y. The lateral load of intensity $p(x,y)$ is applied to the element and the moments due to this load acting on the element. To derive differential equation of the plate straining due to combined effect of direct and lateral loads.



Rectangular plate



Considering the equilibrium of the element subjected to inplane forces N_x, N_y & N_{xy} as well as to the lateral load $P(x,y)$

We apply $\Sigma F_x = 0$ from the equilibrium of $N_x\,dy$ forces in x-direction we obtain

$$-N_x\,dy + \left(N_x + \frac{\partial N_x}{\partial x}dx\right)dy = 0$$

$\div\, dx\,dy$

$$-\frac{N_x\,\cancel{dy}}{\cancel{dx}\,dy} + \left(\cancel{N_x} + \frac{\partial N_x}{\partial x}\cancel{dx}\right)\frac{dy}{dx\,dy}$$

$$\left(N_x + \frac{\partial N_x}{\partial x}\right)dx\,dy\cos\beta' - N_x\,dy\cos\beta$$

$\beta' = \beta + \partial\beta$ $\quad \beta \approx \theta$ $\quad = \beta + \frac{\partial \beta}{\partial x}\partial x$, Noting that deflections are assumed to be very small and hence $\cos\beta = 1$

$\Sigma f_x = 0$

$$-N_x\,dy\overset{\cos\beta}{} + \left(N_x + \frac{\partial N_x}{\partial x}\partial x\right)dy\cos\beta' - N_{yx}\,dx\overset{\cos\beta}{} + N_{yx} + \frac{\partial N_{yx}}{\partial y}dy\,\overset{\cos\beta'}{}\,dx = 0$$

$\cos\beta = 1$

$$-N_x\cancel{dy} + N_x\cancel{dy} + \frac{\partial N_x}{\partial x}dx\,dy - N_{yx}\cancel{dx} + N_{yx}\cancel{dx} + \frac{\partial N_{yx}}{\partial y}dy\,dx = 0$$

$\div\, dx\,dy$

$$\boxed{\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0} \longrightarrow a$$

Illy $\Sigma Y = 0$

$$\boxed{\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0} \longrightarrow b$$

Considering the projection of all the forces on the Z-axis, the plate deflection must be taken in to account. Due to bending of plate in the xz plane, the z component of the Normal force N_x is

$$-N_x\, dy \sin\beta + \left(N_x + \frac{\partial N_x}{\partial x} dx\right) \partial y \sin\beta'$$

β and β' are small $\quad \sin\beta = \beta = \partial \omega / \partial x \qquad \sin\beta' = \beta'$

$$\beta' = \beta + \partial\beta \quad = \frac{\partial \omega}{\partial x} + \frac{\partial^2 \omega}{\partial x^2} dx \quad = \beta + \frac{\partial \beta}{\partial x} dx$$

$$-N_x\, dy \frac{\partial \omega}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\frac{\partial \omega}{\partial x} + \frac{\partial^2 \omega}{\partial x^2} dx\right) dy$$

$$= -N_x\, dy \frac{\partial \omega}{\partial x} + N_x \frac{\partial \omega}{\partial x} dy + N_x \frac{\partial^2 \omega}{\partial x^2} dx\, dy + \frac{\partial N_x}{\partial x} \frac{\partial \omega}{\partial x} dx\, dy +$$

$$\frac{\partial N_x}{\partial x} dx \cdot \frac{\partial^2 \omega}{\partial x^2} dx\, dy \qquad dx^2 \cdot dy$$

Neglecting higher order terms

$$= N_x \frac{\partial^2 \omega}{\partial x^2} dx\, dy + \frac{\partial N_x}{\partial x} \frac{\partial \omega}{\partial x} \partial x\, dy \longrightarrow c$$

The Z-component of inplane shear forces N_{xy} on the x edges of the elements are determined as follows. The slope of the deflection surface in the y-direction on the x-edge are equal to $\partial\omega/\partial y$ and $\frac{\partial \omega}{\partial y} + \left(\frac{\partial^2 \omega}{\partial x\, \partial y}\right) \partial x$. The Z directed component of N_{xy} is then

$$N_{xy}\, dy \sin\beta + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx\right) dy \sin\beta'$$

$$N_{xy} \frac{\partial^2 \omega}{\partial x\, \partial y} dx\, dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial \omega}{\partial y} dx\, dy \longrightarrow e$$

As expression identical to the above is found for the Z projection of the in-plane shear forces N_{yx} acting on the y-edges

$$N_{yx} \frac{\partial^2 \omega}{\partial x \partial y} dx\, dy + \frac{\partial N_{yx}}{\partial y} \cdot \frac{\partial \omega}{\partial x} dx\, dy \longrightarrow f$$

