ANNAMACHARYA UNIVERSITY
EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
ESTD, UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)
Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Lecture Notes
on
Analysis of Shells and Folded Plates

Written by
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Course Structure for M. Tech-Structural Engineering

Title of the Course: Analysis of shells and folded plates
Category: Program Elective-IV
Couse Code: 24DSTE2DT
Branch/es: Structural Engineering
Semester: II Semester

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	-	-	3

Course Description:

This course provides an in-depth exploration of the analysis and design of shell structures and folded plates, which are widely used in modern engineering for their efficient load-bearing capabilities and aesthetic appeal

Course Objectives:

1. To understand the basic equations, bending effects of plates.. 4. To study the concepts of orthotropic plates, numerical, approximate methods, large deflection theory of plates. 6. To apply the numerical techniques and tools for the complex problems in shells
2. To understand the symmetrical loading and various loading conditions of circular and annular plates.
3. To understand the simultaneous bending and stretching of plates and to develop governing equation
4. To study the concepts of orthotropic plates, numerical, approximate methods, large deflection theory of plates.
5. To understand the analytical methods for the solution of shells.
6. To apply the numerical techniques and tools for the complex problems in shells

Course Outcomes:

At the end of the course, the student will be able to

1. Understand behaviour of plates for UDL, hydrostatic, concentrated load.
2. Perform cylindrical bending of long rectangular plates, pure bending of rectangular and circular plates, and deflection theories
3. Understand bending theory for structural behaviour of plates.
4. Implement numerical and approximate methods for plate problems.
5. Use analytical methods for the solution of shells.

Unit 1 10

Equations of equilibrium:Introduction, classification, derivation of stress Resultants, Principles of membrane theory and bending theory.

Unit 2 10

Cylindrical shells: Derivation of governing DKJ equation for bending theory, details of Scherer‟s theory, Applications to the analysis and design of short shells and long shells, Introduction of ASCE manual co-efficient for design.

Unit 3 10

Introduction to shells of double curvature: (other than shells of revolution) Geometry and analysis of ellipti
membrane theory.

Unit 4 10

Folded Plates: Folded plate theory, plate and slab action, Whitney‟s theory, Simpson‟s theory for the analysis of different types of folded plates (Design is not included)

Unit 5 10

Shells of double Curvature: Surfaces of revolution. Derivation of equilibrium equations by membrane theory, Applications to spherical shell and rotational Hyperboloid

Prescribed Text books:

1.Design and construction of concrete shell roofs by G.S. Rama Swamy – CBS Publishers & Distributors, 485, Jain BhawanBholaNath Nagar, Shahotra, Delhi.
2. Fundamentals of the analysis and design of shell structures by VasantS.Kelkar Robert T.Swell – Prentice hall, Inc., Englewood cliffs, new Jersy -02632.
3. N.k.Bairagi, Shell analysis, Khanna Publishers, Delhi, 1990.
4. Bollington, Ithin shell concrete structures, McGraw Hill Book company, New York, St. Louis, Sand Francisco, Toronto, London.
5. ASCE Manual of Engineering practice No.31, design of cylindrical concrete shell roofs ASC, New York.

CO-PO Mapping

Course Outcomes	PO1	PO2	PO3	PO4	PO5	PO6
24BCIV11T.1	3	-	3	-	3	3
24BCIV11T.2	3	-	3	-	3	3
24BCIV11T.3	3	-	3	-	3	3
24BCIV11T.4	3	-	3	-	3	3
24BCIV11T.5	3	-	3	-	3	3

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CIVIL ENGINEERING

Analysis of Shells and Folded Plates

UNIT-1

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UNIT- I

9- June-2015

EQUATIONS OF EQUILIBRIUM

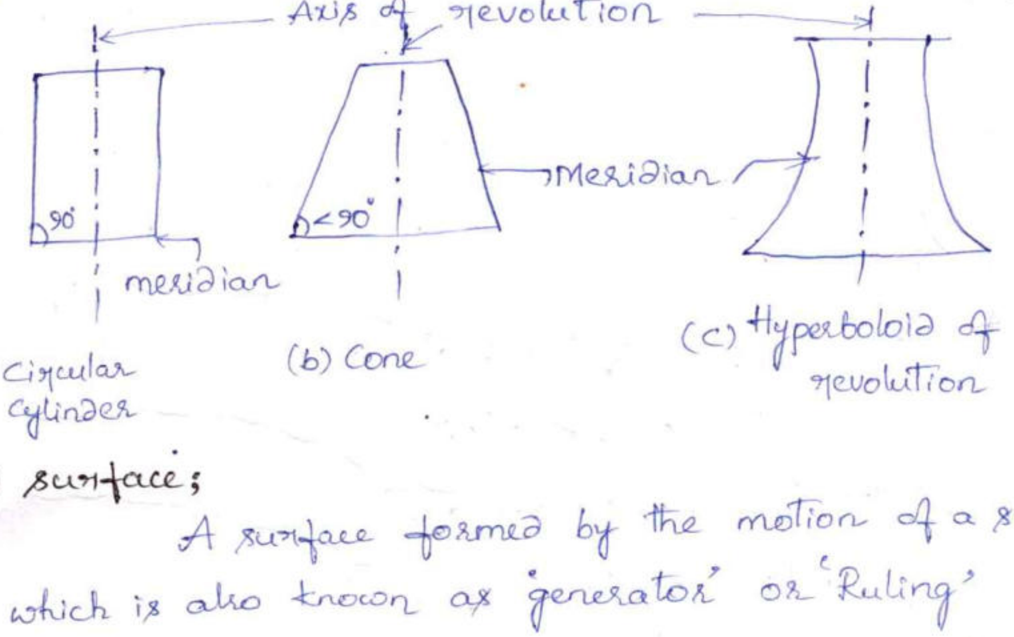
Shells;

Shells are stressed skin structures which by virtue of their Geometry and small flexural rigidity, tend to carry the loads by direct stress with little or no bending.

Shells and folded plates are usually adopted for covering the large spans with little thickness and where the constructions of normal beams and slabs become difficult and costly.

Surface of Revolution;

Surface of revolution obtained by rotation of a plane curve called the meridian about an axis lying in the plane of the curve. This plane is known as meridian plane



(a) Circular cylinder (b) Cone (c) Hyperboloid of revolution

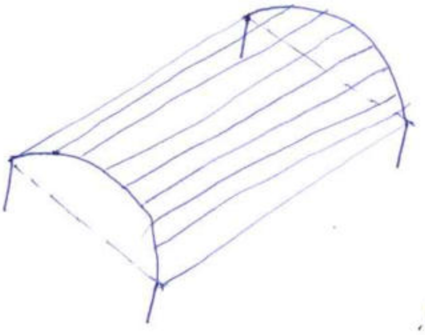
Ruled surface;

A surface formed by the motion of a single straight line, which is also known as 'generator' or 'Ruling'

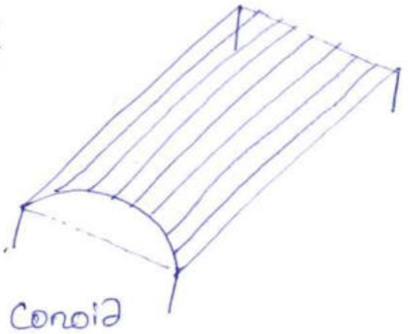
A surface is said to be singly ruled, if at every point only a single straight line is ruled and Doubly ruled if at any point two straight lines can be ruled.

Example; conical shells, conoids and cylinders - Singly ruled

Hyperbolic paraboloid and hyperboloid of revolution

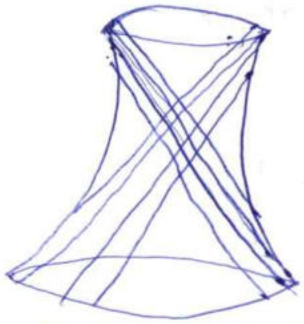


Cylindrical shell



Conoid

Singly ruled



Doubly ruled

Hyperboloid of revolution

Surface of translation;

A surface of Translation is generated by the motion of a plane curve parallel to itself over the another curve such that the planes containing two curves being at right angles to each other, one of the curves may be a straight line as in the case of a cylindrical surface.

The elliptic paraboloid, generated by a convex parabola moving over another convex parabola or by a concave parabola, is the surface of Transilation. Here the two parabola's involved are disimilar parabola's](wrong)

moving over another concave parabola, is a surface of transilation. Here the two parabola's involved eith both concave or convex.

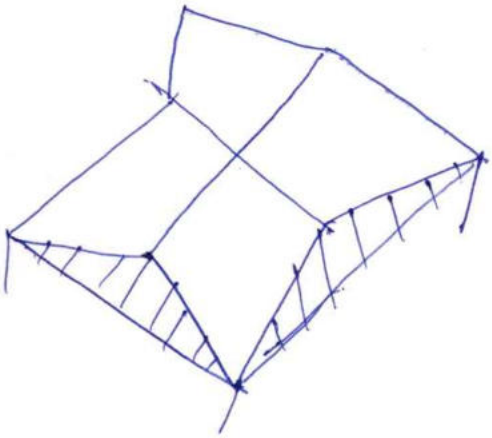
→ A special case of the paraboloid of revolution for which both the parabola's involved are 'identical parabolas'.

→ If a convex parabola moves over a concave parabola or viceversa, a hyperbolic paraboloid is formed.

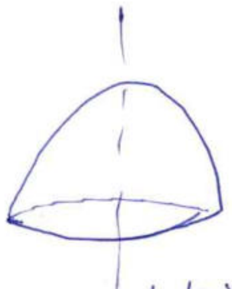
PT

The general equation of any translational surface can be represented as

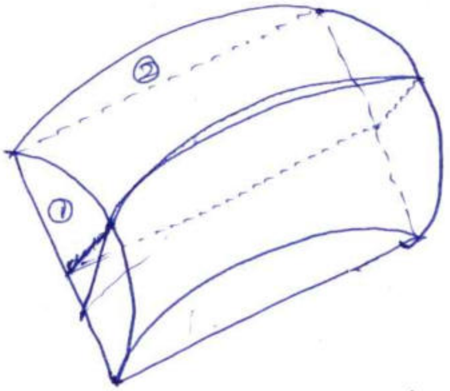
$$z = f_1(x) + f_2(y)$$



Hyperbolic paraboloid



paraboloid of revolution



Elliptic paraboloid

Singly curved surfaces & Doubly curved surface;

Singly curved surfaces will have curvature in only one direction. Eg; cylinders and cones

These surfaces are developable surfaces which means that they will try to flatten out easily under external loading.

Doubly curved surfaces are the surface with curvature in both the directions. Eg; Elliptic paraboloid, circular dome (foot ball), Hyperbolic paraboloid.

These are usually non developable surfaces which means that they wont get easily flattened out under normal loading.

Gauss curvature;

The product of two radii of curvature at any point of a surface is called Gauss curvature. If $\frac{1}{r_1}$ and $\frac{1}{r_2}$ are two radii of curvature of surface at a point then the Gauss curvature is given by

$$G.C = \frac{1}{r_1} * \frac{1}{r_2}$$

Synclastic, Anticlastic & Developable surfaces;

If a gauss curvature of a surface is greater than zero or positive quantity such a surface is called synclastic surface (Both concave & convex)

If the Gauss curvature of a surface at a particular point is -ve such a surface is called Anticlastic surface (one concave & one convex)

When the Gauss curvature of a surface is '0' which happens when any one of the two curves involved is a straight line and is called developable surface.

Ex; cylindrical surface.

15-06-15
Absenties
1
2
3
5
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8
10
11
12
15

16-06-15
Abs
2
3
8
10
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12
15

19-06-15
Absenties
1
3
5
6
8
10
11
14

19/2/19 Ab
2
4
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11
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16/2/17
Abs 4&9

26/2/19 Ab
2
3
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~~12~~

13/3/18
P
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5/3/19 Ab
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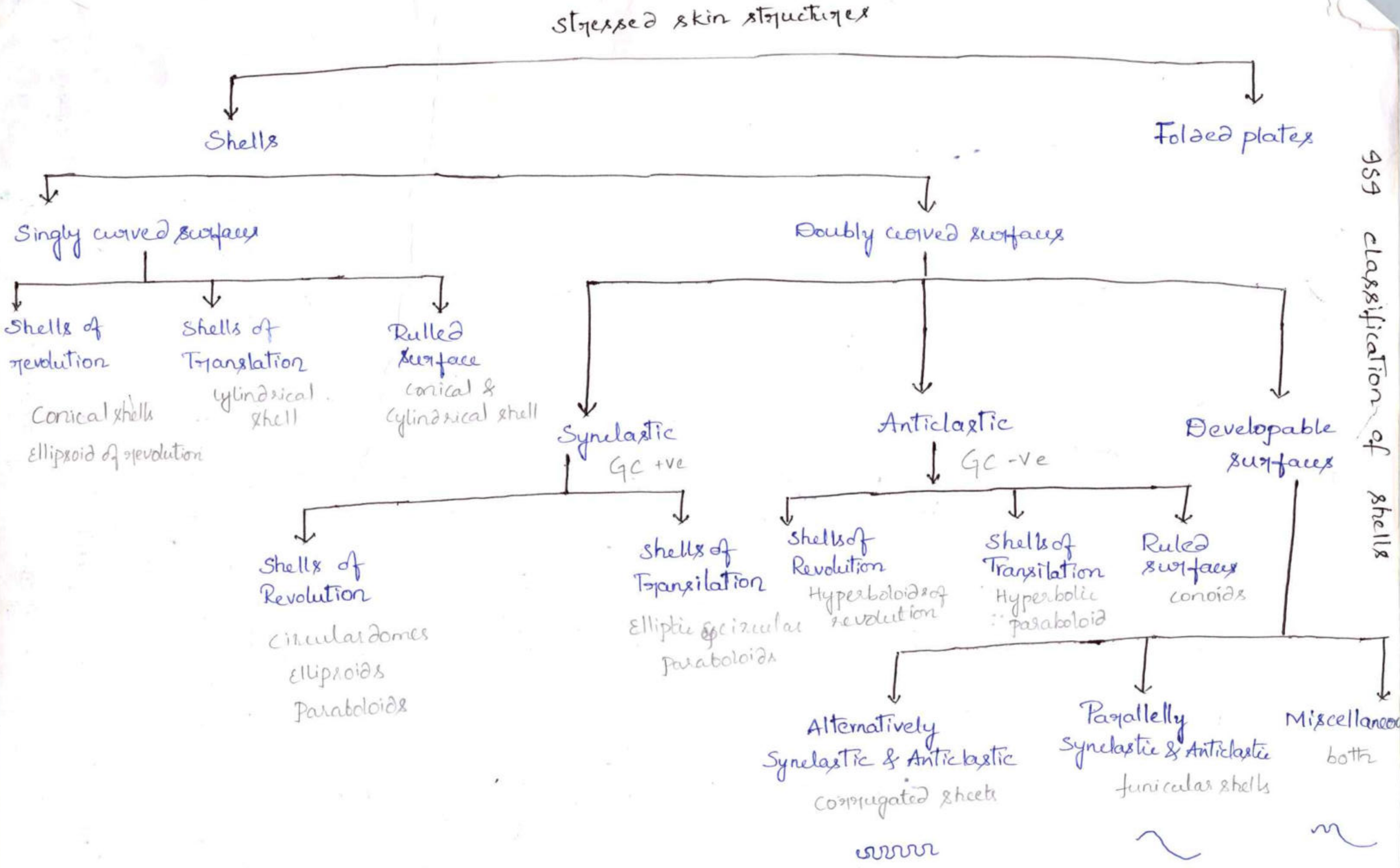
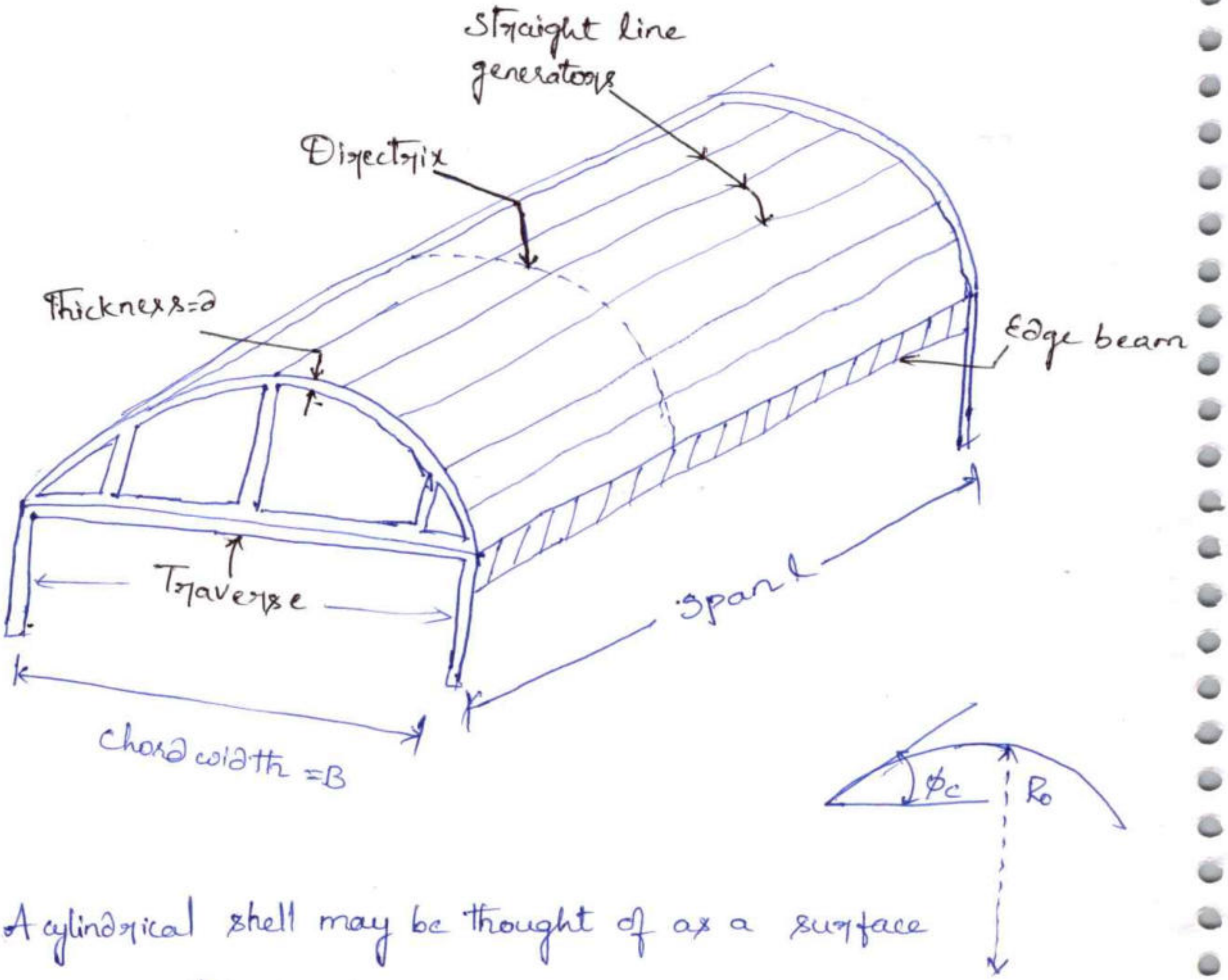
13

954 classification of shells



Parts of cylindrical shells; IS(2210 - 1962)



A cylindrical shell may be thought of as a surface generated by a straight line moving over a plane curve. The straight line generating the surface is known as the generator and the plane curve that guides it is known as "directrix".

The directrices usually employed are the arc of a circle, the semi ellipse the parabola, the conoid, and the catenary.

A cylindrical shell may or may not be provided with an edge beam or edge member. The supporting members at the two ends of a shell are known as traverse. The traverse may be a solid diaphragm or a brick wall or a rigid frame or a truss.

The distance b/n adjacent traverses is known as the span of shell. The projection of the arc of the shell is generally called chord wid

Assumptions made in the analysis of a cylindrical shell ;

→ Shell is assumed to be simply supported over the traverse.

→ It is also assumed that the Traverse's are rigid in their own plane but they are flexible out of their plane's such that they cannot recieve any load normal to them.

Loads ;

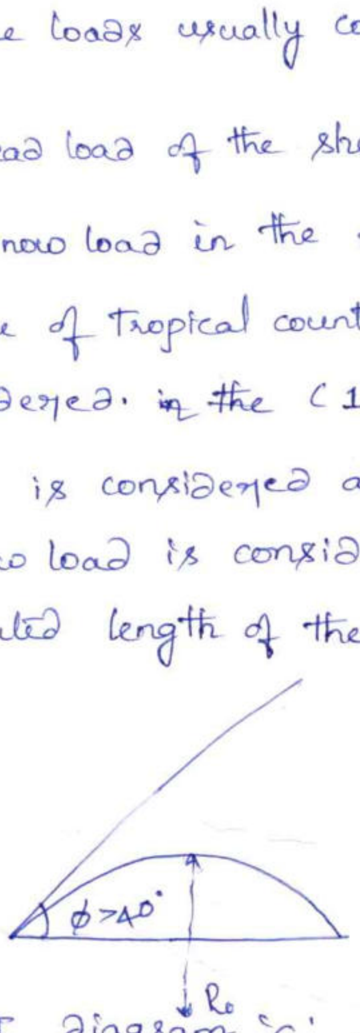
The loads usually considered in the design of cylindrical shell are

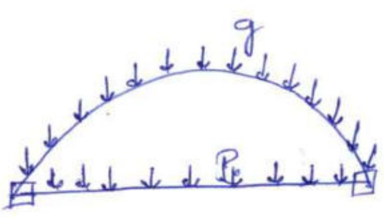
a) Dead load of the shell represed by (g)

b) Snow load in the regions subjeted to snow fall (P_0)

Incase of tropical countries instead of snow load, live load is considered. ~~in~~ the (10 - 15 PSF)

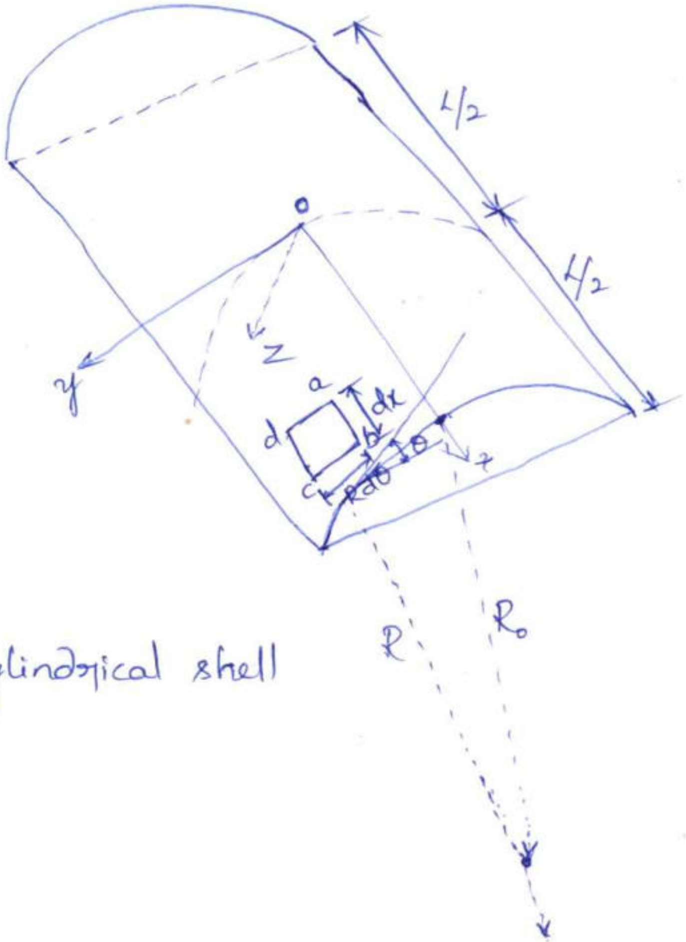
D.L is considered as U.D.L over the shell surface and Snow load is considered as UDL of the horizontally Projected length of the surface.





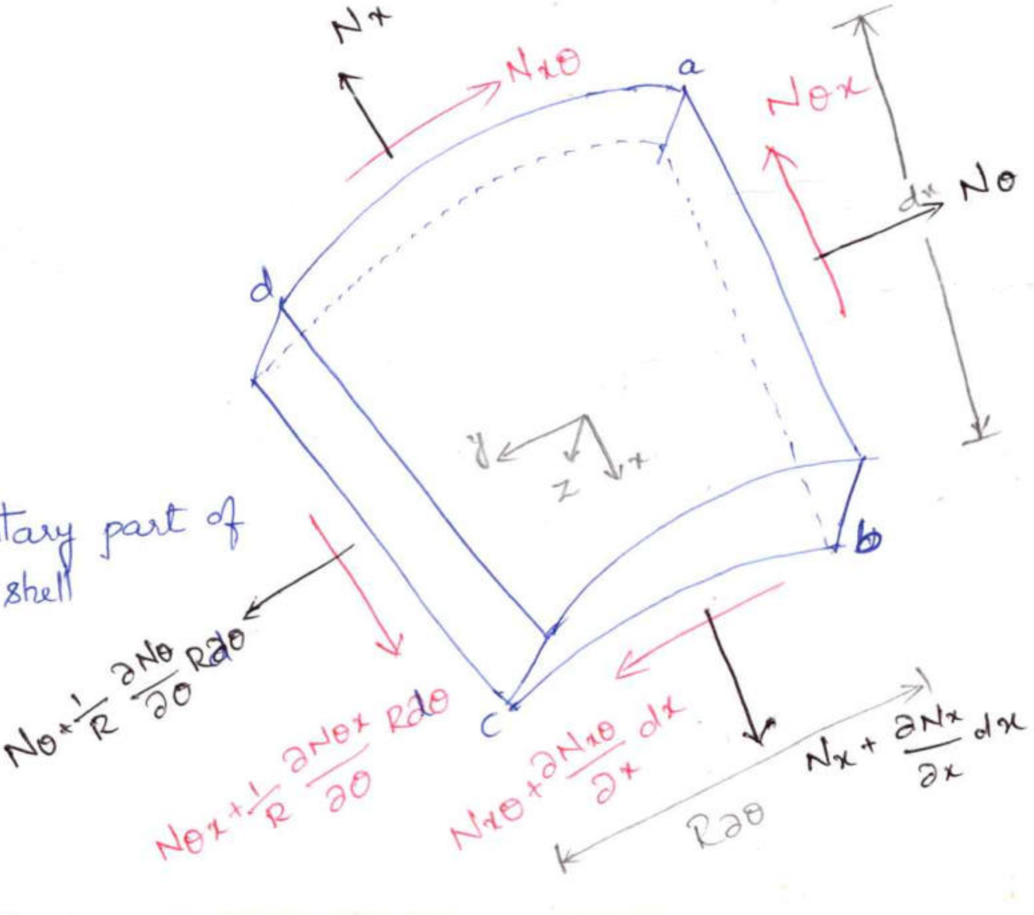
W.R.T diagram 'C'. The semicentral angle ϕ which is made by the Tangent at any point of the shell should not exceed 40°. other wise suction pressure's will develop.

