ANNAMACHARYA UNIVERSITY
EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
ESTD, UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)
Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Lecture Notes
on
Advanced Structural Analysis

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Advanced Structural Analysis

UNIT-1

Unit-1

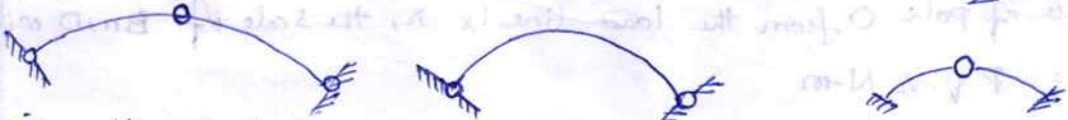
ARCHES

An arch is a curved structure that spans a space and may or may not support weight above it.

Structure capable of spanning a space while supporting significant weight eg; doorway.

A plane curved beam either a bar or a rib, supported at its ends and carrying transverse loads which are frequently vertical. Since the transverse loading at any section normal to the axis of the axis of the girder is at an angle to the normal face, an arch is subtended to three forces Thrust, shearforce & B.M.

- fixed Arch



Three Hinged Arch — Two Hinged Arch — Single Hinged Arch

Indeterminacy - 3-3=0 — 3-2=1 — 3-1=3

A Three hinged Arche is statically determinate

An arch is pure compression form. It can span a large area by resolving forces into compressive stresses and in turn eliminating tensile stresses

nged $I = m-(2j-r)$ $\quad 2-(2\times3-4)$ $\quad 2-(6-4) = 2-2=0$

inged $m-(2j-r)$ $\quad 1-(2\times2-4)$ $\quad 1-(4-4) = 1$

inged $= 2-(2\times3-6) = 2$

xed $= 1-(2\times2-6) = = 3$

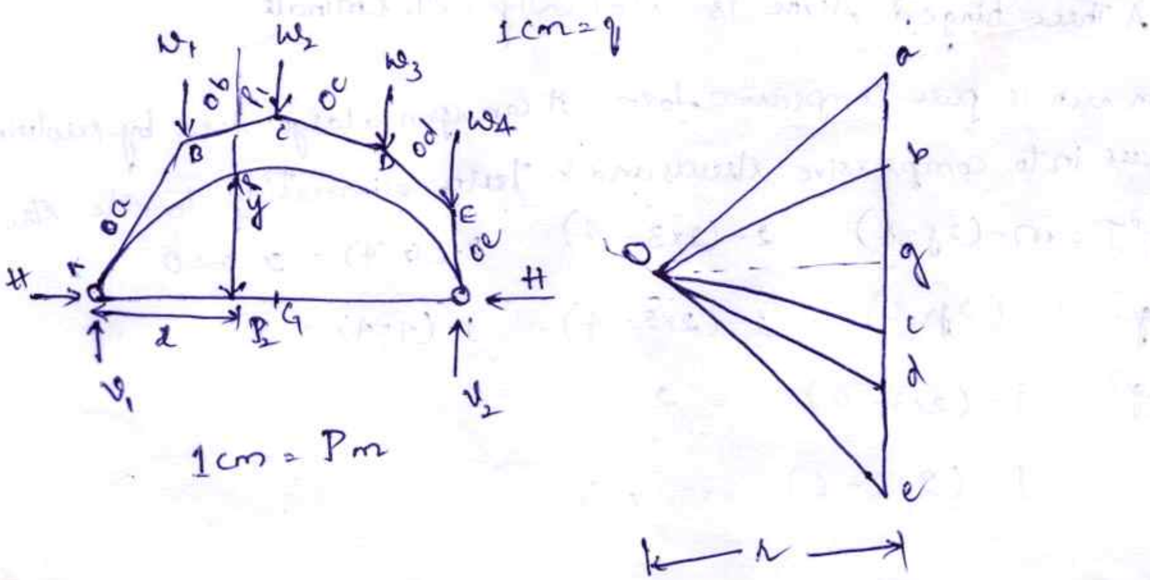
Eddy's Theorem;

Bending moment at any section of an arch is equal to the vertical intercept between the linear arch and the centre line of the actual arch.

Consider a section P at A distant x from A, of an arch. Let the co-ordinate of P be y. For the given system of loads, the linear arch can be constructed. Since funicular polygon represents the BMD to some scale, the vertical intercept $P_1 P_2$ at the section P will given the B.M due to external load system. If the arch is drawn to scale of 1cm = Pm, load diagram is plotted to scale 1cm = q N and if the distance of pole O from the load line is r, the scale of BMD will be

1cm = P. q. r N-m

Now theoretically B.m at P is given by



$$M_P = V_1 x - W_1(x-a) - H \times y = M_x - H \times y$$

where $M_x = V_1 x - W_1(x-a)$ usual B.M at a section due to load system on a simply supported beam

$$M_x = + (P_1 P_2) \times \text{Scale of B.M.D}$$

$$= (P_1 P_2)\ (P \times q \times r)$$

$$H \times y = (P P_2) \times \text{scale of BMD}$$

$$= P P_2\ (P \times q \times r)$$

Hence $M_P = M_x - H \cdot y = + P_1 P_2 (P q r) - P P_2 (P q r)$

$$= (P P_1)(P q r)$$

Hence the ordinate between the linear arch and the actual arch gives the bending moment.

This is known as Eddy's theorem Hence proved.

Three Hinged Arch:

A Three hinged arch is a statically determinate structure having a hinge at each abutment or springing, and also at the crown. There are in all 4 reaction components available. Three equations are available from static equilibrium and one additional equation is available from the fact that B.M at hinge and the crown is zero. Thus the value of H can be easily calculated for any given load system.

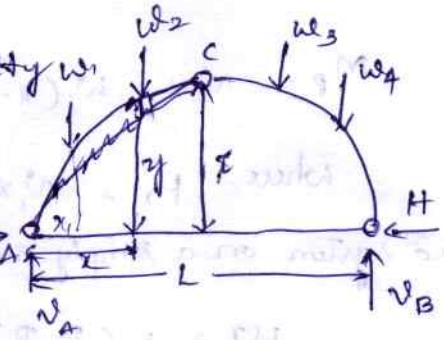
Let the arch is subjected to a number of load W_1, W_2 etc. Let the reactions at A and B be (H, V_A) and (H, V_B) respectively. Since the

BM at C is zero, we have

$M_C = V_A \times L/2 - W_2(L/2 - x) - W_1(L/2 - x_1) - H y$

$M_C = \mu_c - H \times y$

$H = \frac{\mu c}{y}$

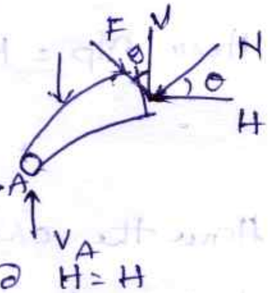


The value of H is thus known.

The value of V_A can be know by taking moments of all force about B

$\Sigma M_B = 0$ Similarly V_B at can be known by $\Sigma M_A = 0$

After having the known reaction components the value of radial shear (F) and Normal thrust (N) at any section P can be calculated with fig (b)



where equilibrium of left portion PA has been shown. The vertical and horizontal reactions on the section P are $V = V_A - W_1$ and $H = H$

Now resolving along the section at P, we get

$$F = V\cos\theta - H\sin\theta$$

Similarly, resolving normal to the section, we get

$$N = V\sin\theta + H\cos\theta$$

Three Hinged Parabolic Arch;

The equation of a parabola, with origin at the left hand hinge A can be written as

$$y = kx(L - x) \longrightarrow (1) \quad k \text{ is Constant}$$

$x = L/2$ Let $y = r$ = Central rise

Substituting in (1), we get

$$r = k\,\frac{L}{2}\left(L - \frac{L}{2}\right)$$

$$r = \frac{kL^2}{4}$$

$$k = \frac{4r}{l^2}$$

$$\tan\theta = \frac{dy}{dx} = \frac{4r}{l^2}(l - 2x)$$

$$\theta = \tan^{-1}\left(\frac{4r}{l^2}(l - 2x)\right)$$

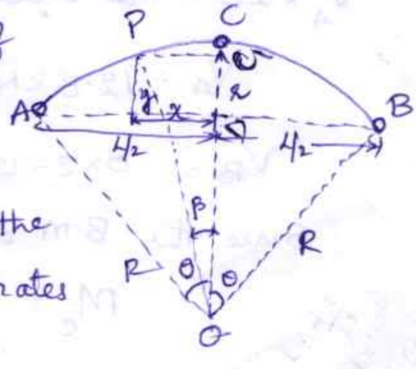
$$y = \frac{4r}{l^2}\,x(L - x)$$

Equation is called parabolic arch.

A/c to Eddy's theorem, the vertical intercept between the linear arch and the centre line of the actual arch gives the B.M at a section. Due to uniformly distributed load, the linear arch will be parabola. It will pass through the hinge at the crown. The centre line of the actual arch is also parabolic, passing through the central hinge. These two parabola's pass through three common points and hence they overlap each other. Therefore a parabolic arch will not have B.M due to UDL. It will be subjected to pure compression.

Three hinged circular Arch;

Let us now consider the centre line of the arch to be segment of a circle of radius R, subtending an angle of 2θ at the centre. It is always convenient to have the origin O, the middle of the span. Let (x, y) be the coordinates of the point P. Draw line PC_1 parallel to AB



Then $OP^2 = OC_1^2 + PC_1^2$

$$R^2 = \{y + (R - r)\}^2 + x^2$$

Connects y with x

Also $r(2R-r) = \frac{L}{2} \times \frac{L}{2} = \frac{L^2}{4}$

The values of radius can be calculated for the known values of the span and the rise

The coordinates of P can also be expressed as trigonometric functions. Thus, if OP makes an angle β with OC.

$$x = OP\sin\beta = R\sin\beta$$

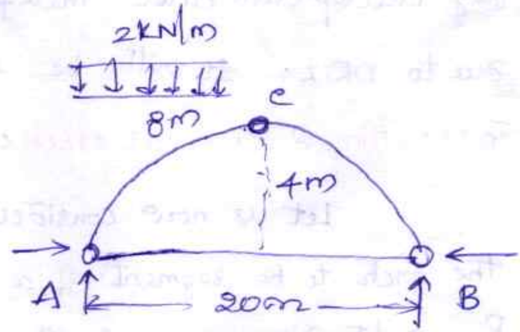
$$y = C_1D = OC_1 - OD = R\cos\beta - R\cos\theta = R(\cos\beta - \cos\theta)$$

A Parabolic arch hinged at the springings and crown has a span of 20m. The central rise of arch is 4m. It is loaded with a UDL of intensity 2kN/m on the left 8m length. Calculate the direction and magnitude of reactions at the hinges

b) BM, Normal thrust and shear at 4m and 15m from the left end and (c) maximum +ve & -ve BM.

Reaction at the hinges

for vertical reaction at A, Taking moments @ B. Thus $\Sigma M_B = 0$



$$V_A \times 20 - 2 \times 8 (20-4) \quad (8/2+12)$$

$$V_A = 12.8 kN$$

$$V_B = 8 \times 2 - 12.8 = 3.2 kN$$

Since the B.M at the hinges c is zero we have

$$M_C = 3.2 \times 10 - H \times 4 = 0$$

$$H = \frac{32}{4} = 8 kN$$

Reaction at A $R_A = \sqrt{V_A^2 + H^2} = \sqrt{12.8^2 + 8^2} = 15.09\ kN$

Its inclination with the horizontal

$$\tan\theta_A = \frac{V_A}{H} = \frac{12.8}{8} = 1.6 \quad \theta_A = 58^\circ$$

Reaction at B $R_B = \sqrt{V_B^2 + H^2} = \sqrt{3.2^2 + 8^2} = 8.62\ kN$

$$\tan\theta_B = \frac{V_B}{H} = \frac{8.62}{8} = 0.4 \quad \theta_B = 21^\circ 48'$$

BM, thrust and shear;

Equation of parabola $y = \frac{4h}{l^2} x(L-x)$

$$y = \frac{4 \times 4}{20^2} x(20-x) = \frac{x}{25}(20-x)$$

$$\frac{dy}{dx} = \frac{20-2x}{25}$$

At $x = 4m$, $y = \frac{4}{25}(20-4) = 2.56m$

$$\tan\theta = \frac{dy}{dx} = \frac{20-2\times4}{25} = 0.48 \quad \theta = 25^\circ 38'$$

$$M_4 = (12.8 \times 4) - (4 \times 2 \times 2) - (8 \times 2.56) = 14.72\ kNm$$

Vertical shear at the section

$V = 12.8 - 2\times4 = 4.8\ kN$ & $H = 8\ kN$

$F = V\cos\theta - H\sin\theta = 4.8 \times 0.901 - 8 \times 0.433 = 0.861\ kN$

$N = V\sin\theta + H\cos\theta = 4.8 \times 4.33 + 8 \times 0.901 = 9.286\ kN$

At $x = 15m$ $y = \frac{x}{25}(L-x) = \frac{15}{25}(20-15) = 3.0m$

$$\frac{dy}{dx} = \tan\theta = \frac{20 - 2\times15}{15} = 0.4$$

$\theta = 21°48'$ $\sin\theta = 0.3714$ and $\cos\theta = 0.9285$

$M_{1r} = (3.2\times5) - 8\times3 = -8\,KNm$

from 16.7(b)

$I = V\cos\theta - H\sin\theta$

$= 3.2\times0.9285 - 8\times0.3714 = 0$

$N = V\sin\theta + H\cos\theta$

$= 3.2\times0.3174 + 8\times0.9285 = 8.616\,kN$

(C) Maximum +ve and -ve B.M

Maximum +ve BM will occur some where under the UDL.

$$M_x = (12.8\,x) - \frac{2x^2}{2} - 8y$$

$$= 12.8x - x^2 - \frac{8x}{25}(20-x)$$

$$\frac{\partial M_x}{\partial x} = 12.8 - 2x - \frac{32}{5} + \frac{16}{25}x = 0 \quad @\ x = 4.7m$$

$$M_{max}(+ve) = 12.8\times4.7 - 4.7^2 - \frac{8}{25}(4.7)(20-4.7)$$
$$= 15\,kNm$$

Maximum -ve B.M

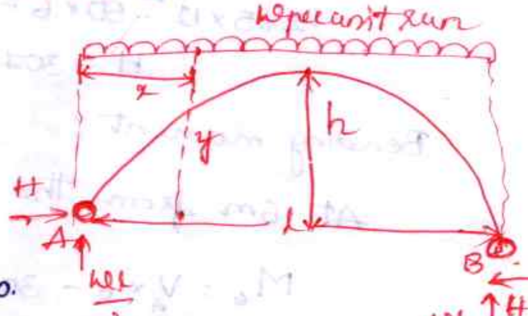
$$M_x = 3.2x - 8y = 3.2x - \frac{8x}{25}(20-x)$$

$$\frac{\partial M}{\partial x} = 3.2 - \frac{32}{5} + \frac{16x}{25}$$ from which $x = 5m$

Hence max -ve B.M occurs where the radial shear is zero

$$M_{-ve} = 3.2\times5 - \frac{8\times5}{25}(20-5) = -8\,kNm$$

A Three-hinged arch of span l and rise h carries a UDL of w per unit run over the whole span show that the horizontal thrust at each support is $\frac{wl^2}{8h}$.



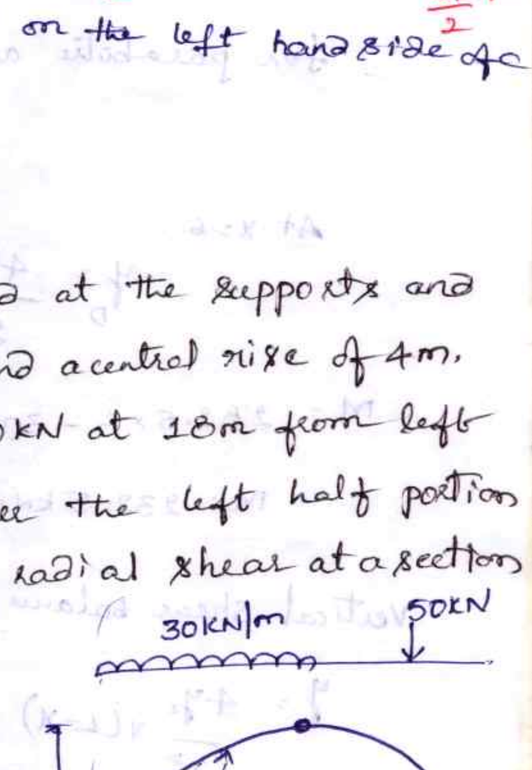
Each vertical = $\frac{wl}{2}$

let the horizontal thrust at each support be H. The BM at the crown hinge c is zero. Hence taking moments @C, of the forces on the left hand side of c we have

$$\frac{wl}{2} \times \frac{l}{2} = Hh + \frac{wl}{2} \times \frac{l}{4}$$

$$H = \frac{wl^2}{8h}$$

A Three hinged parabolic arch hinged at the supports and at the crown has a span of 24m and a central rise of 4m. It carries a concentrated load of 50 kN at 18m from left support and a UDL of 30 kN/m over the left half portion. Determine the moment, thrust and radial shear at a section of 6m from left support.



Support Reactions ;

Taking moments @ B $\Sigma M_B = 0$

$$V_A \times 24 - 30 \times 12 \left(\frac{12}{2} + 12\right) - 50 \times 6 = 0$$

$$V_A = 282.5 \text{ kN}$$

$$V_A + V_B = (30 \times 12) + 50$$

$$V_B = 410 - 282.5 = 127.5 \text{ kN}$$

Horizontal thrust

Taking moments @ crown 'c' from right hand side

$V_B \times 12 - 50 \times 6 - H \times 4 = 0$

$127.5 \times 12 - 50 \times 6 - H \times 4 = 0$

$H = 307.5\ kN$

Bending moment :

At 6m from the left support

$$M_6 = V_A \times 6 - 30 \times 6 \times \frac{6}{2} - H \times y_D$$

for parabolic arch

$$y = \frac{4y_c}{l^2} x(L-x)$$

At $x = 6$

$$y_D = \frac{4 \times 4}{24^2} \times 6(24-6) = 3m$$

$$M = 282.5 \times 6 - 307.5 \times 3 - 30 \times \frac{6^2}{2}$$

$$M = 232.5\ kNm$$

Vertical shear balance force $V = V_A - 30 \times 6 = 102.5\ kN$

$$y = \frac{4y_c}{L^2} x(L-x) = \frac{4y_c}{L^2}(Lx - x^2)$$

$$\tan\theta = \frac{dy}{dx} = \frac{4y_c}{L^2}(L-2x)$$

At $x = 6m$

$$\tan\theta = \frac{4 \times 4(24 - 2 \times 6)}{24^2}$$

$$\theta = 18.435'$$

Normal thrust $N = H\cos\theta + V\sin\theta$

$= 3075\cos 18°43.5' + 102.5\cos 18°43'$

Radial shear $F = V\cos\theta - H\sin\theta$

$=$

A Three hinged parabolic arch of span l and rise h carries a UDL of w per unit over the whole span. Show that arch is not subjected to any B.M at any section.