Finally the forces from $\Sigma f_z = 0$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} +$$

$$\left[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\frac{\partial w}{\partial y} = 0\right.$$

llr

$$N_y \frac{\partial^2 w}{\partial y^2} dx\,dy + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y} dx\,dy \rightarrow e$$

Summing up of e and f equation's

$$N_{xy}\frac{\partial^2 w}{\partial x \partial y} dx\,dy + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y} dx\,dy + N_{yx}\frac{\partial^2 w}{\partial x \partial y} dx\,dy +$$

$$\frac{\partial N_{yx}}{\partial y} \times \frac{\partial w}{\partial x} \times dx \times dy = 0$$

$$2N_{xy}\frac{\partial^2 w}{\partial x \partial y} dx\,dy + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y} dx\,dy + \frac{\partial N_{yx}}{\partial y}\cdot\frac{\partial w}{\partial x} dx\,dy \rightarrow g$$

Summing up all forces e, f and g

$$\cancel{N_{xy}\frac{\partial^2 w}{\partial x \partial y} dx\,dy + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y} dx\,dy + N_{yx}\frac{\partial^2 w}{\partial x \partial y} dx\,dy + \frac{\partial N_{yx}}{\partial y} \times \frac{\partial w}{\partial x} dx\,dy}$$

$$2N_{xy}\frac{\partial^2 w}{\partial x \partial y} \partial x\,dy + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y} dx\,dy + \frac{\partial N_{yx}}{\partial y}\cdot\frac{\partial w}{\partial x} dx\,dy + q\,dx\,dy =$$

$$+ N_x \frac{\partial^2 w}{\partial x^2} dx\,dy + \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}\partial x\,\partial y + N_y\frac{\partial^2 w}{\partial y^2}\partial x\,\partial y + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}\partial x\,\partial y$$

Considering all the second order term's

$$N_x \frac{\partial^2 w}{\partial x^2} dx\,dy + N_y \frac{\partial^2 w}{\partial y^2} dx\,dy + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} dx\,dy + q\,dx\,dy = 0$$

$$\left[N_x \times \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + q\right] dx\,dy = 0 \rightarrow h$$

From theory of pure bending of plate, use condition of equilibrium

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \text{ (Total load)}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left[N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + q\right] dx\,dy \quad \text{(9)}$$

From theory of pure bending of plates

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \quad M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} = -M_{yx}$$

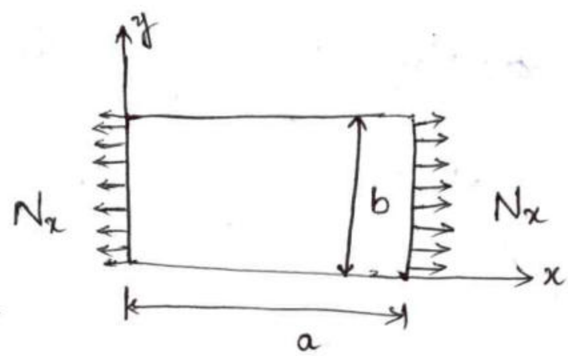
Substituting M_x M_y and M_{xy} in the above (9) equation

$$\frac{\partial^2}{\partial x^2}\left[-D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)\right] - 2\frac{\partial^2}{\partial x \partial y}\left[D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}\right] + \frac{\partial^2}{\partial y^2}\left[-D\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] =$$

$$-\left[\left[q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right] dx\,dy\right]$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left[q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) dx\,dy$$

□ular Simply Supported plates under combination of uniform lateral load under uniform tension



Assume a rectangular plate having length 'a' along x-direction and having width 'b' along y-direction. The plate is subjected to uniform tension N_x in the middle ~~plate~~ plain of the plate. The uniform lateral load may be taken as

$$q = \frac{16 q_0}{\pi^2}\sum\sum\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

As we know the governing equation for the □ular plate to uniform tension & uniform lateral load is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left[q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right]$$

Since there is no N_y and N_{xy} we can neglect both that term's in the above expression. Then we get

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left[q + N_x\frac{\partial^2 w}{\partial x^2}\right]$$

Rearranging the above terms

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x}{D}\left[\frac{\partial^2 w}{\partial x^2}\right] = \frac{q}{D} \longrightarrow 2$$

Above expression is the governing equation. Considering uniform tension & uniform lateral load.