Equations of Equilibrium;



1(a) A cylindrical shell

Angle = $\frac{\text{arc}}{\text{radius}}$



1(b) Elementary part of cylindrical shell

In the above fig 1(a) shows a cylindrical shell with central radius R_0 and orbitrary Radius 'R'. For this shell, take the x-axis along the crown generator and y-axis along the tangent at 'O', and z-axis along the inward normal direction.

Let us consider a small element abcd (hatched) as shown in fig 1.(a)

Fig1(b) shows the enlarged view of the element abcd. Here it is to be observed that the sides ad and bc are slightly curved and accordingly the size of the element is $\partial x \times R\partial\theta$.

let N_x, N_θ and N_r are the normal forces acting per unit length in x and y directions respectively.

$N_{x\theta}$ & $N_{\theta x}$ are the shear forces acting along x and y directions per unit length.

It is observed that $\partial y = R \times \partial\theta$ means R & θ depending on each other.

Summing up of forces in the x-direction and equating them to zero,

$$\left(\cancel{N_x} + \frac{\partial N_x}{\partial x} dx - \cancel{N_x}\right) \times R\partial\theta + \left(\cancel{N_{\theta x}} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial\theta} R\partial\theta - \cancel{N_{\theta x}}\right) dx + X R d\theta dx = 0$$

$$\frac{\partial N_x}{\partial x} dx R\partial\theta + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial\theta} R d\theta dx + X R dx d\theta = 0$$

In the above equation 'X' denotes the component of external load acting along x direction per unit area

Dividing the above equation by dx R dθ which is a common term

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial\theta} + X = 0 \longrightarrow (1)$$

Similarly summing up of all forces in y-direction and equating them to zero, we get

$$\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + Y = 0 \longrightarrow [2]$$

||ly summing up of all forces in z-direction or inward normal direction and equating them to zero,

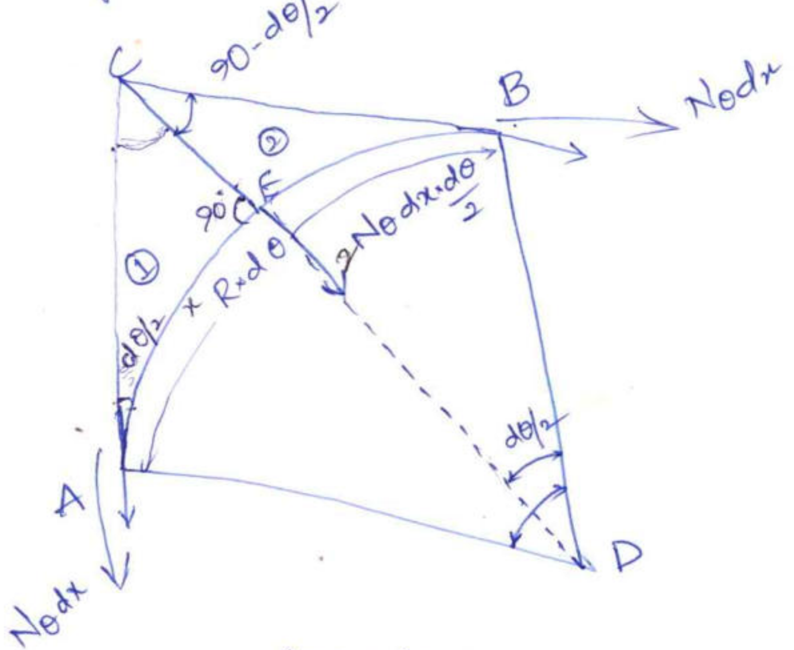


fig (1)(C)

observing the fig 1(b) on both sides of element ($\frac{1}{R}\frac{\partial N_\theta}{\partial \Theta}$ $R\cdot d\theta$) and protruding the element, we can get fig 1(c) which is c/s of given shell element in a particular direction WRT fig 1(c) draw the tangents at A & B to meet at C and CD is a angular ⊥r bisector, so that $\angle D = \theta$

$\angle ADC = d\theta/2 = \angle BDC$

||ly $\angle AEC = 90° = \angle BEC$

From ||le law of forces

$\angle CAE = \angle CBE = d\theta/2$

So that other angles $(90 - d\theta/2)$ in the Triangle.

The net forces acting along CA & CB are each $N_\theta dx$

From Δ① $\cos\left(90 - \frac{d\theta}{2}\right) = \frac{x}{N_\theta dx}$

$$x = N_\theta dx \cos\left(90 - \frac{\partial\theta}{2}\right)$$

$$= N_\theta dx \cancel{\cos} \sin\frac{d\theta}{2} \simeq N_\theta dx \frac{d\theta}{2}$$

Neglecting sin for small angles.

from Δ② in the IIr manner, the force along inward normal direction (3) can be proved to be same as Δ①

Adding the net forces acting along 3-direction due to

$N_\theta dx$ = Result due to Δ① + Resultant force due to Δ②

$$= 2 N_\theta dx \times \frac{d\theta}{2}$$

Forces along 3-direction is

$$2 N_\theta dx \times \frac{d\theta}{2} + Z(R\, d\theta\, dx) = 0$$

Dividing the above equation by common term $R \times dx \times d\theta$

we get

$$\frac{\cancel{2} N_\theta \cancel{dx} \times \cancel{d\theta} / \cancel{2}}{R \times \cancel{dx} \times \cancel{d\theta}} + \frac{Z(R \cancel{d\theta} \cancel{dx})}{R \cancel{d\theta} \cancel{dx}}$$

$$N_\theta + ZR = 0 \longrightarrow [3]$$

From the equation $\boxed{2}$

$$\frac{1}{R}\frac{\partial N_\theta}{\partial\theta} + \frac{\partial N_{x\theta}}{\partial x} + Y = 0$$

Isolating the $N_{x\theta}$ component

$$\frac{\partial N_{x\theta}}{\partial x} = -\left[\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + Y\right]$$

Integrating the above equation on both sides WRT 'x' we get

$$N_{x\theta} = -\left\{\int \frac{1}{R}\frac{\partial N_\theta}{\partial \theta}\,dx + \int y\,dx + f_1(\theta)\right\} \longrightarrow 4$$

$f_2(\theta)$, $f_1(\theta)$ is arbitrary function of 'θ' and serves as a constant.

Illy from equation1, Isolating the N_x component

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} + X$$

$$\frac{\partial N_x}{\partial x} = -\frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} + X$$

Integrating the above equation once WRT 'x' we can have

$$N_x = -\left\{\int \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta}\,dx + \int X\,dx + f_2(\theta)\right) \downarrow 5$$

In many cases of practical interests x, y & z are functions of θ only and they did not vary along x-axis with this assumption from eq ③, we have

$$N_\theta = -ZR$$

Generally Z and R are the functions of θ, N_θ should be a function of θ from equation ④,

$$N_{x\theta} = -\left(\frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + y\right) \cdot x + f_1(\theta)$$

$$N_{x\theta} = -Kx + f_1(\theta) \longrightarrow [6]$$

Now substituting the value of $N_{x\theta}$ (or) $N_{\theta x}$ from 6 in equation ⑤ with in the first term and simplifying the entire thing, we can get the following equation:

$$N_x = -\left(\frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} - X\right)_{x} + f_2(\theta)$$

$$= -\frac{1}{R} \cdot x^2$$

$$N_x = \left[\frac{x^2}{2R} \cdot \frac{dK}{d\theta} - \frac{1}{R}\frac{\partial F_1(\theta)}{\partial \theta} \times x - Xx + F_2(\theta)\right] \longrightarrow [7]$$

Stresses in a simply supported shell;

For a Simply supported shell over the Traverses following boundary conditions will exist

(a) $N_x = 0$ @ $x = \pm l/2$. This boundary condition follows from the assumptions that traverses will not recieve any loads applied norm to them.

(b) $N_{x\theta} = 0$ @ $x = 0$. This follows from the symmetry of shell. For most of the shells occuring in practice, normally $X = 0$. $f_1(\theta) = 0$

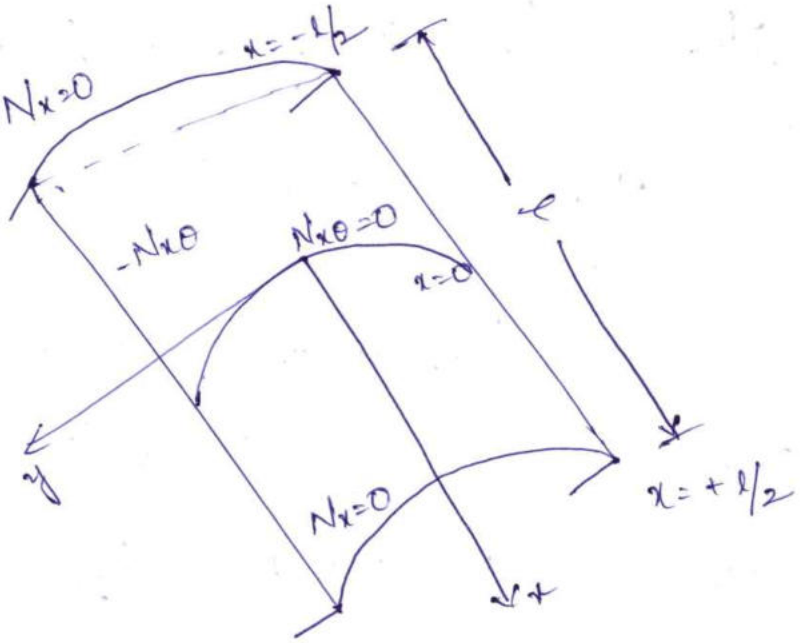
From 2nd boundary condition,

Inserting in eq 7

$$N_{x\theta} = -Kx + f_1(\theta)$$

$$\therefore f_{(1\theta)} = 0$$

$$N_{x\theta} = -Kx$$



Inserting the 1st boundary condition i.e., ⓐ in the equation ⑦ after noting that $f_1(\theta) = 0$ i.e., $X = 0$

$$N_x = \frac{x^2}{2R} \times \frac{dk}{\partial\theta} - \frac{1}{R}\frac{\partial f_1(\theta)}{\partial\theta} \times x - Xx + f_2(\theta)$$

$N_x = 0$; $f_1(\theta) = 0$ then $f_2(\theta)$ at $x = l/2$

$$f_2(\theta) = \frac{l^2 dk}{8R\,d\theta}$$

Substituting values of $f_1(\theta)$, $f_2(\theta)$ as calculated above

~~in 3, 6 and~~ 7 equations

~~$N_\theta + ZR = 0$~~

~~$N_{x\theta} = -kx + f_1(\theta)$~~

To obtain N_x values substituting $f_1(\theta) = 0$ $f_2(\theta) = \frac{l^2 dk}{8R d\theta}$

Then $X = 0$ at x

$$N_x = \frac{x^2}{2R} \times \frac{dk}{d\theta} - f_2(\theta) \qquad \text{at } x = l/2$$

$$= \frac{x^2}{2R} \times \frac{dk}{d\theta} - \frac{l^2 dk}{8R d\theta}$$

$$N_x = -\frac{1}{2}\left[\frac{l^2}{4} - x^2\right] \frac{1}{R}\frac{dK}{d\theta}$$

$$\therefore \; N_\theta = -ZR \longrightarrow [8\text{-}a]$$

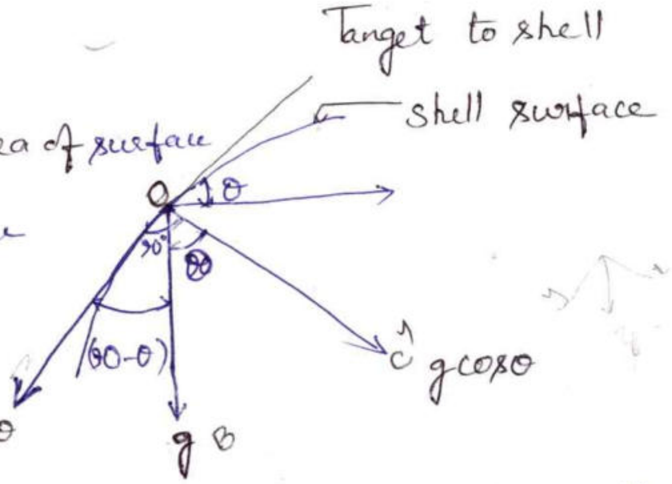
$$N_{x\theta} = -Kx \longrightarrow [8\text{-}b]$$

$$N_x = -\frac{1}{2}\left[\frac{l^2}{4} - x^2\right] \frac{1}{R}\frac{dK}{d\theta} \longrightarrow [8\text{-}c]$$

Value of K for D.L ;

Value of dead load = g per unit area of surface

θ = Angle with horizontal surface



As shown in above fig consider a unit of shell such that at any point 'O', the Tangent is drawn. let this Tangent makes an angle θ with horizontal. g be the D.L per unit area and let it act as shown in fig, Perpendicular to tangential direction,

From the geometry of we seen that ∠BOC = θ, So that its components of 'g' along two perpendicular directions y & z are g sinθ & g cosθ. Hence, due to D.L components of External load

$$\left.\begin{aligned} X &= 0 \\ Y &= g\sin\theta \\ Z &= g\cos\theta \end{aligned}\right\} \quad (8\text{-}d)$$

From 8 Equation

$$N_\theta = -ZR$$

$$= -gR\cos\theta$$

$N_{x\theta} = -Kx$

From Equation 6 $\quad K = \frac{1}{R}\frac{\partial N_\theta}{\partial \theta} + y$

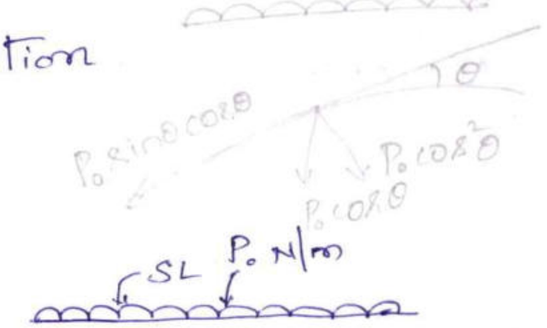
$$K = \frac{1}{R} \times \frac{\partial \cancel{N\theta}}{\partial \theta} * (-gR\cos\theta) + g\sin\theta$$

$$K = -\frac{1}{R}\frac{\partial R}{\partial \theta} g\cos\theta + g\sin\theta$$

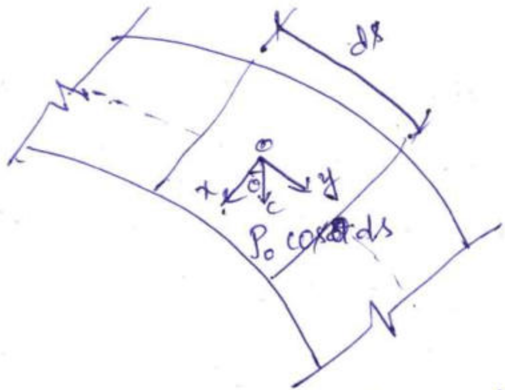
~~$N_{x\theta} = \left[\frac{1}{R}\frac{\partial R}{\partial \theta} g\cos\theta + g\sin\theta\right] *$~~

$$K = gg\sin\theta - \frac{1}{R}\frac{\partial R}{\partial \theta} g\cos\theta$$

Value of K for D.L. Condition



* K for Snow load condition;



let external load = P N/m

Small curvilinear distance along shell = ds

Small horizontally projected distance = ds cos θ

External load for distance ds. cos θ = $P_o.(ds\cos\theta)$

Total external load per distance of ds cos θ = $P_o\cos\theta$

The direction of action of $P_o\cos\theta$ is along 'oc'. Now assuming that the direction of $P_o\cos\theta$ makes an angle 'θ' with 'x', (90 - θ) with y-axis, its components along y, z axis are

$P_o \sin\theta * \cos\theta$

$P_o * \cos\theta * \cos\theta = P\cos^2\theta$

Hence for snow load conditions the external loads are

$$X = 0 \longrightarrow [11\text{-}a]$$

$$Y = P_o \sin\theta \cos\theta \rightarrow [11\text{-}b]$$

$$Z = P_o \cos^2\theta \rightarrow [11\text{-}c]$$

From equation 8 $N_\theta = -ZR$ but $Z = P_o \cos^2\theta$

$$\therefore N_\theta = -P_o R \cos^2\theta$$

From equation 6 $K = \frac{1}{R} \times \frac{\partial N_\theta}{\partial \theta} + Y$

$$= \Bigg[\frac{1}{R} \times \frac{\partial}{\partial \theta} \Big[-P_o R \cos^2\theta\Big) + P_o \sin\theta \cos\theta\Bigg]$$

$$= \left[\frac{-P_o}{R} \times \frac{\partial}{\partial \theta}(R \cdot \cos^2\theta)\right] + P_o \sin\theta \times \cos\theta$$

∴ After simplification

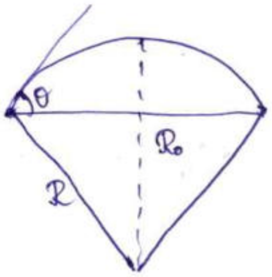
$$K = \left[(3 P_o \sin\theta \cos\theta) - \left(P_o \cos^2\theta \frac{1}{R} \times \frac{\partial R}{\partial \theta}\right)\right] \longrightarrow [12]$$

Expressions for stress resultants under DL and S.L for various direction's; (circular, parabolic, catenary and cycloidal Directrices

Dead weight ;

Let $R = R_o \cos^n\theta \longrightarrow$ [13] be the equation for any given direction

From Equation (8) $N_\theta = -ZR$



From Eq 9-C

$Z = g\cos\theta$

$N\theta = -g\cos\theta \cdot R = -gR\cos\theta$

Substituting in 13 Eq

$$N_\theta = -g R_o \cos^{n+1}\theta \longleftrightarrow [14]-a$$

$g\cos\theta \cdot R_o\cos^n\theta$

$g \cdot R_o \cos^{n+1}\theta$

we know that from Eq (6

$$K = \frac{1}{R} \cdot \frac{\partial N_\theta}{\partial \theta} + Y$$

Now substituting value of N_θ from 14 and y from 9-b

$$y = g\sin\theta$$

$$K = \frac{1}{R}\frac{d}{d\theta}\left[-gR_0\cos^{n+1}\theta\right] + g\sin\theta$$

$$k = \left[\frac{-gR_0}{R}\times\frac{d}{d\theta}\cos^{n+1}\theta\right] + g\sin\theta$$

Differentiating WRT n

$$K = \frac{-gR_0}{R}(n+1)\cos^n\theta \overset{\sin\theta}{\times} + g\sin\theta$$

$$= \frac{g(n+1)R_0\cos^n\theta\sin\theta}{R} + g\sin\theta$$

$$= \frac{g(n+1)R\sin\theta}{R} + g\sin\theta = g(n+1)\sin\theta + g\sin\theta$$

$$K = (n+2)g\sin\theta$$

Substituting these values of K in equations 8-a, b, c

$N_\theta = -ZR \longrightarrow$ [14-a]

$N_{x\theta} = -Kx$

$= -(n+2)g\sin\theta\times x \longrightarrow$ [14-b]

$$N_x = -\frac{1}{2}\left[\frac{l^2}{4}-x^2\right]\frac{1}{R}\frac{dK}{d\theta}$$

$= \frac{(n+2)}{2}g\left(\frac{l^2}{4}-x^2\right)\frac{1}{R}\cos^{n-1}\theta \longrightarrow$ [14-c]

Stress resultant for snow load condition;

we have $R = R_0\cos^n\theta$

$N_\theta = -ZR$

For Snow load condition from 11-c

$$Z = P_0 \cos^2\theta$$

$$N_\theta = -P_0 R \cos^2\theta$$

Substituting $R = R_0 \cos^n\theta$

$$N_\theta = -P_0 R_0 \cos^n\theta \cos^2\theta$$

$$= -P_0 R_0 \cos^{n+2}\theta$$

From 6 Eq

$$K = \frac{1}{R}\frac{\partial N_\theta}{\partial\theta} + Y$$

Substituting N_θ and $y = P_0 \sin\theta \cos\theta$ in above equation

$$K = \frac{1}{R}\frac{\partial}{\partial\theta}\left[-P_0 R_0 \cos^{n+2}\theta\right] + P_0 \sin\theta \cos\theta$$

$$= \frac{1}{R} P_0 R_0\, n+2 \cos^{n+1}\theta \sin\theta + P_0 \sin\theta \cos\theta$$

$$= (n+3) P_0 \sin\theta \cos\theta \quad R_0 \cos^n\theta = R$$

Substituting K value in 8-a, b, c

$$N_\theta = -ZR \longrightarrow 15\text{-}a = -P_0 R_0 \cos^{n+2}\theta$$

$$N_{x\theta} = -Kx = -(n+3) P_0 \sin\theta \cos\theta\, x \longrightarrow 15b$$

$$N_x = -\frac{1}{2}\left[\frac{l^2}{4} - x^2\right]\frac{1}{R}\frac{dK}{d\theta}$$

$$= (n+3) P_0 \left[\frac{l^2}{4} - x^2\right] \frac{\cos^2\theta - \sin^2\theta}{\cos^n\theta} \longrightarrow 15\text{-}c$$

$$= -\frac{1}{2}\left[\frac{l^2}{4} - x^2\right]\frac{1}{R}\frac{d}{d\theta}\left[(n+3) P_0 \cos\theta \sin\theta\, x\right.$$

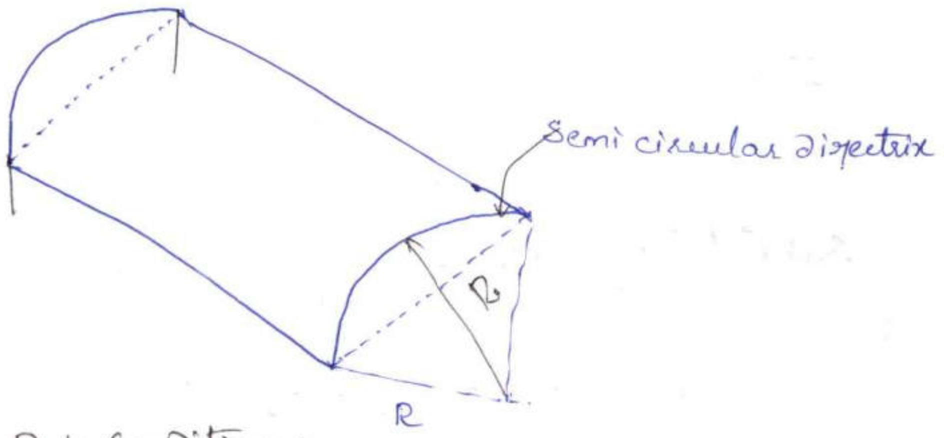
$$= \frac{\frac{n+3}{\cancel{2}} P_0 \left(\frac{l^2}{4} - x^2\right)}{2R_0} \times \frac{\cos^2\theta - \sin^2\theta}{R}$$

$$= \frac{-(n+3)}{2R_0} P_0 \left(\frac{l^2}{4}-x^2\right)\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^n\theta}\right) \longrightarrow 15\text{-c}$$

Cylindrical shell with a circular directrix;

For a cylindrical shell with a circular directrix

(say) $R = R_0 = a$ and also $n = 0$



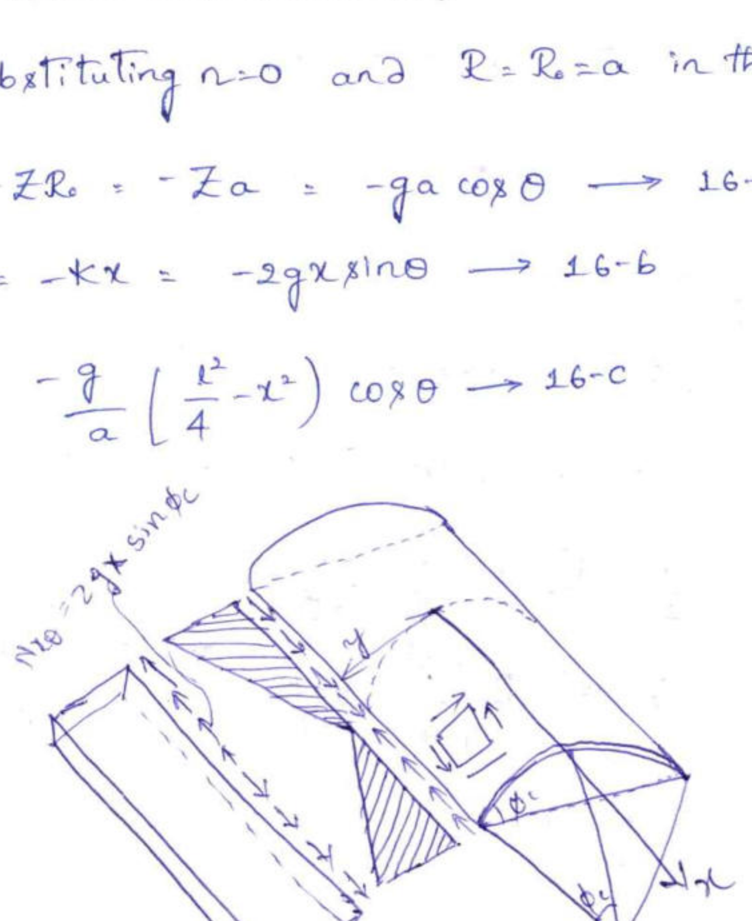
Stresses under D.L. Condition;

Substituting $n=0$ and $R = R_0 = a$ in the set of eq 14

$$N_\theta = -ZR_0 = -Za = -ga\cos\theta \longrightarrow 16\text{-a}$$

$$N_{x\theta} = -kx = -2gx\sin\theta \longrightarrow 16\text{-b}$$

$$N_x = -\frac{g}{a}\left(\frac{l^2}{4}-x^2\right)\cos\theta \longrightarrow 16\text{-c}$$



Free body diagram of an edge beam with shear forces transferred by shell

At the centre $x=0$ because of symmetry of the problem, let shear stress at that point be zero

At $x = \pm l/2$ making use of formula, we get

$$N_{x\theta} = -2qx\sin\theta \quad \text{at } x = -l/2$$

$$= -\cancel{2}q \times l/\cancel{2} \times \sin\theta = ql\sin\theta$$ at the edges as shown in fig.