Horizontal thrust at each support $= \frac{wl^2}{8h}$

Equation to the arch with the end A as origin $y = \frac{4h}{l^2}x(l-x)$

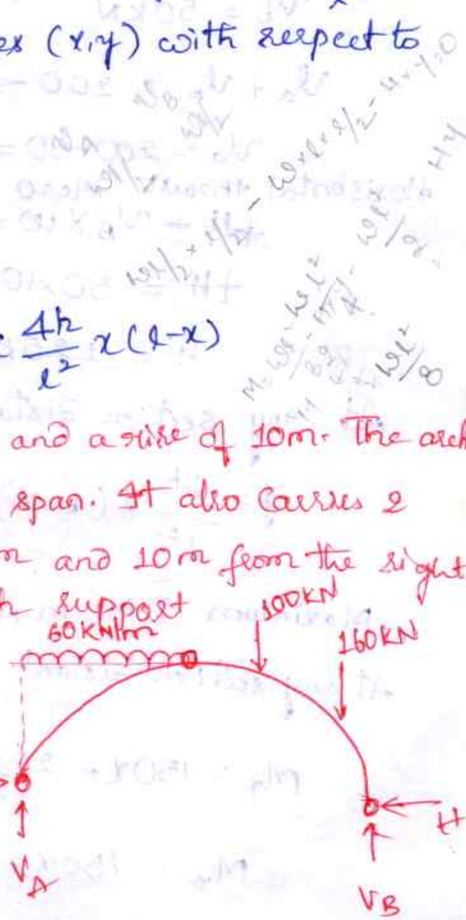
B.M at any section X having coordinates (x, y) with respect to A as origin is given by

$$M_x = \frac{wl}{2}x - \frac{wx^2}{2} - Hy$$

$$0 = \frac{wl}{2}x - \frac{wx^2}{2} - \frac{wl^2}{8h} \times \frac{4h}{l^2}x(l-x)$$

A Three hinged Arch has a span of 30m and a rise of 10m. The arch carries UDL of 60 kN/m on the left half of its span. It also carries 2 concentrated loads of 160 kN and 100 kN at 5m and 10m from the right end determine the horizontal thrust at each support

Let Support Reactions



$\Sigma M_A = 0$

$V_b \times 30 = 60 \times 15 \times \frac{15}{2} + 100 \times 20 + 160 \times 25$

$V_b = 425$ kN

$V_a + V_b = 60 \times 15 + 100 \times 10 \times 160 \times 5$ $V_a = 735$ kN

Taking moments about C of the forces on the left hand side of C, we have

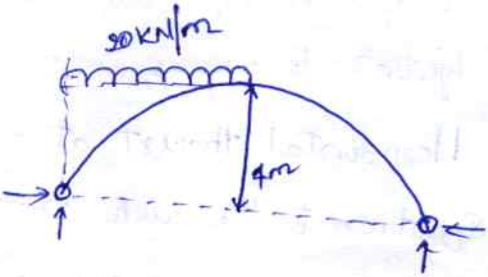
$M_C = 0$

$735 \times 15 - 60 \times 15 \times \frac{15}{2} + H \times 10 = 0$

$H = 4275\ kN$

A Three hinged parabolic arch of span 20 m and rise 4 m Carries a UDL of 20 kN per meter run on the left half of the span. find the max B.M for the arch.

Support Reaction



$\Sigma M_A = 0$

$V_b \times 20 - 20 \times 10 \times 5 = 0$

$V_b = 50\ kN$

$V_a + V_b = 200$

$V_a = 200 - 50 = 150\ kN$

Horizontal thrust $M_C = 0$

$4H - V_B \times 10 = 0$

$4H = 50 \times 10$

$H = 125\ kN$

At any section distance x from A or B

$$y = \frac{4h}{l^2} x(l - x) = \frac{4 \times 4}{20 \times 20} x(20 - x) = \frac{1}{25} x(20 - x)$$

Maximum B.M in AC

At any section distant x from A the B.M is given by

$$M_x = 150x - 20\frac{x^2}{2} - 125 \times \frac{1}{25} x(20 - x)$$

$$M_x = 150x - 10x^2 - 100x + 5x^2$$

$$= 5x(10 - x)$$

$150 - 20x - 100 + 10x$

$50 - 10x = 0$

$x = 5$

for the Condition of Max BM

$$\frac{\partial M_x}{\partial x} = 0 = 50 - 10x = 0 \quad x = 5m$$

$$M_{max} = 5 \times 5(10-5) = 125 \text{ kNm}$$

At any section distance x from B, the BM is given by

$$M_x = 50x - 125 \times \frac{1}{25} x(20-x)$$

$$M_x = 50x - 100x + 5x^2$$

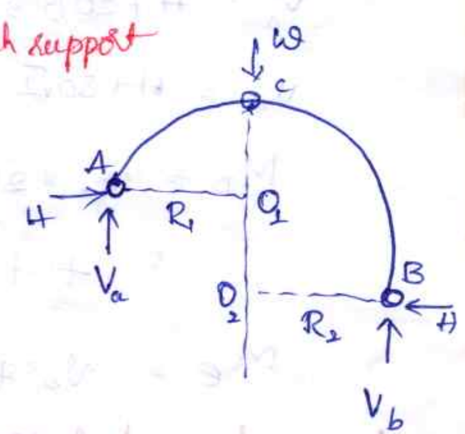
$$M_x = -50x + 5x^2 = -5x(10-x)$$

for the Condition of maximum B·M is given by

$$\frac{\partial M_1}{\partial x} = 50x + 10x = 0 \quad \therefore x = 5m$$

$$M_{max} = -5 \times 5(10-5) = -125 \text{ kNm.}$$

A three hinged arch consisting of two quadrantal parts AC and CB of radii R_1 and R_2 the arch carries a concentrated load W on the crown. Find the horizontal thrust, at each support



$\Sigma M_c = 0$

$$V_a R_1 = H R_1$$

$$V_a = H$$

Similarly taking moment @ C $\Sigma M_c = 0$

$$V_b R_2 = H R_2$$

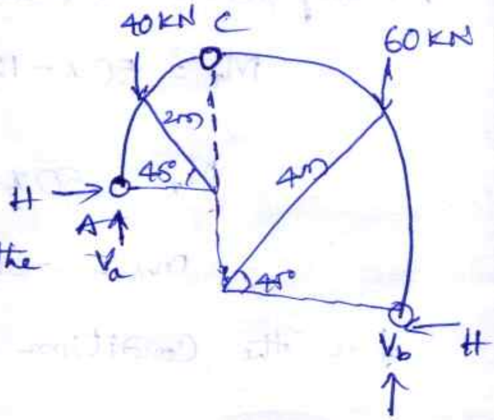
$$V_b = H$$

$$V_a = V_b = H$$

$$V_a + V_b = W \qquad V_a = V_b = H = W/2$$

A three hinged arch consisting of two quadrantal parts AC and CB of radii 2m and 4m respectively. For the load system acting on the arch, calculate the reactions at the supports and the Bm under the load 8.

let $V_a + V_b = 100KN$



$V_a \times 2 - 2H - 40\sqrt{2}$

$V_a = H + 20\sqrt{2}$

Taking moments @ C of the forces on the right side of C

$V_b \times 4 = 4H + 60 \times 2\sqrt{2}$

$V_b = H + 30\sqrt{2}$

$V_a + V_b = 2H + 50\sqrt{2} = 100$

$H = 50 - 25\sqrt{2} = 25(2 - \sqrt{2})$

$V_a = H + 20\sqrt{2} = 50 - 25\sqrt{2} + 20\sqrt{2} = 50 - 5\sqrt{2}$

$V_B = H + 30\sqrt{2} = 50 - 25\sqrt{2} + 30\sqrt{2} = 50 + 5\sqrt{2}$

$M_D = V_a \times 2(1 - \cos 45°) - H \times 2\sin 45°$

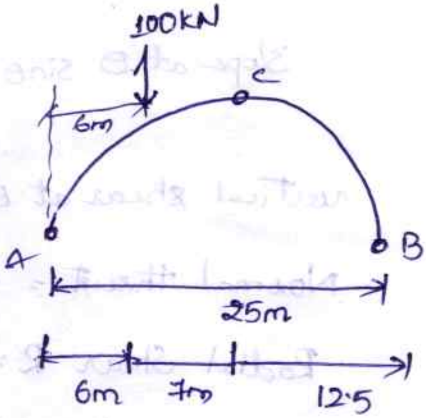
$= 4.44$ kNm

$M_E = V_b \times 4(1 - \cos 45°) - H \cdot 4\sin 45° = 25.44$ KNm

A circular arch of span 25m with a central rise of 5m is ~~subjected~~ hinged at the crown and springing. It carries a point load of 100KN at 6m from the left support. Calculate the reactions at the supports, the reactions at crown and the moment at 5m from the left support

Support Reactions

Taking moments @ B, we get



$V_A \times 25 = 100(25-6)$

$V_A = 76\,kN$

$V_A + V_B = 100$

$V_B = 100 - 76 = 24\,kN$

Horizontal thrust

Considering the moments @ C $\quad \Sigma M_c = 0$

$V_B \times 12.5 - H \times 5 = 0 \quad \therefore H = 60\,kN$

Bending moment @ 5m from the left support

$y_c(2R - r) = L^2/4$

$5(2R - 5) = \frac{25^2}{4}$

$R = 18.125\,m$

$x = R\sin\theta = (12.5 - 5) = 7.5$

$\sin\theta = \frac{7.5}{18.125} \quad \theta = 24.44$

Bending moment at D $= V_A \times 5 - H \times y_D$

$y_D = y_c - R(1 - \cos\theta)$

$= 5 - 18.125(1 - \cos 24.443) = 3.375$

$M_D = 76 \times 5 - 60 \times 3.375 = 177.25\,kNm$

let D be the point 10m from the left support where the normal thrust and shear are to be found

$r(2R - r) = L^2/4$

$8(2R - 8) = 40^2/4$

$R = 29 m$

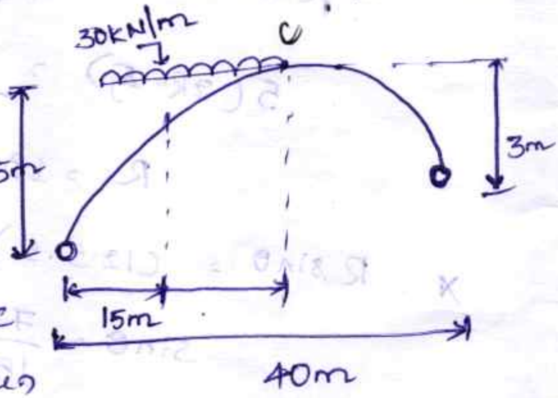
Slope at θ $\sin\theta = \frac{10}{R}$ $\pm \sin^{-1}\left(\frac{10}{29}\right) = 20.171$

vertical shear at θ $V = V_A - 20 \times 10 = 325 - 200 = 125 kN$

Normal thrust $= H\cos\theta + V\sin\theta = 336.437 kN$

Radial Shear $R = V\cos\theta - H\sin\theta = 9.575 kN$

A Three hinged arch parabolic having supports at different levels shown carries a UDL of intensity 30kN/m over the portion left to the crown. Determine the horizontal thrust developed. Find also B.M, Normal thrust and radial shear force developed at a section 15m from the left support.



Taking 'C' as the origin the eq of parabola is

$\frac{x^2}{y} = a$ where a = Constant

let the horizontal distance b/n A and C be L_1 and that of C & B be L_2. Then

$$\frac{L_1^2}{y} = \frac{L_2^2}{y}$$

$$\frac{L_1^2}{5} = \frac{L_2^2}{3}$$

$$\frac{L_1}{\sqrt{5}} = \frac{L_2}{\sqrt{3}}$$

$$= \frac{L_1 + L_2}{\sqrt{5}+\sqrt{3}} = \frac{L}{\sqrt{5}+\sqrt{3}}$$

$$\frac{L_1}{\sqrt{5}} = \frac{L}{\sqrt{5}+\sqrt{3}}$$

$$L_1 = \frac{L\sqrt{5}}{\sqrt{5}+\sqrt{3}} = \frac{40\sqrt{5}}{\sqrt{5}+\sqrt{3}} = 22.54m$$

Horizontal thrust $\quad L_2 = 40 - 22.54 = 17.46m$

$\Sigma M_C = 0$

$V_B \times 17.46 - H \times 3 = 0$

$17.46 V_B = 3H$

$H = 5.82 V_B$

$\Sigma M_A = 0$

$V_B \times 40 - H(5-3) = 30 \times 22.54 \times \frac{22.54}{2}$

$40V_B - 2H = 7620.774$

From (1) and equation

$V_B 40 - 5.82 \times 2V_B = 7620.774$

$V_B = 147.58 KN$

$H = 5.82 \times 147.58 = 858.92$

$V_A + V_B =$ Total load

$V_A + 147.58 = 30 \times 22.54$

$V_A = 528.02 KN$

The portion left of 'c' may be treated as a parabola of span

$L' = 2L_1 = 2 \times 22.54 = 45.08m$

A/c to 3hinged parabolic arch $\quad y_c = 5m$

$$y = \frac{4y_c}{L^2} x(L-x)$$

At $x = 15m \quad y = 4.44m$

BM at section D is

$$M_D = V_A \times 15 - H \times 4.44 - 30 \times 15 \times \frac{15}{2}$$

$$= 740 \text{ KNm}$$

$$\frac{dy}{dx} \quad \tan\theta = \frac{4y_c}{l^2}(l - 2x)$$

$$V = V_A - W$$
$$= 528.62 - 30 \times 15$$
$$= 78.62 \text{ KN}$$

At $x = 15$ $\quad \theta = 8.44°$

Normal thrust $N = H\cos\theta + V\sin\theta$

$$= 858.92\cos(8.44) + 78.62\sin(8.44)$$

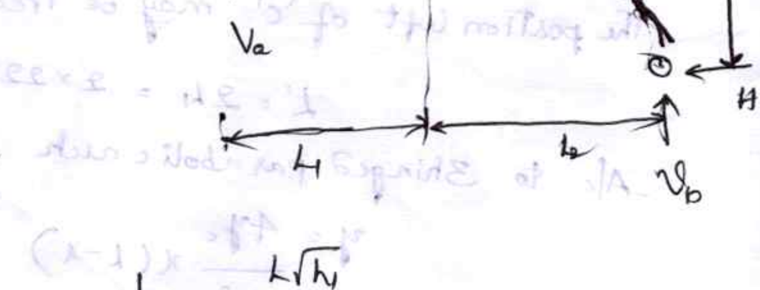
$$= 861.25 \text{ KN}$$

Radial shear $F = V\cos\theta - H\cos\theta$

$$= 78.62\cos(8.44) - 858.92\sin(8.44)$$

$$= 48.30 \text{ KN}$$

A Three hinged parabolic arch of span l has its abutments A and B at depth h_1 and h_2 below the crown C. Thes carries a concentrated load w at the crown. Determine horizontal thrust at each support



$$\frac{L_1}{\sqrt{h_1}} = \frac{L_2}{\sqrt{h_2}}$$

$$= \frac{L_1 + L_2}{\sqrt{h_1} + \sqrt{h_2}} = \frac{L}{\sqrt{h_1} + \sqrt{h_2}}$$

$$\frac{L_1}{\sqrt{h_1}} = \frac{L}{\sqrt{h_1} + \sqrt{h_2}} \qquad L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}}$$

Taking moments @ C of forces on the left hand side of C, we have

$$V_a l_1 = H h_1 \quad \therefore V_a = H \frac{h_1}{l_1}$$

Taking moments @ C on the forces

$$V_b l_2 = H h_2 \qquad V_b = H \times \frac{h_2}{l_2}$$

$$V_a + V_b = H\left[\frac{h_1}{l_1} + \frac{h_2}{l_2}\right]$$

$$V_a + V_b = W$$

$$W = H\left[\frac{h_1}{l_1} + \frac{h_2}{l_2}\right]$$

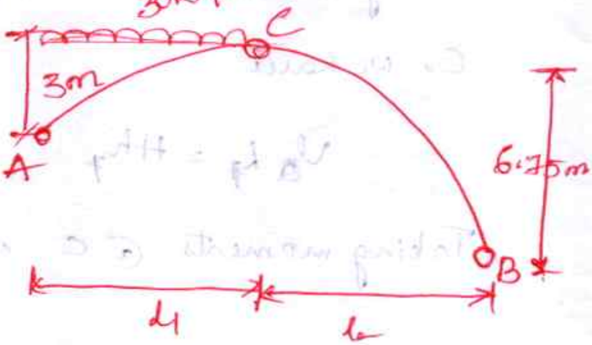
$$H = \frac{W}{\left(\frac{h_1}{l_1} + \frac{h_2}{l_2}\right)}$$

$$H = \frac{W}{\frac{h_1(\sqrt{h_1}+\sqrt{h_2})}{l\sqrt{h_1}} + \frac{h_2(\sqrt{h_1}+\sqrt{h_2})}{l\sqrt{h_2}}}$$

$$= \frac{Wl}{\sqrt{h_1}(\sqrt{h_1}+\sqrt{h_2}) + \sqrt{h_2}(\sqrt{h_1}+\sqrt{h_2})}$$

$$H = \frac{Wl}{(\sqrt{h_1}+\sqrt{h_2})^2}$$

A Three-hinged parabolic arch



$$\frac{l_1}{\sqrt{3}} = \frac{l_2}{\sqrt{6.75}}$$

$$\frac{l_1 + l_2}{\sqrt{3} + \sqrt{6.75}} = \frac{22.5}{\sqrt{3} + \sqrt{6.75}}$$

$$l_1 = \frac{22.5\sqrt{3}}{\sqrt{3} + \sqrt{6.75}} = 9m$$

$$l_2 = 22.5 - 9 = 13.5m$$

Vertical Reactions

@ $\Sigma M_c = 0$ from left

$+V_a \times 9 - 3H - 30 \times 9 \times 4.5$ $\quad V_a = \frac{H}{3} + 135$

$\Sigma M_c = 0$ from Right

$V_b \times 13.5 - 6.75H = 0$ $\quad V_b = \frac{H}{2}$

$V_a + V_b = 30 \times 9 = 270$

$\frac{H}{3} + 135 + \frac{H}{2} = 270$ $\quad \frac{5H}{6} = 135$ $\quad \therefore H = 162kN$