By using expression (1) in and (2) becomes

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = \frac{16 q_0}{D\pi^2}\Sigma\Sigma\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} = \frac{16 q_0}{D\pi^2}\sum_{m=1,3,5}^{\infty}\frac{1}{m}\sin\frac{m\pi x}{a}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\frac{m\pi y}{b} \quad 3$$

Boundary conditions at the simply supported edges will be satisfied if we take the deflection 'w' is in the form of

$$\therefore w = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

In which m & n are odd numbers if m or n both are even number. The coefficient a_{mn} will be zero

$$\frac{\partial w}{\partial x} = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} a_{mn}\cos\frac{m\pi x}{a}\frac{m\pi}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = -\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} a_{mn}\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^3 w}{\partial x^3} = -\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} a_{mn}\left(\frac{m\pi}{a}\right)^3\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\frac{\partial^4 w}{\partial x^4} = \Sigma\Sigma\, a_{mn}\left(\frac{m\pi}{a}\right)^4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \longrightarrow a$$

$$\frac{\partial w}{\partial y} = \sum_{m=1,3,5}^{\alpha}\sum_{n=1,3,5}^{\alpha} a_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cdot\frac{n\pi}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = \sum_{m=1,3,5}^{\alpha}\sum_{n=1,3,5}^{\alpha} a_{mn}\sin\frac{m\pi x}{a}\left(-\cos\frac{n\pi y}{b}\right)\left(\frac{n\pi}{b}\right)^2$$

$$\frac{\partial^3 w}{\partial y^3} = \sum_{m=1,3,5}^{\alpha}\sum_{n=1,3,5}^{\alpha} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(\frac{n\pi}{b}\right)^3$$

$$\frac{\partial^4 w}{\partial y^4} = \Sigma\Sigma\, a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\frac{n\pi}{b}\right]^4 \longrightarrow b$$

$$\frac{\partial^2 w}{\partial x^2} = -\Sigma\Sigma\, a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\frac{m\pi}{a}\right]^2$$

$$\frac{\partial^3 w}{\partial x^2\partial y} = +\Sigma\Sigma\, a_{mn}\sin\frac{m\pi x}{a}\left(\frac{m\pi}{a}\right)^2\cos\frac{n\pi y}{b}\times\frac{n\pi}{b}$$

$$\frac{\partial^4 w}{\partial x^2\partial y^2} = \Sigma\Sigma\, a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \longrightarrow c$$

Substituting abc in eq ③ we get

$$a_{mn}\sin\frac{m\pi x}{a}\cancel{\sin\frac{n\pi y}{b}}\left[w\left[\frac{m\pi}{a}\right]^4 + 2w\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right] + w\left[\frac{n\pi}{b}\right]^4 + \frac{N_x}{D}w\left[\frac{m\pi}{a}\right]^2\right]$$

$$= \frac{16 q_0}{D\pi^2}\Sigma\Sigma\frac{1}{mn}\sin\frac{m\pi x}{a}\cancel{\sin\frac{n\pi y}{b}}$$

$$a_{mn}\left(\left(\frac{m}{a}\right)^4 + 2\left(\frac{mn}{ab}\right)^2 + \left(\frac{n}{b}\right)^4 + \frac{N_x}{D}\frac{m^2}{\pi^2 a^2}\right) = \frac{16 q_0}{D\pi^6 mn}$$

$$a_{mn} = \frac{16 q_0}{D\pi^6 mn\left(\left(\frac{m}{a}\right)^4 + 2\left(\frac{mn}{ab}\right)^2 + \left(\frac{n}{b}\right)^4 + \frac{N_x m^2}{D\pi^2 a^2}\right)}$$

$$a_{mn} = \frac{16q_0}{D\pi^6 mn\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{N_x m^2}{D\pi^2 a^2}\right]}$$

Bending of plates with a small initial curvature;

let us assume a plate with some initial deflection of the middle surface. So that at any point there is an initial deflection 'w_0'

If such plate is subjected to lateral loads, additional deflection 'w_1' will occurs. Therefore the total deflection at any point will be $w = w_0 + w_1$

where w_1 = deflection due to lateral loads

In calculating the deflection 'w_1' we can use

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \longrightarrow (1)$$

In addition to lateral loads, there are forces acting in the middle of the plate (N_x or N_y or N_{xy}). The effect of these forces on bending depends on total deflection.

To apply this concept, we can use the total deflection w on the RHS of the governing equation. It will be remembered that the L.H.S of governing equation was obtained from the expression for B.M in the plate M_x, M_y, M_{xy}

let us assume that the effect of an initial curvature on the deflection is equivalent to the effect of a fictitious lateral load of an intensity

$$N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} \longrightarrow (2)$$

Since the moments depend not on the total curvature but only on the change in curvature of the plate, the deflection 'w_1' should be used instead of 'w' in the L.H.S of the governing equation. Therefore, the governing equation becomes

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{1}{D}\left[q + N_x \frac{\partial^2 (w_0+w_1)}{\partial x^2} + N_y \frac{\partial^2 (w_0+w_1)}{\partial y^2} + 2N_{xy}\frac{\partial^2 (w_0+w_1)}{\partial x \partial y}\right]$$

where q is total lateral load.