At any point 'a', at a distance 'x' from the y-axis, let the shear force transferred from the shell to the edge beam, considering the shell to the edge beam, considering only one half can be calculated as follows

$$P = \int_{x=0}^{l/2} (N_{x\theta})\, dx \qquad \text{where } N_{x\theta} = -2qx\sin\theta$$

we have $$P = \int_{x=0}^{l/2} (-2qx\sin\theta)\, dx$$

$$P = q\left(\frac{l^2}{4} - x^2\right)\sin\theta$$

Stresses under snow load condition;

Substituting $R = R_0 = a$ and $n=0$ in the set of equations 15

$$N_\theta = -P_0\, a\cos^2\theta \longrightarrow 17\text{-a}$$

$$N_{x\theta} = -1.5\, P_0\, x\sin 2\theta \longrightarrow 17\text{-b}$$

$$N_x = -1.5\frac{P_0}{a}\left(\frac{l^2}{4} - x^2\right)\cos 2\theta \longrightarrow 17\text{-c}$$

Cylindrical shell with catenary directrix;

Here $n = -2$. $R = R_0 = 0$

stresses under dead load;

Substituting $n = -2$ in the set of eq 14 we can get

$$N_{\cancel{x}\theta} = 0 - gR_0\cos^{n+1}\theta - gR_0\cos^{-1}\theta = -\frac{gR_0}{\cos\theta}$$

$$N_{x\theta} = 0 = -(n+2)\, g\sin\theta \times x = 0$$

$$N_x = \frac{-g\cancel{R_0}}{\cancel{\cos}\theta} = \left(\frac{n+2}{2}\right) g\left(\frac{l^2}{4} - x^2\right)\frac{1}{R\cos^{n+1}\theta} = 0$$

observing the above results, it leads to an important conclusion, that a catenary shell would degenerate in to number of independent arches and transfer the entire load on their edge beams and no loads to traverses. Such a shell is usually termed as 'funicular curve of applied loading

stresses under snow load condition;

Substituting $n = -2$ in set of equations 15

$$N_\theta = -P_0R_0\cos^{n+2}\theta = -P_0R_0 \longrightarrow 1$$

$$N_{x\theta} = -(n+3)\,P_0\sin\theta\cos\theta\, x = -P_0\, x\sin\theta\cos\theta$$

$$N_x = (n+3)\frac{P_0}{4}\left(\frac{l^2}{4} - x^2\right)\frac{\cos^2\theta - \sin^2\theta}{\cos^n\theta} =$$

$$= 0.50\,\frac{P_0}{R_0}\left[\frac{l^2}{4} - x^2\right]\cos 2\theta\cos^2\theta$$

$$N_\theta = -P_0R_0$$

$$N_{x\theta} = -P_0\, x\sin\theta\cos\theta$$

$$N_x = 0.5\frac{P_0}{R_0}\left[\frac{l^2}{4} - x^2\right]\cos 2\theta\cos^2\theta$$

Cylindrical shell with parabolic directrix;

Here $n = -3$

stresses under D.L;

Substituting $n = -3$ in the set of equations 14, we get

$$N_\theta = \frac{-qR_o}{\cos^2\theta}$$

$$N_{x\theta} = qx\sin\theta$$

$$N_x = 0.5\frac{q}{R_o}\left[\frac{l^2}{4} - x^2\right]\cos^4\theta$$

stresses under Snow load;

Substituting $n = -3$ in the set of eq- 15, we get

$$N_\theta = \frac{-P_o R_o}{\cos\theta}$$

$$N_{x\theta} = 0$$

$$N_x = 0$$

Simultaneously in the present case, the shell behaves in the same manner as for the catenary directrix under D.L. co.

Cylindrical shell with cycloidal directrix;

Here $n = +1$

Stresses under D.L;

Substituting $n = +1$ in the set of equations- 14

$$N_\theta = -qR_o\cos^{n+1}\theta = -qR_o\cos^2\theta$$

$$N_{x\theta} = -(3n+2)q\sin\theta\, x = -3q\sin\theta\, x$$

$$N_x = 1.5q\left[\frac{l^2}{4} - x^2\right]\frac{1}{R}$$

Stresses under the Snow load;

$$N_\theta = -P_0 R_0 \cos^{n+2}\theta = -P_0 R_0 \cos^3\theta$$

$$N_{x\theta} = -(n+3) P_0 \sin\theta \cos\theta \, x = -4 P_0 \sin\theta \cos\theta \, x$$

$$N_x = (n+3) \frac{P_0}{4}\left(\frac{l^2}{4}-x^2\right) \frac{\cos^2\theta-\sin^2\theta}{\cos^n\theta}$$

$$= \cancel{4}\frac{P_0}{\cancel{4}}\left[\frac{l^2}{4}-x^2\right] \frac{\cos^2\theta-\sin^2\theta}{\cos\theta}$$

$$N_x = P_0\left[\frac{l^2}{4}-x^2\right] \frac{\cos^2\theta-\sin^2\theta}{\cos\theta}$$

Principle's of Membrane theory and Bending theory

In this case shell is assumed like a perfect ~~bendin~~ membrane in which only direct stresses are considered and Bending moments This is an ideal case because any shell will be if any are neglected. Subjected to some kind of moment's and here they are neglected.

A/c to membrane theory any thin shell acts partially as an arch and partially as a beam. The arch action is responsible for the Transfer of loads to the edge beam's - and the beam action is responsible for the Transfer of loads to the Traverse's

Abs 23/6
1
3
11
12
13

Abs 24/6
3
8
10
12
13

Abs 25/6
7
11
13

Abs 26/6
1
9
11
13

Abs 29/6 Present
~~1~~
~~2~~
~~3~~
5
6
13
14

30/6 Abs
3
7
8
10
11
12
15

11/3/19 Ab
1
5
7

12/3/19 Ab
2
5

14/3/19 Ab
1

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CIVIL ENGINEERING

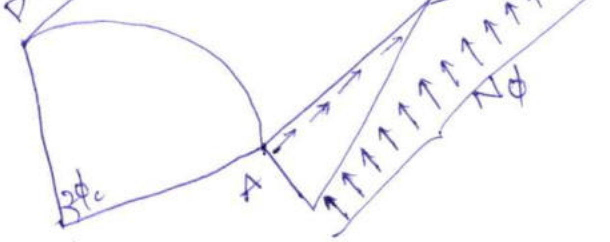
Analysis of Shells and Folded Plates

UNIT-2

UNIT - II

CYLINDRICAL SHELLS - BENDING THEORY

Most of the RCC cylindrical shells used in practice do not behave like membrane's. Along the edge of the shell, stresse's and displacements, different from those given by membrane theory usually exist. This depend upon the manner in which the shell is supported or in other words, type of physical boundary conditions that exist along the supporting edges. Consider, shell with four edges, the membrane theory would indicate the presence of stresses N_ϕ and $N_{x\phi}$. But it is evident, from boundary conditions these stresses cannot exist at edges being free. The actual boundary conditions can be realized by applying corrective line loads. But the application of such line loads would cause the shell to bend and depart from its membrane state. The shell now seeks a new equilibrium and in that process brings in to play bending moments, twisting moments and radial shears. A bending theory is essential to account for this

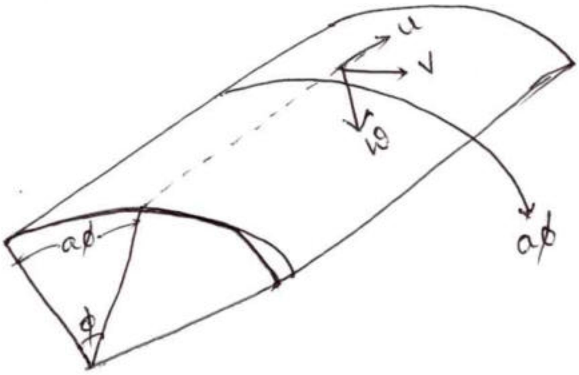


Expression for strains and change in curvature to applied loads

when a $\square^{le}$ body is subjected to plane state stress. The strain components developed under these state of stress.

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \quad 18\ a, b, c$$

Stresses in acircular cylindrical shells;



Consider a circular cylindrical shell of radius 'a' as shown in fig. If φ is semicentral angle AB = aφ Here the x axis is choosen as axis passing through one of the edges and y-direction along the tangential direction at origin 'o'. Z-direction is along inward normal direction. Here y-axis is along tangential direction and usually it is represented as y = aφ direction. If U, V, W are components of displacement along x, y and z directions due to external load applied, the expression for strain is as follows.

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right) \end{aligned}\right\} \quad 19(a,b)$$

Strain in y = aφ direction will consists of two parts as follows

$$\frac{\partial v}{\partial y} = \frac{1}{a}\frac{\partial v}{\partial \phi}\, a = -w/a$$

a) strain corresponding to plane state of stress

(b) circumferential strain caused by w.

To calculate the circumferential strain assume that the radius shinks from "a to a-w". Hence circumferential strain is equal to

$$\text{Circumferential strain} = \frac{\text{change in length}}{\text{original length}} = \frac{(a-w)d\phi - ad\phi}{ad\phi}$$

$$= -w/a$$

Here the strain along Y-direction is equal to a+b

$$\varepsilon_\phi \text{ or } \varepsilon_{y\phi} = \left(\frac{1}{a}\frac{\partial v}{\partial \phi} - \frac{w}{a}\right) \longrightarrow 19(c)$$

Stress Resultants.

When a cylindrical shell is loaded and subjected to bending also various stress resultants per unit length are as follows

N_x, N_ϕ → direct

$N_{x\phi}$ $N_{\phi x}$ → shear

M_x M_ϕ → Moment

$M_{x\phi}$ $M_{\phi x}$ → Twisting moments

Q_x Q_ϕ → Radial moments

DKJ Theory for the Bending analysis of shells; Exact theory

[DKJ - Dormon korman Jenking]

Assumptions;

-> Material is homogenous, isotropic obeys hooke's law.

-> An element normal to the middle surface of the shell remains normal even after deformation.

-> All displacements of the shell surface are assumed to be small.

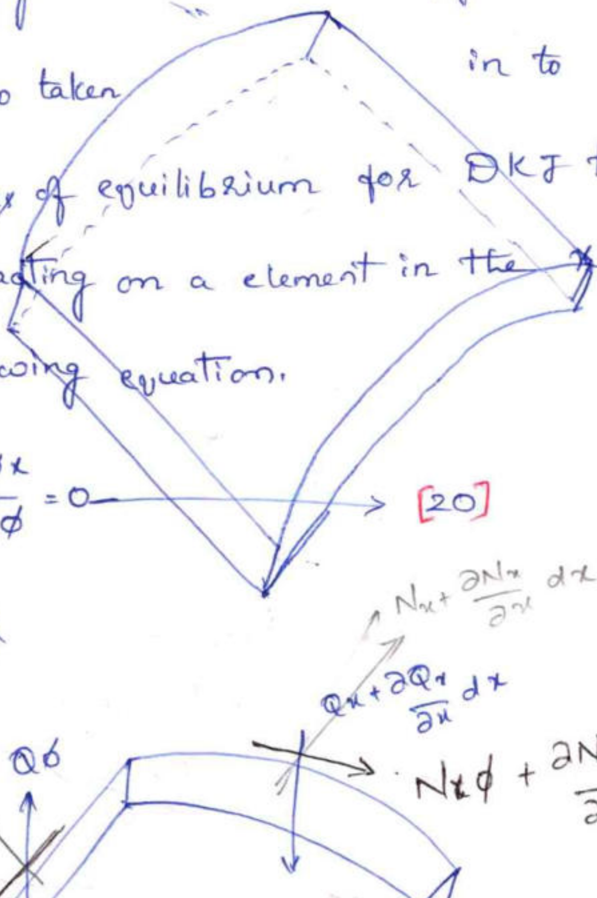
-> The Quantities M_x, Q_x & $M_{x\phi}$ are neglected considered in the analysis.

~~Equation's of Equilibrium;~~ Shell as a combination of disc, plate and membrane;

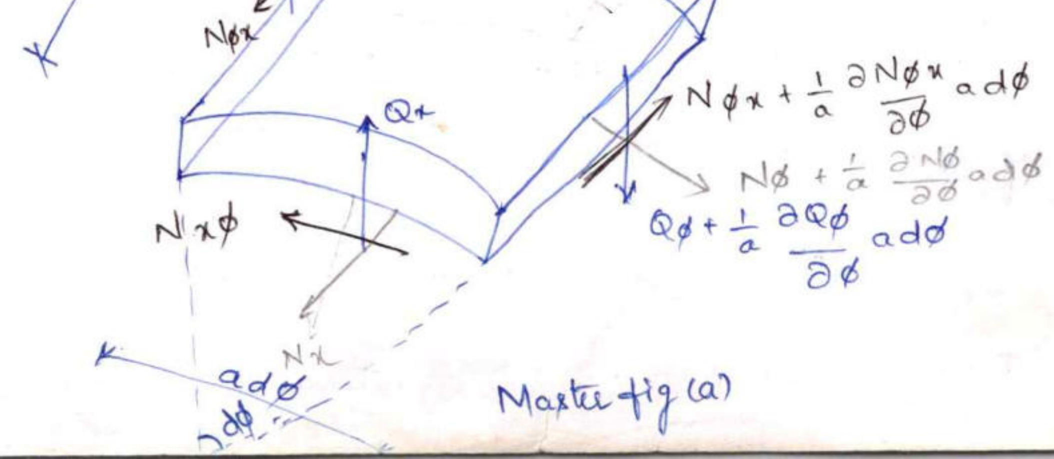
Here the structural action of a cylindrical shell under ben ding can be approximated by combining the structural action of a disk formed by a developed shell loaded in its own plane, of a plate, formed by the developed shull loaded at right angles to the plane, and of the shell regarded, as a flexible membrane. There these actions are described as 'disc action', 'plate action' and membrane action'. Omitting in terms not figuring in equations for the disc, the plate or the membrane offers as eligant approach for the derivation of DKJ Equation.

Equations of Equilibrium;

Refering to the master fig's and noting that M_x, Q_x & $M_{x\phi}$ are also taken in to consideration to get the following equations of equilibrium for DKJ theory. Equating all forces acting on a element in the x direction be zero to get the following equation.

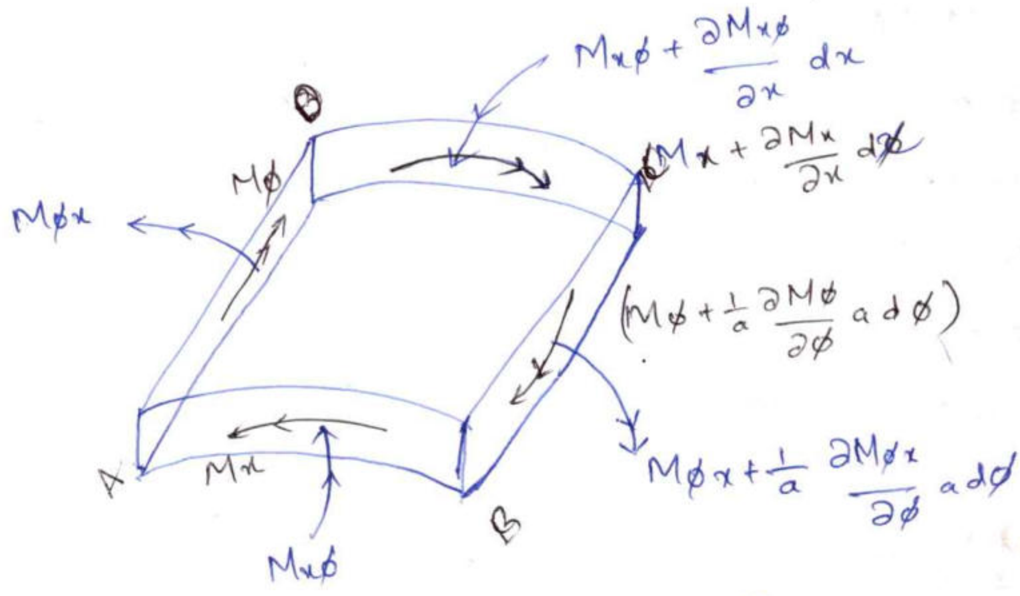


$$\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} = 0 \longrightarrow [20]$$



Master fig (a)

$\partial N\phi dx d\phi \frac{}{2}$



Equating sum of all forces in the ϕ- direction

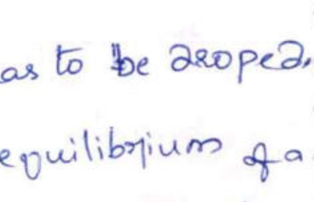
$$\frac{\partial N\phi}{\partial \phi} - Q_\phi + \frac{a\partial N_{x\phi}}{\partial x} = 0 \longrightarrow 21$$

Now the Term Q_ϕ in the above equation has to be droped, as it does not occur in the corresponding equations of equilibrium of a disc or a plate or a membrane in the above equation can be written as

$$\frac{\partial N\phi}{\partial \phi} + a\frac{\partial N_{x\phi}}{\partial x} = 0 \longrightarrow 22$$

The third equation of equilibrium derived by equating all forces acting on the element in a normal direction (z-direction) zero. This equation takes the following form

$$a\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi = 0 \longrightarrow 23$$



Equating sum of all moments on the element above the generator AD to zero

$$\frac{a\partial M_{x\phi}}{\partial x} + \frac{\partial M_\phi}{\partial \phi} - aQ_\phi = 0 \qquad 24$$

Now taking moments WRT AB and equating them to zero to get the following equation

$$\frac{\partial M_{\phi x}}{\partial \phi} + a\frac{\partial M_x}{\partial x} - aQ_x = 0 \qquad 24(a)$$

Stress strain Relations ;

$$\frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{N_x}{Ed} - \frac{\nu N_\phi}{Ed} \qquad 25(a)$$

$$\varepsilon_\phi = \frac{1}{a}\left[\frac{\partial v}{\partial \phi} - w\right] = \frac{N_\phi}{Ed} - \nu\frac{N_x}{Ed} \qquad 25(b) \quad G = \frac{E}{2(1+\nu)}$$

$$\tau_{x\phi} = \frac{1}{a}\left[\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right] = \frac{N_{x\phi}}{Gd} = \frac{2(1+\nu)N_{x\phi}}{Ed} \qquad 25(c)$$

Moment curvature relation ;

As per DKT Theory following moment curvature relation's are neglecte

$$M_x = -D\frac{\partial^2 w}{\partial x^2} \longrightarrow 26a$$

$$M_\phi = -\frac{D}{a^2}\frac{\partial^2 w}{\partial \phi^2} \longrightarrow 26b$$

$$M_{x\phi} = \frac{-D}{a}\frac{\partial^2 w}{\partial x\,\partial \phi} \longrightarrow 26c$$

let u v w be the components of displacement x, y & z directions of the shell. Refering to stress-strain relations for poisson's ratio to be zero, we get

$$N_x = Ed\frac{\partial u}{\partial x}$$

$$N_\phi = \frac{Ed}{a}\left[\frac{\partial v}{\partial \phi} - w\right]$$

$$N_{x\phi} = \frac{Ed}{2}\left[\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right]$$

Now substituting the above values in the equation's 20 and 22 to get the following equation

$$\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} = 0$$

$$\frac{\partial}{\partial x} Ed\left[\frac{\partial u}{\partial x}\right] + \frac{1}{a}\frac{\partial}{\partial \phi}\frac{Ed}{2}\left[\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right] = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left[\frac{1}{a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{a}\frac{\partial^2 v}{\partial x\,\partial \phi}\right] = 0 \rightarrow \times a^2$$

$$a^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial \phi^2} + a\frac{\partial^2 v}{\partial x\,\partial \phi}\right] = 0 \longrightarrow 27 \checkmark$$

$$\frac{\partial N_\phi}{\partial \phi} + a\frac{\partial N_{x\phi}}{\partial x} = 0$$ ~~substit~~

$$\frac{\partial}{\partial \phi}\frac{Ed}{a}\left(\frac{\partial v}{\partial \phi} - w\right] + a\frac{\partial}{\partial x}\left[\frac{Ed}{2}\left(\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right)\right]$$

$$\frac{1}{a}\left[\frac{\partial^2 v}{\partial \phi^2} - \frac{\partial w}{\partial \phi}\right] + a\left[\frac{1}{2a}\frac{\partial^2 u}{\partial \phi \partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\right] = 0$$

$$\frac{1}{a}\frac{\partial^2 v}{\partial \phi^2} - \frac{1}{a}\frac{\partial w}{\partial \phi} + \frac{1}{2}\frac{\partial^2 u}{\partial \phi}\partial x + \frac{a}{2}\frac{\partial^2 v}{\partial x^2} = 0 \longleftrightarrow \times a$$

$$\left(\frac{\partial^2 v}{\partial \phi^2} - \frac{\partial w}{\partial \phi}\right) + \frac{1}{2}\left[a\frac{\partial^2 u}{\partial \phi\,\partial x} + a^2\frac{\partial^2 v}{\partial x^2}\right] = 0 \longrightarrow 28$$

From the equation's 24 and 24(a) isolating the values of Q_x and Q_ϕ we get

$$\cancel{Q}\quad a\frac{\partial M_{x\phi}}{\partial x} + \frac{\partial M_\phi}{\partial \phi} + aQ_\phi = 0$$

$$\frac{\partial M_{\phi x}}{\partial \phi} + a\frac{\partial M_x}{\partial x} - aQ_x = 0$$

Isolating Q_x and Q_ϕ we get

$$Q_x = \frac{1}{a}\left[\frac{\partial M_{\phi x}}{\partial \phi} + a\frac{\partial M_x}{\partial x}\right]$$

$$Q_\phi = \frac{1}{a}\left[a\frac{\partial M_{x\phi}}{\partial x} + \frac{\partial M_\phi}{\partial \phi}\right]$$

Now substituting the moment curvature relations M_x M_ϕ $M_{x\phi}$ from eq ㉕ in the above equation Q_x Q_ϕ takes the following form

$$Q_x = \frac{-\Theta}{a}\left[\frac{1}{a}\frac{\partial^3 w}{\partial x \partial \phi^2} + a\frac{\partial^3 w}{\partial x^3}\right]$$

$$Q_\phi = \frac{-D}{a}\left[\frac{\partial^3 w}{\partial x^2 \partial \phi} + \frac{1}{a^2}\frac{\partial^3 w}{\partial \phi^3}\right]$$

Now substituting these values of Q_x and Q_ϕ in eq 23 and after simplification we get the following equation.

$$a\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi = 0$$

$$N_\phi = -\left[a\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi}\right]$$

$$N_\phi = -\left[a\frac{\partial}{\partial x}\left[\frac{-\Theta}{a}\left[\frac{1}{a}\frac{\partial^3 w}{\partial x \partial \phi^2} + a\frac{\partial^3 w}{\partial x^3}\right] + \frac{\partial}{\partial \phi}\left[\frac{-D}{a}\left(\frac{\partial^3 w}{\partial x^2 \partial \phi} + \frac{1}{a^2}\frac{\partial^3 w}{\partial \phi^3}\right)\right]\right]\right.$$

$$N_\phi = +\frac{D}{a^3}\left[a^4\frac{\partial^4 w}{\partial x^4} + 2a^2\frac{\partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{\partial \phi^4}\right] \longrightarrow 29$$

From equation for poisson's ratio $\nu = 0$

$$N_\phi = \frac{Ed}{a}\left[\frac{\partial v}{\partial \phi} - w\right] \longrightarrow 29(a)$$

Now substituting above values N_ϕ in above

$$D = \frac{Ed^3}{12(1+\nu)} \quad \text{for } \nu = 0 \quad D = \frac{Ed^3}{12} \qquad \left(\frac{\dot{v}-w}{w}\right)\frac{12a^2}{d^2}$$

Eq $\overset{29}{\cancel{30}}$ takes the following form after simplification

$$\left[\frac{\partial v}{\partial \phi} - w\right] + \frac{d^3}{12a^2}\left[\frac{a^4\partial^4 w}{\partial x^4} + 2a^2\frac{\partial^4 w}{\partial x^2 \partial \phi^2} + \frac{\partial^4 w}{\partial \phi^4}\right] = 0 \;\downarrow a$$

(or) 30

$$\left[\frac{\partial v}{\partial \phi} - w\right] + \frac{d^2}{12a^2}\left[\frac{a^2\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2}\right]^2 w = 0 \rightarrow b$$

Now the equation's 27, 28 and 30 are usually termed as fluggies equation.

DKJ Equation;

Here a single differential equation of 8th order interms of displacements w. Eliminating u and v is formulated from the above set of equations Differentiating eq 28 twice with respect to x will get the following equation

$$\left(\frac{\partial^2 v}{\partial\phi^2}-\frac{\partial w}{\partial\phi}\right)+\frac{1}{2}\left[a\frac{\partial^2 u}{\partial x\,\partial\phi}+a^2\frac{\partial^2 v}{\partial x^2}\right]=0$$

$$\frac{\partial^4 v}{\partial\phi^2\partial x^2}-\frac{\partial^3 w}{\partial x^2\partial\phi}+\frac{1}{2}\left[a\frac{\partial^4 u}{\partial x^3\partial\phi}+a^2\frac{\partial^4 v}{\partial x^4}\right]=0$$

$$v^{||\cdot\cdot}-w^{||\cdot}+\frac{1}{2}\left[a\,u^{|||\cdot}+a^2 v^{||||}\right]=0\longrightarrow 31$$

where | represents $\frac{\partial}{\partial x}$ & $\cdot$ represents $\frac{\partial}{\partial\phi}$.