$V_a = \frac{162}{3} + 135 = 189kN$

$V_b = \frac{162}{2} = 81kN$ $\quad l = 2l_1 = 2 \times 9$

Max +ve BM occurs under AC. Equation of Arch from A to C

$$y = \frac{4h}{l^2}x(l-x) = \frac{4 \times 3}{18^2}x(18-x) = \frac{12}{18^2}x(18-x)$$

$$M_x = V_a x - 30\frac{x^2}{2} - H \times \left\{\frac{12}{18^2}x(18-x)\right.$$

$$= 189x - 15x^2 - 108x + 6x^2$$

$M_x = 81x - 9x^2$

For the condition of max BM, $\frac{\partial M_x}{\partial x} = 0$

$81 - 18x = 0 \quad \therefore x = 4.5m$

$M_{max} = 81 \times 4.5 - 9 \times 4.50^2 = 182.25$ kNm

Maximum -ve BM; occurs at section BC $\quad l = 2l_2 = 2 \times 13.5 = 27$

$y = \frac{4 \times 6.75}{27^2} x(27-x) \quad y = \frac{1}{27} x(27-x)$

$M_x = 81x - 162 \times \frac{1}{27} x(27-x)$

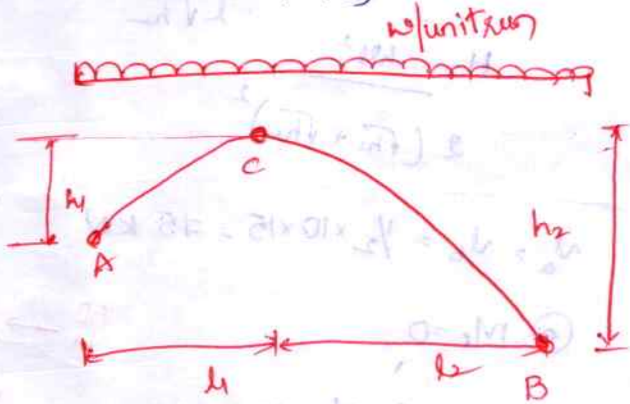
$= 6x^2 - 81x$

$\frac{\partial M_x}{\partial x} = 12x - 81 = 0 \quad \therefore x = 6.75m$

$M_{max} = 6 \times 6.75^2 - 81 \times 6.75 = 273.375$ kNm

$\frac{l_1}{\sqrt{h_1}} = \frac{l_2}{\sqrt{h_2}} = \frac{l_1 + l_2}{\sqrt{h_1} + \sqrt{h_2}} = \frac{l}{\sqrt{h_1} + \sqrt{h_2}}$

$l_1 = \frac{l\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}} \quad l_2 = \frac{l\sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}}$



Taking moment @ $\Sigma M_C = 0$

from left hand side

$V_a l_1 = \frac{wl_1^2}{2} + Hh_1 \quad V_a = \frac{wl_1}{2} + \frac{Hh_1}{l}$

Ily from Right hand side

$V_b l_2 = \frac{wl_2^2}{2} + Hh_2 \quad V_b = \frac{wl_2}{2} + H\frac{h_2}{l_2}$

$$V_a + V_b = \frac{w}{2}(l_1 + l_2) + H\left[\frac{h_1}{l_1} + \frac{h_2}{l_2}\right]$$

$$= \frac{wl}{2} + H\left[\frac{h_1}{l_1} + \frac{h_2}{l_2}\right]$$

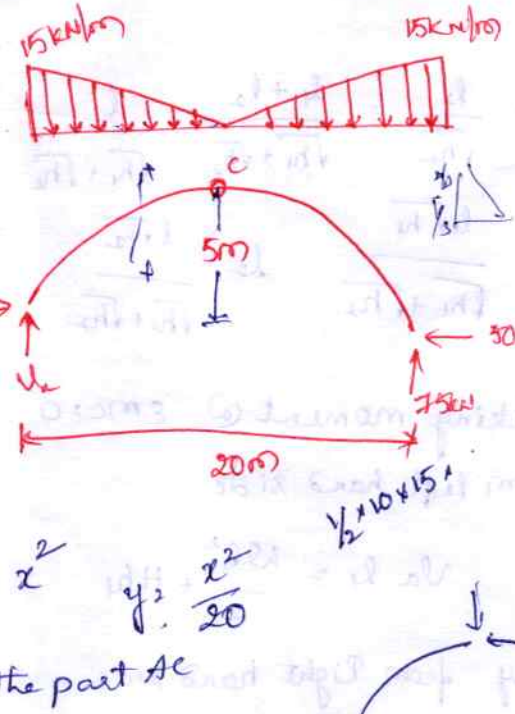
$$V_a + V_b = wl$$

$$wl = \frac{wl}{2} + H\left[\frac{h_1}{l_1} + \frac{h_2}{l_2}\right]$$

$$H = \frac{wl/2}{\frac{h_1}{l_1} + \frac{h_2}{l_2}}$$

$$H = \frac{wl/2}{\frac{h_1(\sqrt{h_1}+\sqrt{h_2})}{l\sqrt{h_1}} + \frac{h_2(\sqrt{h_1}+\sqrt{h_2})}{l\sqrt{h_2}}} = \frac{wl^2/2}{h_1/\sqrt{h_1}(\sqrt{h_1}+\sqrt{h_2}) + h_2/\sqrt{h_2}(\sqrt{h_1}+\sqrt{h_2})}$$

$$H = \frac{wl^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}$$



$V_a = V_b = 1/2 \times 10 \times 15 = 75$ kN

@ $M_c = 0$,

$$-H \times 5 + 75 \times 2/3 \times 10 + 75 \times 10$$

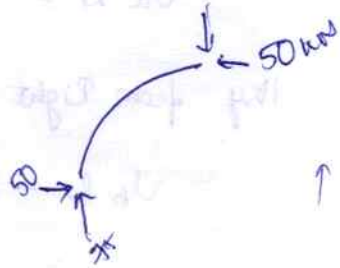
$$1/2 \times 15 \times 10 \times 2/3 \times 10$$

$$H = 50$$

$1/2 \times 10 \times 15$

$$y = \frac{4h}{l^2}x(l-x) = \frac{4 \times 5}{20^2}x^2 \qquad y = \frac{x^2}{20}$$

Considering the equilibrium of the part AC



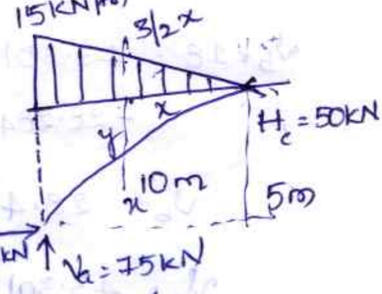
$H_c = 50$ KN

$V_c = 0$

Bending moment at a section distant x from C

$$M_x = 50\frac{x^2}{20} - \frac{1}{2}x\,\frac{3}{2}x \times \frac{x}{3} \qquad H \times y - \frac{1}{2}x \cdot \frac{3}{2}x \times x/3$$

$$M_x = \frac{5}{2}x^2 - \frac{x^3}{4}$$



For B.M to be max $\frac{\partial M_x}{\partial x} = 0$

$$\frac{\partial M_x}{\partial x} = 5x - \frac{3}{4}x^2 = 0$$

$$x\left[5 - \frac{3}{4}x\right] = 0 \quad \therefore x = 20/3\ \text{m}$$

$$M_{max} = \frac{5}{2}\left[\frac{20}{3}\right]^2 - \frac{1}{4}\left[\frac{20}{3}\right]^3 = 37.037\ \text{KN/m}$$

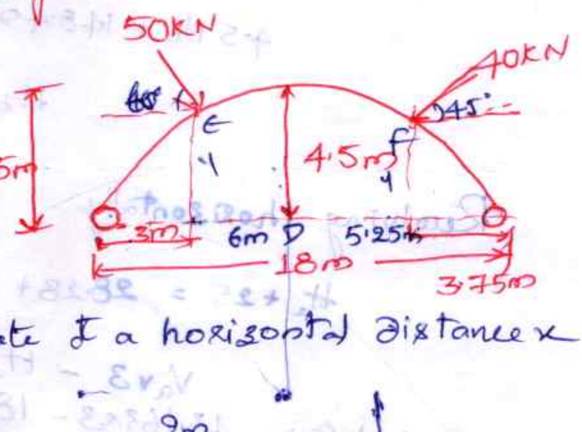
A circular Segmental three hinged arch

let R be the Radius of the arch

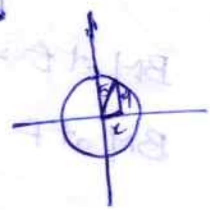
$$r(2R - r) = l^2/4$$

$$4.5(2R - 4.5) = 18^2/4$$

$$R = 11.25\ \text{m}$$



With D as origin, the ordinate at a horizontal distance x from D is given by



$$y = \sqrt{11.25^2 - x^2} = \sqrt{11.25^2 - 9^2} = 6.75$$

Ordinate at E $= \sqrt{11.25^2 - 6^2} - 6.75 = 2.76\text{m}$

Ordinate at f $= \sqrt{11.25 - 5.25^2} - 6.75 = 3.2\text{m}$?

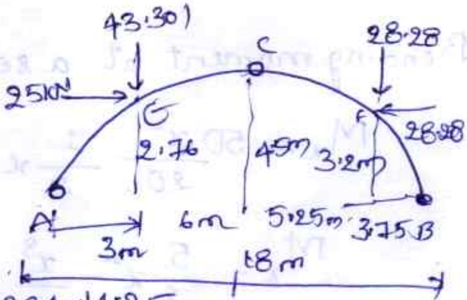
The given inclined loads are replaced by V and H components

$$H = 50\cos 60^\circ = 25\text{KN}$$

$$V = 50\sin 60^\circ = 43.301\ \text{KN}$$

$H_3 = 40\cos 45° = 28.28\,kN$

$H_4 = 40\sin 45° = 28.28\,kN$



Taking moments about the end A,

$V_b \times 18 = 43.301 \times 3 + 25 \times 2.76 + 28.284 \times 14.25 - 28.284 \times 3.20$

$V_b = 28.42\,kN$

$V_a = 43.301 + 28.284 - 28.42 = 43.16\,kN$

Taking moments @ C from right

$-H_b \times 4.5 + 28.284 \times 5.25 + 28.284\,(4.5 - 3.2) = 28.42 \times 9$

$4.5 H_b + 148.497 + 36.77 = 255.79$

$H_b = 15.675\,kN$

Resolving horizontally

$H_a + 25 = 28.28 + 15.675 \qquad H_a = 18.959$

$V_a \times 3 - H_a \times 2.76$

BM at E = $43.163 \times 3 - 18.959 \times 2.76 = 77.04\,kNm$

BM at F = $28.42 \times 3.75 - 15.675 \times 3.2 = 56.423\,kNm$

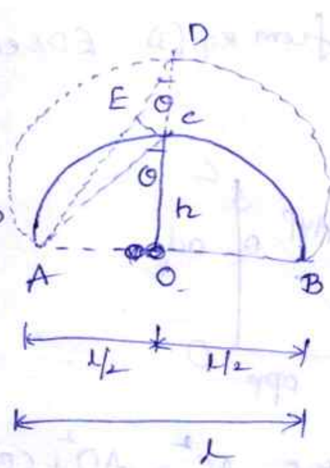
Effect of temperature on three hinged Archs

Three hinged arch of span l and vertical rise h. let the arch be subjected to a rise of temperature of t°c. The rise of temperature increases the length of arch. Since the ends A & B do not move and since the hinge C is not connected to any permanent object, the crown hinge will rise from C to D. Now AD represents the new position of AC so that

Arc AD = Arc AC + Arc AC α T

Arc AC $[1+\alpha T]$

where α = coefficient of linear expansion of material



let us make the approximation that

chord AD = chord AC $(1+\alpha T)$

Increase in length of chord AC = chord AD − chord AC

chord AC + chord AC α T − chord AC

$\therefore$ AD − AC = chord ACαT ⟶ ①

let CE be the ⊥r to AD. Since CD is small, AC & AE are nearly equal

AD − AC = AD − AE

AD − AC = ED ⟶ ②

from ① & ②

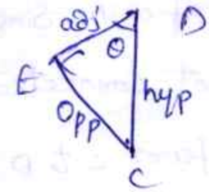
ED = ACαT ⟶ ③

Increase in the rise of the arch δ = CD

Now Consider the Δ^{le} EDC

$$\cos\theta = \frac{ED}{CD}$$

$$CD = ED \times \frac{1}{\cos\theta}$$

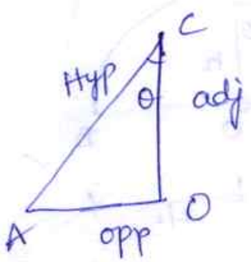


$$\delta = CD = ED\sec\theta$$

$$= ED\sec EDC$$

from eq (3) $ED\sec\theta = ED\sec EDC = ED\sec ACO$

$$= AC\alpha T \cdot \frac{AC}{CO}$$



$$\frac{AC^2 \alpha T}{CO} \rightarrow (4)$$

WKT $AC^2 = AO^2 + CO^2$

$$AC^2 = L^2/4 + h^2 \longrightarrow (5)$$

By Eq (4) & (5)

$$\delta = \frac{AC^2}{CO}\alpha T$$

$$\frac{L^2/4 + h^2 \alpha T}{h}$$

$$\therefore \delta = \frac{L^2 + 4h^2}{4h} \cdot \alpha \cdot T$$

A Three hinged Arch of span length 30m & rise 6m is subjected to a rise of temperature of 40°c. Determine the change in the rise of the Arch. Take $\alpha = 12\times10^{-6}$ °C

Given data

$$Span\ l = 30m,\ h = 6m,\ T = 40°c,\ \alpha = 12\times10^{-6}$$

$$\therefore \delta = \frac{l^2+4h^2}{4h}\times\alpha\times T$$

$$= \frac{30^2+4\times6^2}{4\times6}\times12\times10^{-6}\times40 = 0.02m.$$

Effect of temperature rise on the horizontal thrust for a three hinged arch carrying a load;

While no stresses are produced in a three hinged arch due to temperature change alone, since the rise of the arch is altered as a consequence of the temperature change, the horizontal thrust for the arch already carrying a load will also alter. Suppose a 3-hinged arch of span l and rise h carries a UDL of w per unit run over the whole span. In this condition the horizontal thrust for the arch

$$H = \frac{wl^2}{8r}$$

let due to rise in temperature, the increase in the rise of the arch

$$\partial H = \frac{-wl^2}{8r^2}\partial r$$

from equations (i) and (ii)

$$\frac{\partial H}{H} = -\frac{\partial r}{r} \qquad \partial H = -\frac{\partial r}{r}(H)$$

This is the decrease in the horizontal thrust due to rise in temperature.

A Three hinged arch of span 20m and rise 4m Carries a UDL of 25kN/m. Find the horizontal thrust for the arch. If now the arch is subjected to a rise in Temperature of 40°c. Find what change in the horizontal thrust will occur. Take $\alpha = 12\times10^{-6}\,°C$

Before the rise in temperature

Horizontal thrust $H = \frac{wl^2}{8h} = \frac{25\times20^2}{8\times4} = 312.5\,kN$

Increase in the rise of the arch due to rise in Temperature $= \frac{l^2+4h^2}{4h}\alpha T$

$$= \frac{20^2+4\times4^2}{4\times4}\times12\times10^{-6}\times40$$

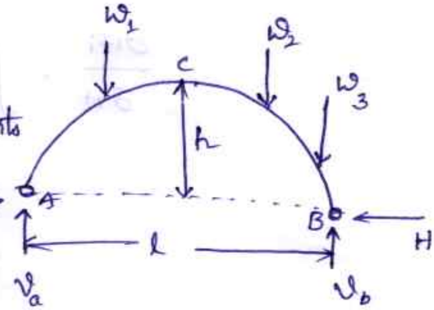
$$\delta h = 0.01392\,m$$

Decrease in horizontal thrust due to inc in the rise of the arch

$$\frac{\partial h}{h}H = \frac{0.01392}{4}\times312.5$$

$$= 1.0875\,kN.$$

Two Hinged Arches;



Two hinged arch hinged only at the abutments A and B. The conditions of equilibrium are insufficient to find the horizontal thrust for this arch. The vertical reactions V_a and V_b of the course, may be determined by taking moments about either hinge.

The horizontal thrust at each support may be determined from the condition that the horizontal displacement of either hinge WRT the other is zero.

let M be the beam moment at any section X (i.e., the BM at any section ignoring the horizontal thrust at the supports). let the coordinate of X be (x, y) with the end A as origin.