By expanding the above expression, we get

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} = \frac{1}{D}\left[q + N_x \frac{\partial^2 (w_0+w_1)}{\partial x^2} + N_y \frac{\partial^2 (w_0+w_1)}{\partial y^2} + 2N_{xy}\frac{\partial^2 (w_0+w_1)}{\partial x \partial y}\right]$$

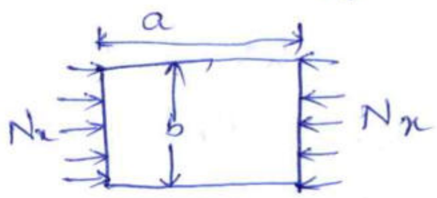
$$= \frac{1}{D}\left[q - \left[N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2N_{xy}\frac{\partial^2 w_0}{\partial x \partial y}\right] - N_x \frac{\partial w_1}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w_1}{\partial x \partial y}\right]\right] \rightarrow 3$$

let us assume that the initial deflection of the plate is

$$w_0 = a_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \rightarrow 4$$

As an example, the case of $\square^{\text{ular}}$ plate subjected to N_x are acting on the edges of the plate as shown in figure

let us take the deflection "w_1" is in the form of



$$w_1 = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \longrightarrow 5$$

Considering

$$w_1 = \alpha A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_0 = a_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial w_1}{\partial x} = +A\cos\frac{\pi x}{a} \times \frac{\pi}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial w_0}{\partial x} = \frac{\pi}{a} a_{11} \cos\frac{\pi}{a}x \sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w_1}{\partial x^2} = -A\frac{\pi^2}{a^2} \sin\frac{\pi}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{\pi^2}{a^2} a_{11} \cancel{\cos} -\sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b}$$

$$\frac{\partial^3 w_1}{\partial x^3} = -A\frac{\pi^3}{a^3} \cos \cancel{\sin} \frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w_0}{\partial y^2} = \frac{\pi^2}{b^2} a_{11} - \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^4 w_1}{\partial x^4} = A\frac{\pi^4}{a^4} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w_0}{\partial x \partial y} = a_{11}\frac{\pi^2}{ab} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$\frac{\partial w_1}{\partial y} = A\sin\frac{\pi x}{a} \times \frac{\pi}{b} \cos\frac{\pi y}{b}$$

$$\frac{\partial^2 w_1}{\partial y^2} = A\sin\frac{\pi x}{a}\left[-\frac{\pi^2}{b^2}\right) \sin\frac{\pi y}{b}$$

$$\frac{\partial^3 w_1}{\partial y^3} = A\sin\frac{\pi x}{a}\left[-\frac{\pi^3}{b^3}\right] \cos\frac{\pi y}{b}$$

$$\frac{\partial^4 w_1}{\partial y^4} = A\sin\frac{\pi x}{a}\left[\frac{\pi^4}{b^4}\right] \sin\frac{\pi y}{b}$$

$$\frac{\partial^4 w_1}{\partial x^2 \partial y^2} = \frac{\partial^2 w_1}{\partial x^2} = \frac{-\pi^2 A}{a^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$\frac{\partial^3 w_1}{\partial x^2 \partial y} = -A\frac{\pi^2}{a^2} \sin\frac{\pi x}{a} \times \frac{\pi}{b} \cos\frac{\pi y}{b}$$

$$\frac{\partial^4 w_1}{\partial x^2 \partial y^2} = \frac{A\pi^4}{a^2 b^2} \sin\frac{\pi x}{a} \cdot \sin\frac{\pi y}{b}$$

Since there is no N_y & N_{xy} forces and initial deflection is present in the plate then the governing equation will be reduced to

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} = \frac{1}{D}\left[q - \left(N_x \frac{\partial^2 w_0}{\partial x^2} + q\right] - N_x \frac{\partial^2 w_1}{\partial x^2}\right)$$

Let $q = \frac{\partial^2 w_0}{\partial y^2} N_y + N_{xy}\frac{\partial^2 w_0}{\partial x \partial y}$

$$= \frac{1}{D}\left[-N_x \frac{\partial^2 w_0}{\partial x^2} - N_x \frac{\partial^2 w_1}{\partial x^2}\right]$$

$$= \frac{1}{D}\left(N_x\, a_{11} \frac{\pi^2}{a^2} \sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b} + N_x \times A \times \frac{\pi^2}{a^2} \sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b}\right]$$

$$= \frac{1}{D}\left[N_x\, a_{11} \frac{\pi^2}{a^2}\psi + N_x A \times \frac{\pi^2}{a^2}\psi\right]$$

$$\psi = \sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b}$$

$$A\left(\frac{\pi}{a}\right)^4 \psi + 2A\left(\frac{\pi}{a} \times \frac{\pi}{b}\right)^2 \psi + A\psi\left[\frac{\pi}{b}\right]^4 = \frac{N_x}{D}\left[a_{11}\frac{\pi^2}{a^2}\psi + A \times \frac{\pi^2}{a^2}\psi\right]$$