Now apply the operator $a\times\frac{\partial^2}{\partial x\,\partial\phi}$ and the eq 27 above to get the following equation

$$a^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\left[\frac{\partial^2 u}{\partial\phi^2}+a\frac{\partial^2 v}{\partial x\,\partial\phi}\right]=0\quad\times\ a\frac{\partial^2}{\partial x\,\partial\phi}$$

$$\left[a^3\frac{\partial^4 u}{\partial x^3\partial\phi}+\frac{1}{2}\left[a\frac{\partial^4 u}{\partial\phi^3\partial x}+a^2\frac{\partial^4 v}{\partial x^2\partial\phi^2}\right]=0\right.$$

$$a^3 u^{|||\cdot}=-\tfrac{1}{2}\left[a\,U^{\overset{\cdot\cdot\cdot|}{\cancel{|||\cdot}}}+a^2 v^{||\cdot\cdot}\right]\qquad 32$$

Now apply operation $\partial^2/\partial\phi^2$ on 28 again we get

$$\frac{\partial^2 v}{\partial\phi^2}-\frac{\partial w}{\partial\phi}+\frac{1}{2}\left[a\frac{\partial^2 u}{\partial x\,\partial\phi}+a^2\frac{\partial^2 v}{\partial x^2}\right]=0\longrightarrow\times\,\partial^2/\partial\phi^2$$

$$\left(v^{\cdot\cdot\cdot\cdot}-w^{\cdot\cdot\cdot}\right)+\frac{1}{2}\left[a\,U^{\cdot\cdot\cdot|}+a^2 v^{||\cdot\cdot}\right]=0\longrightarrow 33$$

From equations 33 and 32 equating the similar terms

$$a^3 u^{'''\cdot} = \frac{1}{2}\left[a u^{\cdot\cdot\cdot'} + a^2 v^{''\cdot\cdot} \right]$$

$$+ \left(v^{::} - w^{\cdot\cdot\cdot} \right] = \frac{1}{2}\left[a u^{\cdot\cdot\cdot'} + a^2 v^{''\cdot\cdot} \right]$$

$$a^3 u^{'''\cdot} = v^{::} - w^{\cdot\cdot\cdot}$$

$$u^{'''\cdot} = v^{::} - w^{\cdot\cdot\cdot} \longrightarrow 34$$

Neglecting

Substituting $u^{'''\cdot}$ in eq 31 to get the following

$$\left(v^{''\cdot\cdot} - w^{''\cdot} \right) + \frac{1}{2}\left[a u^{'''\cdot} + a^2 v^{'''} \right] = 0$$

Simplifying the above expression it can be rewritten as follows

$$\left(a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)^2 v = \left[w^{\cdot\cdot\cdot} + 2a^2 w^{''\cdot} \right] \longrightarrow 35$$

Differentiating the above expression once WRT ϕ on both sides and adding substracting the term $a^4 \underline{w^{''''}}$ RHS

$$\left(a^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)\right)^2 \dot{v} = \left(w^{::} + 2a^2 w^{''\cdot\cdot} \right) + a^4 w^{''''} - a^4 w^{''''} \quad 36$$

Above expression can be rewritten as follows

$$\left(a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)^2 \dot{v} = \left(a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)^2 w - a^4 w^{''''} \quad 37$$

Again simplifying the above equation

$$a^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)^2 \left(\dot{v} - w \right) = a^4 w^{''''} \longrightarrow 38$$

Now applying the operator $\left(a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right)^2$ in 30 eq

$$\left(a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^2(\dot{v}-w)\ -\cancel{+}\ \frac{d^2}{12\,a}\left(a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^4 w=0 \quad \downarrow 39$$

observing the equations 38 and 39.

$$\left(a\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^2(\dot{v}-w) = \frac{d^2}{12a^2}\left[a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right]^4 w$$

$$\left(a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^2(\dot{v}-w) = a^4 w'''' \checkmark$$

$$\left(a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^4 w + a^4\,\frac{12a^2}{d^2}\,w''''=0$$

$$K = \frac{d^2}{12a^2}$$

$$\left(a^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \phi^2}\right)^4 w + a^4\,\frac{w''''}{K}=0 \rightarrow 40$$

Equation usually termed as Donal's equation in w

DKJ characteristic Equation;

In order to get DKJ equation for the given equation assume the solution for w

$$w = H \times e^{m\phi}\cos\frac{\lambda_n x}{a}$$

Now substituting these values of w in Donal's equation above after expanding 1st Term in the bracket

$$\left[a^8\frac{\partial^8}{\partial x^8} + 4\,\frac{a^6\partial^8}{\partial x^6\partial\phi^2} + 6\,a^4\frac{\partial^8}{\partial x^4\partial\phi 4} + 4\frac{a^6\partial^8}{\partial x^2\partial\phi^6} + \frac{\partial^8}{\partial\phi^8}\right]w$$

$$+\ a^4\,\frac{w''''}{K}=0 \qquad \lambda_n = a\frac{\partial}{\partial x}$$

$$\lambda_n^8 - 4\lambda_n^6 m^2 + 6\lambda_n^4 m^4 - 4\lambda_n^2 m^6 + m^8) + \frac{\lambda n^4}{k} = 0$$

(or)

$$(m^2 - \lambda_n^2)^4 + \frac{\lambda n^4}{k} = 0 \longrightarrow 41$$

Equation usually termed as DKJ characteristic equation.

Roots of this characteristic equation

For this equation there will be 8 roots which occur in conjugate pairs as,

$$\left.\begin{aligned} m_1 &= \alpha_1 \pm i\beta & m_5 &= -m_1 \\ m_2 &= \alpha_1 - i\beta_1 & m_6 &= -m_2 \\ m_3 &= \alpha_2 + i\beta_2 & m_7 &= -m_3 \\ m_4 &= \alpha_2 - i\beta_2 & m_8 &= -m_4 \end{aligned}\right\} \longrightarrow 42$$

where α_1, α_2, β_1, β_2 are the constants. By substituting the above values of roots $m_1 m_2 \ldots m_8$ in eq 41 to get value of w then substituting this value in eq 29 to get the value of one of the stress resultants making use of these value and equilibrium equation arrived for DKJ theory (starting from 2 equilibrium equation (22) other stress resultants can be found out.

Single Expression for w in terms of ϕ;

In the expression $w = Hx\, e^{m\phi} \cos\frac{\lambda_n x}{a} \longrightarrow 43$

Same substituting ㊷ equation in the above we get

$$w = \Big[H_1 e^{(\alpha_1 + i\beta_1)\phi} + H_2 e^{(\alpha_1 - i\beta_1)\phi} + H_3 e^{(-\alpha_1 + i\beta_1)\phi} + H_4 e^{(-\alpha_1 - i\beta_1)\phi} + H_5 e^{(\alpha_2 + i\beta_2)\phi} + H_6 e^{(\alpha_2 - i\beta_2)\phi} + H_7 e^{(-\alpha_2 + i\beta_2)\phi} + H_8 e^{(-\alpha_2 - i\beta_2)\phi}\Big] \cos\frac{\lambda_n x}{a}$$

Expanding the imaginary terms by means of demoduleu theorem

$$w = \Big\{ H_1 e^{\alpha_1\phi}(\cos\beta_1\phi + i\sin\beta_1\phi) + H_2 e^{\alpha_1\phi}(\cos\beta_1\phi - i\sin\beta_1\phi) + H_3 e^{-\alpha_1\phi}$$

$$(\cos\beta_1\phi + i\sin\beta_1\phi) + H_4 e^{-\alpha_1\phi}(\cos\beta_1\phi - i\sin\beta_1\phi) + H_5 e^{+\alpha_2\phi}(\cos\beta_2\phi + i\sin\beta_2\phi)$$

$$+ H_6 e^{\alpha_2\phi}(\cos\beta_2\phi - i\sin\beta_2\phi) + H_7 e^{-\alpha_2\phi}(\cos\beta_2\phi + i\sin\beta_2\phi) + H_8 e^{-\alpha_2\phi}(\cos\beta_2\phi$$

$$-i\sin\beta_2\phi)\Big] \cos\frac{\lambda n x}{a}$$

observing the above expression it can be seen that there are four terms with +ve exponential index and 4 more with -ve exponential index. The moment M_ϕ being the nature of disturbance eminating from the edge $\phi = 0$, it is critically observed that its value decay exponentially as we move away from the edge as the other words increase value of ϕ.

This condition is satisfied by ~~the~~ multiplying the 4 terms by -ve exponential index. The other four terms are neglected in a such a case. The value of M_ϕ takes the following form.

$$W = \Big[H_3 e^{-\alpha\phi}(\cos\beta_1\phi + i\sin\beta_1\phi) + H_4 e^{-\alpha\phi}(\cos\beta_1\phi - i\sin\beta\phi)$$
$$+ H_7 e^{-\alpha_2\phi}(\cos\beta_2\phi + i\sin\beta_2\phi) + H_8 e^{-\alpha_2\phi}(\cos\beta_2\phi - i\sin\beta_2\phi)\Big] \cos\frac{\lambda n x}{a}$$

The above expression can be rearranged as follows

$$W = \Big[e^{-\alpha_1\phi}(H_3 + H_4)\cos\beta_1\phi + e^{-\alpha_1\phi}\, i(H_3 - H_4)\sin\beta_1\phi +$$
$$e^{-\alpha_2\phi}(H_7 + H_8)\cos\beta_2\phi + e^{-\alpha_2\phi}\, i(H_7 - H_8)\sin\beta_2\phi\Big] \cos\frac{\lambda n x}{a}$$

Designated the terms with in the brackets as A_n B_n C_n D_n the above expression still can be written as

$$M_\phi = \Big[\Big(e^{-\alpha_1\phi} A_n \cos\beta_1\phi + e^{-\alpha_1\phi} B_n \sin\beta_n\phi + e^{-\alpha_2\phi} C_n \cos\beta_2\phi$$
$$+ e^{-\alpha_2\phi} D_n \sin\beta_2\phi\Big) \cos\frac{\lambda n x}{a}\Big]$$

$$W = e^{-\alpha_1\phi}(A_n \cos\beta_1\phi + B_n\sin\beta_1\phi) + e^{-\alpha_2\phi}\, C_n\cos\beta_2\phi + D_n\sin\beta_2\phi$$
$$D_n\sin\beta_2) \longrightarrow 44$$

SCHORER'S THEORY ;

Assumptions ;

-> Material is homogenous , isotropic and obeys hooke's law.

-> An element normal to the middle surface of the shell remains normal even after deformation.

-> All displacements of the shell surface are assumed to be small.

-> The quantities M_x, Q_x & $M_{x\phi}$ are neglected in the analysis.

-> Tangential strain ε_ϕ is also known as strain in ϕ direction and the shear strain $\gamma_{x\phi}$ are assumed to be very small when comparing to longitudinal strain. Hence they are assumed to be zero.

3/7/15 All present

Tangential strain $\epsilon_\phi = \frac{1}{a}\left[\frac{\partial v}{\partial \phi} - w\right] \simeq 0$

$$w = \frac{\partial v}{\partial \phi} \longrightarrow 45$$

From the strains in a cylindrical shell $\gamma_{x\phi} = \left[\frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x}\right] \simeq 0$

$$\frac{1}{a}\frac{\partial u}{\partial \phi} = -\frac{\partial v}{\partial x} \longrightarrow 46$$

Moment curvature relations

Here the moment curvature relation is assumed to be same as DKJ Theory.

$$M_\phi = \frac{-D}{a^2}\frac{\partial^2 w}{\partial \phi^2} \longrightarrow 47$$

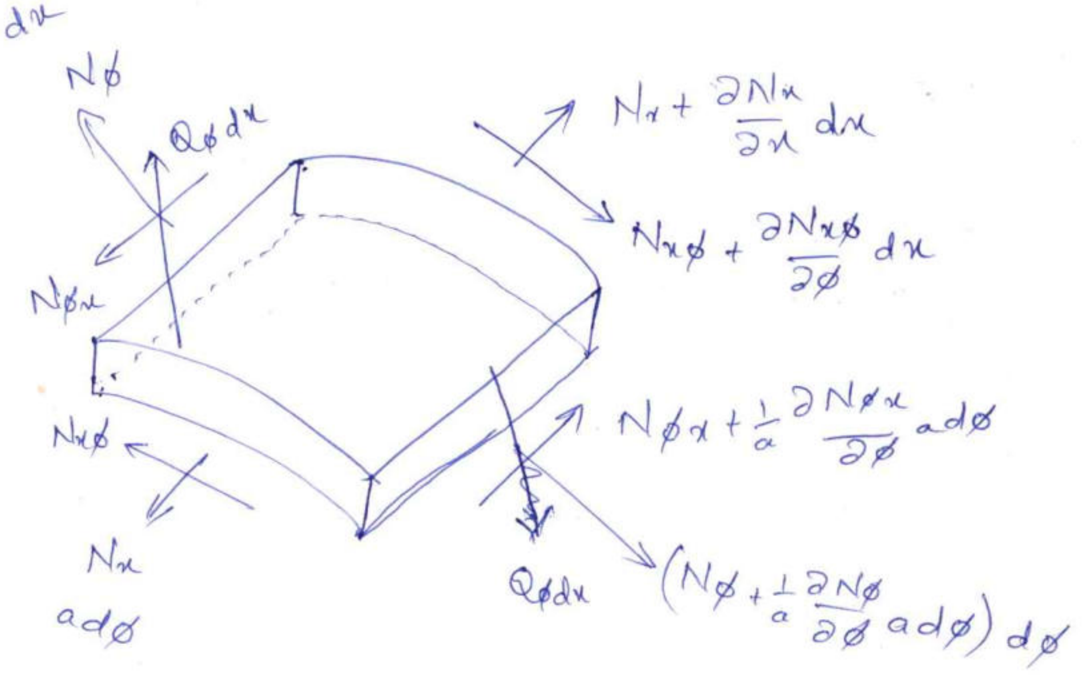
Equilibrium equations;

Equating all forces acting along x-direction to zero, we get the following equations. Refer master fig (a)

$\Sigma X = 0$

$$\left[\frac{\partial N_x}{\partial x} dx\right] a\, d\phi + \left[\frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} a\, d\phi\right] dx = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} a\, d\phi = 0 \longrightarrow 48$$



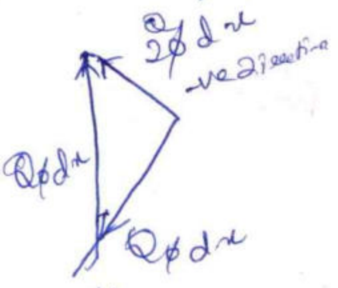
$\Sigma\phi = 0$

Suming up of all forces in φ-direction to zero

$$\left(+N_\phi \mp N_\phi + \frac{1}{a}\frac{\partial N_\phi}{\partial \phi} a\,d\phi\right) dx + \left(N_{x\phi} - N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x} dx\right) a\,d\phi -$$

$$2 Q_\phi\, dx \frac{d\phi}{2} = 0 \qquad \frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} a \cancel{\phi} = 0$$

In the above equation the first two terms obtained by algebraic summation of corresponding force where as third term needs some explanation as follows for getting 3rd as above equation the shears Qφ dx on opp edges which are inclined at an angle dφ with one another. Taking the triangle law of forces as shown in fig the resultant should be 2Qa dx dφ which is acting along -ve φ direction.



while considering this it may be observed that the effect of small increment of Qφ on the side BC of element master fig (a) ABCD is neglected as it is very small

Dividing the above eq (a) by a∂x dφ we get from eq, 22

$$\frac{1}{a}\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + Q_\phi = 0 \longrightarrow 49$$

Equating to zero the sum of all normal forces acting in the inward normal direction i.e., Z- direction

$$2N_\phi\,dx\frac{d\phi}{2} + \left[-Q_\phi + Q_\phi + \frac{1}{a}\frac{\partial Q_\phi}{\partial \phi}a\,d\phi\right]dx = 0$$

$$2N_\phi\,dx\,d\phi/2 + \frac{1}{a}\frac{\partial Q_\phi}{\partial \phi}a\,dx\,d\phi.$$

Dividing $dx\,d\phi$ we get

$$N_\phi + \frac{\partial Q_\phi}{\partial \phi} = 0 \longrightarrow 50 \qquad N_\phi = -\dot{Q}_\phi = \frac{D}{a^3}\overset{::}{w} \longrightarrow 53$$

Refering to master fig (b) Equating sum of all moments of all respective forces about AD to zero

$$-M_\phi + M_\phi + \left[\frac{1}{a}\frac{\partial M_\phi}{\partial \phi}a\,d\phi\right]dx - Q_\phi(a\,d\phi)\,dx = 0$$

$$\left(\frac{1}{a}\frac{\partial M_\phi}{\partial \phi}a\,d\phi\right)dx - Q_\phi\,a\,d\phi\,dx = 0$$

Dividing above equation by $a\,\partial\phi\,dx$

$$\frac{1}{a}\frac{\partial M_\phi}{\partial \phi} - Q_\phi = 0 \longrightarrow 51 \qquad Q_\phi = \frac{1}{a}\frac{\partial M_\phi}{\partial \phi} = \frac{1}{a}\dot{M}_\phi = -\frac{D}{a^3}\dddot{w} \longrightarrow 52$$

Schorer theory is applicable only for longshells

Transverse moment curvature relationwill be

$$\frac{\partial N_{\phi x}}{\partial x} = -\frac{1}{a}\dot{N}_\phi = \frac{-D}{a^4}\overset{\cdot\cdot\cdot\cdot\cdot}{w}$$

From equation $\frac{\partial N_\phi}{\partial \phi} + a\frac{\partial N_{x\phi}}{\partial x} = 0$

$$\frac{\partial N_\phi}{\partial \phi} = -a\frac{\partial N_{x\phi}}{\partial x} \longrightarrow 54$$

$$\frac{\partial N_x}{\partial x} = -\frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi}$$

Differentiating above expression once WRT x on bothsides

$$\frac{\partial^2 N_x}{\partial x^2} = -\frac{1}{a}\frac{\partial^2 N_{\phi x}}{\partial x\,\partial \phi} \longrightarrow 5\cancel{4}(a)$$

From stress strain relation

$$\frac{N_x}{Ed} = \frac{\partial N_\phi}{Ed} \quad \epsilon_x = \frac{\partial u}{\partial x} \text{ for } \partial = 0$$

$$\frac{N_x}{Ed} = \frac{\partial u}{\partial x}$$

$$N_x = Ed \frac{\partial u}{\partial x}$$

Differentiating above expression twice WRT x on both sides

$$\frac{\partial^2 N_x}{\partial x^2} = Ed \frac{\partial^3 u}{\partial x^3} = Ed\, u''' \longrightarrow 54(b)$$

As comparing a and b R.H.S of two equations are equal

L.H.S = R.H.S

$$Ed\, v''' = -\frac{1}{a} \frac{\partial^2 N_{\phi x}}{\partial x \partial \phi}$$

$$= \frac{1}{a} \frac{\partial}{\partial \phi} \left[\frac{\partial N_{x\phi}}{\partial x} \right] \longrightarrow (55)$$

From 54 eq $\frac{\partial N_{x\phi}}{\partial x} = -\frac{1}{a} \frac{\partial N_\phi}{\partial \phi}$ substituting this value on R.H.S of 55 equation we get from 53

$$Ed\, v''' = \frac{1}{a} \frac{\partial}{\partial \phi} \left[-\frac{1}{a} \left(\frac{\partial N_\phi}{\partial \phi} \right) \right]$$

$$N_\phi = \frac{D}{a^3} w^{::}$$

$$Ed\, v''' = \frac{D}{a^5} w^{:::}$$

$$v''' = \frac{D}{Ed\, a^5} w^{:::} \longrightarrow 56$$

CIVIL ENGINEERING

Analysis of Shells and Folded Plates

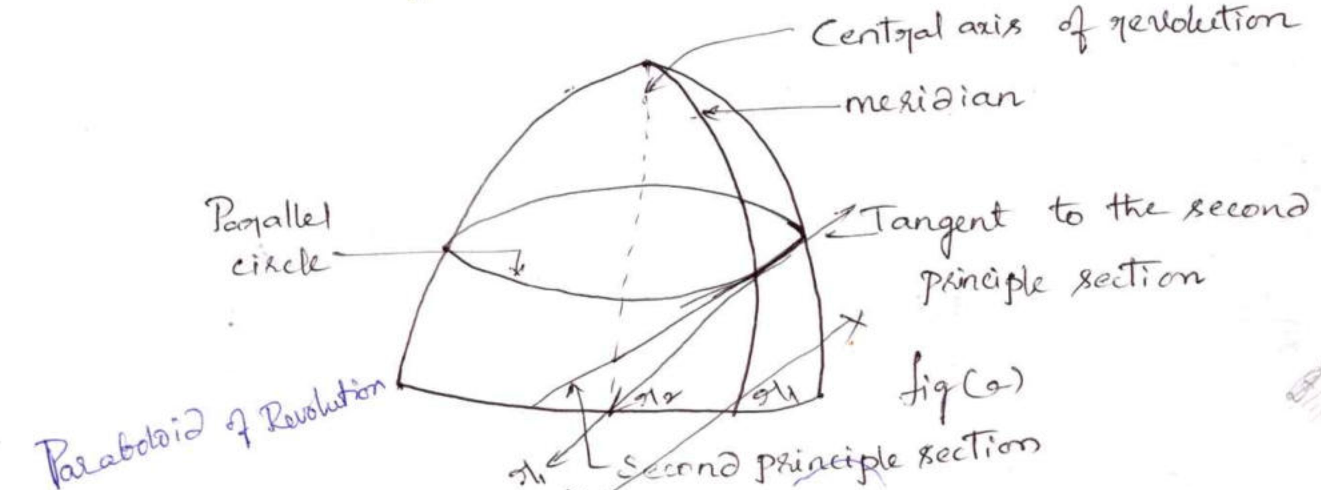
UNIT-3

UNIT - IV

14/8

Introduction to Shells of double Curvature

Membrane theory (~~other than~~ shells of revolution)



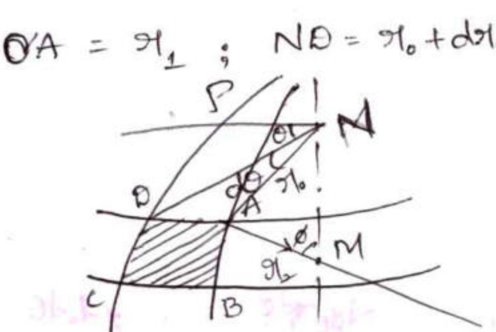
Shell of double curvature formed by surface of revolution

As shown in the above diagram consider any shell of double curvature formed by surface of revolution
From the mathematics, for a given surface of revolution it can be proved that meridian is one of the principle sections and its curvature is one of the principle curvatures
let r_1 be the radius of curvature of meridian or 1st principal section WRT to origin O
The other orthogonal section, which is known as 2nd principle section is as shown in fig and its radius of curvature is 'r_2' this orthogonal principle section is obtained by the intersection of the given surface with a plane i.e., at right angles to the plane of meridian.

Parallel circle; Is the usual circular curve obtained by intersection of horizontal plane with given surface of revolution. Its radius is usually represented by 'R_0'.

$MA = r_2$; $AN = r_0$

$OA = r_1$; $ND = r_0 + dr_0$



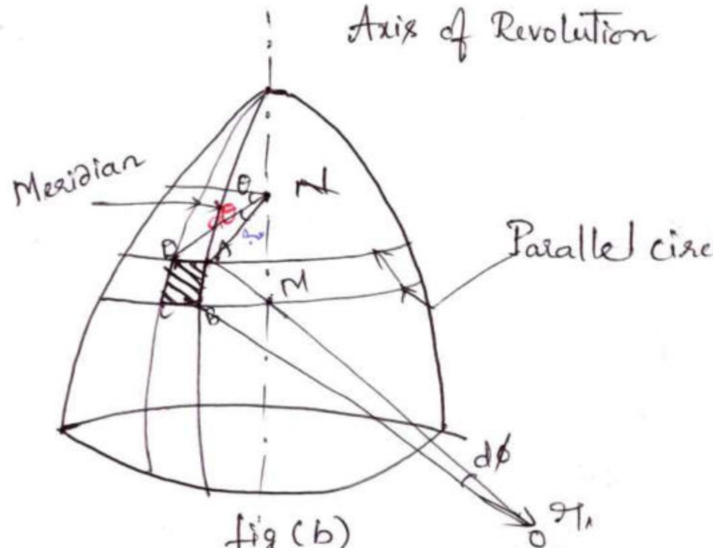


fig (b)

As shown in fig (b) consider a surface of revolution. ABCD is a small element formed by the intersection of Two meridians and two parallel circles.

let 'O' be the origin

Now $OA = r_1$ and the angle made by OA with the central vertical axis of revolution is 'ϕ'.

Also $AM = r_2$

$\underline{AOB} = d\phi$

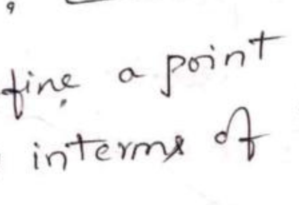
$AN = r_0$, which is the radius of arbitrary ||ele and hence

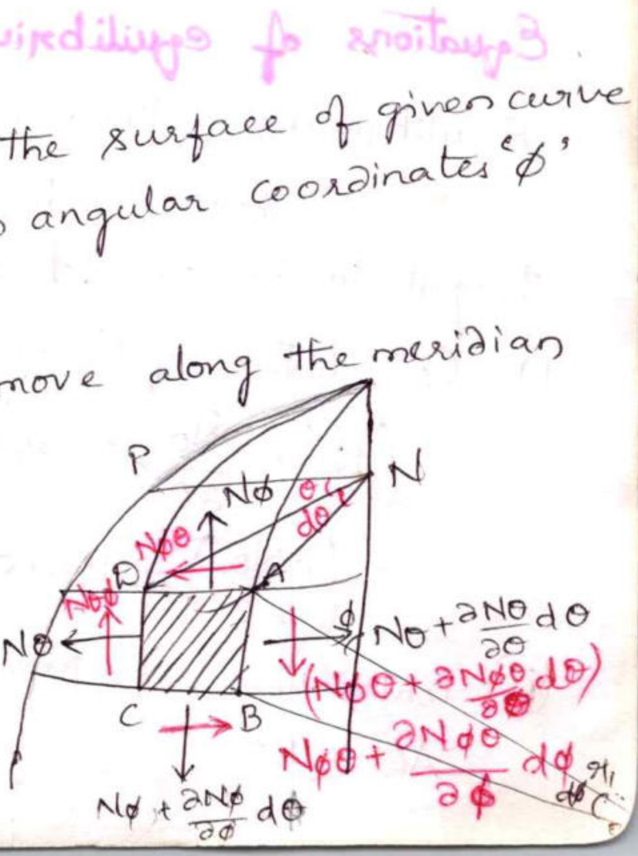
$ND = r_0 + \partial r_0$

let $\underline{DNP} = \theta$, $\underline{AND} = d\theta$

In order to define a point on the surface of given curve usually it is defined interms of two angular coordinates 'ϕ' and 'θ'

The angle 'ϕ' increases as we move along the meridian from top to bottom



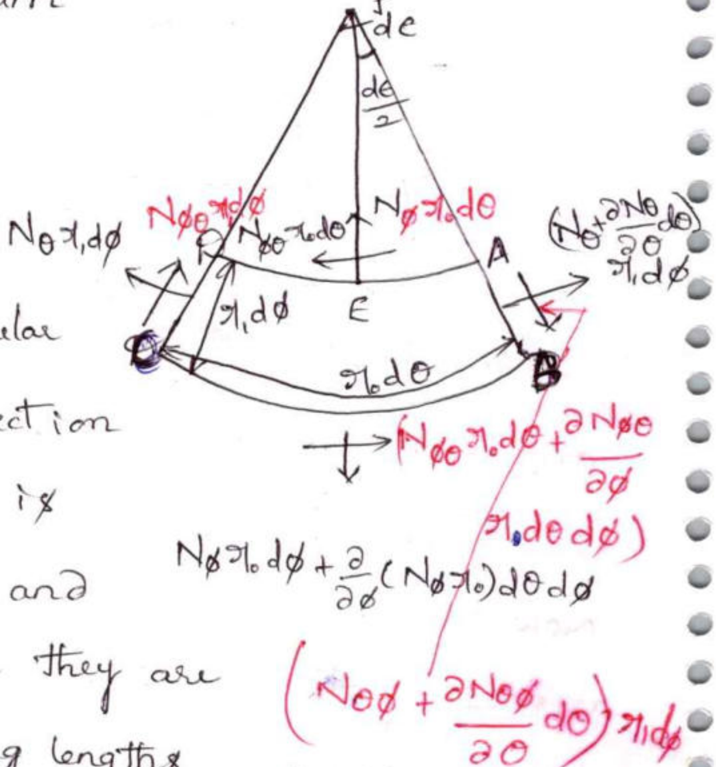


Various forces acting over an element of surface of revolution

Asshown in above fig over an element ABCD, since we are analysis the shell using membrane theory the forces acting over shell element are N_θ, $N_{\theta\phi}$, N_ϕ, $N_{\phi\theta}$ which are per unit length acting as shown in diagram

The size of the element is
$r_1 d\phi \times r_0 d\theta$

fig(d) shown the details of the element on extending to the perpendicular to the paper. let the point of the intersection be 'P'. Such that $\angle APD = \epsilon$ 'PE' is ⊥r bisector. Since N_θ, N_ϕ, $N_{\theta\phi}$ and $N_{\phi\theta}$ are the forces per unit length they are to be multiplied by the corresponding lengths over which they act. The net forces acting over the extended element as shown in fig (d) above.



fig(D)

Equations of equilibrium ;

Equilibrium equation in x- direction

This direction is usually termed as direction of tangent to the parallel circle (or) Circle of latitude.