Actual BM at the section is given by

$$M_x = (M - Hy)$$

Total strain energy stored by the whole arch

$$W_i = \int M_x^2 \frac{\partial s}{2EI}$$

$$= \int (M - Hy)^2 \frac{\partial s}{2EI}$$

By the first Castigliano theorem, the horizontal movement of either end relative to other is given by $\frac{\partial W_i}{\partial H}$. Since such a relative horizontal displacement of one end with respect to the other end is not possible in the two hinged arch.

$$\frac{\partial w_i}{\partial H} = \int 2\,(M - Hy)\,(-y)\,\frac{\partial s}{2EI}$$

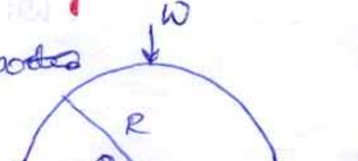
$$\int \frac{My\,ds}{EI} - H\int \frac{y^2}{2EI}\,\partial s = 0$$

$$H = \frac{\int \frac{My}{EI}\,\partial s}{\int \frac{y^2}{2EI}\,ds}$$

If the arch is of uniform flexural rigidity EI, we have

$$H = \frac{\int My\,\partial s}{\int y^2\,\partial s}$$

A two hinged semicircular arch of radius R carries a concentrated load W at the crown. Show that the horizontal thrust at each support is $\frac{W}{\pi}$. Assume uniform flexural rigidity



Taking the span as that of a simply supported beam, the beam moment at any section & the radius vector corresponding to which makes an angle θ with the horizontal given by

$$M = \frac{W}{2}R(1-\cos\theta) \text{ and } y = R\sin\theta$$

$$\text{Horizontal thrust} = \frac{\int My\,ds}{\int y^2\,ds}$$

$$H = \frac{2\int_0^{\pi/2} \frac{\omega}{2} R(1-\cos\theta)\, R\sin\theta\, R\,d\theta}{\int_0^{\pi/2} R^2 \sin^2\theta \cdot R\,d\theta}$$

$$= \frac{\frac{\omega}{2}\int_0^{\pi/2} (1-\cos\theta)\sin\theta\,d\theta}{\int_0^{\pi/2} \sin^2\theta\,d\theta}$$

If $(1-\cos\theta) = u$

$\sin\theta\,d\theta = du$ $\quad \int (1-\cos\theta)\sin\theta\,d\theta = \int u\,du = \frac{u^2}{2} = \frac{(1-\cos\theta)^2}{2}$

$$\therefore \int_0^{\pi/2} (1-\cos\theta)\sin\theta\,d\theta = 1/2$$

$$\int_0^{\pi/2} \sin^2\theta\,d\theta = \pi/4$$

$$H = \frac{\omega}{2}\,\frac{1/2}{\pi/4} = \frac{\omega}{\pi}$$

Actual bending moment at any section X and radius vector corresponding to which makes an angle θ with horizontal, is given by

$$M_x = \frac{\omega}{\pi} R(1-\cos\theta) - HR\sin\theta$$

$$M_x = \frac{\omega}{\pi} R(1-\cos\theta) - \frac{\omega}{\pi} R\sin\theta$$

$$= \frac{\omega R}{2\pi}\left(\pi - \pi\cos\theta - 2\sin\theta\right)$$

strain energy stored by the arch

$$\omega_1 = \int \frac{M_x^2}{2EI}\, ds$$

$$\frac{\omega^2 R^2}{4\pi^2}\int_0^{\pi/2} \left(\pi - \pi\cos\theta - 2\sin\theta\right)^2 \times \frac{R\,d\theta}{2EI}$$

$$= \frac{\omega^2 R^3}{4\pi^2 EI}\int_0^{\pi/2}\left(\pi^2 + \pi^2\cos^2\theta - 2\pi^2\cos\theta + 4\pi\sin\theta\cos\theta - 4\pi\sin\theta\right)d\theta$$

$$= \frac{\omega^2 R^3}{4\pi^2 EI}\left[\pi^2 + \pi^2 \quad \pi^2 \times \pi/2 + \pi^2 \times \pi/4 + 4\frac{\pi}{4} - 2\pi^2 + 4\pi \times \frac{1}{2} - 4\pi\right]$$

$$= \frac{\omega^2 R^3}{4\pi^2 EI}\left[\frac{3}{4}\pi^3 - 2\pi^2 - \pi\right]$$

$$= \frac{\omega^2 R^3}{4\pi^2 EI} \times \frac{\pi}{4}\left[3\pi^2 - 8\pi - 4\right]$$

$$= \frac{\omega^2 R^3}{16\pi EI}\left[3\pi^2 - 8\pi - 4\right]$$

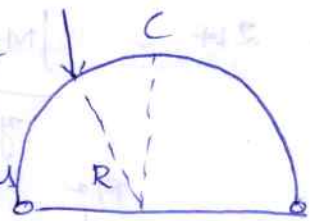
By the first theorem of Castigliano

Vertical deflection of the crown of the arch

$$\delta = \frac{\partial \omega_i}{\partial \omega} = \frac{\omega}{8\pi}\frac{R^3}{EI}[3\pi^2 - 8\pi - 4]$$

A Two-hinged semi circular arch of radius R carries a load w at a section the radius vector corresponding to which makes an angle α with horizontal. Find the horizontal thrust at each support. Assume uniform flexural rigidity.

Arc carrying the load W at D so that the radius OD makes an angle α with the horizontal. let the horizontal thrust be H at each support. let the vertical reactions at A and B be V_1 and V_2 respectively



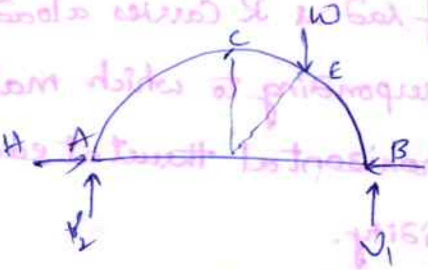
Considering the span So that of a simply supported beam, the beam moment at any section X, the Radius vector at which makes an angle θ with the horizontal is given by

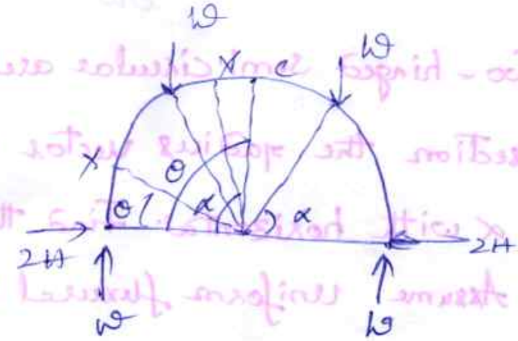
$$M = WR(1-\cos\theta) \text{ when } \theta < \alpha$$

$$M = WR(1-\cos\theta) - WR(\cos\alpha - \cos\theta) \text{ when } \theta > \alpha < \pi/2$$

$= WR - WR\cos\theta - WR\cos\alpha + WR\cos\theta$

$= WR(1-\cos\alpha)$





Horizontal thrust

$$2H = \frac{\int My\,ds}{y^2\,ds} = \frac{2\left[\int_0^{\alpha} WR(1-\cos\theta)\cdot R\sin\theta R\,d\theta + \int_{\alpha}^{\pi/2} WR(1-\cos\alpha)R\sin\theta\cdot R\,d\theta\right]}{2\int_0^{\pi/2} R^2\sin^2\theta\; R\,d\theta}$$

$$= \frac{W/2\,(1-\cos\alpha)^2 + W(1-\cos\alpha)(-\cos\theta)_{\alpha}^{\pi/2}}{\pi/4}$$

$$= \frac{\frac{W}{2}(1-\cos\alpha)^2 + W(1-\cos\alpha)\cos\alpha}{\pi/4}$$

$$= \frac{\frac{W}{2}(1-\cos\alpha)\,(1-\cos\alpha+2\cos\alpha)}{\pi/4}$$

$$= \frac{\frac{w}{2}(1-\cos^2\alpha)}{\pi/4} = \frac{2w}{\pi}\sin^2\alpha$$

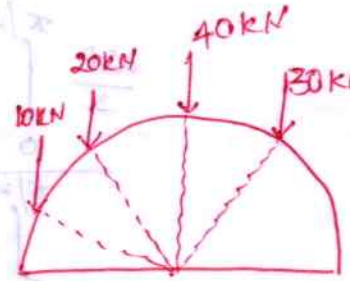
$$2H = \frac{2w}{\pi}\sin^2\alpha$$

Horizontal thrust at each support when one of the concentrated load is present

$$H = \frac{w}{\pi}\sin^2\alpha$$

when $\alpha = \pi/2$ $\quad H = \frac{w}{\pi}$

Find the horizontal thrust for the two hinged semicircular arch



$$H = \Sigma \frac{w}{\pi}\sin^2\alpha$$

$$= \frac{10}{\pi}\sin^2 30^\circ + \frac{20}{\pi}\sin^2 45^\circ +$$

$$\frac{40}{\pi} + \frac{30}{\pi}\sin^2 60^\circ$$

$$= 23.873\ \text{KN}$$

A Two hinged semicircular arch of radius R carries a UDL of w per unit run over the whole span. Determine the horizontal thrust at each support. Assume uniform flexural rigidity

Each vertical reaction $= wR$

The beam moment at any section (the BM at any section ignoring the horizontal thrust)

$$M = \omega R \cdot R(1-\cos\theta) - \omega R(1-\cos\theta) \cdot \frac{R(1-\cos\theta)}{2}$$

$$= \frac{\omega R^2}{2}(1-\cos\theta)(2-1+\cos\theta)$$

$$= \frac{\omega R^2}{2}(1-\cos^2\theta)$$

$$= \frac{\omega R^2}{2}\sin^2\theta$$

Horizontal thrust at the support $H = \dfrac{\int My\,dx}{\int y^2\,dx}$

$$= \frac{2\int_0^{\pi/2} \frac{\omega R^2}{2}\sin^2\theta \cdot R\sin\theta \, R\,d\theta}{2\int_0^{\pi/2} R^2 \sin^2\theta R\,d\theta}$$

$$\frac{\omega R}{2}\frac{\int_0^{\pi/2}\sin^3\theta\,d\theta}{\int_0^{\pi/2}\sin^2\theta\,d\theta}$$

$$H = \frac{4}{3}\frac{\omega R}{\pi}$$

A Two hinged parabolic semi circular arch carries a UDL of w per unit run over the left half of its span. Determine the horizontal thrust at each support. Assume uniform flexural rigidity

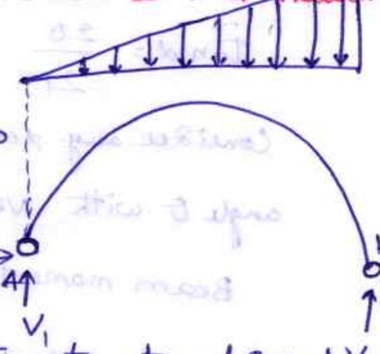
The arch is subjected to UDL

$$2H = \frac{4}{3} \times \frac{\omega R}{\pi}$$

Hence the horizontal thrust at each support when one half is load

$$\frac{4}{3} \cdot \frac{wR}{\pi} \times \frac{1}{2} = \frac{2}{3} \frac{wR}{\pi}$$

→ A Two hinged semicircular arch of radius R carries a distributed load uniformly varying from zero on the left end to w per unit run at the right end. Determine the horizontal thrust at each support.



Arch carrying the varying load from zero at the left end to w per unit run at the right end. let the vertical reactions at A and B be V_1 and V_2. let H be the horizontal thrust, at each support V_2

Arch carrying a load varying from w per unit run at the left end to zero at the right end. The vertical reactions at A and B will be V_2 and V_1 respectively. The horizontal thrust will still be H, at each support The same arch carrying both above load systems. The horizontal thrust will now be 2H at each support But the above two load systems together constitute a UDL of w per unit run over the whole span.

Horizontal thrust at each support $2H = \frac{4}{3} \frac{wR}{\pi}$

$$H = \frac{2}{3} \frac{wR}{\pi}$$

A segmental arch has a span of 40m and a rise of 8m and is hinged at springings Both the hinges are at same level. The arch

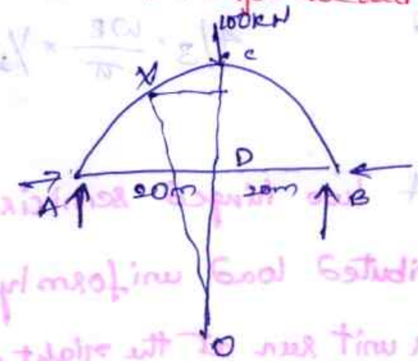
· supports a load of 100KN at the crown. find horizontal thrust at each support and the max B.M for the arch.

$x(2R-x) = L^2/4$

$R = 29m$

$AOC = \alpha$

$AD = 20m, \ OD = 21m$



$R\sin\alpha = 20m$

$R\cos\alpha = 21m$

$\therefore \tan\alpha = \frac{20}{21} \quad \alpha = 0.761 \text{ radians}$

Consider any section x whose radius vector makes an angle θ with vertical

Beam moment $= V_a(L/2 - R\sin\theta) = 50(20 - R\sin\theta)$

$y = R\cos\theta - 21$

H = Horizontal thrust at each support

$$= \frac{2\int_{\theta=0}^{\theta=\alpha} My\,ds}{2\int_{\theta=0}^{\alpha} y^2\,ds} = \frac{\int_0^{\alpha} My\,ds}{\int_0^{\alpha} y^2\,ds}$$

$$= \int_0^{\alpha} My\,ds$$

$$\int_0^{\alpha} 50(20 - R\sin\theta)\ (R\cos\theta - 21)\ R\,d\theta$$

$$= 50R\int_0^{\alpha}\left(20R\cos\theta - R^2\sin\theta\cos\theta + 21R\sin\theta - 420\right)d\theta$$

$$= 50R\left[20R\sin\alpha - \frac{R^2}{2}\sin^2\alpha - 21R(\cos\theta)_0^{\alpha} - 420\alpha\right]$$

$$= 50R\left[20R\sin\alpha - \frac{R^2}{2}\sin^2\alpha - 21R\cos\alpha + 21R - 420\alpha\right]$$

$$= 50R\left[20\times 20 - \frac{20\times 20}{2} - 21\times 21 + 21\times 29 - 420\times 0.76\right]$$

$$= 50\times 29\left[400 - 200 - 441 + 609 - 319.62\right]$$

$$= 70151$$

$$\int_0^{\alpha} y^2\,dx = \int_0^{\alpha}\left(R\cos\theta - 21\right)^2 dx$$

$$= \int_0^{\alpha}\left(R^2\cos^2\theta - 42R\cos\theta + 441\right) R\,d\theta$$

$$= R\int_0^{\alpha}\left[\frac{R^2(1+\cos 2\theta)}{2} - 42R\cos\theta + 441\right] d\theta$$

$$= R\left[\frac{R^2}{2}\alpha + \frac{R^2}{2}\cdot\frac{1}{2}\sin 2\alpha - 42R\sin\alpha + 441\alpha\right]$$

$$= 29\left[\frac{29\times 29}{2}\times 0.761 + \frac{20\times 21}{2} - 42\times 20 + 441\times 0.76\right]$$

$$= 29\left[320 + 210 - 840 + 335.6\right] = 742.4$$

$$H = 70151/742.4 = 94.49\text{ kN}$$

B.M at any section X

$$M_x = 50(20 - R\sin\theta) - 94.49(R\cos\theta - 21)$$

$$\frac{dM_x}{d\theta} = 0$$

$$\frac{dM_x}{d\theta} = 50R\cos\theta + 94.49\,R\sin\theta$$

$$\tan\theta = \frac{50}{94.49} = 0.5292$$

$$\theta = 27^\circ 53'$$

$\sin\theta = 0.467$ $\cos\theta = 0.884$

$$\text{Max B.M} = 50(20 - 29 \times 0.467) - 94.49(29 \times 0.8839 - 21)$$

$$= -115.946\,\text{kNm}$$

Max sagging moment occurs at crown

$$= 50 \times 20 - 94.49 \times 8 = 244.02\,\text{kNm}$$

A two hinged parabolic arch of span l and rise h carries a UDL of w per unit run

A Two hinged parabolic arch of span l and rise h carries a UDL of w per unit run over the whole span. find horizontal thrust at each support

Arch carrying UDL over the whole span

The BM at any section x ignoring the horizontal thrust i.e. the beam moment at any section x.

$$M = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{w}{2}x(l-x)$$

$$y = \frac{4h}{l^2}x(l-x)$$

$$H = \frac{\int My\,dx}{\int y^2\,ds}$$

$$= \frac{\int_0^l \frac{wx}{2}(l-x)\frac{4h}{l^2}x(l-x)\,dx}{\int_0^l \frac{16h^2}{l^4}x^2(l-x)^2\,dx}$$

$$= \frac{wl^2\int_0^l x^2(l-x^2)\,ds}{8h\int_0^l x^2(l-x)^2\,ds}$$

$$= \frac{wl^2}{8h}$$

Two hinged parabolic arch carrying UDL over half span $= \frac{wl^2}{16h}$

$UVLH = \frac{wl^2}{16H}$

Three hinged parabolic arch subjected to any general load system;

Suppose the arch is uniform flexural rigidity, the horizontal thrust due to any given load system

$$H = \frac{\int My\,ds}{\int y^2 dx}$$

where M = Bending moment at any section ignoring the horizontal thrust

$$y = \frac{4h}{l^2} x(l-x)$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

$$= \sqrt{1+\left(\frac{4h}{l^2}(l-2x)\right)^2} dx$$

$$H = \frac{\int M \cdot \frac{4h}{l^2} x(l-x) \sqrt{1+\left[\frac{4h}{l^2}(l-2x)\right]^2} dx}{\int \frac{16h^2}{l^4} x^2 (l-x)^2 \sqrt{1+\left[\frac{4h}{l^2}(l-2x)\right]^2} dx}$$

If the load system be known the beam moment M also can be expressed in terms of x. Expressions for integration in the Num & deno are not integrable

Hence analysis can be done by approximation

Let us introduce an approximation that the moment of inertia of the arch section is not constant throughout and that its value at any section is given by

$$I = I_0 \sec\theta$$

θ = Inclination of tangent to the arch at any point with horizontal

I_0 = a constant

$\theta = 0$ and at this section

$I = I_0$

General expression for horizontal thrust $H = \dfrac{\int \frac{My\,dx}{EI}}{\int \frac{y^2\,dx}{EI}}$

$\frac{dy}{dx} = \tan\theta \quad dy/ds = \sin\theta \quad \frac{dx}{ds} = \cos\theta$

$ds = dx\sec\theta$

$I = I_0\sec\theta$

$$H = \frac{\int \frac{My\,dx\sec\theta}{EI_0\sec\theta}}{\int \frac{y^2\,dx\sec\theta}{EI_0\sec\theta}}$$

$$H = \frac{\int My\,dx}{\int y^2\,dx}$$

A two hinged parabolic arch of span L and rise h carries a concentrated load W at the crown. Show that the horizontal thrust equals

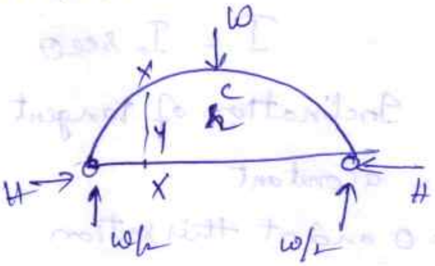
$\frac{25}{128} \frac{Wl}{h}$ at each support.