$$A\psi\left[\left(\frac{\pi}{a}\right)^4 + 2\left[\frac{\pi}{a} \times \frac{\pi}{b}\right]^2 + \left(\frac{\pi}{b}\right)^4\right] - \left(A \times \frac{\pi^2}{a^2}\psi \frac{N_x}{D}\right) = \frac{N_x}{D} a_{11} \frac{\pi^2}{a^2}\psi$$

$$A\cancel{\psi}\left[\left(\frac{\pi}{a}\right)^4 + 2\left(\frac{\pi}{a} \times \frac{\pi}{b}\right)^2 + \left(\frac{\pi}{b}\right)^4\right] - \left(\frac{\pi^2}{a^2} \times \frac{N_x}{D}\right) = \frac{N_x}{D} a_{11} \frac{\pi^2}{a^2}\cancel{\psi}$$

$$A\cancel{\pi}\left[\frac{\pi^2}{a^4} + 2\left[\frac{\pi}{a^2} \times \frac{\pi}{b^2}\right] + \frac{\pi^2}{b^4}\right] - \left[\frac{N_x}{Da^2}\right] = \frac{N_x}{D}\frac{a_{11}}{a^2}\cancel{\pi}$$

$$A\cancel{\pi}\left[\left(\frac{\pi}{a^2} + \frac{\pi}{b^2}\right)^2 - \frac{N_x}{Da^2}\right] = \frac{N_x a''}{Da^2}$$

$$A\pi^2\left[\left(\frac{\cancel{\pi}}{a^2} + \frac{\cancel{\pi}}{b^2}\right)^2 - \frac{N_x}{Da^2}\right] = \frac{N_x a''}{Da^2}$$

$$A = \frac{N_x a_{11}}{Da^2\pi^2\left[\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 - \frac{N_x}{Da^2}\right]}$$

$$A = \frac{N_x a_{11}}{\frac{\pi^2 D}{a^2}\left[1 + \frac{a^2}{b^2}\right]^2 - N_x}$$

$$w_1 = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_1 = \frac{N_x a_{11}}{\frac{\pi^2 D}{a^2}\left[\left(1+\frac{a^2}{b^2}\right)^2\right] - N_x} \sin\frac{\pi x}{a} \times \sin\frac{\pi y}{b}$$

$$w = w_0 + w_1$$

$$= (a_{11}+A)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} = \frac{a_{11}}{1-\alpha}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

where $$\alpha = \frac{N_x}{\frac{\pi^2 D}{a^2}\left[1+\frac{a^2}{b^2}\right)^2}$$

at $x = a/2 \quad y = b/2$

$$w_{max} = \frac{a_{11}}{1-\alpha} \sin\frac{\pi \frac{\alpha}{2}}{\alpha} \sin\frac{\pi \frac{b}{2}}{b}$$

$$w_{max} = \frac{a_{11}}{1-\alpha}$$

Simply supported □ular plates under the combined action of lateral loads and of forces in the middle plane of the plate;

Let us begin with □le plate uniformly stretched in the x direction and carrying a concentrated load P at a point with coordinates ξ and η. The general expression for the deflection that satisfies the boundary conditions is

$$w = \sum_{m=1,2,3}^{\alpha} \sum_{n=1,2,3}^{\alpha} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

To obtain the coefficients a_{mn} in this series we use the circular plates under combined action of lateral load and tension or compression general equation. Since $N_y = N_{xy} = 0$

$$\frac{1}{2}\int_0^a\int_0^b N_x\left(\frac{\partial w}{\partial x}\right)^2 \partial x\,\partial y = \frac{ab}{8} N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \frac{m^2\pi^2}{a^2}$$

Strain energy of bending representing

$$V = \frac{\pi^4 ab}{8} D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^2$$

To obtain a virtual deflection δw we give to a coefficient a_{mn} an increase $\delta a_{m_1n_1}$. The corresponding deflection of the plate is

$$\delta_w = \delta a_{m_1n_1} \sin\frac{m_1\pi x}{a} \sin\frac{n_1\pi y}{b}$$

workdone during this virtual displacement by the lateral load P

$$P\,\delta a_{m_1n_1} \sin\frac{m_1\pi\xi}{a} \sin\frac{n_1\pi\eta}{b} \longrightarrow (d)$$