(1) Contribution of N_θ forces ;

$$\left(\cancel{N_\theta} + \frac{\partial N_\theta}{\partial\theta} d\theta\right) r_1 d\phi - \cancel{N_\theta r_1 d\phi} = 0$$

$$\left(\frac{\partial N_\theta}{\partial\theta}\right) r_1 d\theta\, d\phi \longrightarrow a$$

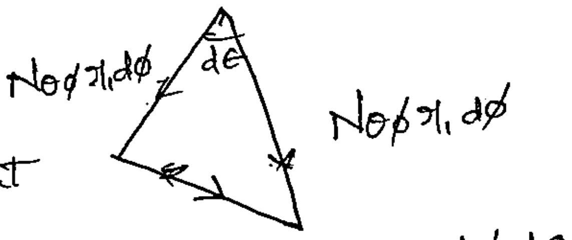
(ii) contribution of pair of shear forces acting on side AD & BC

$$\left(\cancel{N_{\phi\theta}\, r_0\, d\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi}\, r_0\, d\theta\, d\phi\right) - \cancel{N_{\phi\theta}\, r_0\, d\theta} = 0$$

$$\frac{\partial N_{\phi\theta}}{\partial \phi}\, r_0\, d\theta\, d\phi \longrightarrow b$$

iii Now the contribution of SF acting on AB and CD i.e., $N_{\theta\phi}$ forces along x-axis

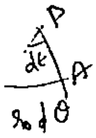
Making use of Triangular law of forces, the net force due to SF acting on AB and CD are



$$N_{\theta\phi}\, r_1\, d\phi\, d\epsilon \longrightarrow C$$

From the fig it may be observed that

Resultant $= N_{\theta\phi}\, r_1\, d\phi\, d\epsilon$

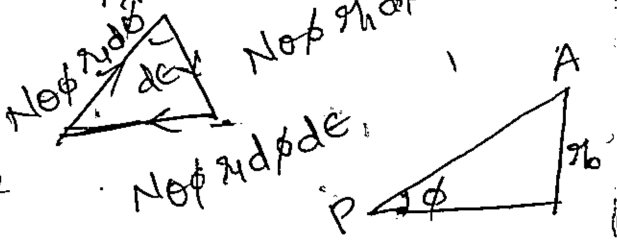


$$d\epsilon = \frac{r_0\, d\theta}{PA}$$

Also from fig, it is established that $\frac{r_0}{PA} = \cos\phi$

substituting this in above

$$d\epsilon = \cos\phi\, d\theta$$



Sub this in (C) above, we can have

$$N_{\theta\phi}\, r_1\, d\theta\, d\phi\, \cos\phi$$

Now ~~substituting~~ summing up of all forces AB and CD and equating to zero we can get equation along x-direction

$$\frac{\partial N_\theta}{\partial \theta} r_1\, d\theta\, d\phi + \frac{\partial}{\partial \phi}(N_{\phi\theta}\, r_0)\, d\theta\, d\phi + N_{\phi\theta}\, r_1\, d\theta\, d\phi \cos\phi$$

$$+ X(r_0\, d\theta)(r_1\, d\phi) = 0$$

In the above eq 'X' is component of external load along X-axis per unit area.

Dividing the above eq by common term $d\theta\, d\phi$ the above eq takes the following form:

$$\frac{\partial N_\theta}{\partial \theta} r_1 + \frac{\partial (N_{\phi\theta}\, r_0)}{\partial \phi} + N_{\theta\phi}\, r_1 \cos\phi + X r_0 r_1 = 0 \longrightarrow$$

Equilibrium Equation along Y- direction (or) meridianal tangent direction φ- direction

Contribution of N_ϕ

Algebraically adding the Terms on φ- direction, we get

$$-N_\phi r_0 d\theta + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right) r_0 d\theta$$

$$\frac{\partial N_\phi}{\partial \phi} d\phi\, r_0\, d\theta$$

Contribution of $N_{\theta\phi}$;

Along faces AB & CD we get

$$\left(N_{\theta\phi} r_1 d\phi + \frac{\partial N_{\phi\theta}}{\partial \theta} d\theta\, r_1 d\phi\right) - N_{\theta\phi} \cancel{r_1 d\phi} = 0$$

$$\frac{\partial N_{\phi\theta}}{\partial \theta} d\theta\, r_1 d\phi$$

Contribution of N_θ;

$N_\theta r_1 d\phi d\theta \cos\phi$

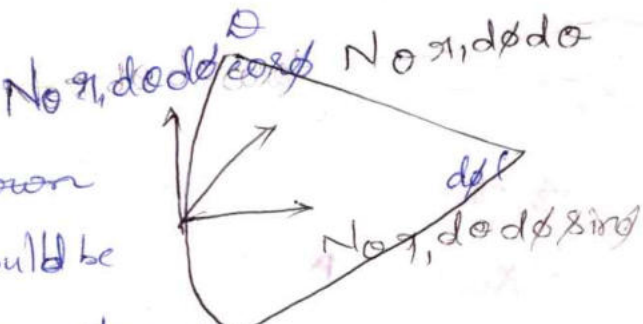
$N_\theta r_1 d\theta d\phi \sin\phi$

Summing of N_θ force along y-direction

These force $N_\theta r_1 d\phi$ acting on CD and its component force acting on AB or acting along x-direction

To find a component along the required meridianal tangential direction or y-direction

Neglecting small increment of force in the identified forces which are known shown in the blocks the result would be



Here it is assumed that sides of these elements have been protruded or projected ⊥e to paper to meet at a point making an angle dθ from these diagram the resultant force

force would be $N_\theta r_1 d\phi d\theta$

Geometrically these resultant force is found to be inclined at angles with the meridional tangent direction as shown in fig(e). Resolving these forces along y and z direction the component along y direction is equal to $N_\theta r_1 d\theta d\phi \cos\phi \rightarrow c$

Now summing up of all forces acting along y-direction i.e., adding a+b+c

$$-\frac{\partial}{\partial\phi}(N_\phi r_0)d\theta d\phi + \frac{\partial N_{\theta\phi}}{\partial\theta} r_1 d\theta d\phi - N_\theta r_1 d\theta d\phi \cos\phi + Y(r_1 d\theta \times r_0 d\phi) = 0$$

÷ entire equation by $d\theta d\phi$

$$\therefore \frac{\partial}{\partial\phi} N_\phi r_0 + \frac{\partial N_{\theta\phi}}{\partial\theta} r_1 - N_\theta r_1 \cos\phi + Y r_1 r_0 = 0 \rightarrow 55$$

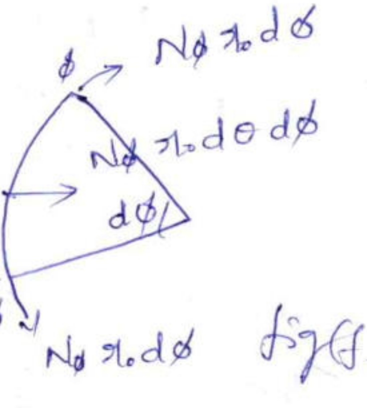
Equilibrium equation along z-direction;

From fig(e) Contribution of N_θ

$N_\theta r_1 d\theta d\phi \sin\phi \rightarrow a$

Contribution of N_ϕ;

From fig(a) identify the forces $N_\phi r_0 d\theta$ acting on AD and BC. For the force acting on BC neglect the incremental portion then assume the two forces $N_\phi r_0 d\theta$ are going form a small sector of a circle by their projection making an angle $d\phi$ at O. As shown in fig(f). The resultant force should be $N_\phi r_0 d\theta$ which is found to act along z-direction



fig(f)

$N_\phi r_0 d\theta d\phi \rightarrow (b)$

Equilibrium equation along Z direction can be obtained by adding a b & c

$\Sigma Z = 0$

$$N_\theta r_1 \, d\theta \, d\phi \sin\phi + N_\phi r_0 \, d\theta \, d\phi + Z r_1 r_0 \, d\theta \, d\phi = 0$$

$\div \, d\theta \, d\phi$

$$N_\theta r_1 \sin\phi + N_\phi r_0 + Z r_1 r_0 = 0 \longrightarrow 56$$

Symmetrically loaded shells ;

From practical observations for shells of revolution which are symmetrically loaded it is found that usually

$$X = 0 \quad N_{\theta\phi} = N_{\phi\theta} = 0 \longrightarrow 57$$

Inserting this equation in 2 and 3 equations of equilibrium

$$\frac{\partial}{\partial\phi}(N_\phi r_0) \cancel{+\partial x} - N_\theta r_1 \cos\phi + Y r_1 r_0 = 0$$

$$N_\theta r_1 \cos\phi - \frac{\partial N_\phi}{\partial\phi} r_0 = Y r_0 r_1 \longrightarrow 58$$

$$N_\phi r_0 + N_\theta r_1 \sin\phi + Z r_1 r_0 = 0$$

Dividing the ~~entire~~ term by $r_0 r_1$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_0} = \cancel{-Z} \longrightarrow$$

Since from the geometry of the shell it has been assumed that $r_0 = r_2 \sin\phi$

$$N_\phi r_2 \sin\phi + N_\theta r_1 \sin\phi + Z r_1 r_2 \sin\phi = 0$$

Divide by $r_1 r_2 \sin\phi$

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = -Z \longrightarrow 5\cancel{8}9$$

Isolating N_θ in the above equation we get

$$N_\theta = -r_2\left[Z + \frac{N_\phi}{r_1}\right] \longrightarrow 58(a)$$

Substituting value of N_θ in 58 equation, we can have

$$N_\theta r_1 \cos\phi - r_1 r_2 \cos\phi\left[Z + \frac{N_\phi}{r_1}\right) - \frac{d}{d\phi}(N_\phi r_0) = Y r_0 r_1$$

Re arranging terms the above equation takes the following form

$$-N_\phi r_2 \cos\phi - \frac{d}{d\phi}(N_\phi r_0) = (Y r_0 r_1 + Z r_0 r_1) \cos\phi) \quad x \div N$$

$$-r_1 r_2 \cos\phi\left[Z + \frac{N_\phi}{r_1}\right) - \frac{d}{d\phi}(N_\phi r_0) = Y r_0 r_1$$

Multiplying by $\sin\phi$ on both sides and noting that $r_0 = r_2\sin\phi$, we can get the following

$$N_\phi r_2 \sin\phi \cos\phi + \frac{d}{d\phi}(N_\phi r_0 \sin\phi) = -r_1 r_2 (Y\sin\phi + Z\cos\phi)\sin\phi$$

The above expression can be expressed in the following compact form

$$\frac{d}{d\phi}(N_\phi r_2 \sin^2\phi) = -r_1 r_2 (Y\sin\phi + Z\cos\phi)\sin\phi$$

Integrating on both sides above and solving for N_ϕ we can having

$$N_\phi = -\frac{1}{r_2\sin\phi}\int r_1 r_2\left[Y\sin\phi + Z\cos\phi\right]\sin\phi\, d\phi + C$$

The above expression becomes physically more meaning full if it is multiplied and ÷ by 2π. Then above expression takes the following form

$$N_\phi = -\frac{1}{2\pi r_2 \sin\phi}\int 2\pi r_1 r_2\left[Y\sin\phi + Z\cos\phi\right]\sin\phi\, d\phi + C \longrightarrow 59$$

A simple physical explanation of the above integral can be given like this. The term $2\pi r_1 r_2 \sin\phi\, d\phi$ denotes the surface area of the elementary strip of dome.

The term with in the brackets of integral i.e., $(Y\sin\phi + Z\cos\phi)$ represent vertical component of forces/unit area acting on a elemental strip of the dome. In such a case the entire term with in the integral $2\pi r_1 r_2 (Y\sin\phi + Z\cos\phi)\sin\phi\, d\phi$ represents vertical component of force acting over the elemental strip of dome. Hence finally

$$\int 2\pi r_1 r_2 \, (Y\sin\phi + Z\cos\phi)\sin\phi\, d\phi$$

Represents the Total vertical load acting over the entire dome this is usually ~~represented~~ seperated by letter 'w'.

$$\therefore N_\phi = \frac{-w}{2\pi r_2 \sin^2\phi} \longrightarrow 60$$

$$\text{where } w = \int 2\pi r_1 r_2 \, (Y\sin\phi + Z\cos\phi)\sin\phi\, d\phi \longrightarrow$$

After getting value of N_ϕ from the above relation substituting in 58(a) to get the value of N_θ

$$N_\theta = -r_2\left[Z + \frac{N_\phi}{r_1}\right] \longrightarrow 60(a)$$

Stresses in a Spherical shell;

Stresses under own weight;

from $X = 0$

$Y = g\sin\phi$

$Z = g\cos\phi$

For a spherical shell $r_1 = r_2 = a$ from ⑥⓪

$$N_\phi = \frac{-w}{2\pi r_2 \sin^2\phi} \longrightarrow 60(b)$$

where w = weight of load acting on a spherical shell or dome

$$= \frac{1}{2\pi r_2 \sin^2\phi}\int 2\pi r_1 r_2 \, (Y\sin\phi + Z\cos\phi)\sin\phi\, d\phi + c$$

Substituting all the above conditions given by 'a' in the expression 'w' above we can have

$$w = \int_0^{\phi} 2\pi a^2 g \left(\sin^2\phi + \cos^2\phi \right) \sin\phi \, d\phi \longrightarrow$$

$$= \int_0^{\phi} 2\pi a^2 g \sin\phi \, d\phi$$

$$= 2\pi a^2 g \int_0^{\phi} \sin\phi \, d\phi$$

$$= -2\pi a^2 g (\cos\phi - 1)$$

$$-2\pi a^2 g (1 - \cos\phi)$$

Substituting these value of w in

$$N\phi = \frac{-2\pi a^2 g (1-\cos\phi)}{2\pi a \sin^2\phi} = \frac{-ag(1-\cos\phi)}{\sin^2\phi}$$

$$= \frac{-ag(1-\cos\phi)}{1-\cos^2\phi} = \frac{-ag}{(1+\cos\phi)}$$

$$N\phi = \frac{-ag}{(1+\cos\phi)}$$

from eq 60 (a)

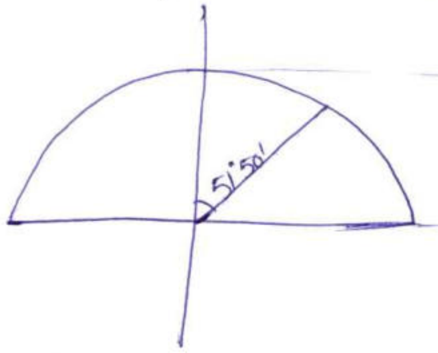
$$N_\theta = -r_2 \left[Z + \frac{N\phi}{r_1} \right] \quad \text{where } r_1 = r_2 = a$$

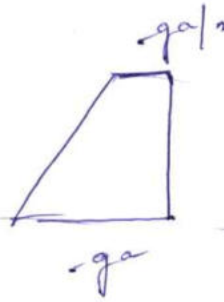
$$N_\phi = \frac{-ag}{(1+\cos\phi)}$$

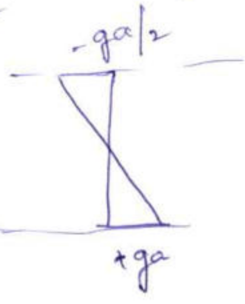
$$N_\theta = -a \left[g\cos\phi + \frac{-ag}{a(1+\cos\phi)} \right]$$

$$= -a \left[g\cos\phi - \frac{g}{1+\cos\phi} \right] = -ag \left(\frac{\cos\phi + \cos^2\phi - 1}{(1+\cos\phi)} \right)$$

Representation of variation of stress resultants through diagram







Spherical shell

Variation of 'N$_\phi$'

Variation of 'N$_\theta$'

At ϕ

At $\phi = 0$	At $\phi = 0$
$N_\phi = -ag/2$	$N_\theta = -ag/2$
@ $\phi = 90°$	@ $\phi = 90°$
$N_\phi = -ag$	$N_\theta = ag$

Observing the above set of values, N_ϕ is found to be compressive between 0 to 90° on the other hand 'N_θ' value changes from $-ag/2$ to ag. This means that b/n 0 and 90°. N_θ value becomes 0 for ϕ equate the above N_θ value to zero, which gives

$$\cos^2\phi + \cos\phi - 1 = 0$$

From this equation it can be observed that $\phi = 51°50'$ at $N_\theta = 0$

Stresses under S.L condition;

For snow load $X = 0$

$$Y = P_o \sin\phi \cos\phi$$

$$Z = P_o \cos^2\phi$$

For spherical shells $r_1 = r_2 = a$

substituting all these values for expression 'w' above we get following result

$$W = \int_0^\phi 2\pi r_1 r_2 (Y\sin\phi + Z\cos\phi)\sin\phi \, d\phi \longrightarrow$$

$$\int_0^\phi 2\pi a^2 (P_0 \sin^2\phi \cos\phi + P_0 \cos^3\phi) \sin\phi \, d\phi$$

$$\pi a^2 P_0 \int_0^\phi (\sin^2\phi + \cos^2\phi) \, 2\sin\phi \cos\phi \, d\phi$$

$$= \pi a^2 P_0 \int_0^\phi \sin 2\phi \, d\phi$$

$$= -\pi a^2 P_0 \left[\frac{\cos 2\phi}{2} \right]_0^\phi$$

$$= \frac{\pi a^2 P_0}{2} \left[\cos 2\phi - 1 \right]$$

$$= \frac{\pi a^2 P_0}{2} (1 - \cos 2\phi]$$

$$= \frac{\pi a^2 P_0}{2} \left[(1 - (1 - 2\sin^2\phi)) \right]$$

$$= \pi a^2 P_0 \sin^2\phi$$

$$N_\phi = \frac{-W}{2\pi r_2 \sin^2\phi}$$

Substituting 'w' value in above expression

$$N_\phi = \frac{-P_0 \pi a^2 \sin^2\phi}{2\pi a \sin^2\phi} = \frac{-P_0 a}{2}$$

Substituting there value of N_ϕ in 60(a) we get

$$N_\theta = -r_2 \left[Z + \frac{N_\phi}{r_1} \right)$$

$$= -a \left[P_0 \cos^2\phi + \frac{-P_0 a/2}{a} \right]$$

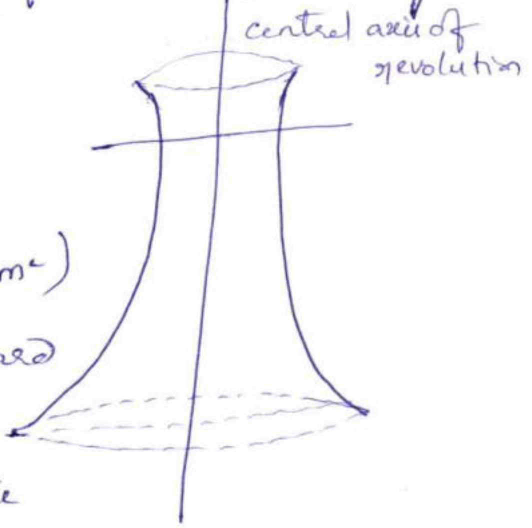
$$= -aP_0 \left[\cos^2\phi + \frac{-\alpha/2}{\alpha} \right]$$

$$= \frac{-aP_0}{2} \left[2\cos^2\phi - 1 \right]$$

$$\frac{-aP_0}{2} \cos 2\phi$$

8. Rotational Hyperboloid / Hyperboloid of Revolution / Cooling Tower

Shell stress resultants ;



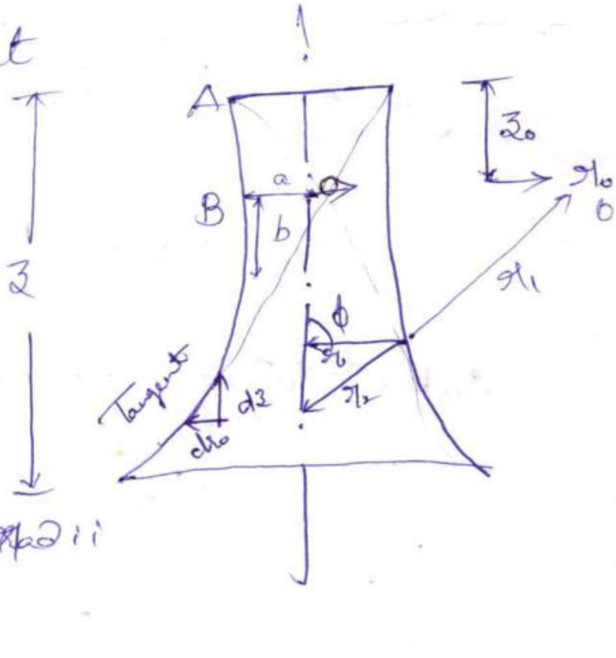
Normal height of cooling tower shells are >100m made of concrete and choose the soil with good bearing capacity (20-22 N/mm²) and also upward pressure should have downward pressure.

A hyperboloid of revolution formed by the revolution of a hyperbola with respect to central axis of revolution. This shell is also called cooling tower shell because it is mainly used for cooling the water that is being discharged from thermal power plants, steel manufacturing units, cement manufacturing units, refractory units etc.,

Stresses under own- weight

Consider a hyperboloid of revolution as shown in the above diagram with $r_0 O z$ plane.

Let the axis of revolution be along z-axis. With respect to any centre O' let in general r_1, r_2 & r_0 be the radii



of the 1st principle section, 2nd principle section and parallel circle

let in general ϕ & ϕ_0 be the ~~axis of revolution at any~~ instant. angles made by the line containing r_1 & r_2 be axis of revolution at any instant. let the corresponding heights be z and z_0. let the tangents shown by dotted line drawn at the bottom of the surface intersect the axis of revolution at 'O' let $OB = a$

let BD be the perpendicular drawn from B and to the tangent such that $BD = b$.

The general equation for any cooling tower shell can be mathematically represented as

$$\frac{r_0^2}{a^2} - \frac{z^2}{b^2} = 1 \longrightarrow 61$$

Similarly the values of r_1 and r_2 can also be represented mathematically as follows

$$\left.\begin{aligned} r_2 &= a\left[1 + \frac{z^2}{a^2}\left(\alpha + \alpha^2\right)\right]^{1/2} \longrightarrow a \\ r_1 &= a^2 b^2 \left[\frac{r_0^2}{a^4} + \frac{z^2}{b^4}\right]^{3/2} \longrightarrow b \\ \alpha &= a^2/b^2 \\ r_1 &= \frac{-r_2^3}{\alpha a^2} \longrightarrow c \end{aligned}\right\} 62$$

From (61) equation we have

$$\frac{r_0^2}{a^2} - 1 = \frac{z^2}{b^2}$$

$$\frac{z^2}{b^2} = \frac{r_0^2}{a^2} - 1 = \frac{r_0^2 - a^2}{a}$$

$$z^2 = \frac{b^2}{a^2}\left[r_0^2 - a^2\right]$$

$$z = \pm \frac{b}{a}\sqrt{r_0^2 - a^2} \longrightarrow 63$$

Differentiating above equation WRT r_0

$$\frac{dz}{dr_0} = \tan\phi = \pm \frac{b}{a}\sqrt{\frac{r_0^2}{(r_0^2 - a^2)}}$$

From there Squaring on both sides

$$\tan^2\phi = \pm \frac{b^2}{a^2}\sqrt{\frac{r_0^2}{(r_0^2 - a^2)}}$$

$$\tan^2\phi = \pm \frac{b^2}{a^2}\left[\frac{r_0^2}{(r_0^2 - a^2)}\right]$$

$$\frac{\sin^2\phi}{\cos^2\phi} = \frac{b^2}{a^2}\left[\frac{r_0^2}{r_0^2 - a^2}\right]$$

$$a^2(r_0^2 - a^2)\sin^2\phi = b^2 r_0^2 \cos^2\phi$$

$r_0 = r_2 \sin\phi$

$r_2 = r_0/\sin\phi$

$a^2 r_0^2 \sin^2\phi - a^4 \sin^2\phi = b^2 r_0^2 \cos^2\phi$

$r_0^2 (b^2 \sin^2\phi - b^2\cos^2\phi) = a^4 \sin^2\phi$

$r_0 = \frac{\sqrt{a^4\sin^2\phi}}{\sqrt{a^2\sin^2\phi - b^2\cos^2\phi}}$

Isolating r_0

$$r_0 = \frac{a^2\sin\phi}{\left(a^2\sin^2\phi - b^2\cos^2\phi\right)^{1/2}} \longrightarrow 64$$

$$r_0 = r_2 \sin\phi$$

$$r_2 = \frac{r_0}{\sin\phi}$$

$$r_2 = \frac{a^2}{\left(a^2\sin^2\phi - b^2\cos^2\phi\right)^{1/2}} \longrightarrow 65$$

Making use

$$r_1 = \frac{-r_2^{\,3}}{\alpha a^2}$$

Substituting equation above we get

$$r_1 = \frac{-a^2b^2}{(a^2\sin^2\phi - b^2\cos^2\phi)^{3/2}} \longrightarrow 66$$

The total vertical load W in a present problem case given by

$$W = \int 2\pi r_1 r_2 (Y\sin\phi + Z\cos\phi)\sin\phi \, d\phi$$ from

From self weight condition

$$Y = g\sin\phi$$

$$Z = g\cos\phi$$

Substituting the values of r_1 r_2 from eq 64 & 65 and Y Z we get

$$W = 2\pi g a^4 b^2 \int_{\phi}^{\phi_1} \frac{\sin\phi \, d\phi}{(a^2\sin^2\phi - b^2\cos^2\phi)^2}$$

This integral is solved by integration by substitution is as follows

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}\xi \qquad \xi. = C\xi Y$$

Substituting these in the above integral it takes the following form

$$W = \frac{-2\pi g a b^2}{\sqrt{a^2+b^2}} \int_{\xi}^{\xi.} \frac{d\xi}{(1-\xi^2)^2}$$

Solving the above integral we can have

$$w = \frac{\pi g}{2} \frac{ab^2}{\sqrt{a^2+b^2}} \left| \frac{2\xi_1}{1-\xi_1^2} + I_n\left[\frac{1+\xi_1}{1-\xi_1}\right] \right|_{\xi_1}^{\xi_0}$$

However we know that

$$N_\phi = \frac{-w}{2\pi r_2 \sin^2\phi} \quad \text{from } 60(b)$$

Substituting the value of w & r_2 in the above equation, we can get the value of N_ϕ from (60)

$$N_\theta = -r_2\left[Z + \frac{N_\phi}{r_1}\right]$$

Making use of above equation N_θ value also found

26/8 Present
9
13
15
12
7
2

3/9 Absenties
1
2
3
6
7
8
9
10
11
12
13
15

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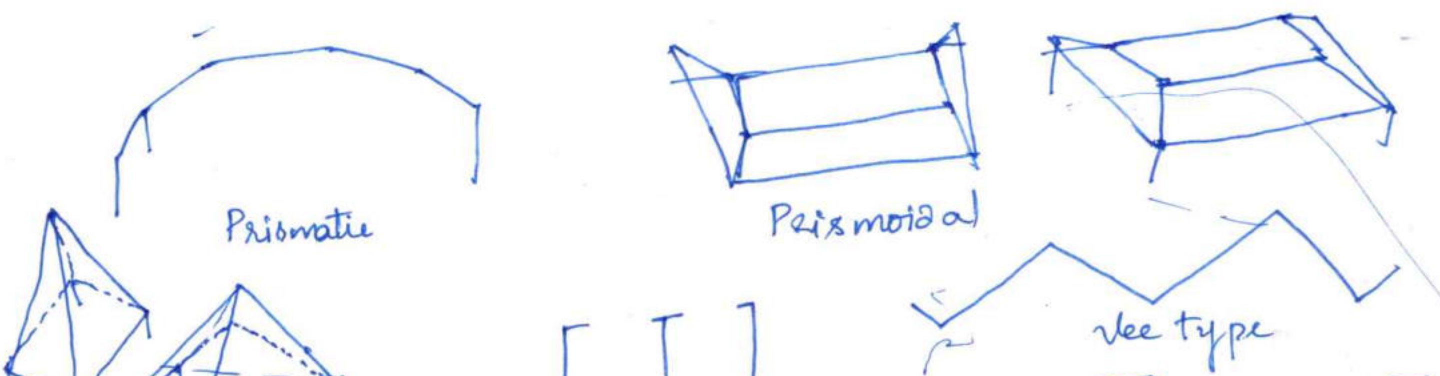
CIVIL ENGINEERING

Analysis of Shells and Folded Plates

UNIT-4

FOLDED PLATES

Almost skin to shells in structural actions are folded plates or hipped plates. They consume a little more material than continously curved cylindrical shells. Extra cost on this account in many times offset by the saving effected on forms. Prismatic, prismoidal, Pyramidal or curved in plan, they find application as roofs Coal bunkers, cooling towers, staircases etc



Types of folded plates

A thin walled building structure of the shell type. folded plate consists of flat components of plates that are interconnected at some dihedral angles

Shell's	Folded plates
→ Follows particular curved geometry	→ Here entire folded plates appears as though No of thin rectangular plates are connected in a particular fashion other than curved geometry.
→ They involve skilled labour Skilled engineer's and complicated centering & shuttering.	→ Simple normal centering & shuttering is essential to finish the work.
→ Centering & shuttering cost is usually very high.	→ Centering and shuttering cost is normal.
→ Curved geometry, they consume slightly lesser quantities of steel / concrete	→ Prismatic rectangular plates are involved slightly higher quantities of steel & concrete are involved.
→ Considering the & -ve points of both shells and folded plates	→ Same

It ~~is~~ may be observed that final cost of both shells and folded plates is more or less same.