The two hinged parabolic arch

Beam ~~section~~ moment at any section x

Beam moment $M = W/2\, x$

$y = \frac{4h}{l^2} x(l-x)$



$$H = \frac{\int My\,dx}{\int y^2\,dx} = \frac{2\int_0^{l/2} W/2\, x \,\frac{4h}{l^2} x(l-x)\,dx}{2\int_0^{l/2} \frac{16h^2}{l^4} x^2 (l-x)^2 dx}$$

$$H = \frac{\frac{2Wh}{l^2}\int_0^{l/2} x^2(l-x)\,dx}{\frac{16h^2}{l^4}\int_0^{l/2} x^2(l-x)^2\,dx}$$

$$N = \frac{2Wh}{l^2}\int_0^{l/2} x^2(l-x)\,dx$$

$$= \frac{2Wh}{l^2}\left[l \times \frac{l}{3} \times \frac{l^3}{8} - \frac{1}{4}\frac{l^4}{16}\right]$$

$$= \frac{2Wh}{l^2}\, l^4 \left[\frac{l}{24} - \frac{l}{64}\right]$$

$$= \frac{2whl^2}{192}(8-3)$$

$$= \frac{5}{96} whl^2$$

Denominator $$D = \frac{16h^2}{l^4}\int_0^{l/2}\left[x^2(l-x)^2\right]dx$$

$$= \frac{16h^2}{l^4}\int_0^{l/2}(l^2x^2 - 2lx^3 + x^4)\,dx$$

$$= \frac{16h^2}{l^4}\left[l^2 \times \frac{1}{3}\frac{l^3}{8} - 2l \times \frac{1}{4} \times \frac{l^4}{16} + \frac{1}{5}\frac{l^5}{32}\right]$$

$$= \frac{16h^2}{l^4} \times \frac{l^5}{480}\,[20 - 15 + 3]$$

$$= \frac{4}{15} h^2 l$$

$$H = \frac{N}{D} = \frac{5}{96} whl^2 \times \frac{15}{4h^2 l}$$

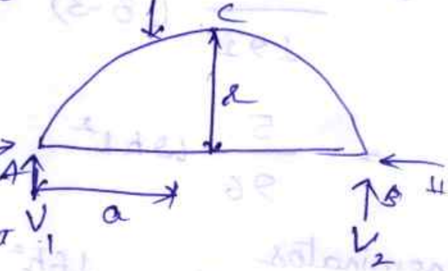
$$= \frac{25}{128}\frac{wl}{h}$$

A Two hinged parabolic arch of span l and rise h carries a concentrated load w at a distance from the left end. Show that the horizontal thrust at each support is given by

$$\frac{5}{8}\frac{w}{hl^3}a(l-a)(l^2 + al - a^2)$$

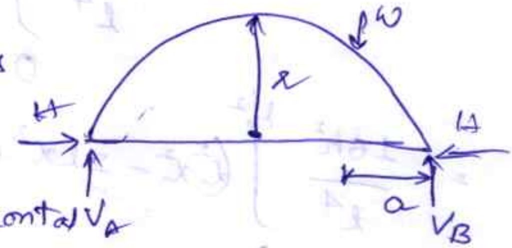
Parabolic arch carrying the load w at D at a distance from the end A.

let the vertical reactions at A and B be V_1 and V_2 respectively. let H be the horizontal thrust at each support



fig

Considering two concentrated loads acting C & D



for this load system the horizontal thrust would be 2H, at each support

Each vertical reaction $= w$

At any section distant x from $A (x<a)$ the beam moment at the section $M = wx$

But at any section X distance x from A $x>a$ and $< l/2$

$$M = wx - w(x-a) = wa$$

$$y = \frac{4r}{l^2}x(l-x)$$

$$2H = \frac{\int My\,dx}{\int y^2 dx}$$

$$2H = \frac{2\int_0^a wx\frac{4h}{l^2}x(l-x)\,dx + \int_0^{l/2} wa\frac{4h}{l^2}x(l-x)\,dx}{\int_0^{l/2}\frac{16h^2}{l^4}x^2(l-x)^2\,dx}$$

$$2H = \frac{\frac{4wh}{l^2}\int_0^a x^2(l-x)\,dx + \frac{4wha}{l^2}\int_0^{l/2} x(l-x)\,dx}{\frac{16h^2}{l^4}\int_0^{l/2} x^2(l-x)^2\,dx}$$

Numerator

$$N = 2\left[\frac{4wh}{l^2}\int_0^a x^2(l-x)\,dx + \frac{4wha}{l^2}\int_a^{l/2} x(l-x)\,dx\right]$$

$$= \frac{4wh}{l^2}\left[\frac{la^3}{3} - \frac{a^4}{4}\right] + \frac{4wha}{l^2}\left[\frac{l}{2}\left[\frac{l^2}{4} - a^2\right] - \frac{1}{3}\left[\frac{l^3}{8} - a^3\right]\right]$$

$$= \frac{4wha}{l^2}\left[\frac{la^3}{3} - \frac{a^4}{4} + \frac{l^3}{8} - \frac{la^2}{2} - \frac{l^3}{24} + \frac{a^3}{3}\right]$$

$$= \frac{4wha}{l^2} \times \frac{1}{24}\left[8la^2 - 6a^3 + 3l^3 - 12la^2 - l^3 + 8a^3\right]$$

$$= \frac{wha}{6l^2}\left[2l^3 - 4la^2 + 2a^3\right] = \frac{wha}{6l^2} \times 2\left[l^3 - 2la^2 + a^3\right]$$

$$= \frac{wha}{3l^2}\left(l^3 - 2la^2 + a^3\right) = \frac{wha}{3l^2}\left[l^3 - la^2 - la^2 + a^3\right]$$

$$= \frac{wha}{3l^2}\left[l(l^2 - a^2) - a^2(l-a)\right]$$

$$= \frac{wha}{3l^2}(l-a)(l^2 + la - a^2)$$

Denominator

$$D = \frac{16h^2}{l^4}\int_0^{l/2} x^2(l-x)^2\,dx$$

$$= \frac{16h^2}{l^4}\int_0^{l/2}(l^2x^2-2lx^3+x^4)\,dx$$

$$= \frac{16h^2}{l^4}\left[l^2\times\frac{1}{3}\times\frac{l^3}{8} - 2l\times\frac{1}{4}\times\frac{l^4}{16} + \frac{1}{5}\times\frac{l^5}{32}\right]$$

$$= 4/_{15}\, h^2 l$$

Horizontal thrust

$$2H = \frac{N}{D} = \frac{Wha}{3l^2}(l-a)(l^2+la-a^2)\times\frac{15}{4h^2l}$$

$$= \frac{5}{4}\,\frac{W}{hl^3}\,a(l-a)\,(l^2+al-a^2)$$

Horizontal thrust when one of the two point loads is present

$$H = \frac{5}{8}\times\frac{W}{hl^3}\,a(l-a)\,(l^2+la-a^2)$$

Above expression for the horizontal thrust maybe rewritten

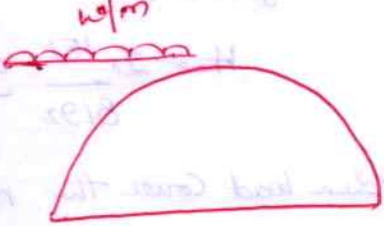
$$H = \frac{5}{8}\,\frac{Wl}{h}\left[\frac{a}{l} - 2\left(\frac{a}{l}\right)^3 + \left(\frac{a}{l}\right)^4\right]$$

If $a = nl$

$$H = \frac{5}{8}\,\frac{Wl}{h}\,n(1-n)\,(1+n-n^2)$$

$$H = \frac{5}{8}\,\frac{Wl}{h}\,(n-2n^3+n^4)$$

A Two hinged parabolic arch of span l and rise h carries a UDL of w/mt



$$dH = \frac{5}{8} \cdot \frac{w\,dx}{hl^3} x(l-x)(l^2+lx-x^2)$$

$$= \frac{5}{8} \frac{w}{hl^3} (l^3x - 2lx^3 + x^4)\,dx$$

Total horizontal thrust

$$H = \frac{5}{8} \frac{w}{hl^3} \int_0^a (l^3x - 2lx^3 + x^4)\,dx$$

$$= \frac{5}{8} \frac{w}{hl^3} \left[\frac{l^3a^2}{2} - \frac{2la^4}{4} + \frac{a^5}{5} \right]$$

$$= \frac{5}{8} \frac{w}{hl^3} \int_0^a (l^3x - 2lx^3 + x^4)\,ds$$

$$= \frac{5}{8} \frac{w}{hl^3} \left[\frac{l^3a^2}{2} - \frac{2la^4}{4} + \frac{a^5}{5} \right]$$

$$= \frac{5}{8} \frac{w}{hl^3} \cdot \frac{a^2}{10} \left[5l^3 - 5la^2 + 2a^3 \right] = \frac{wa^2}{16hl^3} \left[5l^3 - 5la^2 + 2a^3 \right]$$

When the load covers a distance $l/4$ from one end

for this case $a = l/4$

$$H = \frac{w}{16hl^3} \left[l/4 \right]^2 \left[5l^3 - 5l \times \frac{l^2}{16} + \frac{2l^3}{64} \right]$$

$$= \frac{1}{256} \left[5 - \frac{5}{16} + \frac{1}{32} \right] \frac{wl^2}{h}$$

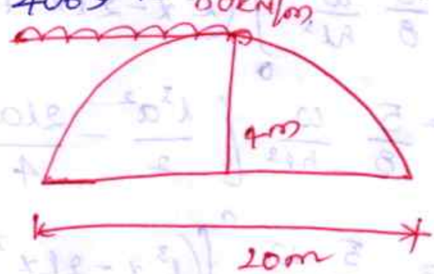
$$= \frac{151}{8192} \frac{wl^2}{h}$$

When the load covers a distance $l/4$ from each end for this case

$$H = 2 \times \frac{151}{8192}\frac{\omega l^2}{h} = \frac{151}{4096}\frac{\omega l^2}{h}$$

when load cover the middle half of span

$$H = \frac{\omega l^2}{8h} - \frac{151}{4096}\frac{\omega l^2}{h} = \frac{361}{4069}\frac{\omega l^2}{h}$$



$\Sigma M_A = 0$

$V_b \times 20 - 50 \times 10 \times 5 = 0$

$V_b = 125\,KN$

$V_a = (50 \times 10) - 125 = 375\,KN$

Since one half $H = \frac{1}{2}\frac{\omega l^2}{8h} = \frac{\omega l^2}{16h} = \frac{50 \times 20^2}{16 \times 4} = 312.5\,KN$

Max +ve BM occurs under AC. At any section AC distance x from A

$M_x = V_A x - \omega \times x \times x/2 - H \times y$

$= 375x - 25x^2 - 312.5 \times \frac{4 \times 4}{20 \times 20} x(20-x)$

$= 125x - 12.5x^2$

for max BM $\frac{\partial M_x}{\partial x} = 0$

$\frac{\partial M_x}{\partial x} = 125 - 25x = 0$

$x = 5m$

$M_{max} = 125 \times 5 - 12.5 \times 5^2$

$= 625 - 312.5 = 312.5\,KNm$

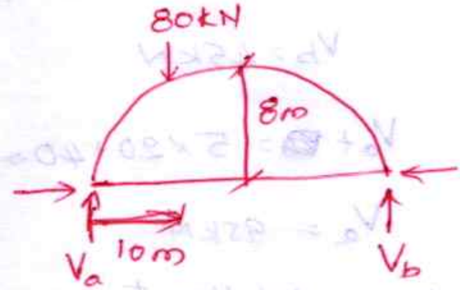
Maximum negative BM @ BC

$$M_x = 125x - 312.5 \times \frac{4 \times 4}{20^2}\, x(20-x)$$

$$M_x = 125x - 12.5x^2$$

$$\frac{dm}{dx} = 0 \quad x = 5\,m$$

$$M_{max\,-ve} = 312.5\ kNm$$



$$\Sigma M_b = 0$$

$$V_a \times 40 - 80 \times 30 = 0$$

$$V_a = 60\,kN \qquad V_a + V_b = 80\,kN$$

$$V_b = 80 - 60 = 20\,kN$$

Horizontal thrust

$$H = \frac{5}{8}\frac{W}{hl^3}\, a(l-a)(l^2 + al - a^2)$$

$$= \frac{5}{8} \times \frac{80}{8 \times 40^3} \times 10 \times 30 (40^2 + 40 \times 10 - 10^2) = 55.664\,kN$$

Max +ve BM occur under load

$$y = \frac{4 \times 8}{40^2} \times 10 \times 30 = 6\,m$$

$$\text{Max +ve BM} = 60 \times 10 - 55.65 \times 6 = 266.016\,kNm$$

Max -ve BM

As a section in DB

$$M_x = V_b x - Hy = 20x - 55.664 \times \frac{4 \times 8}{40^2}\, x(40-x)$$

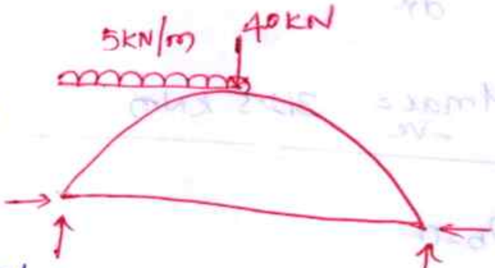
$$M_x = 1.11328\,x^2 - 24.53\,x$$

$$\frac{\partial M}{\partial x} = 0$$

$x = 11.0175 m$

$M_{max} = 1.11328(11.012)^2 - 24.53(11.01) = -135.135 kNm$

$\Sigma M_a = 0$



$V_b \times 40 = 5 \times 20 \times 10 + 40 \times 20$

$V_b = 45 kN$

$V_a + V_b = 5 \times 20 + 40 =$ ~~95kN~~

$V_a = 95 kN$

Horizontal thrust

$$H = \frac{wl^2}{16h} + \frac{25}{128}\frac{wl}{h} - \frac{5 \times 40^2}{16 \times 5} + \frac{25}{128} \times \frac{40 \times 40}{5}$$

$= 100 + 62.5 = 162.5 kN$

+ve BM at AC

$M = 95x - \frac{5x^2}{2} - Hy \qquad y = \frac{4h}{l^2}x(l-x) = \frac{4 \times 5}{40^2}x(40-x)$

$y = \frac{x}{80}(40-x)$

$M = 95x - \frac{5x^2}{2} - \frac{162.5}{80}x(40-x)$

$= 95x - 2.5x^2 - 81.25x + 2.031x^2$

$\frac{dM}{dx} \quad 13.75 - 0.46875 \times 2x \quad \therefore x = 14.67 m$

$M_{max} = 13.75 \times 14.67 - 0.469 \times 14.67^2$

$= 100.84 kNm$

Max -ve BM occurs at a section on the right half of the span

$$M = 45x - Hy = 45x - \frac{162.5}{80} x(40-x)$$

$$= 45x - 81.25x + 2.03125x^2$$

$$= -36.25x + 2.03125x^2$$

$$= -(36.25x - 2.031x^2)$$

$\frac{dM_x}{dx} = 0$ $\quad \frac{dM}{dx} = -(36.25 - 2 \times 2.031x) = 0 \quad \therefore x = 8.925\,m$

Max - Ve BM

$$= -(36.25 \times 8.925 - 2.031 \times 8.925^2) = -161.73\,kNm$$

Temperature effect on two hinged arches

A Two hinged arch subjected to a rise of temperature t. obviously the length of the member tends to increase. If one of the ends say the end B had been placed on rollers there would be a horizontal displacement BB_1 equal to nearly αTL

Since such a horizontal displacement cannot take place when the ends are hinged, Inward horizontal thrust H will be developed at each support. The hinged end B with the horizontal thrust H may be looked upon as a roller end B with an externely applied horizontal force H. Let for this condition the inward horizontal movement due to external force H applied at B be equal to δ.

If $\delta = \alpha TL$ then the net displacement of B = 0. Hence the value of H for which $\delta = \alpha TC$ represents the horizontal thrust developed in the two hinged arch due to rise of temperature

Considering the end B as a roller end with an external horizontal force H applied at B, the BM at any section is given by

$$M_x = -Hy$$

strain energy stored by the arch $W_i = \int \frac{M_x^2 ds}{2EI}$

By the first theorem of castigliano

Inward horizontal movement $B = \delta = \int \frac{H^2 y^2 ds}{2EI}$

$$\delta = \frac{\partial W_i}{\partial H} = \int \frac{2H \cdot y^2 \partial s}{2EI}$$

$$\delta = H \int \frac{y^2 \partial s}{EI}$$

The condition that H may represent the horizontal thrust for the two hinged arch subjected to rise of temperature is

$$\delta = \alpha T L$$

$$H \int \frac{y^2 \partial s}{EI} = \alpha T L$$

$$H = \frac{\alpha T L}{\int y^2/_{EI} \, ds}$$

If the arch section is of uniform EI

$$H = \frac{EI \alpha T L}{\int y^2 ds}$$

for a semicircular two hinged arch

$$\int y^2 ds = 2 \int_0^{\pi/2} (R \sin\theta)^2 R \, d\theta = 2R^3 \times \pi/4 = \frac{\pi R^3}{2}$$

$l = 2R$

$$H = \frac{EI\alpha T(2R)}{\left(\pi R^3/2\right)} = \frac{4EI\alpha T}{\pi R^2}$$

for parabolic two hinged arch

$$H = \frac{EI_0 \alpha T L}{\int y^2 ds}$$

$$\int y^2 ds = 2\int_0^{l/2} \left(\frac{4h}{l^2} x(l-x)\right)^2 dx = 8/15\, h^2 l$$

$$H = \frac{EI\alpha TL}{8/15\, h^2 l} = \frac{15}{8}\frac{EI_0\alpha T}{h^2}$$

$$\text{Max BM } M = Hh = \left(\frac{EI\alpha TL}{\int y^2 ds}\right) h$$

If D is depth of arch

$$\text{Section modulus } Z = I/y = \frac{I}{D/2} = 2I/D$$

Maximum stress due to rise in temperature

$$f = \frac{M}{Z} = \left[\frac{EI\alpha TL}{\int y^2 ds}\right] h \frac{D}{2I}$$

$$f = \frac{EI\alpha TLhD}{2\int y^2 ds}$$

A Two-hinged semicircular arch of radius 10m is subjected to a rise of temperature 40°c. find the maximum stress due to rise of temperature take $E = 2\times10^5\ N/mm^2$, $\alpha = 12\times10^{-6}$ per °c. Depth of the arch section is 1000mm.