The corresponding change in the strain energy consists of two terms $\delta a_{m_1n_1}$

$$\frac{1}{2}\,\delta\int_0^a\int_0^b N_x\left(\frac{\partial w}{\partial x}\right)^2 \partial x\,\partial y = \frac{ab}{8} N_x \frac{\partial}{\partial a_{m_1n_1}}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \frac{m^2\pi^2}{a^2}\right)$$

$$= \frac{ab}{4} N_x a_{m_1n_1} \frac{m_1^2\pi^2}{a^2}\,\delta a_{m_1n_1}$$

and

$$\delta V = \frac{\partial V}{\partial a_{m_1n_1}}\,\delta a_{m_1n_1} = \frac{\pi^4 ab}{4} D\, a_{m_1n_1}\left[\frac{m_1^2}{a^2}+\frac{n_1^2}{b^2}\right]^2 \delta a_{m_1n_1} \quad (e)$$

substituting expression d and e

$$P\,\delta a_{m_1n_1} \sin\frac{m_1\pi\xi}{a}\sin\frac{n_1\pi\eta}{b} = \frac{ab}{4} N_x a_{m_1n_1}\frac{m_1^2\pi^2}{a^2}\,\delta a_{m_1n_1} + \frac{\pi^4 ab}{4} D\, a_{m_1n_1}\left(\frac{m_1^2}{a^2}+\frac{n_1^2}{b^2}\right)^2 \delta a_{m_1n_1}$$

$$a_{m_1n_1} = \frac{4P \sin\frac{m_1\pi\xi}{a}\sin\frac{n_1\pi\xi}{b}}{ab\pi^4 D\left[\left(\frac{m_1^2}{a^2}+\frac{n_1^2}{b^2}\right)^2 + \frac{m_1^2 N_x}{\pi^2 a^2 D}\right]}$$

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CIVIL ENGINEERING

Theory and Analysis of Plates

UNIT-4

$$w = a_{m_1 n_1} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w = \frac{4P \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}}{ab\pi^4 D\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{m_1^2 N_x}{\pi^2 a^2 D}\right]}$$

ORTHOTROPIC PLATES

Orthotropic material;

If a plate has 3 planes of symmetry w.R.T its elastic properties, the plate is said to be an orthotropic plate (xy, yz & zx are 3 planes of symmetry)

To analyze the orthotropic plates four elastic constants are needed. Thus are E_x', E_y', E'' & G

Plane stress;

Differential Equation of a bent plate

Considering the ~~pto~~ material of the plate has three planes of symmetry WRT its elastic properties. Take these planes as the coordinate planes, the relation between the stress and strain components for the case of plane stress in the xy plane can be represented by

$$\sigma_x = E_x' \epsilon_x + E'' \epsilon_y$$

$$\sigma_y = E_y' \epsilon_y + E'' \epsilon_x$$

$$\tau_{xy} = G\gamma_{xy}$$

In case of plane stress, four constants E_x' E_y' E'' and G are needed to characterise the elastic properties of a material

Considering the bending of a plate made of such a material, we assume linear elements perpendicular to the middle plane (xy plane) of the plate before bending remain straight and normal to the deflection surface of the plate after bending. Hence we can use expressions for the strain components

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

corresponding stress components

$$\sigma_x = -z\left[E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right]$$

$$\sigma_y = -z\left[E_y' \frac{\partial^2 w}{\partial y^2} + E'' \frac{\partial^2 w}{\partial x^2}\right]$$

$$\gamma_{xy} = -2G_z \frac{\partial^2 w}{\partial x \partial y}$$

Expressions for stress components the bending and twisting moments are

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \, \partial z$$

$$= \sigma_x \int_{-h/2}^{h/2} z \, \partial z$$

$$= -z\left[E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right] \int_{-h/2}^{h/2} z \, \partial z$$

$$= \left(E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right) \int_{-h/2}^{h/2} z^2 \, \partial z$$

$$= -\left[E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right] \left[\frac{z^3}{3}\right]_{-h/2}^{h/2}$$

$$= \frac{-1}{3}\left[E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right] \left[\left(\frac{h^3}{8}\right) + \left(\frac{h^3}{8}\right)\right]$$

$$= \frac{-h^3}{12}\left[E_x' \frac{\partial^2 w}{\partial x^2} + E'' \frac{\partial^2 w}{\partial y^2}\right]$$ $D_x = E_x' \frac{h^3}{12}$

On simplifying above expression, we get $D_1 = \frac{E' h^3}{12}$

$$M_x = -\left[D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right]$$ $\frac{Eh^3}{12}$ $D_y = \frac{-E_y' h^3}{12}$