Assumptions;

→ Material is homogenous, isotropic and elastic.

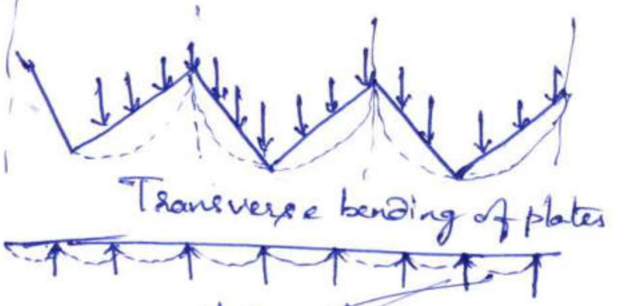
→ Structure is monolithic with rigid joints.

→ length of each plate is more than twice its width.

→ Plane sections remains plane even after deformations of the plate.

Structural Behaviour of folded plates;

The folded plates resist the system of transverse loads by slab action and plates action



Transverse bending of plates

SLAB ~~PLATES~~ ACTIONS;

The loads acting normal to each plates causes transverse bending between the junctions of the plate which can be considered as imaginary supports of a continous slab. This transverse moments developed in the plate can be determined by a continous beam analysis assuming the supports to the junctions of the plates.

Slab action

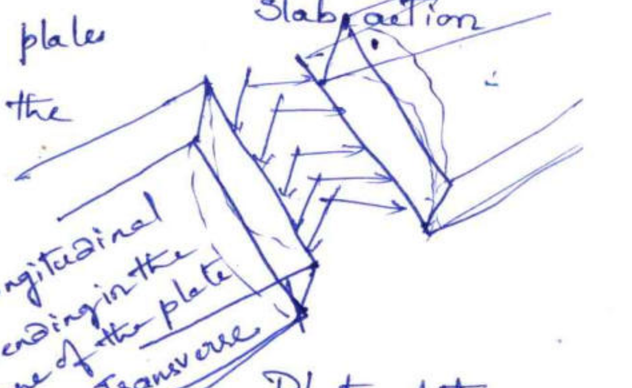


Plate Action

PLATE ACTION;

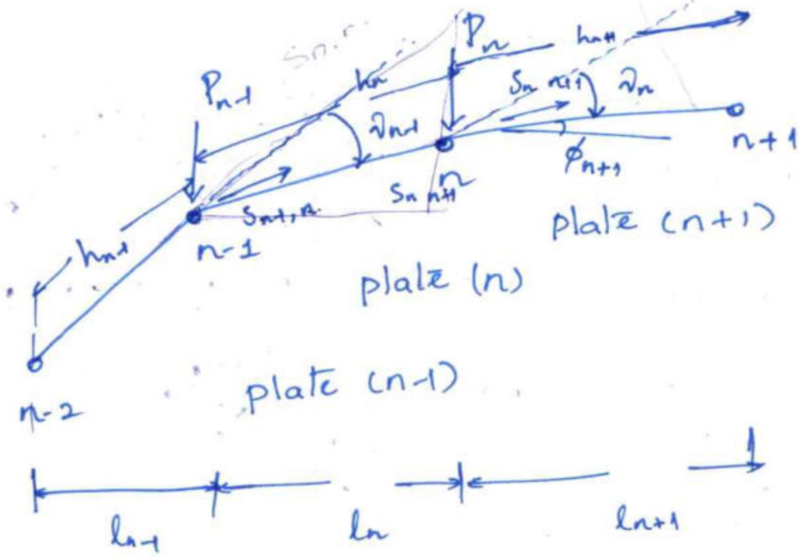
Plate being supported at their ends on the transverses, bend under the action of loads in their own plane as shown in fig b). The longitudinal bending of the plates in their own plane is termed as plate action.

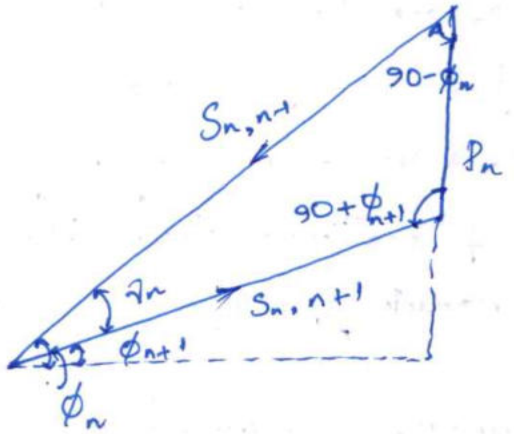
Plate acts as a one-way slab and longitudinal slab action may be ignored. The plates supported at their ends on tranverses bend under the action of loads acting in their plane. This is called plate action.

Because of assumption: The bending stress resulting from plate action may be considered to have a linear distribution across each plate.

Resolution of Ridge loads;

let us consider the plates to be hinged at the joints. The effect of moments at the joints can be super imposed later. Loads such as P_n applied at the ridges where adjacent plates meet are known as ridge loads.





$$\frac{S_n\ n-1}{\sin(90+\phi_{n+1})} = \frac{P_n}{\sin\vartheta_n}$$

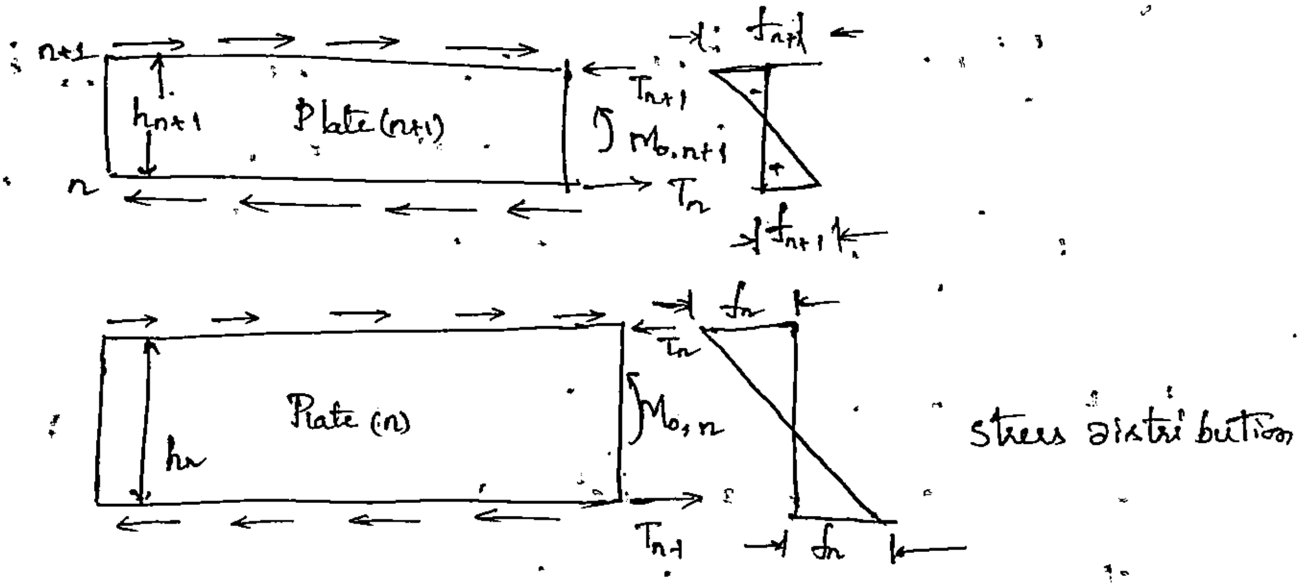
$$S_{n,(n-1)} = \frac{P_n \cos\phi_{n+1}}{\sin\vartheta_n}$$

24|9|15

Absenties

1
3
6
7
8
10

Theorems of three edge shear



Two adjacent plates of a continous folded plates

Consider 2 adjacent plate n, $n+1$, subjected to moments $M_{o,n}$ $M_{o\,n+1}$, shown in fig. let h_n, h_{n+1} are the widths of the plates n, $(n+1)$. let the edges of the plates be $n+1$, n, $n-1$ 'n' being Common edge.

let T_{n-1}, T_n, T_{n+1} denoted shear forces along the edges $(n-1)$ n, $n+1$. Similarly τ_n τ_{n-1} τ_{n+1} are the shear stresses along the edges $(n-1)$, (n) $(n+1)$. stress distribution is ~~disb~~ indicated for the two adjacent plates under independent conditions.

If the plates are assumed to act independently, they would develop different fibre stresses f_n, f_{n+1} at the edge 'n'. But two plates being connected together initially, shear stress (t_n) would develop along the Common edge(n). The magnitude of the shear stress is would be such that the fibre stresses n, $n+1$ would develop along their common edge 'n' are same.

Refering to the above fig, the fibre stress along the common edge 'n+1' is

$$\underbrace{\frac{M_{o\,n+1}}{Z_{n+1}}}_{\text{Bending stress}} + \underbrace{\frac{T_n h_{\frac{n+1}{2}}}{Z_{n+1}} + \frac{T_n}{A_{n+1}}}_{\text{stress due to } T_n} - \underbrace{\frac{T_{n+1}}{A_{n+1}} + \frac{T_{n+1} h_{\frac{n+1}{2}}}{Z_{n+1}}}_{\text{stress due to } T_{n+1}} \longrightarrow$$

where Z_{n+1}, A_{n+1} stands for modulus of section and Area of plate Plate $_{n+1}$

stress is assumed +ve, if tension at bottom and compression at top.

Similarly, the fibre stress along the common edge 'n', calculated with respect to plate 'n' is given by

$$-\frac{M_{o\,n}}{Z_n} - \frac{T_{n-1}\, h_n/_2}{Z_n} + \frac{T_{n-1}}{A_n} - \frac{T_n}{A_n} - \frac{T_n(h_n/_2)}{Z_n} \longrightarrow$$

Z_n A_n are the section modulus and Area of plate n

Since 'n' be the common edge the stress given by so equating and rearranging above two equations we get

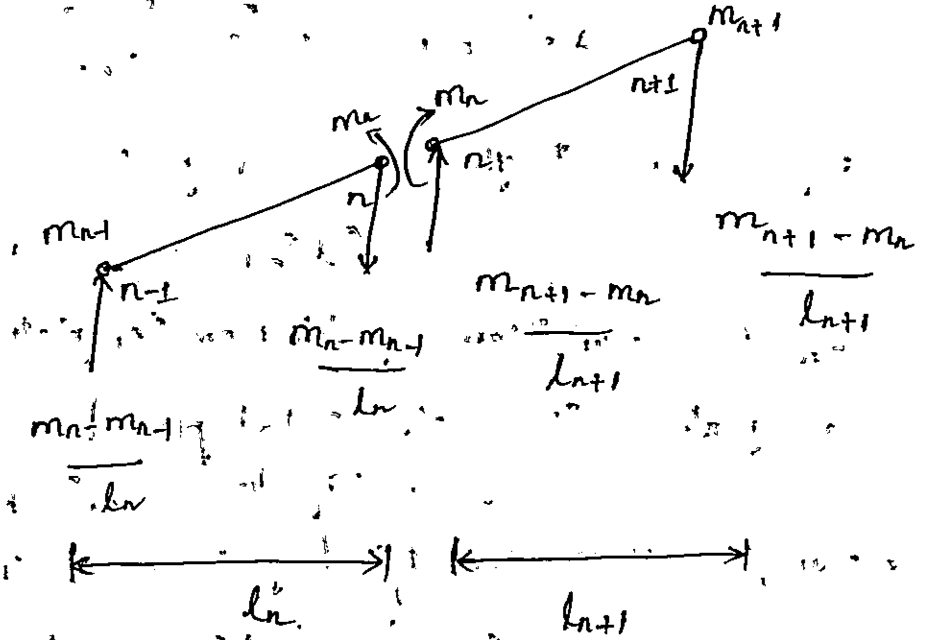
$$\frac{T_{n-1}}{A_n} + 2\left[\frac{T_n}{A_n} + \frac{T_n}{A_{n+1}}\right] + \frac{T_{n+1}}{A_{n+1}} = -\frac{1}{2}\left[\frac{M_{on}}{Z_n} + \frac{M_{o,\,n+1}}{Z_{n+1}}\right] \longleftrightarrow$$

Usually this equation is termed as theorem of three edge shear. 9th to the three moments in structural analysis

let us consider the plates of unit length.

Effect of Joint moments;

Because the plates are rigidly connected together, moments will develop at joints. Let us consider them to be sagging moments. These moments will result in upward reactions at the ridges.



Thus at ridges n, the moments will cause an upward reaction of

$$\frac{M_{n+1} - M_n}{l_{n+1}} - \frac{M_n - M_{n-1}}{l_n}$$

But we know that such forces do not really exist at the ridges. To realize this condition, it is necessary to apply downward forces such as ΔP_n at joints whose magnitudes are given by relation

$$\Delta P_n = \frac{M_{n+1} - M_n}{l_{n+1}} - \frac{M_n - M_{n-1}}{l_n}$$

Thus influence of joint moments can be accounted for by applying additional loads at ridges. These additional ridge loads may be resolving in to plate loads in the same loads manner as the ridge loads P_n. Thus the ridge loads ΔP_n applied at the joint n resolves itself in to plate loads. $\Delta S_n, S_{n-1}$ and $\Delta S_{n\,n+1}$ which are given by the following

$$\Delta S_{n\,n-1} = \Delta P_n \frac{\cos \phi_{n+1}}{\sin \theta_n}$$

$$\Delta S_{n,n+1} = \Delta P_n \frac{\cos \phi_n}{\sin \theta_n}$$

Again the net ~~state~~ load on plate

$$(\Delta S_{n\,n-1} - \Delta S_{n+1\,n})$$

In the calculation of plate moments, plate deflections, edge shears, and plate rotation, the total plate loads caused by ridge loads P_n and the additional ridge loads ΔP_n have to be used. It is thus seen that the total plate load P_n on plate n is

$$(S_{n,n-1} - S_{n-1\,n}) + (\Delta S_{n\,n-1} - \Delta S_{n-1\,n})$$

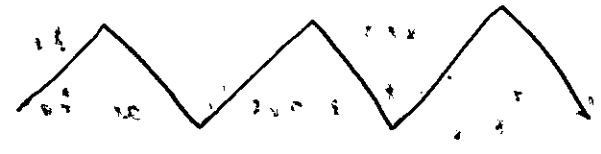
Analysis of folded plates;

The most popular methods for the analysis of folded plates are

1) Whitney's methods
2) Simpson's method

Whitney's method;

→ In this method, in a continous folded plate the end plates are treated as Cantilevers.



→ Calculate the ridge loads (P) from the UDL ($P_1, P_2, - - P_5$) acting over the adjacent plates.

6 where ever possible adopt the quantities by fourier series representation

→ After getting the ridge loads P_n, P_{n+1} arrive at the plate loads such as $S_{n,n+1} = \frac{P_n \cos\phi_{n+1}}{\sin\theta_n}$

→ Resolve the ridge loads at each and every point.

→ Apply the additional ridge loads such as.

$$\Delta P_n = \left[\frac{M_{n+1} - M_n}{l_{n+1}} - \frac{M_n - M_{n+1}}{l_n}\right]$$

At the joints 'n' (say) to account for the joint moments and then calculate additional plate loads by resolving them on the plates such as $\Delta S_{n,n+1} = \frac{\Delta P_n \cos\phi_{n+1}}{\sin\theta_n}$

→ Now, Compute the net plate loads 'R_n' is given by

$$R_n = (S_{n,n+1} - S_{n+1,n}) + (\Delta S_{n,n+1} - \Delta S_{n+1,n})$$

This value of 'R_n' involves unknown joint moments such as M_{n+1}, M_n ..., etc.

→ The longitudinal B.M due to plate action and due to resultant plate load 'R_n' is usually calculated by the formula

$$M_{o\,n} = \left(\frac{l}{\pi}\right)^2 R_n$$

Here usually 'R_n' is expressed as $R_n = R_n \frac{\sin \pi x}{l}$

→ The inplane deflection 'v_n' for the plate 'n' is calculated as follows

$$M_{o,n} = E I_n v_1''$$

$$v_1'' = \frac{-M_{on}}{E I_n}$$

$$v'' = (l/\pi)^2 R_n / E I_n$$

$$\frac{-\left(\frac{L}{\pi}\right)^2 R_n \sin\frac{\pi x}{L}}{E I_n}$$

In order to get the value of v_n, integrating the term twice WRT 'x'

$$v_1 = \frac{1}{EI_n}\left[\frac{l}{\pi}\right]^4 R_n \sin\frac{\pi x}{l} \quad \text{At } x = l/2$$

$$v_1 = \frac{1}{EI_n}\left[\frac{l}{\pi}\right]^4 R_n$$

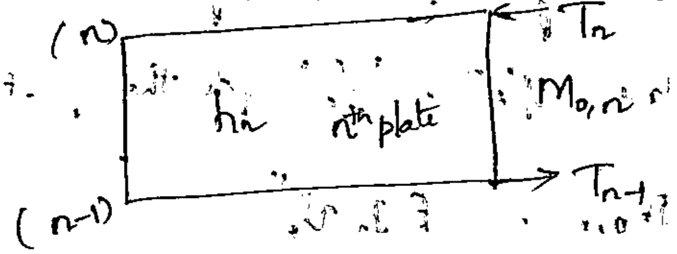
→ Now, the total inplane deflection 'v_n' of n^{th} plate is calculated using the formula

$$v_n = \frac{1}{EI_n}\left(\frac{l}{\pi}\right)^4 \cdot \left[R_n + \frac{T_n + T_{n-1}}{2}\left(\frac{\pi}{l}\right)^2 h_n\right]$$

$$T_n = \frac{d_n h_n^3}{2}$$

In the above expression, first term due to 'R_n' is calculated in the above step the second term of this expression due to edge shear force can be found as follows.

Consider n^{th} plate of a given folded plate having a width of h and subjected to edge shear T_n, T_{n-1} and the edges n, n-1



From the above fig, the moment at central section

$$M = -T_n\left[\frac{h_n}{2\cancel{\pi}}\right] + T_{n-1}\left[\frac{h_n}{2\cancel{\pi}}\right]$$

For the plate n at the joint n;

$$\psi_{n,n-1} = \frac{h_n}{6EI_n}(2m_n + m_{n-1})$$

For the plate (n+1) at the joint 'n'

$$\psi_{n,n+1} = \frac{h_{n+1}}{6EI_{n+1}}(2m_n + m_{n+1})$$

Because of above '2' slopes occuring at the joint 'n' the net slope is given by $(\psi_{n,n+1} - \psi_{n,n-1})$

Since the basic structure is monolithic in nature, at the joints no change in angle can occure Hence $(\phi_{n+1} - \phi_n)$

$$+ (\vartheta_{n,n+1} - \vartheta_{n,n-1}) + (\psi_{n,n+1} - \psi_{n,n-1}) = 0$$ for joint

Such equations are to be formulated at each and every joint. The folded plates connected to 'n' plates; the equations to be formulated (n-3), if the problem does not involve symmetry.

→ Solve such equations simultaneously finally we have to set the simultaneous equations at the joints

→ knowing the joint moments, R_n, M_o, n T_n for the plate 'n' with these values Compute the fibre stresses at the central section. Connect the fibre stress by multiplying the factor $\pi^3/32$, this connection is called first term of fourier series is considered in the analyse

Simpson's method;

The additional assumption is made that Δ_n and hence φ_n vary as the ordinates of a sine curve along the span of the folded plate.

$$\Delta_n = (\Delta_n)_c \sin\frac{\pi x}{l}$$

$$\varphi_n = (\varphi_n)_c \sin\frac{\pi x}{l}$$

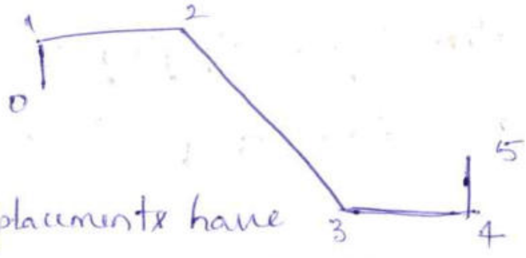
Assumption appears to be reasonable for a SS folded plate symmetrically loaded about its mid span. Perhaps it is more accurate to assume that these functions vary as the ordinates of elastic curve of SS beam loaded in the same manner as folded plate.

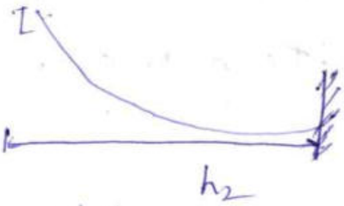
Consider a transverse

Step 1;

Consider a transverse section of unit length at mid span of a given plate. Assuming joints do not deflect, calculate reactions at joints and apply forces equal and opposite to these at joints. Resolve the loads, these applied in to plate loads, calculate bending stresses. Assume each plate to be free to bend independently. These stresses shall be designated as free-edge stresses. Next, establish stress compatibility at the common edges of adjacent plates by stress distribution. resulting stresses in the plates are those which occurs in folded plate if the joint do not deflect. This solution will be referred to as no-rotation solution.

→ Step 2





Effect of joint displacements have now to be accounted for by considering the rotation of plates 2, 3, and 4 the first and last plates being regarded as cantilevers. We start with plate 2 - let joint 2 deflect with, by an arbitrary amount Δ_{20} below the level of joint 1. The fixing moment induced at 2 as a result is equal to

$$\frac{3EJ_2\Delta_0}{h^2} = \frac{3EJ_2\psi_{20}}{h_2}$$

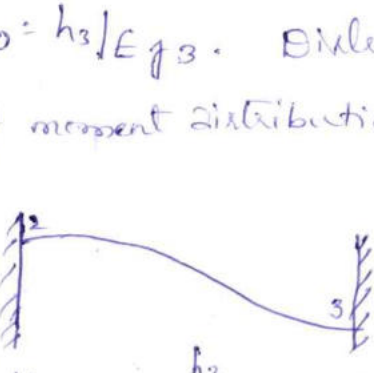
where $\psi_{20} = \Delta_{20}/h_2$

As Δ_{20} is arbitrary, ψ_{20} is an arbitrary rotation of plate 2. Let ψ_{20} be such that the magnitude induced at joint 3. Hence

$$\psi_{20} = h_2/EJ_2$$

The arbitralrary rotation and actual rotation of the plate are clearly related by an unknown constant k_2 such that $\psi_2 = k_2\psi_{20}$. This arbitrary moment of 3 at joint 2 is next distributed by moment-distribution procedure. The resulting joint moments and reactions are found. Forces equal and opposite to reactions are applied at joints and resolved along the plates as plate loads. The free edge stresses caused by loads are next determined.