Horizontal thrust due to rise of temperature $H = \frac{4EI\alpha T}{\pi R^2}$

$$= \frac{4 \times 2 \times 10^5 \times 12 \times 10^{-6} \times 40 I}{\pi (10{,}000)^2}$$

$$= \frac{384 I}{\pi \times 10^8} N$$

Max BM due to rise of temperature $= \frac{384 I}{\pi \times 10^8} \times 10{,}000$

Section modulus $Z = I/y = I/500$

Max Bending stress $= \frac{M}{Z} = \frac{384 I}{\pi \times 10^4} \times \frac{500}{I} = 6.11 N/mm^2$

A two hinged parabolic arch of span 40m and rise 8m is subjected to a rise of temperature 30°C. find the max Bending stress at the crown due to temperature rise. The rib section is 1000 mm deep. Take $E = 2 \times 10^5 N/mm^2$ & $\alpha = 12 \times 10^{-6}$ °C

Horizontal thrust due to rise of temperature

$$H = \frac{15}{8} \frac{I E_0 \alpha T}{h^2}$$

$$= \frac{15}{8} \times \frac{2 \times 10^5 \times 12 \times 10^{-6} \times 30 I_0}{(8000)^2} = \frac{27 I_0}{128 \times 10^5}$$

Max BM due to rise of temperature $M = \frac{27 I_0}{128 \times 10^5} \times 8000$

$$= \frac{27 I_0}{1600} Nmm$$

$Z = \frac{I_0}{500} mm^3$

$$f = \frac{27 I_0}{1600} \times \frac{500}{I_0} = 8.44 N/mm^2$$

Effect of RIB shortening

we know any section of the arch is subjected to a BM, a shear force and a normal thrust. Normal thrust causes a shortening of the actual length of the arch

Normal thrust $N = V\sin\alpha + H\sin\alpha$, $M_x = M - Hy$ $V = Sf$ $M_x = BM$

strain energy stored

$$W_i = \int \frac{M_x^2\, ds}{2EI} + \int \frac{N^2}{2AE}\, ds$$

$$W_i = \int \frac{(M - Hy)^2}{2EI}\, ds + \int \frac{(V\sin\alpha + H\cos\alpha)^2}{2AE}\, ds$$

for the condition that the strain energy stored is a minimum

$$\frac{\partial W_i}{\partial H} = 0$$

$$\frac{\partial W_i}{\partial H} = \int \frac{(M - Hy)(-y)\, ds}{EI} + \int \frac{(V\sin\alpha + H\cos\alpha)\cos\alpha\, ds}{AE} = 0$$

$$-\int \frac{My\, ds}{EI} + H\int \frac{y^2\, ds}{EI} + \int \frac{V\sin\alpha\cos\alpha\, ds}{AE} + H\int \frac{\cos^2\alpha\, ds}{AE} = 0$$

$$H = \frac{\int \frac{My\, ds}{EI} - \int \frac{V\sin\alpha\cos\alpha\, ds}{AE}}{\int \frac{y^2}{EI}\, ds + \int \frac{\cos^2\alpha}{AE}\, ds}$$

we know in the practical cases, the sectional area is minimum at the crown section and increases towards the supports. The slope α is zero at crown and increases towards support

Hence, the quantity $\frac{A}{\cos\alpha}$ has a min value at crown increases towards the support

A_m = mean value of $\frac{A}{\cos\alpha}$ and putting $ds\cos\alpha = dx$

The quantity $\int \frac{\cos^2\alpha\, ds}{AE} = \int \frac{dx}{A_m E} = \frac{l}{A_m E}$ l = span

we also find the effect of V in shorting the rib is ignorable

$$H = \frac{\int \frac{My\, ds}{EI}}{\int \frac{y^2 ds}{EI} + \frac{l}{A_m E}} = \frac{\int \frac{My}{EI_0} dx}{\int \frac{y^2}{EI_0} dx + \frac{l}{A_m E}}$$

Horizontal thrust due to temperature change and rib shorten

$$H = \frac{\alpha T L}{\int \frac{y^2}{EI} ds}$$

To consider the effect of rib shortening the horizontal thrust

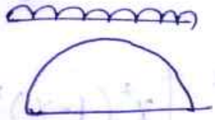
$$H = \frac{\alpha T L}{\int \frac{y^2}{EI} ds + \frac{l}{A_m E}} = \frac{\alpha T L}{\frac{y^2 dx}{EI_0} + \frac{l}{A_m E}}$$

Normal thrust and Radial shear;

Normal thrust $P_n = H_d \cos\theta + V_d \sin\theta$

Radial shear $S = H_d \sin\theta - V_d \cos\theta$

Two hinged parabolic arch of span l or rise h carries a UDL of 10kN/m unit run over the entire span. Calculate the horizontal thrust at each support.



By taking moments @ A from B we get,

$$V_b \times l - w \cdot l \times l/2 = 0$$

$$V_b \times l = wl^2/2$$

$$V_b = wl/2 \qquad V_a = wl/2$$

Horizontal thrust $H = \dfrac{\int_A^B My\, dx}{\int_A^B y^2\, dx}$

Beam moment or moment at any section of the arch can be given as

$$M_x = V_a x - w \cdot x \cdot x/2$$

$$= \frac{wl}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(l-x)$$

Vertical rise $y = \dfrac{4h}{l^2}x(l-x)$

Substituting values of M & y in horizontal thrust we get

$$H = \frac{\int_A^B \frac{wx}{2}(l-x) \times \left[\frac{4h}{l^2}x(l-x)\right] dx}{\int_A^B \left[\frac{4h}{l^2}x(l-x)\right]^2 dx}$$

Now consider the numerator

$$\int_A^B wx(l-x)\,\frac{2h}{l^2}x(l-x)\,dx$$

$$\frac{2wh}{l^2}\int x^2(l-x)^2\,dx$$

Denominator

$$\int_A^B\left(\frac{4h}{l^2}x(l-x)\right)^2 dx$$

$$\int_B^A\left(\frac{16h^2}{l^4}x^2(l-x)^2\right)dx$$

$$\frac{16h^2}{l^4}\int_B^A x^2(l-x)^2\,dx$$

$$\therefore H=\frac{\frac{2wh}{l^2}\int x^2(l-x)^2\,dx}{\frac{16h^2}{l^4}\int x^2(l-x)^2\,dx}=\frac{2wh}{l^2}\times\frac{l^4}{16h^2}$$

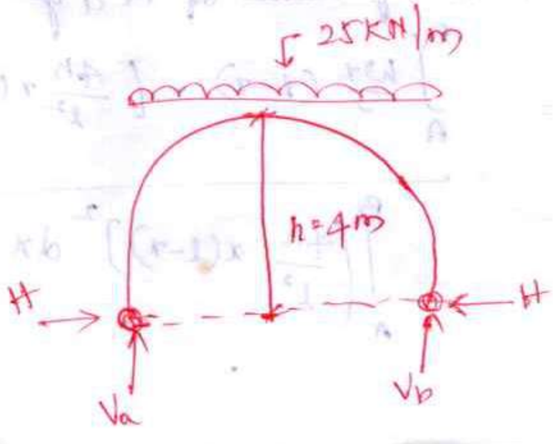
$$H=\frac{wl^2}{8h}$$

A P

$l=20m$

$w=25KN/m$

$h=4m$



By taking moment @A from B

$V_b = V_a = \frac{25 \times 20}{2} = 250 kN$

Horizontal thrust $H = \frac{\int_A^B My\,dx}{\int_A^B y^2\,dx}$

$M = V_a x - w x^2/2 = 250x - 25x^2/2 = 250x - 12.5x^2$

$y = \frac{4h}{l^2} x(l-x) = \frac{4 \times 4^2}{20} x(20-x) = \frac{1}{25} x(20-x)$

$$H = \frac{\int_0^{20} 250x - 12.5x^2 \left[\frac{1}{25} x(20-x)\right] dx}{\int_0^{20} \frac{1}{25} x(20-x)\,dx}$$

$$= \frac{12.5\int_0^{20} 20x - x^2\,dx}{\frac{1}{25}\int_0^{20} x(20-x)\,dx} = \frac{12.5\int_0^{20} x(20-x)\,dx}{\frac{1}{25}\int_0^{20} x(20-x)\,dx}$$

$$= \frac{12.5}{1/25} = 312.5 kN$$

$H = \frac{wl^2}{8h} = \frac{25 \times 20^2}{8 \times 4} = 312.5 kN$

Maximum B-M $M_x = V_a x - w x \times x/2 - Hy$

$= 250x - 25x^2/2 - 312.5y$

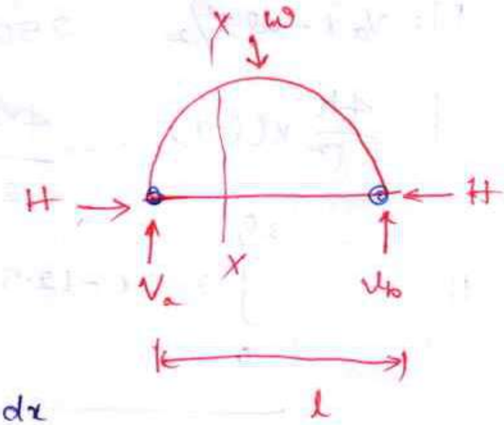
$= 250x - 12.5x^2 - 312.5\left[\frac{1}{25}x(20-x)\right]$

$= 250x - 12.5x^2 - 312.5\,(0.8x - 0.04x^2)$

$250x - 12.5x^2 - 250x + 12.5x^2$

$= 0$

Support Reaction



$\Sigma M_B = 0$

$V_a \times l - wl/2 = 0$

$V_a l = wl/2$

$V_a = w/2 = V_b$

Horizontal thrust $H = \dfrac{\int_A^B M y\,dx}{\int_A^B y^2\,dx}$

$M = V_a x = \frac{w}{2}x = \frac{wx}{2}$

Vertical rise $y = \frac{4h}{l^2}x(l-x)$

$H = \dfrac{\int_A^B M y\,dx}{\int_A^B y^2\,dx} = \dfrac{\int_0^{l/2} M y\,dx}{\int_0^{l/2} y^2\,dx}$

$$= \frac{\int_0^{l/2} \frac{wx}{2}\left[\frac{4h}{l^2}x(l-x)\right]dx}{\int_0^{l/2}\left[\frac{4h}{l^2}x(l-x)\right]^2 dx}$$

Numerator - Consider

$$\int_0^{l/2} \frac{wx}{2}\left[\frac{4h}{l^2}x(l-x)\right]dx = \int_0^{l/2}\frac{2wx}{2}\left[\frac{2h}{l^2}x(l-x)\right]dx$$

$$= \frac{2wh}{l^2}\int_0^{l/2} x^2(l-x)\,dx = \frac{2wh}{l^2}\int_0^{l/2}(x^2 l - x^3)\,dx$$

Integrating the above eq

$$\frac{2wh}{l^2}\left[\frac{x^3}{3}l - \frac{x^4}{4}\right]_0^{l/2} = \frac{2wh}{l^2}\left[\frac{(l/2)^3}{3}l - \frac{(l/2)^4}{4}\right]$$

$$= \frac{2wh}{l^2}\left[\frac{l^4}{24}-\frac{l^4}{64}\right] = \frac{2wh}{l^2}\times\frac{5l^4}{192}$$

$$= \frac{10whl^2}{192} = \frac{5whl^2}{96}$$

Consider the Denominator

$$\int_0^{l/2}\left[\frac{4h}{l^2}x(l-x)\right]^2dx = \int_0^{l/2}\frac{16h^2}{l^4}x^2(l-x)^2\,dx$$

$$= \frac{16h^2}{l^4} \int_0^{l/2} x^2 (l-x)^2 \, dx$$

$$= \frac{16h^2}{l^4} \int_0^{l/2} x^2 (l^2 + x^2 - 2lx) \, dx$$

$$= \frac{16h^2}{l^4} \int_0^{l/2} (x^2 l^2 + x^4 - 2lx^3) \, dx$$

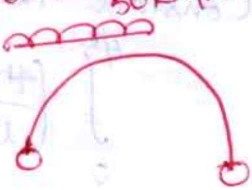
$$= \frac{16h^2}{l^4} \left[\frac{x^3}{3} l^2 + \frac{x^5}{5} - 2l \frac{x^4}{4} \right]_0^{l/2}$$

$$= \frac{16h^2}{l^4} \left[\frac{l^5}{24} + \frac{l^5}{160} - \frac{l^5}{30} \right] = \frac{16h^2}{l^4} \times \frac{l^5}{60} = \frac{4h^2 l}{15}$$

By substituting Denominator & numerator

$$H = \frac{5whl^2/96}{4h^2 l/15} = \frac{5wl/96}{4h/15} = \frac{25wl}{128h}$$

A Two hinged parabolic arch of span 20m & rise 4m carries UDL of 50 kN/m over the left half of the portion. find the reactions at supports & position and also magnitude of max BM



Support Reactions

$V_b \times 20 - 50 \times 10 \times 10/2 = 0$

$V_b = 125 \text{ kN}$

$V_a + V_b = 50 \times 10$

$V_a = 375 kN$

Horizontal thrust $H = \dfrac{\int_A^B M y \, dx}{\int_A^B y^2 \, dx}$

$M = V_a x - 50x^2/2 = 375x - 50x^2/2 = 375x - 25x^2$

Vertical rise $y = \dfrac{4h}{l^2} x(l-x) = \dfrac{4 \times 4}{20^2} x(20-x) = \dfrac{16}{20^2} x(20-x)$

$= \dfrac{1}{25} x(20-x)$

By substituting M & y values in horizontal thrust H

$$\therefore H = \frac{\int_0^{l/2} (375x - 25x^2)\left(\frac{1}{25} x(20-x)\right)}{\int_0^{l/2} \frac{1}{25} x(20-x)^2 \, dx}$$

Numerator

$$= \int_0^{l/2} (375x - 25x^2) \, dx$$

$$= \left[\frac{375x^2}{2} - \frac{25x^3}{3}\right]_0^{l/2}$$

$$= \left(\frac{375 \times 10^2}{2} - \frac{25 \times 10^3}{3}\right) = 10,416.66$$

Denominator

$$\int_0^{l/2} \frac{1}{25} x(20-x) \, dx$$

$$\int_0^{10} 0.8x - 0.04x^2 \, dx = \left[\frac{0.8x^2}{2} - \frac{0.04x^3}{3}\right]_0^{10}$$

$$= \left(\frac{0.8(10)^2}{2} - \frac{0.04(10)^3}{3}\right) = 26.66$$

$$H = \frac{10416.66}{26.66} = 390.72 \text{ kN}$$

$$M_x = V_a x - wx^2/2 - Hy$$

$$= 375x - 50x^2/2 - 390.72\left[\frac{4h}{l^2}x(l-x)\right]$$

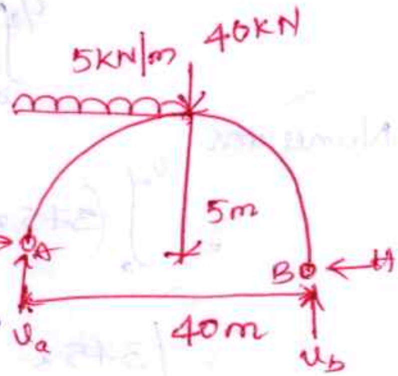
$$y = 0.8x - 0.04x^2$$

$$= 375x - 50x^2/2 - 390.72(0.8x - 0.04x^2)$$

$$= 375x - 25x^2 - 312.50x + 15.62x^2$$

$$\frac{\partial M_x}{\partial x} = 0 \quad x = 3.34 \text{ m}$$

$$M_x = 104.15 \text{ kN}$$



Support Reactions

$$V_a \cdot 40 - 5\times20\times\left[\frac{20}{2}+20\right] - 40\times20 = 0$$

$$V_a = 95 \text{ kN}$$

$$V_a + V_b = 5\times20+40 \quad V_b = 45 \text{ kN}$$

Horizontal thrust $H = \frac{wl^2}{16h} + \frac{25Wl}{128h} = \frac{5\times40^2}{16\times5} + \frac{25\times40\times40}{128\times5}$

$$= 162.5 \text{ kN}$$

maximum +ve B.M @ section X-X
at section

$$M_x = V_a x - \frac{wx^2}{2} - Hy$$

$$= 95x - \frac{5x^2}{2} - 162.5\left[\frac{4h}{l^2}x(l-x)\right]$$

$$= 95x - \frac{5x^2}{2} - 162.5\left[0.5x - 0.012x^2\right)$$

$$= 95x - 2.5x^2 - 81.25x + 1.95x^2$$

$$\frac{\partial M_x}{\partial x} = 95 - 2\times 2.5x - 81.25 + 2\times 1.95x$$

$$x = 12.5m$$

$$M_x = 81.93 \text{ kNm}$$

Max -ve Bm @ section Y-Y right from B

$$V_b x - Hy = 0$$

$$45x - 162.5\left[\frac{4h}{l^2}x(l-x)\right] = 0$$

$$45x - 81.25x + 1.95x^2$$

$$\frac{\partial M_x}{\partial x} = 45 - 81.25 + 3.9x = 0$$

$$x = -9.29$$

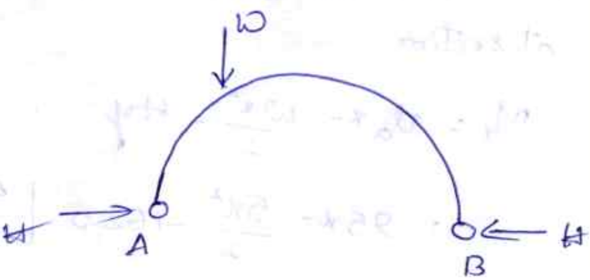
$$M_x = 45\times(9.29) - 81.25(9.29) + 1.95(9.29)^2$$

$$= -166.147 \text{ kNm}$$

Show that the horizontal thrust developed in a parabolic arch of span l vertical rise h subjected to a concentrated load at a distance a from the springing is given by $H = \frac{5}{8}\left[\frac{w}{hl^3}\right] a(l-a)\cdot(l^2+al-a^2)$