Illy

$$M_y = -\left[D_y \left[\frac{\partial^2 w}{\partial y^2}\right] + D_1 \frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z \, \partial z = \int_{-h/2}^{h/2} 2 G z \frac{\partial^2 w}{\partial x \partial y} \partial z$$

$$= -2G \frac{\partial^2 w}{\partial x \partial y} \int_{-h/2}^{h/2} z^2 \partial z = \frac{2}{3} G \frac{\partial^2 w}{\partial x \partial y}\left[\frac{h^3}{8} + \frac{h^3}{8}\right]$$

$$M_{xy} = \frac{2 h^3}{3 \times 4_2} G \frac{\partial^2 w}{\partial x \partial y}$$

$$= \frac{h^3}{6} G \frac{\partial^2 w}{\partial x \partial y}$$ By using notation $D_{xy} = \frac{G h^3}{12}$

$$M_{xy} = -2 D_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

From equilibrium equation becomes

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q$$

Substituting M_x, M_y and M_{xy} values in the above expression we get

$$\frac{\partial^2}{\partial x^2}\left[-\left[D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right]\right] + \frac{\partial^2}{\partial y^2}\left[-\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right)\right] - 2\frac{\partial^2}{\partial x \partial y}\left(2 D_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) = -q$$

$$D_x \frac{\partial^4 w}{\partial x^4} + D_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + D_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 D_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} = q$$

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2 D_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 D_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} = q$$

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(D_1 + 2D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q$$

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q$$

Corresponding expression for shearing force are readily obtained from the condition of equilibrium of an element of the plate

$$Q_x = -\frac{\partial}{\partial x}\left[D_x \frac{\partial^2 w}{\partial x^2} + H \frac{\partial^2 w}{\partial y^2}\right]$$

$$Q_y = -\frac{\partial}{\partial y}\left[D_y \frac{\partial^2 w}{\partial y^2} + H \frac{\partial^2 w}{\partial x^2}\right]$$

In the Isotropy case

$$E_x' = E_y' = \frac{E}{1-\nu^2} \qquad E'' = \frac{\nu E}{1-\nu^2} \qquad G = \frac{E}{2(1+\nu)}$$

$$D_x = D_y = \frac{Eh^3}{12(1-\nu^2)}$$

$$H = D_1 + 2D_{xy} = \frac{h^3}{12}\left[\frac{\nu E}{1-\nu^2} + \frac{E}{1+\nu}\right] = \frac{Eh^3}{12(1-\nu^2)}$$

Application

$$E_x' = \frac{E}{1-\mu^2} = E_y'$$

$$E_x'' = \frac{\mu E}{1-\mu^2}$$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu \sigma_y}{E}$$

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$$E\epsilon_x = \sigma_x - \mu\sigma_y$$

$$\sigma_x = E\epsilon_x + \mu\sigma_y$$

$$\text{lly } \sigma_y = E\epsilon_y + \mu\sigma_x$$

$$\sigma_x + \mu\sigma_y = E\epsilon_x + \mu\sigma_y + \mu E\epsilon_y + \mu^2\sigma_x$$

$$\sigma_x = E\epsilon_x + \cancel{\mu\sigma_y} + \mu E\epsilon_y + \mu^2\sigma_x - \cancel{\mu\sigma_y}$$

$$= E\epsilon_x + \mu E\epsilon_y + \mu^2\sigma_x$$

$$\cancel{E\epsilon_x + (1-\mu^2)} E\epsilon_y$$

$$\sigma_x - \mu^2\sigma_x = E\epsilon_x + \mu E\epsilon_y$$

$$(1-\mu^2)\sigma_x = E\epsilon_x + \mu E\epsilon_y$$

$$\sigma_x = \frac{E\epsilon_x}{1-\mu^2} + \frac{\mu E\epsilon_y}{1-\mu^2}$$

$$\frac{E}{1-\mu^2} = E'_x, \quad \frac{\mu E}{1-\mu^2} = E''_x$$

Application of orthotropic plate theory to the Grid work system

The differential equation of the orthotropic plate theory to the grid work system.

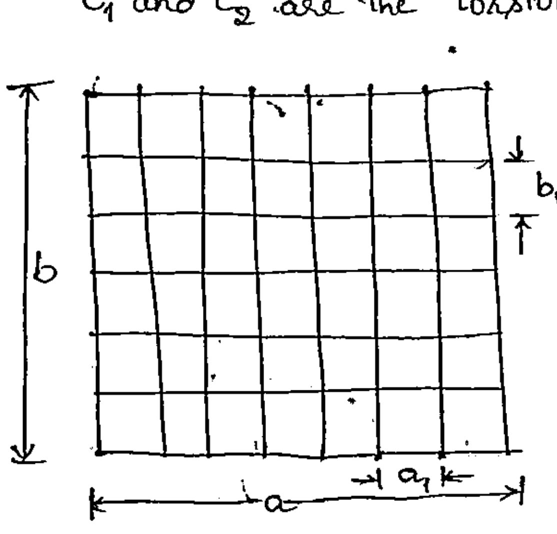
$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q$$