→ <u>Step 3</u>; Consider the effect of arbitrary rotation of plate 3. As before $\psi_3 = k_3\psi_{30}$. The moment induced at joint 2 and 3 is $6EJ_3\Delta_{30}/h_3^2 = 6EJ_3\psi_{30}/h_3$. Let the arbitrary rotation ψ_{30} be such that the magnitude of moments induced is 6. Hence $\psi_{30} = h_3/EJ_3$. Distribute the moments of 6 units each at joints 2 and 3 by moment distribution. Arrive at the reactions at the joints and apply forces opposite to these at joints. Resolve these forces into plate loads



and compute the free edge stresses. Correct these by stress distribution to secure stress compatibility at common edges. The resulting stresses shall be refered as case III solution

Step 4;

The ~~plate deflections v_n~~ case IV solution corresponding to an abitrary rotation Ψ_{40} of plate 4 is worked in the same manner as the Case III solution.

$$\Psi_4 = k_4 \Psi_{40} \quad \& \quad \Psi_{40} = \frac{h_4}{Ej_4}$$

Step 5;

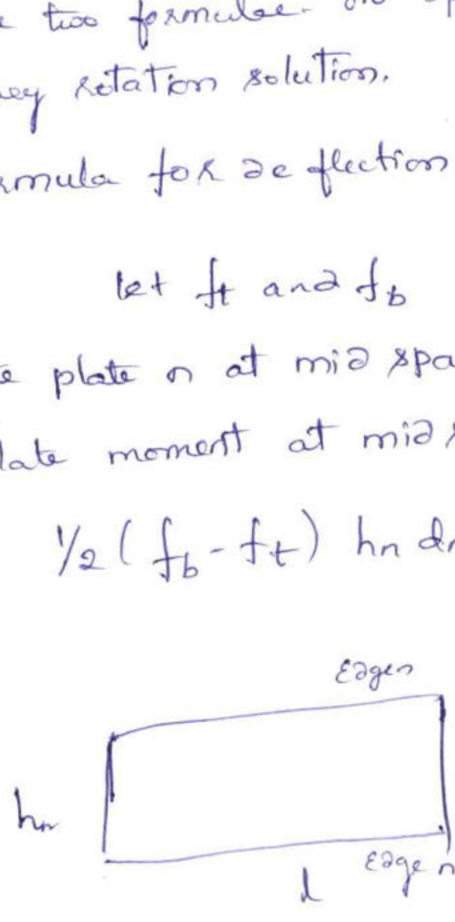
The plate deflections v_n are next worked out. The deflection v_n will consist of deflection corresponding to no-rotation solution plus k_2 times the deflection due to case II solution plus k_3 times the deflection resulting from the case III solution plus k_4 times deflection corresponding to case IV solution. To compute these deflections it is essential to have two formulae- one applicable to no. rotation solution and the other to orbitrary rotation solution.

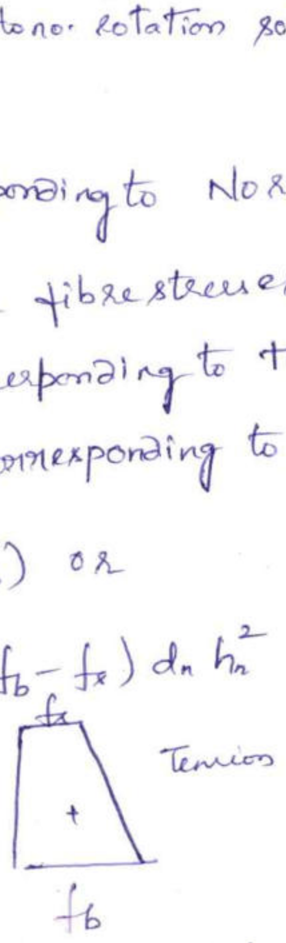
Formula for deflection corresponding to No rotation solution.

let f_t and f_b be the fibre stresses at top and bottom of the plate n at mid span, corresponding to the no-rotation solution The plate moment at mid span corresponding to stress distribution

is $\frac{1}{2}(f_b - f_t)\, h_n d_n (h_n/6)$ or

$$\frac{1}{2}(f_b - f_t)\, d_n h_n^2$$





Let us uniformaly distribute plate load 'P'. equating the bending and resisting moments at midspan

$$\frac{P' l^2}{8} = \frac{1}{12}(f_b - f_t)\, d_n h_n^2 .$$

The down ward plate deflection at midspan is $\frac{5}{384}\left[\frac{Pl^4}{EI_n}\right]$. Substituting for 'P' from (12·18) and noting that $I_n = 1/12$ $d_n h_n^3$, the plate deflection at mid span may be written as

$$\frac{5}{48E}\,\frac{l^2}{h_n}(f_b - f_t)$$

Formula for plate deflection corresponding to Arbitrary rotation

Consider again the same plate n with fibre stresses at midspan of f_b and f_t at its top and bottom fibres caused by an arbitrary rotation of that plate or any other plate. The resisting moment developed at the central section which is equal to the BM is again equal to $1/12\,(f_b - f_t)\, d_n h_n^2$. The load on the plate is proportional to φ_{no} for which a sine variation along the span has been assumed. Two integrations of this loading will yield the BM at the centre of the span which will be proportional to $\varphi_{no}\left[\frac{l^2}{\pi^2}\right]$. similarly the deflection is proportional to $(l^4/\pi^4 EI_n)\,\varphi_{no}$. Hence the deflection at mid-span obtained by multiplying the BM at the section by $l^2/\pi^2 EI_n$. But we have already seen that the BM is $\frac{1}{12}(f_b - f_t)\, d_n h_n^2$. The deflection at midspan is

$$\frac{l^2}{\pi^2 E h_n}(f_b - f_t)$$

Step 6;

From the plate deflections in step 5 we may now arrive at the transverse joint displacements $\omega_{n,n-1}$, $\omega_{n-1,n}$, etc., using formula. It is to be noted that the plate deflection calculated in step 5 and the transverse joint deflections computed from them in step 6 will involve the unknown constants k_2 k_3 k_4.

Step 7;

From the results of step 6, the plate rotations ψ_2, ψ_3 and ψ_4 may be calculated by using formula

$$\psi_n = \frac{1}{h_n} (\omega_{n,n-1} - \omega_{n+1,n})$$

$$\psi_{n+1} = \frac{1}{h_{n+1}} (\omega_{n,n+1} - \omega_{n+1,n})$$

step 8; Equate ψ_2, ψ_3. and ψ_4 calculated in step 7 to $k_2 \psi_{20}$, $k_3 \psi_{30}$ and $k_4 \psi_{40}$ to obtain a set of three linear simultaneous equations in the unknowns k_2 k_3 and k_4

Step 9;

Compute the fibre stresses in folded plate by combining the stresses of the no-rotation solution with k_2 times the stresses of the case II solution, k_3 times the stresses of the case III solution and k_4 times the stresses of the case IV solution. For an unsymmetrical problem, the simpson method leads to (n-2) simultaneous equations, if n is the number of plates. Even if the c/s of folded plate is symmetrical and unsymmetrical problem would result. If it is not symmetrically loaded wrt to its axis of symmetry

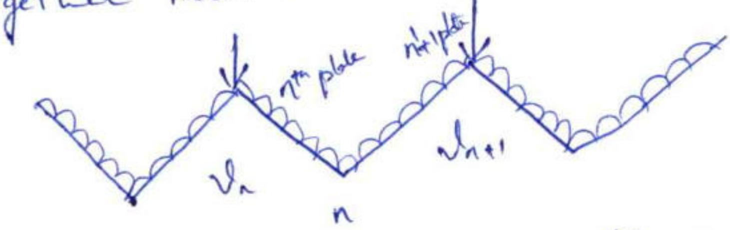
Plate deflection & Plate Rotation's

When a Continous folded plate is loaded by external loading, Two types of deflections are noticed.

(a) The deflections Taking place with in the plane of the plate and are called in plane deflections. They are designated as follows. For the Two adjacent plates n & n+1, the inplane deflection is designated as v_n & v_{n+1}

(b) The deflections that are Taking place at right angles to the plane of the plate occuring at joints. They are designated by 'w' with two suffixes.

For Example; $w_{n,n+1}$ the first suffix indicates, the joint at which outer plane deflections are calculated and the two suffixes put together indicate the two joints of the given plate.



Because of the above deflections of the plates, they tend to rotate and these rotations are calculated as follows

Ex; For plates n & n+1 if θ_n & θ_{n+1} are the rotation of the plate. Then

$$\theta_n = \frac{1}{h_n}(w_{n,n+1} - w_{n+1,n})$$

$$\theta_{n+1} = \frac{1}{h_{n+1}}(w_{n,n+1} - w_{n+1,n})$$

Finally, the change of the included angle of the joint 'n' is given by $(\theta_{n+1} - \theta_n)$

from relation

$$\frac{\partial v}{\partial x} = -\frac{1}{a}\frac{\partial u}{\partial \phi}$$

$$V' = -\frac{1}{a} U^{\cdot}$$

Differentiating thrice WRT x we get

$$V'''' = -\frac{1}{a} U^{\cdot'''} \longrightarrow 57$$

substituting value of U''' in equation (57) from (56) eq

$$V'''' = -\frac{1}{a} \times \frac{D}{Eda^5} w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot} \longrightarrow 58$$

From 45 $w = \frac{\partial v}{\partial \phi}$ consider $w'''' = v''''^{\cdot}$

Differentiating $w = \frac{\partial v}{\partial \phi}$ WRT x for 4 times $w'''' = v''''^{\cdot}$

substituting value v'''' of (58) in the above equation we have

$$w'''' = \frac{-D}{Eda^6} w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$D = \frac{Ed^3}{12}$$

Simplifying the above

$$w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot} + 12\frac{a^6}{d^2} w'''' = 0$$

Since $D = \frac{Ed^3}{12(1-\nu^2)}$ $\nu = 0$ $D = \frac{Ed^3}{12}$

$$K = \frac{d^2}{12a^2}$$

$$w^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot} + \frac{a^4}{K} w'''' = 0 \longrightarrow 59$$

Known as schorer's diff equation for cylindrical shell

3. Classification of long shells and short shells
~~Bea~~

ASCE manual classifies shells as short if $l/a < 1.60$

$l/a > \pi$ Long shells

$$\frac{B}{(l^2 ad)^{1/4}} < 3$$ shell regarded as Long

$$\frac{B}{(l^2 ad)^{1/4}} > 5$$ shell considered as short .

shells witha parameter between 3and 5 are called as quasi short

Jakobsen parameters l and k b/n 4 and 7

with corresponding k values of 0.03 and 0.12 are regarded as long

l values b/n 10 and 20, with corresponding k values of 0.15 and 0.3 considered short.

shells with l values b/n 7 and 10, known as intermediate shells are very rarely used in practice

Comments on DKJ. Theory

Exact theory. There are adequate for the design of most reinforced concrete then shells met with in practice. To make we have

$$E I_n \frac{\partial^2 v}{\partial x^2} = M = \left(\frac{T_n \pm T_{n-1}}{2}\right) h_n$$

$$= \left(\frac{T_n + T_{n-1}}{2}\right) h_n \frac{\sin \pi n}{l}$$

Integrating, the above expression twice WRT 'x'

$$v_2 = \frac{1}{E I_n}\left[\frac{T_n + T_{n-1}}{2}\right]\left(\frac{l}{\pi}\right)^2 h_n \sin \frac{\pi x}{l}$$

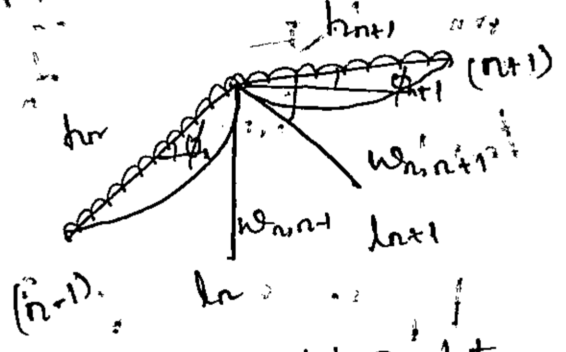
At $x = l/2$

$$v_2 = \frac{1}{E I_n}\left[\frac{T_n + T_{n-1}}{2}\right]\left(\frac{l}{\pi}\right)^2 h_n$$

With these values of inplane plate deflections 'v_n', calculate the outer plate deflections 'w' and calculate the rotations and from them calculate the change in the included angle at a particular joint WRT two adjacent plates !!

→ **Effect of slab Action;**

Here the change in angle at a particular joint due to slab action is calculated as follows



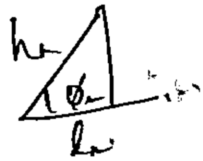
Consider 2 adjacent plates of a folded plate, assumed them as S.S at joints $(n-1)$, n; $(n+1)$. let P_n, P_{n+1} are the UDL's acting at n & $(n+1)^{th}$ plates. Due to external loading, the slopes developed at the joint 'n' for plates n & $n+1$ are given by $w_{n,}$ ~~θ~~$_{n-1}$ & $w_{n\,n+1}$ as shown in fig.

Here, the slopes are directly calculated from theorem of of three moments Area

$$w_{n\,n-1} = \frac{(P_n \cos\phi_n)\, h^3}{24 E I_n}$$

$$I = \frac{1 \times d_n^3}{12}$$

from the geometry. $\cos\phi_n = l_n / h_n$



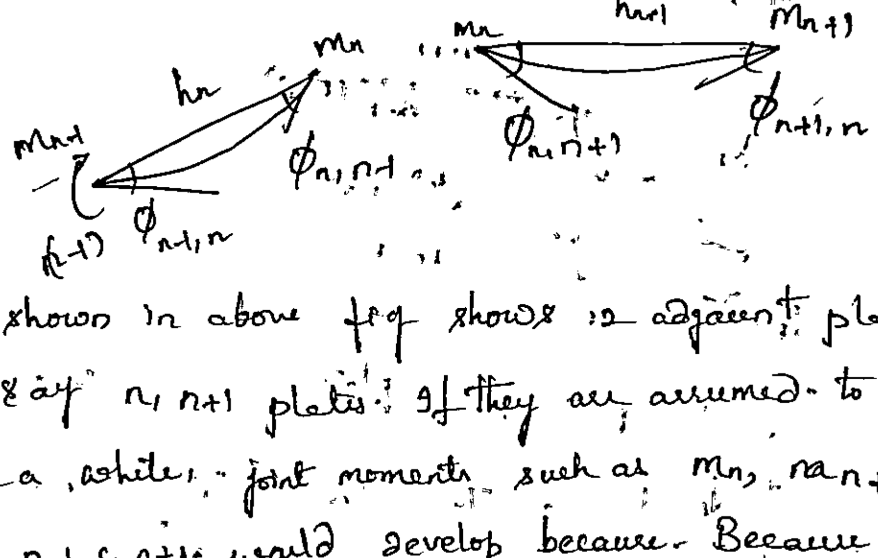
$$w_{n,n-1} = \frac{P_n\, l_n\, h_n^2}{2 \cdot E d_n^3}$$

Ily the other plate of slope of the same joint $(n+1)^{th}$ plate can be found to be

$$w_{n,n+1} = \frac{P_{n+1}\, l_{n+1}\, h_{n+1}^2}{2 E\, d_{n+1}^3}$$

Hence the net change is included angle at the joint n is given by $(w_{n,n+1})$ $(w_{n,n-1})$

(b) Here in this step the net change in angle caused by the joint moments is calculated as



As shown in above fig shows 2 adjacent plates of a continuous folded plate say n, $n+1$ plates. If they are assumed to be acting independently for a while, joint moments such as M_n, M_{n+1}, M_{n-1} at the joints n, $n-1$ & $n+1$ would develop because. Because of the joint moments the slopes are developed at the joints are represented φ with two suffix as before. In present case these values of slopes are calculated directly from Area moment theorem

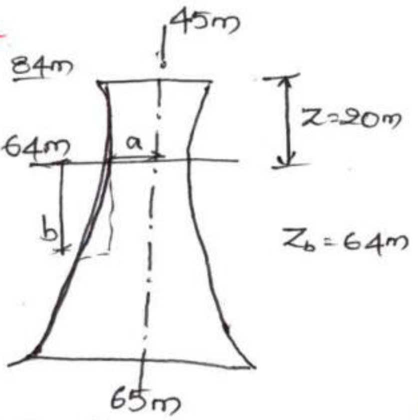
A hyperbolic cooling tower of height 84m has the following data

Top diameter = 45m

Throat diameter = 42m

Density of concrete = 24kN/m³

referring to fig 14.5



$t = 22.5m \quad Z_t = 20m$

$a = 21m \quad Z_b = 64m$

$$\left[\frac{r_0^2}{a^2} - \frac{Z^2}{b^2}\right] \quad \left[\frac{t^2}{a^2} - \frac{Z^2}{b^2}\right] = 1$$ where r_0 = radius at any section

$$b = \left[\frac{aZ_t}{\sqrt{t^2-a^2}}\right] \quad \therefore \; b = \left[\frac{21\times 20}{\sqrt{22.5^2-21^2}}\right] = 53m$$

S = Radius of base section

$$S = a\sqrt{1+\frac{Z_b^2}{b^2}} \text{ or } 21\sqrt{1+\frac{64^2}{53^2}} = 32.5m$$ Diameter at base = 65m

For base section, we have

$$\tan\phi = \frac{b}{a}\sqrt{\frac{r_0^2}{r_0^2-a^2}} \text{ or } \frac{53}{21}\sqrt{\frac{32.5^2}{32.5^2-21^2}} = 3.32$$

$$\cos\phi = \left[\frac{1}{\sqrt{1+\tan^2\phi}}\right] = 0.28$$

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}\xi_1 \quad \xi_1 = \frac{\cos\phi\sqrt{a^2+b^2}}{a} = \frac{0.28\sqrt{21^2+53^2}}{21} = 0.75$$

$$t = \sqrt{1+\frac{Z^2}{b^2}} \quad 21\sqrt{1+\frac{20^2}{53^2}} = 22.5m$$

$$\tan\phi = \frac{b}{a}\sqrt{\frac{r_0^2}{r_0^2-a^2}}, \quad \frac{53}{21}\sqrt{\frac{22.5^2}{22.5^2-21^2}} = 7.1 \quad \cos\phi = 0.14$$

$$\xi_0 = \frac{0.14\sqrt{21^2+53^2}}{21} = 0.38$$

Membrane forces

At top section

$$N_\theta = \left[\frac{ga^2}{\sqrt{a^2+b^2}}\right]\left[\frac{\xi_0}{\sqrt{1-\xi_0^2}}\right] = -\left[\frac{24\times 21^2}{\sqrt{21^2+53^2}}\right]\left(\frac{0.375}{\sqrt{1-0.375^2}}\right) = -74kN/m$$

At base section

$$N_\phi = \frac{-g}{4}b^2\sqrt{a^2+b^2}\left[\frac{1-\xi_1^2}{a^2+b^2-a^2\xi_1^2}\right]\left[f(\xi_1)-f(\xi_0)\right]$$ $N_\phi = -810 kN/m$ compressive

$g = 24kN/m^3$, $a = 21m$, $b = 53m$, $\xi_1 = 0.75$, $f(\xi_1) =$ ~~0.7~~ 55.374 $f(\xi_0) = 1.661$

III^{ly} $N_\theta = -264 kN/m$

Design of shell section and reinforcement

$N_\phi = 810 kN/m$ Using M_{20} grade concrete and fe415 steel $\sigma_{cc} = 0.5\times 5 = 2.5 N/mm^2$

$$\frac{N_\phi}{1000t} = \sigma_{cc} \quad t = 324mm$$

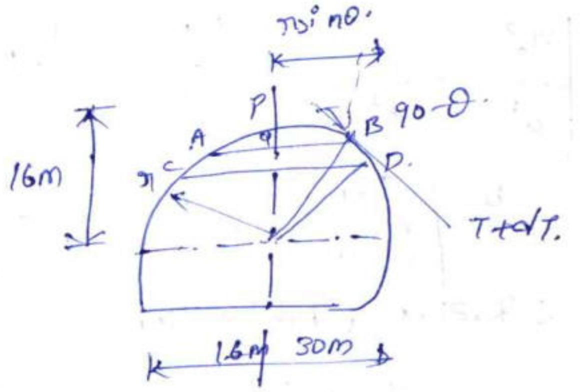
Base dia (a) = 30m

Rise of done (b) = 16m

~~eg~~ = $\cos\theta = \frac{d}{\sqrt{d^2+r^2}}$

$= \frac{30}{\sqrt{(30)^2+(16)^2}}$

$\cos\theta = 0.882$; ~~at~~.



T.L (W) = D.L + L.L + F.F.

F.L will take as $1 N/m^2$.

$w = 2.150 + 1.500 + 1.0$

$w = 4.65\ KN/m^2$ $\rightarrow$ M.T = 39.532

$h = \frac{wr \times (\cos\cancel{\phi} - 1)}{(1+\cos\phi)}$

$= \frac{4.65 \times 16 \times (0.882-1)}{1+0.882}$

circumferential force, h = KN.

$\frac{-8.7792}{1.882}$

hoop ~~stress~~ = $\frac{TH}{\cancel{bd}}$

Adopt a shell thickness of 350m at base gradually reducing to 150mm at top

min reinforcement = 1% of c/s area = $\frac{1}{100} \times 1000 \times 350 = 3500\ mm^2$

Provide 20mm ϕ bars @ 175 mm c/c on both faces

Min reinforcement in circumfer

ANNAMACHARYA UNIVERSITY

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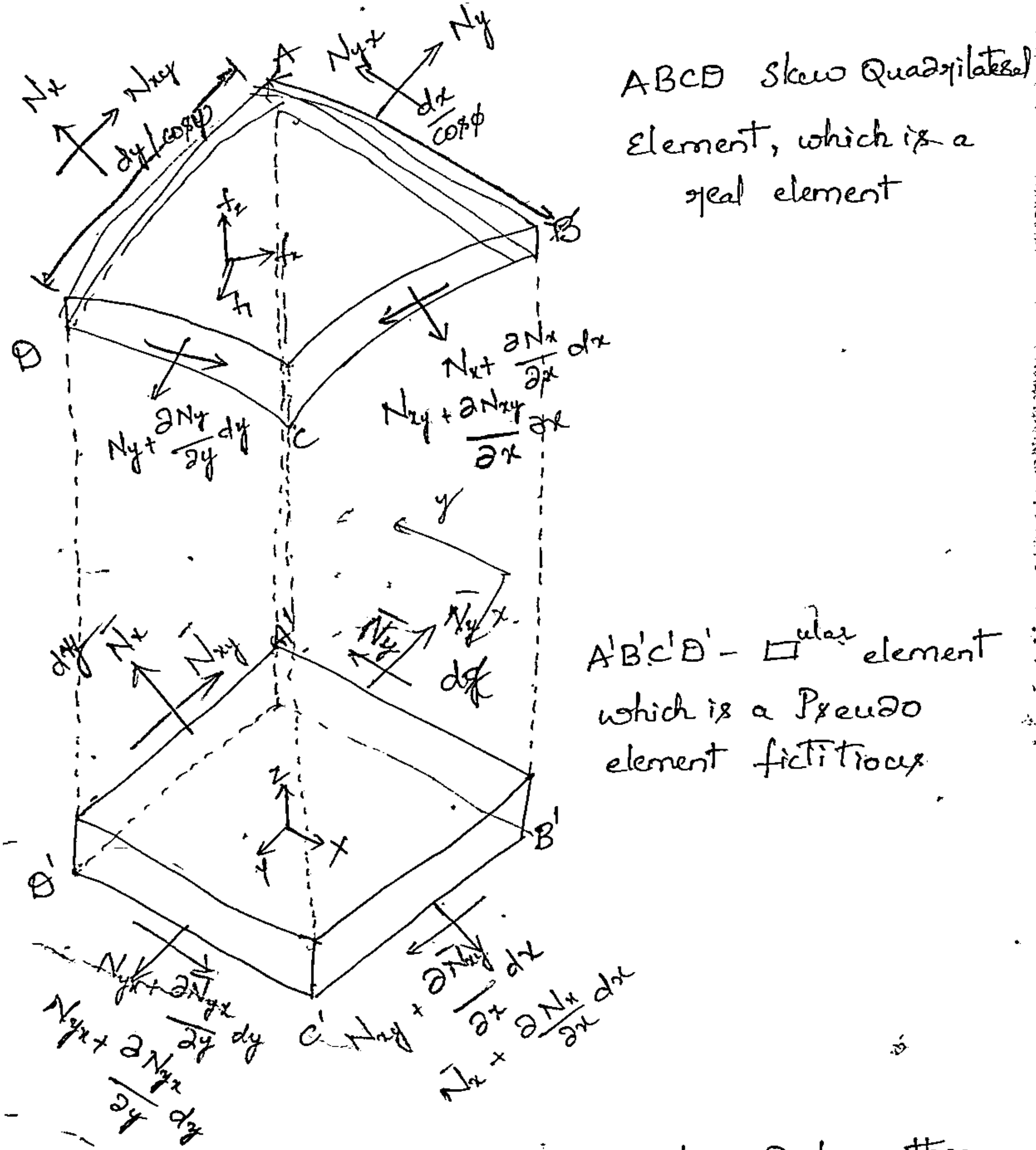
CIVIL ENGINEERING

Analysis of Shells and Folded Plates

UNIT-5

UNIT- IV

other than shells of Revolution



ABCD Skew Quadrilateral Element, which is a real element

A'B'C'D' - □ular element which is a Pseudo element fictitious

Let any shell of double curvature which is formed by other than shells of revolution be represented by the equation

$$z = f(x, y)$$

Consider any small element cut out from the above shell as shown in the above fig. which is represented by ABCD. θ is angle

Subtended at A other than 90°.

Hence the shell is doubly curved both the sides of element are also curved and let the sides make the angles ø and φ with x and y-axis as shown in fig. This is called real element.

Let the real stress resultants acting ~~each~~ over this real element with XOY plane be N_x N_y N_{xy} & N_{yx} acting per unit length.

Let f_x, f_y and f_z be the real load components acting per unit area. As shown in fig.

The dispersion of various stress resultants acting over the real element as shown in fig.

let A'B'C'D' be the normal projection of the real element ABCD over the horizontal element, called □ [illegible] element, which is called Pseudo or fictitious element. The angle subtended at A is 90°.

The sides of this element is parallel to the coordinate axis.

The Pseudo stress resultants $\bar{N}_x$ $\bar{N}_y$ $\bar{N}_{xy}$ and $\bar{N}_{yx}$ acting on the element as shown in fig, acting per unit length.