Taking moments @A from B

$V_b x - W(x-a) - Wa = 0$



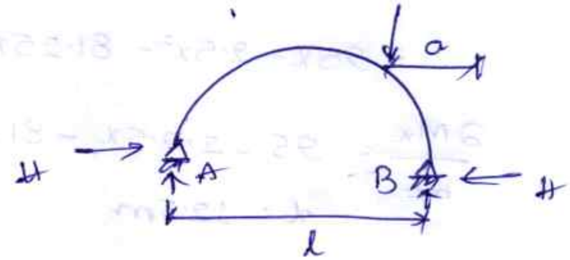
@ $V_b \times l - W(l-a) - Wa = 0$

$V_b l = W(l-a) + Wa$

$V_b l = Wl - Wa + Wa$

$V_b = Wl/l = W$

$V_a = V_b = W$



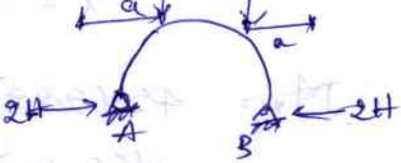
$$H = \frac{\int_0^l M y \, dx}{\int y^2 dx}$$

$$2H = \frac{2\left[\int_0^a M y\, dx + \int_a^{l/2} M y \, dx\right]}{2\left(\int_0^{l/2} y^2 dx\right)}$$

Beam moment at a point $P(x,y)$ in the portion $x = 0$ to a

$M = V_a \times x$ $(x = 0 \text{ to } a)$

$M = Wx$



Beam moment @ $x = a$ to $l/2$

$M = V_a x - W(x-a) = Wx - Wx + Wa$

$M = Wa$ $(x = a \text{ to } l/2)$

Now consider numerator

$$H = \frac{4h}{l^2} x(l-x)$$

Numerator

$$2\int_0^a wx \cdot \frac{4h}{l^2} x(l-x)\,dx + 2\int_a^{l/2} wa \cdot \frac{4h}{l^2} x(l-x)\,dx$$

$$= \frac{8wh}{l^2}\int_0^a x^2(l-x)\,dx + \frac{8wha}{l^2}\int_a^{l/2} x(l-x)\,dx$$

$$= \frac{8wh}{l^2}\int_0^a (lx^2 - x^3)\,dx + \frac{8wha}{l^2}\int_a^{l/2}(lx - x^2)\,dx$$

$$= \frac{8wh}{l^2}\left[\frac{lx^3}{3} - \frac{x^4}{4}\right]_0^a + \frac{8wha}{l^2}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_a^{l/2}$$

$$= \frac{8wh}{l^2}\left\{\left[\frac{lx^3}{3} - \frac{x^4}{4}\right]_0^a + a\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_a^{l/2}\right\}$$

$$= \frac{8wh}{l^2}\left(\left[\frac{la^3}{3} - \frac{a^4}{4}\right] + a\left[\frac{l(l/2)^2}{2} - \frac{(l/2)^3}{3} - a\left[\frac{la^2}{2} - \frac{a^3}{3}\right]\right]\right)$$

$$= \frac{8wh}{l^2}\left[\frac{la^3}{3} - \frac{a^4}{4} + a\left(\frac{l^3}{8} - \frac{l^3}{24}\right) - a\left[\frac{la^2}{2} - \frac{a^3}{3}\right]\right]$$

$$= \frac{8wh}{l^2}\cdot\frac{1}{24}\left[8a^3l - 6a^4 + 3al^3 - al^3 - 12a^3l + 8a^4\right]$$

$$a\left[8a^2l - 6a^3 + 3l^3 - l^3 - 12a^2l + 8a^3\right]$$

$$\cdot \frac{wha}{3l^2}\left[2l^3 - 4a^2l + 2a^3\right]$$

$$= \frac{wha}{3l^2}\, 2\left[l^3 - 2a^2l + a^3\right]$$

$$\frac{2}{3}\left[\frac{wha}{l^2}\right] l^3 - a^2l - a^2l + a^3]$$

$$= \frac{2}{3} \frac{wha}{l^2} \left[l(l-a)(l+a) - a^2(l-a) \right]$$

$$= \frac{2}{3} \frac{wha}{l^2} (l-a)(l^2+la-a^2)$$

$$2H = \frac{\int M y \, ds}{\int y^2 \, dx}$$

$$= \frac{2}{3} \frac{wha}{l^2} (l-a)(l^2+la-a^2) \times \frac{15}{8h^2 l}$$

$$= \frac{5}{8} \frac{w}{hl^2} a(l-a)(l^2+la-a^2)$$

$$H = \frac{5}{8} \frac{w}{hl^3} a(l-a)(l^2+la-a^2)$$

Denominator

$$\int y^2 \, dx \qquad \int_0^{l/2} \left[\frac{4h}{l^2} x(l-x) \right]^2$$

$$\frac{16h^2}{l^4} \int_0^{l/2} x^2 (l-x)^2$$

$$\frac{16h^2}{l^4} \int_0^{l/2} x^2 (l^2+x^2-2lx)$$

$$\frac{16h^2}{l^4} \int_0^{l/2} x^2 l^2 + x^4 - 2lx^3$$

$$\frac{16h^2}{l^4}\left[\frac{x^3}{3}l^2 + \frac{x^5}{5} - \frac{2l x^4}{4_2}\right]_0^{l/2}$$

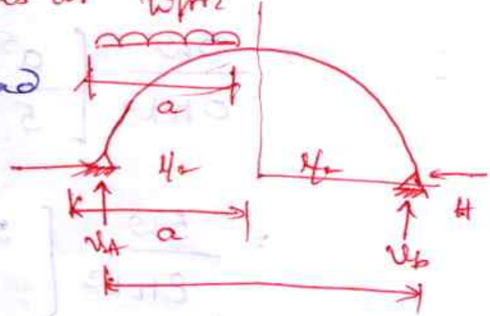
$$\frac{16h^2}{l^4}\left[\frac{l^5}{24} + \frac{l^5}{160} - \frac{l^5}{32}\right]$$

$$\frac{16h^2}{l^4} \times \frac{l^5}{60} = \frac{4h^2 l}{15}$$

$$2H = \frac{8h^2 l}{15}$$

$\frac{38 \times 5}{0}$

A Two hinged parabolic arch of span 30m and rise 6m carries Two point loads 60kN acting at 7.5m and 15m from left end respectively. The moment of inertia varies as



Horizontal thrust due to point load at a distance a from left end

$$H = \frac{5W}{8hl^3} a(l-a)(l^2 + la - a^2)$$

Now the horizontal thrust due to element load Wdx is given by

$$H = \frac{5W\partial x}{8hl^3} x(l-x)(l^2 + xl - x^2)$$

$$\frac{5W}{8hl^3} x(l-x)(l^2 + lx - x^2)\,dx$$

$$\frac{5W}{8hl^3}(xl - x^2)(l^2 + lx - x^2)\,dx$$

$$\frac{5W}{8hl^3}(xl^3 + x^2l^2 - x^3l - x^2l^2 - lx^3 + x^4)\,dx$$

$$\frac{5W}{8hl^3}[x^4 - 2x^3l + xl^3)\,dx$$

Applying limits 0 to a we get

$$H = \int_0^a \frac{5W}{8hl^3}(x^4 - 2x^3l + xl^3)\,dx$$

$$\frac{5W}{8hl^3}\int_0^a (x^4 - 2x^3l + xl^3)\,dx$$

$$= \frac{5}{8}\frac{w}{hl^3}\left[\frac{x^5}{5} - \frac{2x^4}{4}l + \frac{x^2}{2}l^3\right]_0^a$$

$$= \frac{5w}{8hl^3}\left[\frac{a^5}{5} - \frac{a^4 l}{2} + \frac{a^3 l^3}{2}\right]$$

$$= \frac{5w}{8hl^3}\left[\frac{2a^5 - 5a^4 l + 5a^2 l^3}{10}\right]$$

$$= \frac{5w}{8hl^3} \times \frac{1}{10}\left[2a^5 - 5a^4 l + 5a^2 l^3\right]$$

$$= \frac{w}{16hl^3} a^2 \left[5l^3 - 5a^2 l + 2a^3\right]$$

In particular cases if the load is acting at a distance of $l/4$ then horizontal thrust

$$H = \frac{w}{16hl^3}(l/4)^2\left[5l^3 - 5(l/4)^3 l + 2(l/4)^3\right]$$

$$= \frac{wl^2}{16hl^3} \times \frac{1}{16}\left[5l^3 - \frac{5l^3}{16} + \frac{2l^3}{64}\right]$$

$$= \frac{wl^2}{256hl^3}\left[\frac{l^3 - 10l^3 + 32 \times 5l^3}{32}\right]$$

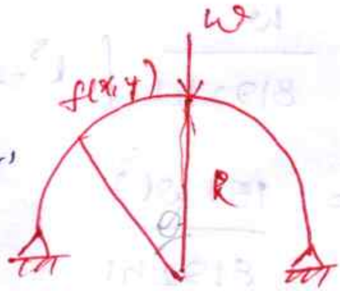
$$= \frac{w}{8192hl}\left[l^3 - 10l^3 + 160l^3\right]$$

$$= \frac{151wl^3}{8192hl}$$

$$= \frac{151wl^2}{8192h}$$

Two hinged circular Arches

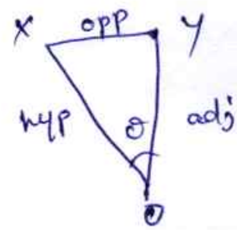
Horizontal thrust developed in a 2 hinged semicircular arch of radius 'r' subjected to a Concentrated load 'W' at the crown.



Due to Symmetry

$$V_a = V_b = \frac{W}{2}$$

Horizontal thrust $H = \dfrac{2\int_A^B \mu y\,ds}{2\int_A^D y^2\,ds}$



Vertical rise

$$\cos\theta = \frac{OY}{OX} \qquad \sin\theta = \frac{XY}{OX}$$

$$\cos\theta = \frac{y}{R} \qquad \sin\theta = \frac{x}{R}$$

$$y = R\cos\theta \qquad x = R\sin\theta$$

Beam moment at section X-X

$$\mu = V_a x$$

$$= \frac{W}{2} \times R - R\sin\theta$$

$$= \frac{W}{2} R(1 - \sin\theta)$$

Substitute all values in horizontal thrust

$$2\int_0^{\pi/2} \frac{W}{2} R(1-\sin\theta)(R\cos\theta) \times R\,d\theta$$

$$2\int_0^{\pi/2} \frac{wR^3}{2}(1-\sin\theta)\cos\theta\, d\theta$$

$$\cancel{2}\cdot \frac{wR^3}{\cancel{2}} \int_0^{\pi/2} (1-\sin\theta)\cos\theta\, d\theta$$

$(1-\sin\theta) = t \qquad dt = -\cos\theta\, d\theta$

By differentiating the value of 't', we get

$$= \int (1-\sin\theta)\cos\theta\, d\theta$$

$$= \int t\,(-dt)$$

$$= \frac{-t^2}{2}$$

$$= \frac{-1}{2}(1-\sin\theta)^2$$

$$= wR^3 \cdot \left[-\tfrac{1}{2}(1-\sin\theta)^2\right]_0^{\pi/2}$$

$$= wR^3\left[-\tfrac{1}{2}(1-\sin\tfrac{\pi}{2})^2\right] - \left[-\tfrac{1}{2}(1-\sin 0)^2\right]$$

$$= -\frac{1}{2}wR^3\,[0-1]$$

$$= -\tfrac{1}{2}wR^3(-1)$$

$$= \frac{wR^3}{2}$$

Denominator

$$2\int_0^{\pi/2} (R\cos\theta)^2 R\, d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$2R^3 \int_0^{\pi/2} (\cos^2\theta)\, d\theta$$

$$2R^3 \int_0^{\pi/2} \left[\frac{1+\cos 2\theta}{2}\right] d\theta$$

$$\frac{2R^3}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

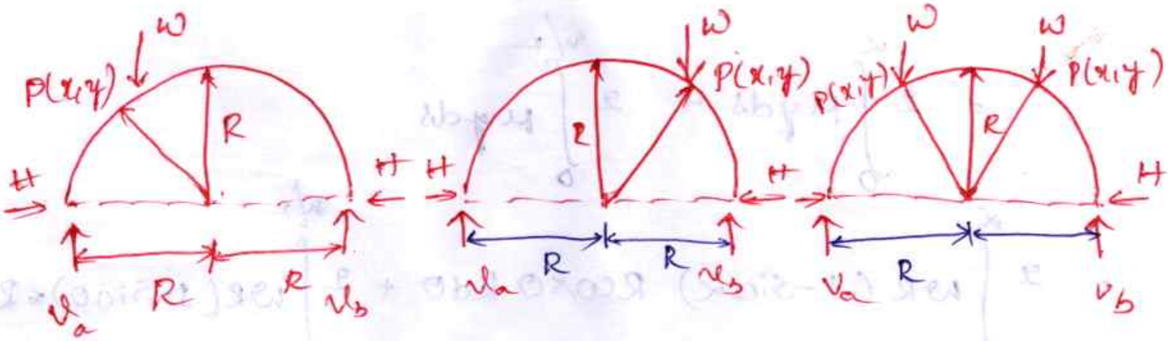
$$= \frac{R^3}{2} \left[2\theta + \sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{R^3}{2} \left[2 \times \pi/2 + \sin 2 \times \pi/2\right] - \left[2(0) + \sin 2(0)\right]$$

$$= \frac{R^3}{2} [\pi] = \pi R^3/2$$

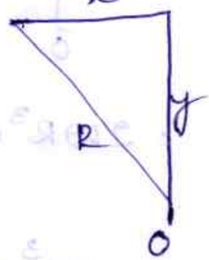
$$\therefore \frac{w R^3/2}{\pi R^3/2} = \frac{w}{\pi}$$

$$H = \frac{w}{\pi}$$



$V_a = V_b = W$

$$H = \frac{2\int_A^B \mu y \, ds}{2\int_A^B y^2 \, ds}$$



$\cos\theta = \frac{OY}{OX}$ $\quad \sin\theta = \frac{XY}{OX}$

$\cos\theta = y/R$ $\quad \sin\theta = x/R$

$x = R\sin\theta$ $\quad ds = R\,d\theta$

$y = R\cos\theta$

$\mu = V_a x - WX$ for $\theta = 0$ to α

$= V_a R(1-\sin\theta) - WX$

$= WR(1-\sin\theta) - W(R\sin\alpha - R\sin\theta)$

$WR - WR\sin\theta - WR\sin\alpha + WR\sin\theta$

$WR(1-\sin\alpha)$

for $\theta = \alpha$ to $\pi/2$

$\mu = V_a x$

$= WR(1-\sin\theta)$

$$= 2\int_0^{\alpha} \mu y\,ds + 2\int_0^{\pi/2} \mu y\,ds$$

$$= 2\int_0^{\alpha} wR(1-\sin\alpha)\,R\cos\theta\,R\,d\theta + 2\int_{\alpha}^{\pi/2} wR(1-\sin\theta)\times R\cos\theta\,R\,d\theta$$

$$= 2\int_0^{\alpha} wR^3(1-\sin\alpha)\cos\theta\,d\theta + 2\int_{\alpha}^{\pi/2} wR^3(1-\sin\theta)\cos\theta\,d\theta$$

$$= 2wR^3(1-\sin\alpha)\int_0^{\alpha}\cos\theta\,d\theta + 2wR^3\int_{\alpha}^{\pi/2}(1-\sin\theta)\cos\theta\,d\theta$$

$$= 2wR^3(1-\sin\alpha)(\sin\theta)_0^{\alpha} + 2wR^3(-1)\left[\frac{(1-\sin\theta)^2}{2}\right]_{\alpha}^{\pi/2}$$

$$= 2wR^3(1-\sin\alpha)(\sin\alpha)_0^{\alpha} - 2wR^3\,\tfrac{1}{2}\left(0-(1-\sin\alpha)^2\right)$$

$$2wR^3(1-\sin\alpha)\sin\alpha - wR^3\left[(1-1)^2-(1-\sin\alpha)^2\right]$$

$$= 2wR^3(1-\sin\alpha)\sin\alpha + wR^3(1-\sin\alpha)^2$$

$$wR^3(1-\sin\alpha)[2\sin\alpha+1-\sin\alpha]$$

$$= wR^3(1-\sin\alpha)(1+\sin\alpha)$$

$$= wR^3(1-\sin^2\alpha) \qquad 2wR^3\,\tfrac{1}{2}\left(0-(1-\sin\alpha)^2\right)$$

$$= wR^3\cos^2\alpha$$

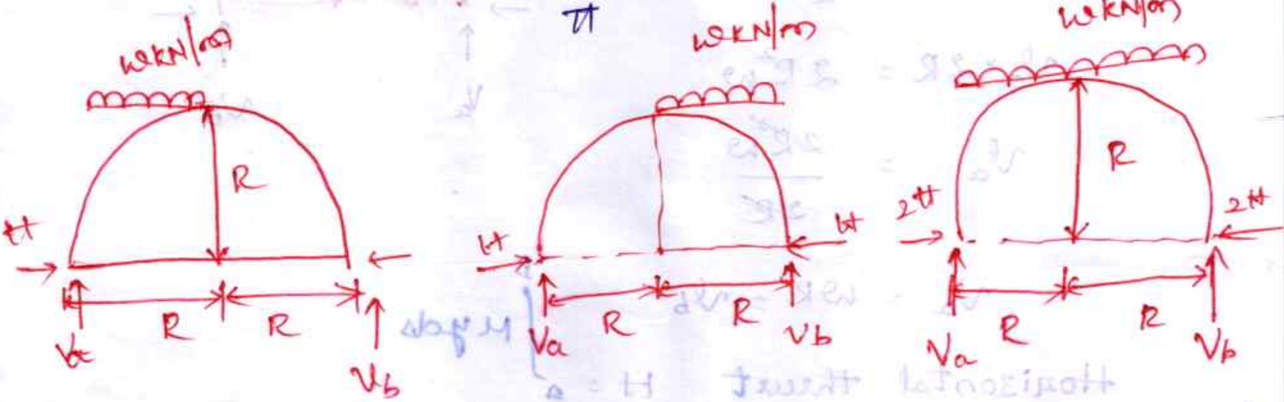
Denominator as above derivation $= \dfrac{\pi R^3}{2}$

$$2H = \frac{wR^3\cos^2\alpha}{\frac{\pi R^3}{2}}$$

$$= wR^3\cos^2\alpha \times \frac{2}{\pi R^3}$$

$$2H = \frac{\not{2} w \cos^2\alpha}{\pi}$$

$$H = \frac{w \cos^2\alpha}{\pi}$$



Case(i) ; In this case an UDL is acting over left half of portion for this condition horizontal thrust be 'H'

Case(ii) ; In this case an UDL has been acting over right half of portion for this Horizontal thrust will also be H