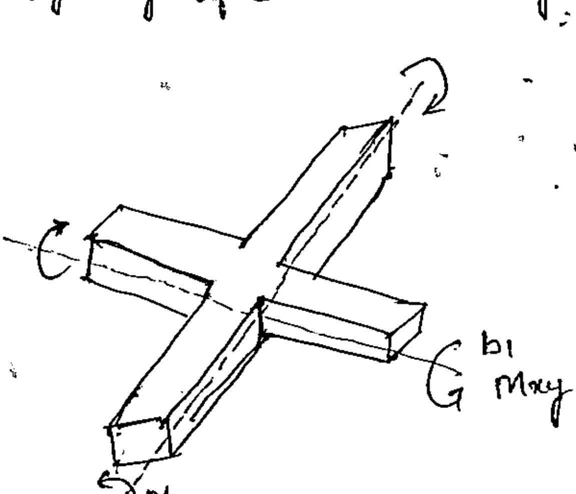
$$H = D_1 + 2 D_{xy}$$

$$D_x = \frac{E_x' h^3}{12} \qquad D_y = \frac{E_y' h^3}{12} \qquad D_{xy} = \frac{G h^3}{12}\,; \quad D_1 = \frac{E'' h^3}{12}$$

This consists of two systems of parallel beams spaced equal distances apart in the x and y directions and rigidly connected at their points of intersections. The beams are supported at the ends, and the load is applied normal to xy plane. Let the distance a_1 and b_1 between the beams are small in comparison with the dimensions a and b of the grid, and if the flexural rigidity of each of the beams parallel to the x-axis is equal to B_1 and that of each of the beams ‖le to y axis is equal to B_2

C_1 and C_2 are the torsional rigidity of each beam along x and y direction





$$\therefore \quad D_x = \frac{B_1}{b_1} \qquad D_y = \frac{B_2}{a_1}$$

$$M_x = -\left[D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right] \qquad D_1 = 0 \ \& \ D_x = \frac{B_1}{b_1}$$

$$M_x = -D_x \frac{\partial^2 w}{\partial x^2} = \frac{-B_1}{b_1} \frac{\partial^2 w}{\partial x^2}$$

$$M_y = \frac{-B_2}{a_1} \frac{\partial^2 w}{\partial y^2}$$

Now the twisting moment M_{xy} can be calculated by using twist $\frac{\partial^2 w}{\partial x \, \partial y}$

$$M_{xy} = \frac{C_1}{b_1} \frac{\partial^2 w}{\partial x \partial y} \qquad M_{xy} = -M_{yx}$$

$$M_{yx} = -\frac{C_2}{a_1} \frac{\partial^2 w}{\partial x \partial y} \qquad M_{yx} = -M_{xy}$$

In equilibrium equations for the grid system which is subjected to twisting moments M_{xy}, M_{yx}

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x \, \partial y} - \frac{\partial^2 M_{yx}}{\partial y \, \partial x} = -q$$

$$\frac{\partial^2}{\partial x^2}\left(-\frac{B_1}{b_1}\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial^2}{\partial y^2}\left(\frac{B_2}{a_1}\right)\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2}{\partial x \partial y}\frac{C_1}{b_1}\frac{\partial^2 w}{\partial x \partial y} - \frac{C_2}{a_1}\frac{\partial^2 w}{\partial x \partial y} = -q$$

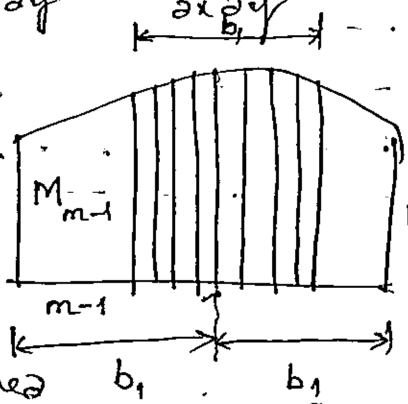
$$-\frac{B_1}{b_1}\frac{\partial^4 w}{\partial x^4} - \frac{B_2}{a_1}\frac{\partial^4 w}{\partial y^4} - \frac{C_1}{b_1}\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{C_2}{a_1}\frac{\partial^4 w}{\partial x^2 \partial y^2} = -q$$

$$\frac{B_1}{b_1}\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2}\left[\frac{C_1}{b_1} + \frac{C_2}{a_1}\right] + \frac{B_2}{a_1}\frac{\partial^4 w}{\partial y^4} = q$$

In order to obtain the final expression for the flexural and torsional moments of a rib, we still have to multiply the moments, such as

$$M_x = D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial x \partial y} \qquad M_y = D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x \partial y}$$

and valid for the unit width of the grid by the spacing of the ribs. The variation of the moments M_x and M_{xy}, may be assumed parabolic between the points $(m-1)$ and $(m+1)$ and the shaded area of the diagram, maybe assigned to the rib (m) running in the direction x. We obtain following approximate formulae for both moments of the rib



$$M_x = -\frac{B_1}{24}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)_{m-1} + 22\left(\frac{\partial^2 w}{\partial x^2}\right)_m + \left(\frac{\partial^2 w}{\partial x^2}\right)_{m+1}\right]$$