Let XYZ are Pseudo loads acting per unit area over the pseudo element.

$$A'B' = C'D' = dx$$

$$A'D' = B'C' = dy$$

Then connecting both the elements from the geometry of the element ABCD

$$AD = BC = \frac{dy}{\cos\psi}$$

$$AB = CD = \frac{dx}{\cos\phi}$$

On the surface of the shell, the radius vector 'r' at any point can be represented by

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$\hat{i}\ \hat{j}\ \hat{k}$ → unit vectors along x, y & z axis

Consider $\frac{\partial r}{\partial x} = \hat{i} + \frac{\partial z}{\partial x}\hat{k}$

$$\frac{\partial r}{\partial y} = \hat{j} + \frac{\partial z}{\partial y}\hat{k}$$

Referring to element, the length AB measured on the shell

$$AB = \left|\frac{\partial r}{\partial x}\right| dx = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2} \times dx$$

length AD measured on the shell is given by $= \sqrt{1+p^2}\ dx \longrightarrow a$ $\quad p = \frac{\partial z}{\partial x}$

$$AD = \left|\frac{dr}{\partial y}\right| dy = \sqrt{1+\left(\frac{\partial z}{\partial y}\right)^2} \times dy \qquad 88$$

$$= \sqrt{1+q^2}\ dy \rightarrow b \quad q = \left(\frac{\partial z}{\partial y}\right)$$

From the above diagram considering the geometry of two elements following relation also holds good.

$$\left.\begin{aligned}\cos\phi &= \frac{A'B'}{AB} = \frac{dx}{\sqrt{1+p^2}dx} = \frac{1}{\sqrt{1+p^2}} \longrightarrow a\\ \cos\psi &= \frac{A'D'}{AD} = \frac{dy}{\sqrt{1+q^2}\,dy} = \frac{1}{\sqrt{1+q^2}} \longrightarrow b\end{aligned}\right\} 89$$

The angle between the sides AB & AD which is θ is found by forming the dot product of two vectors $\frac{\partial r}{\partial x}$ & $\frac{\partial r}{\partial y}$

$$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \left[\hat{i} + \frac{\partial z}{\partial x}\hat{k}\right] \times \left[\hat{j} + \frac{\partial z}{\partial y}\hat{k}\right]$$

Evaluating dot product we can see that

$$\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = pq$$

The dot product is also equal to the length of 2 vectors and cosine of the angle θ included b/n them this gives

$$\sqrt{1+p^2}\,\sqrt{1+q^2}\,\cos\theta = pq$$

11/25

$$\cos\theta = \frac{pq}{\sqrt{1+p^2}\,\sqrt{1+q^2}} \longrightarrow 67$$

The area of the element ABCD can be calculated by forming the cross product of two vectors

$$\left(\frac{\partial r}{\partial x}\right)dx \times \left(\frac{\partial r}{\partial y}\right)dy = \left[\hat{i} + \frac{\partial z}{\partial x}\hat{k}\right]dx \times \left[\hat{j} + \frac{\partial z}{\partial y}\hat{k}\right]dy$$

$\sqrt{1+p^2+q^2}$

Evaluating the cross product and finding the magnitude of this product

$$\sqrt{1+p^2+q^2}\;dx\,dy$$

But from the principles of vector analysis this quantity is nothing but area of element ABCD

$$\text{Area of element ABCD} = \sqrt{1+p^2+q^2}\;dx\,dy \longrightarrow 68$$

Relation b/n Real and Pseudo elements stress resultants;

For example the pseudo stress resultant $\bar{N}_x$ is such that it exerts some force in x-direction on a projected side A'D' as a real stress resultant N_x there on the side AD

Force exerted by $\bar{N}_x$ on the side A'D' WR to x-direction

$\bar{N}_x \, dy \longrightarrow a$

Ily

Force exerted by the real stress resultant on the side AD can be calculated as follows

Force exerted on the side AD of the element is $N_x \times \frac{dy}{\cos\psi}$

But the element is inclined WR to XOY plane. Hence to obtain the horizontal component along XY plane is given by

$$N_x \times \frac{dy}{\cos\psi} \times \cos\phi \longrightarrow b$$

Equating a and b

$$\bar{N}_x \, dy = N_x \cdot \frac{dy}{\cos\psi} \cos\phi \longrightarrow c$$

Substituting values $\cos\phi$ and $\cos\psi$ in the above equation, we get

$$N_x = \bar{N}_x \sqrt{\frac{1+p^2}{1+q^2}} \longrightarrow a$$

$$N_y = \bar{N}_y \sqrt{\frac{1+q^2}{1+p^2}} \longrightarrow b$$

$$N_{xy} = \bar{N}_{xy} \quad \left[N_{xy} \, dy = N_{xy} \frac{dy}{\cos\psi} \cos\phi \right]$$

$$N_{xy} = \bar{N}_{xy}$$

The real loads and fictitious loads are related as follows

Real load × Area of ABCD = fictitious load × Area projected A'B'C'D'

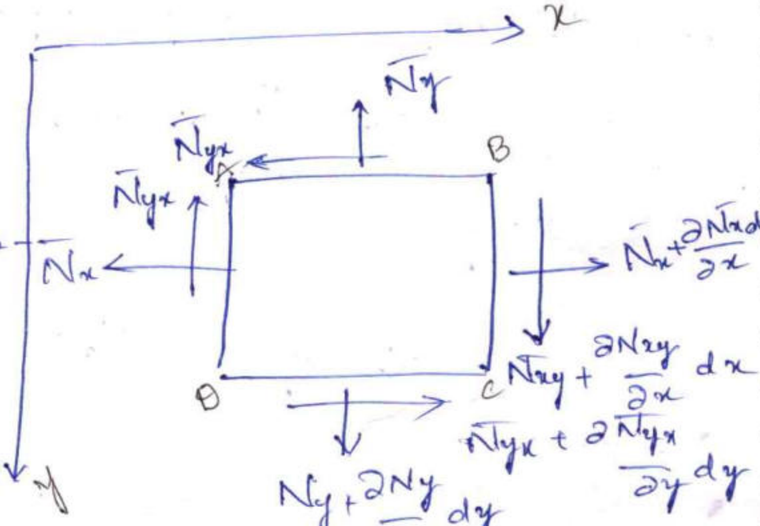
$$f_x \sqrt{1+p^2+q^2}\, dx\, dy = X(dx \times dy)$$

$$\left.\begin{aligned} X &= f_x\sqrt{1+p^2+q^2} \longrightarrow a \\ Y &= f_y\sqrt{1+p^2+q^2} \longrightarrow b \\ Z &= f_z\sqrt{1+p^2+q^2} \longrightarrow c \end{aligned}\right\}$$

||ly

Equations of Equilibrium;

With respect to the forces acting on the projected element A'B'C'D' the equilibrium equations along x and y axis



A/c to theory of elasticity

$$\left.\begin{aligned} \frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{yx}}{\partial y} + X = 0 \longrightarrow a \\ \frac{\partial \bar{N}_y}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial x} + Y = 0 \longrightarrow b \end{aligned}\right\}$$

Projected element on xy plane

fig (b)

For getting equations of equilibrium along 3 axis various procedures to be followed.

Effect of N_x

Vertical component of normal force acting on face AD

$$= \left(N_x \times \frac{dy}{\cos\psi} \right) \sin\phi \uparrow$$ let this face act upwards

$X \div$ by $\cos\phi$ we get

$$= N_x \times \frac{dy}{\cos\psi} \times \sin\phi \times \frac{\cos\phi}{\cos\phi}$$

$$= N_x \times dy \; \frac{\cos\phi}{\cos\psi} \times \frac{\sin\phi}{\cos\phi}$$

$$= N_x \times dy \times \frac{\cos\phi}{\cos\psi} \times \tan\phi$$

$$= N_x \times dy \times \frac{\cos\phi}{\cos\psi} \times \frac{\partial z}{\partial x}$$ $N_x \, dy \times \frac{\cos\phi}{\cos\psi} = \bar{N}_x \, dy$

From © above equation can be written as

$$\bar{N}_x \cdot dy \times \frac{\partial z}{\partial x} \uparrow \longrightarrow ⓓ$$

The vertical component of force acting on the opposite face BC can be written as

$$\bar{N}_x \, dy \times \frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left[\bar{N}_x * \frac{\partial z}{\partial x} \right) dx \, dy$$

Now the net resultant of vertical forces acting on the face AD and BC is given by

$$\frac{\partial}{\partial x}\left(\bar{N}_x \frac{\partial z}{\partial x} \right) dx \, dy \downarrow \longrightarrow d'$$

Effect of Ny;

working out in a similar manner the net resultant of vertical forces acting on the two opposite faces AB and CD can be written by cyclic order as follows

$$\frac{\partial}{\partial y}\left(\bar{N}_y \frac{\partial z}{\partial y}\right) dx\, dy \longrightarrow$$

$$N_y \times \frac{dx}{\cos\phi} \times \sin\psi$$

$$e\, N_y \times \frac{dx}{\cos\phi} \times \sin\psi \times \frac{\cos\psi}{\cos\psi}$$

$$N_y \times dx \times \frac{\sin\psi}{\cos\psi} \times \frac{\cos\psi}{\cos\phi}$$

$$N_y \times dx \times \frac{\partial z}{\partial y}$$

Effect of N_{xy} or $\bar{N}_{xy}$

Vertical component of shear force acting on the side AD is given by

$$N_{xy} \times \frac{\partial y}{\cos\psi} \sin\psi$$

$$N_{xy} \times dy \times \tan\psi$$

$$\bar{N}_{xy} \times dy \times \frac{\partial z}{\partial y},$$

Similarly the vertical component of shear force

N_{yx} o $\bar{N}_{yx}$ acting on face BC given by

Now the resultant of the above two forces is

$$- N_{xy}\, \cancel{dy \frac{\partial z}{\partial y}} + N_{xy}\, \cancel{\partial y} \frac{\partial z}{\partial y} + \frac{d}{dx}\left(\bar{N}_{xy} \times \frac{\partial z}{\partial y}\right) dx\, dy$$

$$\frac{\partial}{\partial x}\left(N_{xy} \times \frac{\partial z}{\partial y}\, dx\, dy\right) \longrightarrow (f)$$

Effect of N_{yx} along Z-axis

Proceeding in a lly manner the resultant of shear force acting on other two opposite sides AB and CD can be found to be

$$\frac{\partial}{\partial y}\left(N_{yx} \frac{\partial z}{\partial x}\right) dx\, dy \downarrow \longrightarrow (g)$$

Due to the component of external load along Z axis is given by $Z\, dx\, dy \longrightarrow (h)$

Equilibrium equation along Z axis can be obtained by summing up of results given in efgh and equating them to zero and dividing by dx dy

$$\frac{\partial}{\partial x}\left(\bar{N_x}\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\bar{N_y}\frac{\partial z}{\partial y}\right)+\frac{\partial}{\partial x}\left(\bar{N_{xy}}\frac{\partial z}{\partial y}\right)+\frac{\partial}{\partial y}\left(\bar{N_{xy}}\times\frac{\partial z}{\partial x}\right)+Z=0$$

Differentiating all the first four terms as UV products simplifying and Re arranging the terms we get the following

$$\left(N_x\times\frac{\partial^2 z}{\partial x^2}+2N_{xy}\frac{\partial^2 z}{\partial x\partial y}+\bar{N_y}\frac{\partial^2 z}{\partial y^2}\right)+\frac{\partial z}{\partial x}\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right]+\frac{\partial z}{\partial y}\left[\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}\right]+Z=0$$

Making use of results of first ⓐ ⓑ we knowing that

$$\frac{\partial z}{\partial x}=P \quad \frac{\partial z}{\partial y}=q$$

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=-X$$

$$\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=-Y$$

Above equation can be re written as follows

$$N_x\frac{\partial^2 z}{\partial x^2}+2N_{xy}\frac{\partial^2 z}{\partial x\partial y}+\bar{N_y}\frac{\partial^2 z}{\partial y^2}=Px+qY\cancel{+}Z\rightarrow$$

$$r=\frac{\partial^2 z}{\partial x^2} \qquad s=\frac{\partial^2 z}{\partial x\partial y} \qquad t=\frac{\partial^2 z}{\partial y^2}$$

Above equation takes following form

$$rN_x+2sN_{xy}+tN_y=PX+qY+Z\longrightarrow$$

$$-Z+PX+qY$$

Stress functions; Pucher's function

In order to solve the various equilibrium equations and to express them in a particular form a scientist by name pucher has introduced stress functions as follows

$$\left.\begin{aligned} \bar{N}_x &= \frac{\partial^2 \phi}{\partial y^2} - \int X\,dx \longrightarrow a \\ \bar{N}_y &= \frac{\partial^2 \phi}{\partial x^2} - \int y\,dy \longrightarrow b \\ \bar{N}_{xy} &= \frac{-\partial^2 \phi}{\partial x\,\partial y} \longrightarrow c \end{aligned}\right\} 72$$

where ϕ is a pucher's function

Subs of thue results back in to equations

$$r \times \frac{\partial^2 \phi}{\partial y^2} - 2s\frac{\partial^2 \phi}{\partial x \partial y} + t \times \frac{\partial^2 \phi}{\partial x^2} = Px + QY - Z + r\int x\,dx + t\int y\,dy$$

Also the above can be written as Assuming

~~$Px + QY - Z + r\int x\,dx + t\int y\,dy$~~ $= L$

$$r \times \frac{\partial^2 \phi}{\partial y^2} - 2s\frac{\partial^2 \phi}{\partial x \partial y} + t \times \frac{\partial^2 \phi}{\partial x^2} = L \longrightarrow$$

After solving the equation 99 we can get the value of ϕ. Substituting thee values ϕ we get N_x N_y N_{xy} again substituting N_x N_y N_{xy} we can get actual ~~e~~ equation

General solution = particular integral + Homogenous equations

$$\phi = \phi_1 + \phi_2$$

$$= -\sum_{n=1,3,5}^{\infty} A_n \cosh \beta_n x \times \overset{\phi_1}{\cos} \lambda_n y - \frac{a^2}{4f_x} \overset{\phi_2}{P_0 y^2} \longrightarrow 73$$

Substituting this value of ϕ in the set of eq (a, b, c -)
Pucher's stress function, we can get the following Pseudo stress resultants

$$\bar{N}_x = \frac{\partial^2\phi}{\partial y^2} = \sum_{n=1,3,5}^{\infty} A_n \lambda_n^2 \cosh\beta_n x * \cos\lambda_n y - \frac{a^2}{2f_x} P_o \longrightarrow a$$

$$\bar{N}_y = \frac{\partial^2\phi}{\partial x^2} = \sum_{n=1,3,5}^{\infty} A_n \beta_n^2 \cosh\beta_n x * \cos\lambda_n y \longrightarrow b \qquad 74$$

$$N_{xy} = \frac{-\partial^2\phi}{\partial x\,\partial y} = -\sum_{n=1,3,5}^{\infty} A_n \beta_n \lambda_n \sinh\beta_n x * \sin\lambda_n y \longrightarrow c$$

where An is a constant which can be obtained from boundary conditions of the shell.

An is a

Observing the eq Above (. (a) Nx the first term is in the series form and the second term is a single term in order to have proper addition of two terms. let us express the second term in the form of a series as follow's

$$P_o = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} (-1)^{\left(\frac{n-1}{2}\right)} P_o \cos\lambda_n y$$

Substituting this in $N_x(a)$ we can get the two terms in the series form

$$P_o = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} (-1)^{\left(\frac{n-1}{2}\right)} P_o \cos\lambda_n y \qquad 75$$

$$\bar{N}_x \frac{\partial^2\phi}{\partial y^2} = \sum_{n=1,3,5}^{\infty} \left[A_n \lambda_n^2 \cosh\beta_n x \cos\lambda_n y - \frac{a^2}{2f_x} \frac{4}{n\pi} (-1)^{\left(\frac{n-1}{2}\right)} P_o \cos\lambda_n y \right.$$

In order to get An we have to make use of earlier boundary condition that transverse cannot recieve any load's normal to their plane which gives the following

$$\bar{N}_x = 0 \ @\ x = \pm a \ ; \ \bar{N}_y = 0 \ @\ y = \pm b$$

Substituting this boundary condition in the 10.7 a. the constant A is found to be

$$A_n = \frac{2\beta_o n^2 (-1)^{\left(\frac{n-1}{2}\right)}}{n\pi f_n \lambda_n^2 \cosh\beta_n} \longrightarrow 76$$

Substituting this value of A_n in 107 a b & c we can get the value of $\overline{N_y}$ $\overline{N_{xy}}$

$$\overline{N_x} = -\sum_{n=1,3,5}^{\infty} \left[\frac{2\beta_o n^2 (-1)^{\left(\frac{n-1}{2}\right)}}{n\pi f_n \lambda_n^2 \cosh\beta_n} \times \lambda_n^2 \cosh\beta_n x \times \cos\lambda_n y \times \frac{-a^2}{2f_x} \times \frac{4}{n\pi} (-1)^{\left(\frac{n-1}{2}\right)} P_o \cos\lambda_n y \right] \longrightarrow a \quad 77$$

$$\overline{N_y} = -\sum_{n=1,3,5}^{\infty} \frac{2\beta_o n^2 (-1)^{\left(\frac{n-1}{2}\right)}}{n\pi f_n \lambda_n^2 \cosh\beta_n} \beta_n^2 \cosh\beta_n x \cos\lambda_n y \longrightarrow b$$

$$\overline{N_{xy}} = +\sum_{n=1,3,5}^{\infty} \frac{2\beta_o n^2 (-1)^{\left(\frac{n-1}{2}\right)}}{n\pi f_n (\lambda_n^2 \cosh\beta_n} \beta_n \lambda_n \sinh\beta_n x \sin\lambda_n y \longrightarrow c$$

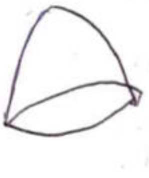
The above are the pseudo stress resultants. By making use of rotations relations a, b, c and a, b c. we canget ~~the~~ actual values of Nx Ny Nxy for the given shell

Rotational Paraboloid

Rotational paraboloid obtained by the involvement of two similar parabolas such that

$$\frac{f_x}{a^2} = \frac{f_y}{b^2} \qquad \cancel{\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2R}}$$

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R}$$



$$r = t = \frac{1}{R}$$

General Equation of rotational paraboloid

$$z = \frac{1}{2R}\left(x^2+y^2\right)$$

Stresses under Snow load

The relevant expressions for $\bar{N}_x$ $\bar{N}_y$ and $\bar{N}_{xy}$ to a, b, c

Because of $\frac{f_x}{a^2} = f_y/b^2$ it is seen ~~from~~ that $\beta_n = \lambda_n$

$$\bar{N}_x = \frac{P_0 a^2}{f_x}\left[\frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{(n-1)/2}\cos\lambda_n x\cos\lambda_n y}{n\cos\lambda_n a} - \frac{1}{2}\right]$$

$$\bar{N}_y = -\frac{P_0 b^2}{f_y}\left[\frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{n-1/2}\cos\lambda_n x\cos\lambda_n y}{n\cos\lambda_n a}\right]$$

$$\bar{N}_{xy} = \frac{-P_0 ab}{\sqrt{f_x f_y}}\left[\frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{(n-1)/2}\sinh\lambda_n x\sin\lambda_n y}{n\cos\lambda_n a}\right]$$

Hence for this shell the final Pseudo stress resultants $\bar{N}_x$ $\bar{N}_y$ N_{xy} can be obtained $\lambda_n = \beta_n$ $f_x = f_y$ $a = b$ in the set of 77c (a, b, c) we can get actual stress resultants

stresses under Dead load g

Provided the D.L g is expanded in double fourier series

$$g = \sum\sum A_{mn}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}$$

Membrane theory of Anticlastic shells;

Gauss curvature is -ve or zero

shells belongs to shells of double curvature other than shells of revolution and mostly shells of translation

Ex; Hyperbolic paraboloid

4. Usually formed by moving one convex parabola over the other concave parabola or vice versa. Sonamed horizontal sections are hyperbolic sections and vertical sections are parabolic section

General Equation of Parabola

$$z = \frac{x^2}{2R_1} - \frac{y^2}{2R_2} \longrightarrow 7$$

z is negligible thickness setting $z=0$

$$\frac{x^2}{2R_1} - \frac{y^2}{2R_2} = 0$$

$$\left(\frac{x}{\sqrt{2R_1}} + \frac{y}{\sqrt{2R_2}}\right)\left(\frac{x}{\sqrt{2R_1}} - \frac{y}{\sqrt{2R_2}}\right) = 0 \longrightarrow$$

Observing the above equation it represents eq of pair of straight lines lying entirely on the surface

From the geometry of the shell, it ϕ is also verified that above two pairs of straight lines from Asymptodes of the hyperbolas obtained by horizontal sections of the surface

From geometry observed that there inclination with

X-axis ~~x~~ $\vartheta = \tan\vartheta = \sqrt{\frac{R_2}{R_1}}$ $\sqrt{\frac{R_2}{R_1}}$ $\tan\vartheta = 1 = \tan 45^\circ$

$\vartheta = 45^\circ$

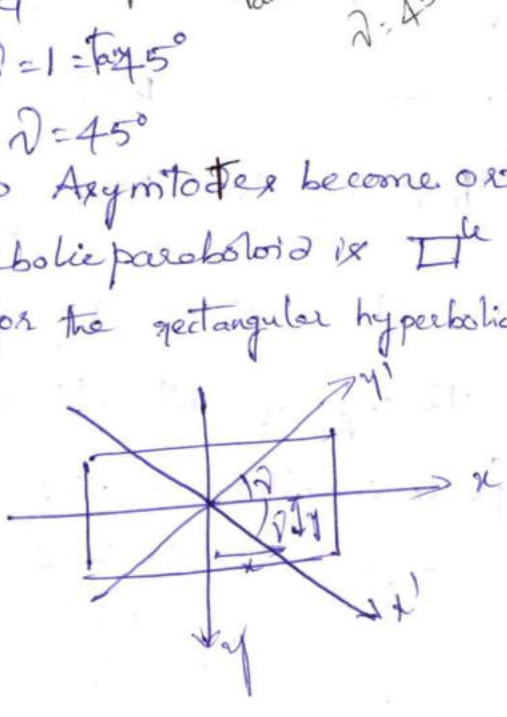
$R_2 = R_1$, $\tan\vartheta = 1 = \tan 45^\circ$

$\vartheta = 45^\circ$

In such a case the two Asymitotes become orthogonal to eachother. The resulting hyperbolic paraboloid is $\boxed{}^{le}$ hyperbolic paraboloid. To get the equation for the rectangular hyperbolic paraboloid the coordinate transformation

xy - WRT 'xy' plane

x'y' - WRT x'y' plane



P= Any point

WRT the above fig, the relation b/n two sets of coordinate axes from geometry as follows

$$x = (x'+y')\cos\vartheta \longrightarrow$$

$$y = (x'-y')\sin\vartheta \longrightarrow$$

Substituting these values in eq z and noting that $\tan\vartheta = \sqrt{\frac{R_2}{R_1}}$ the equation takes following form

$$z = \frac{2\sin^2\vartheta}{R_2} x'y' \longrightarrow$$

When the Asymptotes are orthogonal to each other

$$2\vartheta = 90^\circ \Rightarrow \vartheta = 45^\circ$$

$$\sin\vartheta = \frac{1}{\sqrt{2}} \quad z = \frac{x'y'}{R_2} \longleftrightarrow$$

Generally $z = \frac{xy}{c} \rightarrow$ where c is constant

The above equation is the general equation of called hyperbolic paraboloid with the asymptotes of the surface in terms of geometrical parameters some times constant C expressed as $C = \frac{ab}{f}$

Stress Resultants for Rectangular Hyperbolic paraboloid

$Z = xy/c$ WRt Asymptotes as the surface

$$p = \frac{\partial z}{\partial x} = y/c \qquad t = \frac{\partial^2 z}{\partial y^2} = 0$$

$$q = \frac{\partial z}{\partial y} = x/c$$

$$r = \frac{\partial^2 z}{\partial x^2} = 0$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = 1/c$$

Inserting all the above values in eq takes the following form

$$r\bar{N}_x + 2sN_{xy} + t\bar{N}_y = PX + QY - Z$$

$$\frac{2}{c}N_{xy} = (PX + QY - Z)$$

$$N_{xy} = \frac{cL}{2} \longrightarrow \text{eq} \{ PX + QY - Z = L]$$

From the first equilibrium equation $\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X = 0$

$$\frac{\partial \bar{N}_x}{\partial x} = -\left[\frac{\partial N_{xy}}{\partial y} + X\right]$$

Integrating on both sides, we have

$$\bar{N}_x = -\int\left(\frac{\partial N_{xy}}{\partial y} + X\right) dx + f_1(y)$$

Substituting the value of $\bar{N}_{xy}$ in the above, we get

$$\bar{N}_x = -\int\left[\frac{c}{2}\frac{\partial L}{\partial y} + X\right] dx + f_1(y) \longrightarrow 1\ (a)$$

Similarly, the second equilibrium eq (1. isolating N_y and proceeding in the same manner, we get

$$\bar{N}_y = -\int\left[\frac{c}{2}\frac{\partial L}{\partial x} + Y\right] dy + f_2(x) \longrightarrow (b)$$

Let the given □ular hyperboloid is under the action of D.L which g/unit area. Hence, the value of 'L' corresponding to this case can be obtained as follows

We have

$$Z = f_z\sqrt{1 + p^2 + q^2}$$

$$= g\sqrt{1 + y^2/c^2 + \frac{x^2}{c^2}}$$

$$= \frac{g}{c}\sqrt{c^2+x^2+y^2}$$

But $L = PX + QY - Z$ $[\because X = Y = 0]$ $L = -Z$

$$L = \frac{-g}{c}\sqrt{c^2+x^2+y^2} \longrightarrow$$

$$\bar{N}_{xy} = \frac{CL}{2}$$

$$= \frac{c}{2}\left[\frac{-g}{c}\right]\sqrt{c^2+x^2+y^2}$$

$$\bar{N}_{xy} = \frac{-g}{2}\sqrt{C^2+x^2+y^2}$$

Similarly, Substituting the 'L' value in , we get the values of $\bar{N}_x$, $\bar{N}_y$. After getting the values of Pseudo resultants making using of the relations

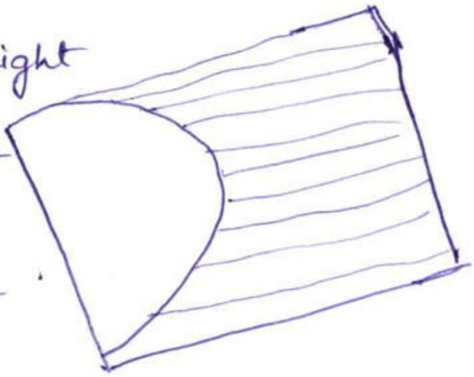
we can get the actual stress resultants.

CONOID;

→ Conoid is generated by a variable straight line moving parallel to

Conoid is another example for Anticlastic shells

A conoid is generated by variable straight line moving with one of its ends on a plane-curve and the other end on a straight line



The plane curve and the straight line are known as 'Directrices'.

→ Depending up on the type of curve of Director used for plane curves, Conoids are known as circular parabolic catenary conoids etc.

General equation of conoid is, $Z = f(y)\frac{x}{L}$

$p = \frac{1}{L} f(y)$ $\quad r = 0 \quad$ $t = f'(y)\frac{x}{L}$

$q = f'(y)\frac{x}{L}$ $\quad$ $s = \frac{1}{l} f'(y)$

Since $rt - s^2 < 0$

where the primes denote differentiation w.r.to y. knowing the values of r, s and t, the two principal curvatures at any point on the shell

$$\frac{1}{R_1} = \frac{r+t}{2} + \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}$$

$$\frac{1}{R_2} = \frac{r+t}{2} - \sqrt{\left(\frac{r-t}{2}\right)^2 + s^2}$$

For a conoid r being zero, $rt - s^2 < 0$. Hence the surface is anticlastic and its gauss curvature is -ve.

Folded plates are assemblies of flat plates rigidly connected together along their edges in such a way that the structural system capable of carrying loads without the need for additional supporting beams along mutual edges.