Case(iii) : It is combination of both the cases (i) & (ii) we already determine the horizontal thrust will be equal to $\frac{4wR}{3\pi}$

$$2H = \frac{4wR}{3\pi}$$

$$H = \frac{4wR}{3\pi/2} = \frac{2wR}{3\pi}$$

$\Sigma M_A = 0$

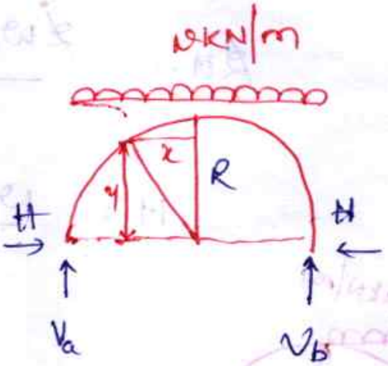
$V_a \times 2R - w \cdot 2R \cdot \frac{2R}{2} = 0$

$V_a \, 2R = \frac{w \cdot 4R^2}{2}$

$V_a \times 2R = 2R^2 w$

$V_a = \frac{2R^2 w}{2R}$

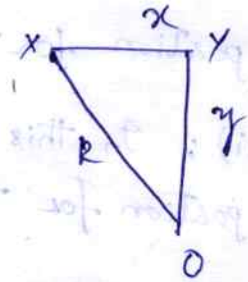
$V_a = wR = V_b$



Horizontal thrust $H = \frac{\int_A^B M y \, ds}{\int_A^B y^2 ds}$

$M = V_a x - \frac{w x^2}{2}$

$= wRx - \frac{w x^2}{2}$



$\sin\theta = XY/OX = x/R$

$\therefore x = R\sin\theta$

$\cos\theta = OY/OX$

$y = R\cos\theta$

Numerator $ds = R\,d\theta$

$2\int_0^{\pi/2} \frac{wR^2}{2}(1-\sin^2\theta)\, R\cos\theta R\, d\theta$

$2\frac{wR^4}{2}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\, d\theta$

$wR^4 \int_0^{\pi/2} \cos^2\theta \cos\theta\, d\theta$

$$wR^4 \int_0^{\pi/2} \cos^3\theta \, d\theta$$

$$I = \int \cos^3\theta \, d\theta = \int (1-\sin^2\theta)\cos\theta \, d\theta$$

$$t = \sin\theta \qquad dt = \cos\theta \, d\theta$$

$$\int (1-t^2)\,dt$$

$$\left(t - \frac{t^3}{3}\right)_0^{\pi/2} dt$$

$$\cancel{3t} = \left[\sin\theta - \frac{\sin^3\theta}{3}\right]$$

$$= \left(\sin\pi/2 - \frac{\sin^3\pi/2}{3}\right) - \left[\sin 0 - \frac{\sin^3 0}{3}\right]$$

$$1 - 1/3 = 2/3$$

$$\frac{2wR^4}{3}$$

Denominator $$2\int_0^{\pi/2} y^2 \, ds$$

$$\frac{\pi R^3}{2}$$

$$H = \frac{2wR^{\cancel{4}}}{3} \times \frac{2}{\pi R^{\cancel{3}}} = \frac{4wR}{3\pi}$$

$$H = \frac{4wR}{3\pi}$$

$wRx - \frac{wx^2}{2}$

$wRR(1-\sin\theta) - \left[\frac{w}{2}R^2(1-\sin\theta)^2\right]$

$wR^2\left[1-\sin\theta - \frac{(1-\sin\theta)^2}{2}\right]$

$= \frac{wR^2}{2}(1-\sin\theta[2-(1-\sin\theta)]$

$= \frac{wR^2}{2}(1-\sin\theta)(1+\sin\theta)$

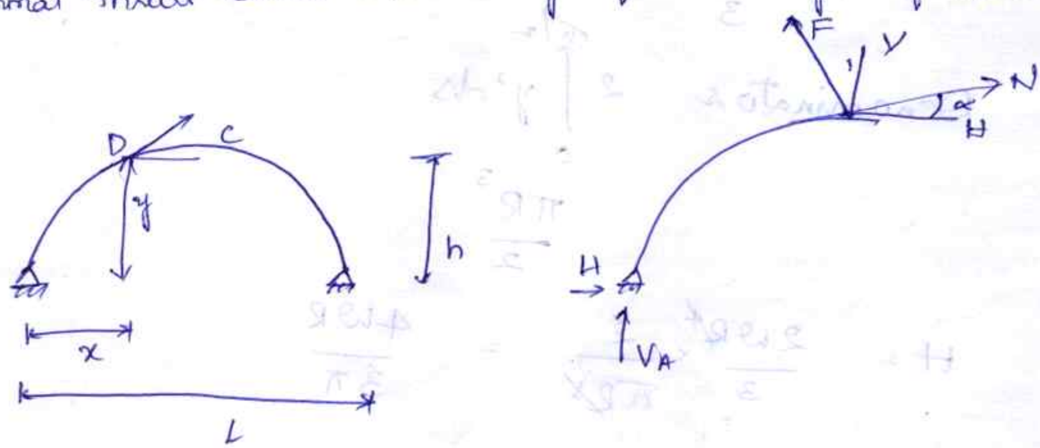
$= \frac{wR^2}{2}(1-\sin^2\theta)$

$= \frac{wR^2}{2}\cos^2\theta$

Effect of Rib shortening;

The cross - section of the arch is subjected to normal thrust Arch being made up of elastic material, shortening of the rib takes place. This shortening reduces the horizontal thrust developed.

Normal thrust causes a shortening of actual length of the arch



Consider any section x of two hinged arch. let the tangent to the arch at the section x-x will be at an angle of α with the horizontal R

M_x = B.M at any section x

V = Shearforce at the section by considering it as a SS beam

N = Normal thrust

$M_x = \mu - Hy$

μ = Beam moment

The strain energy stored by the arch is given by

$$U = \int \frac{M_x^2}{2EI} ds + \int \frac{N^2}{2AE} ds \qquad \frac{\partial M}{\partial H} = -y$$

$$\frac{\partial N}{\partial H} = \cos\alpha$$

A = c/s area at x $\qquad M = M' - Hy$

Now by substituting the M_x and N values in strain energy eq we get

$M_x = \mu - Hy$

$$N = H\cos\theta + V\sin\theta \qquad \int M\left(\frac{\partial M}{\partial H}\right)\left(\frac{\partial s}{EI}\right) + \int N\left[\frac{\partial N}{\partial H}\right]\left[\frac{\partial x}{EA}\right] = 0$$

$= H\cos\alpha + V\sin\alpha$

$$W_i = \int \frac{(\mu - Hy)^2}{2EI} ds + \int \frac{(H\cos\alpha + V\sin\alpha)^2}{2AE} ds$$

But the condition is strain energy stored is minimum

$$\frac{\partial W_i}{\partial H} = 0$$

$$\frac{\partial W_i}{\partial H} = \int \frac{2(\mu - Hy)(-y)}{2EI} ds + \int \frac{2(H\cos\alpha + V\sin\alpha)\cos\alpha}{2AE} ds$$

$$= \int \frac{(\mu - Hy)(-y)}{EI} ds + \int \frac{(H\cos\alpha + V\sin\alpha)\cos\alpha}{AE} ds$$

$$= -\int \frac{\mu y}{EI} ds + \int \frac{Hy^2}{EI} ds + \int \frac{H\cos^2\alpha}{AE} ds + \int \frac{V\sin\alpha\cos\alpha}{AE} ds$$

$$H \int \frac{y^2}{EI} ds + H \int \frac{\cos^2\alpha}{AE} ds = \int \frac{My}{EI} ds - \int \frac{V \sin\alpha \cos\alpha}{AE} ds$$

$$H \left[\int \frac{y^2}{EI} ds + \int \frac{\cos^2\alpha}{AE} ds \right] = \int \frac{My}{EI} ds - \int \frac{V \sin\alpha \cos\alpha}{AE} ds$$

$$H = \frac{\int \frac{My}{EI} ds - \int \frac{V \sin\alpha \cos\alpha}{AE} ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2\alpha}{AE} ds}$$

Now by neglecting the effect of shear in the above eq we get (because S.f at crown point is 0)

$$H = \frac{\int My\, ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2\alpha}{AE} ds}$$

Considering the term $\int \frac{\cos^2\alpha}{AE} ds$. At the crown point $\cos\alpha = 0$ and at supports, it has some definite value. Usually the c/s area is small at the crown and it is large at springings. Hence, by assuming

$$\frac{A}{\cos\alpha} = A_m = \text{Constant and also } ds \cos\alpha = dx$$

$$\int \frac{\cos^2\alpha}{AE} ds = \int \frac{\cos\alpha}{A} \cdot \frac{\cos\alpha}{E} ds$$

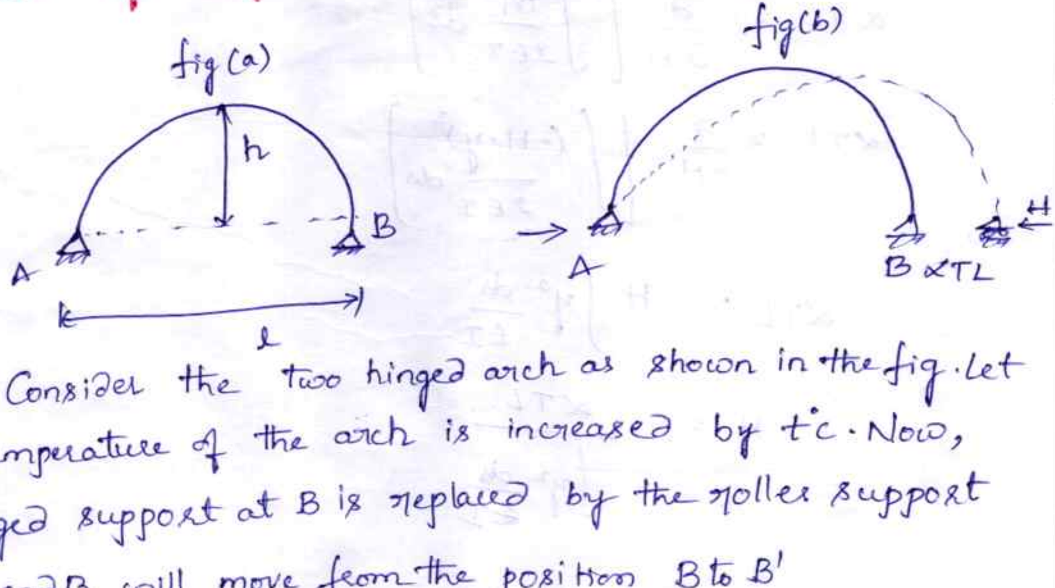
$$= \int_0^L \frac{1}{A_m} \times \frac{dx}{E}$$

$$= \int_0^L \frac{dx}{A_m E} = \frac{L}{A_m E}$$

where L = length of the arch. Substituting the value in H

$$H = \frac{\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds + \frac{L}{A_m E}}$$

Effect of temperature on two-Hinged Arch:



Consider the two hinged arch as shown in the fig. Let the temperature of the arch is increased by $t°C$. Now, the hinged support at B is replaced by the roller support let the end B will move from the position B to B'

$$BB' = \alpha T L$$

α = Coefficient of thermal expansion

T = Rise in temperature

L = Length of the arch

let H be the force Required to bring back the support B to its original position i.e., B which infers that H is the horizontal thrust developed in two hinged arch

A/c to castigliano's theorem ω

$$\frac{\partial U}{\partial H} = \alpha T L$$

Moment $M = -Hy$

$$\alpha T L = \frac{\partial U}{\partial H}$$

Strain energy $U = \int \frac{M^2}{2EI} \partial s$

$$\alpha T L = \frac{\partial}{\partial H}(U)$$

$$\alpha T L = \frac{\partial}{\partial H}\left[\int \frac{M^2}{2EI} ds\right]$$

$$\alpha T L = \frac{\partial}{\partial H}\left[\int \frac{(-H \times y)^2}{2EI} ds\right]$$

$$\alpha T L = H \int y^2 \frac{ds}{EI}$$

$$H = \frac{\alpha T L}{\int y^2 \frac{ds}{EI}}$$

$$H = \frac{EI \alpha T L}{\int y^2 ds}$$

for a Semicircular arch $\int y^2 ds = \frac{\pi R^3}{2}$

$$H = \frac{EI \alpha T L}{\pi R^3 / 2}$$

$L = 2R$

$$\frac{2EI\alpha T 2R}{\pi R^{3}} = \frac{4EI\alpha T}{\pi R^{2}}$$

For parabolic 2-hinged arch w.r.t $y = \frac{4h}{l^2}x(l-x)$

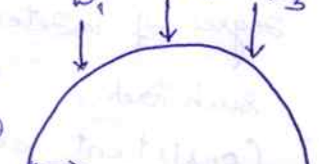
$$\int y^2 ds = \frac{8h^2 l}{15}$$

$$H = \frac{EI\alpha TL}{\frac{8h^2 l}{15}} = \frac{15EI\alpha T}{8h^2}$$

$$\boxed{H = \frac{15EI\alpha T}{8h^2}}$$

Tied Arch

A typical tied arch is shown in fig. In this fig an arch is connected by a tied beam and the system has a hinged end and a roller end. When such a arch is loaded the roller end has tendency to move is restricted due to restraint provided by the tie rod.



The movement of the roller and B will be equal to the extension of tie rod. The tie rod will be subjected to tensile force which is equal to the horizontal thrust produced in the arch.

Temperature rise in two-hinged arch under the effect of rib shortening, is given by

$$H = \frac{\int \frac{My}{EI} ds + \alpha T L}{\int \frac{y^2}{EI} ds + \frac{L}{A_m E} + k}$$

The extension of the rod $k = \frac{HL}{A_t E_t}$

A_t = Cross sectional area of tied beam

E_t = youngs modulus of tied beam

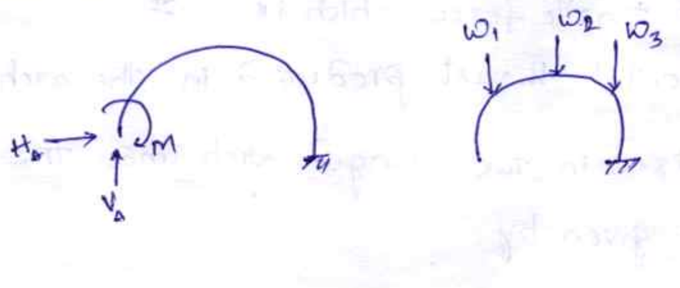
Extension of tied beam = kH

$$H = \frac{\int \frac{My}{EI} ds + \alpha T L}{\int \frac{y^2}{EI} ds + \frac{L}{A_m E} + \frac{L}{A_t E_t}}$$

Fixed Arches ;

A typical fixed arch as shown in fig. At each end there are 3 unknown reaction components, thus giving a total of 6 reaction components. But, we have only 3 equilibrium conditions. Hence degree of indeterminancy or degree of freedom is 3. To analyse such arches

Consistent deformation method



By removing the fixidity at the end A, a cantilever arch is obtained which is determinate, with the given loadings i.e., w_1, w_2, w_3 and also the loading i.e., V_A, H_A and M_A (the reaction components at A with following consistency conditions that represents a fixed arch

$$\Delta V_A = 0, \quad \Delta H_A = 0 \quad M_A (or)\ \theta_A = 0$$

where ΔV_A, ΔH_A and θ_A are the displacements at the end A. let M be moment due to loadings and also consider that m_1, m_2 m_3 be the moments due to loadings in the direction of H_A, V_A & θ_A

$$m_1 = x, \quad m_2 = y; \quad m_3 = 1$$

Due to given loading

$$\Delta V_{A0} = \int \frac{M' m_1 ds}{EI} = \int \frac{M' x\, ds}{EI} \longrightarrow 1(a)$$

$$\Delta H_{A0} = \int \frac{M' m_2 ds}{EI} = \int \frac{M' y\, ds}{EI} \longrightarrow 1(b)$$

$$\theta_{A0} = \int \frac{M' m_3}{EI} ds = \int \frac{M' ds}{EI} \longrightarrow 1(c)$$

Due to vertical reactions V_A alone $M' = V_a x$

$$\Delta V_{A1} = \int \frac{M' m_1 ds}{EI} = \int \frac{V_a x \times x\, ds}{EI} = \int \frac{V_a x^2}{EI} ds$$

$$\Delta V_{A1} = V_a \int \frac{x^2}{EI} ds \longrightarrow 2(a)$$

$$\Delta H_{A1} = \int \frac{M' m_2}{EI} ds = \int \frac{V_a x y}{EI} ds = V_a \int \frac{x y}{EI} ds \longrightarrow 2(b)$$

$$\theta_{A1} = \int \frac{M' m_3}{EI} ds = \int \frac{V_a \times x \times 1 \times ds}{EI} = V_a \int \frac{x}{EI} ds \rightarrow 2(c)$$

Due to horizontal Reaction H_A alone

$$\Delta V_{A2} = \int \frac{M' m . ds}{EI} = \int \frac{(H_A y) x \partial s}{EI} = H_A \int \frac{xy}{EI} ds \rightarrow 3(a)$$

$$\Delta H_{A2} = \int \frac{M' m}{EI} ds = \int \frac{H_A \times y \times y ds}{EI} = H_A \int \frac{y^2}{EI} ds \rightarrow 3(b)$$

$$\theta_{A2} = \int \frac{M' m_3}{EI} ds = \int \frac{H_A y ds}{EI} = H_A \int \frac{y}{EI} ds \rightarrow 3(c)$$

Due to moment M_A alone

$$\Delta V_{A3} = \int \frac{M' m_1}{EI} ds = \int \frac{M_A y ds}{EI} = M_A \int \frac{y}{EI} ds \rightarrow 4(a)$$

$$\Delta H_{A3} = \int \frac{M' m_2}{EI} ds = \int \frac{M_A y ds}{EI} = M_A \int \frac{y}{EI} ds \rightarrow 4(b)$$

$$\Theta_{A3} = \int \frac{M' m_3}{EI} ds = \int \frac{M_A (1) ds}{EI} = M_A \int \frac{\partial s}{EI} \rightarrow 4(c)$$

By solving the above set of equations 1, 2, 3 & 4 we get, the consistency conditions are as follows

$$\Delta V = \Delta V_{A0} + \Delta V_{A1} + \Delta V_{A2} + \Delta V_{A3} = 0$$

$$= \int \frac{M' x ds}{EI} + V_A \int \frac{x^2}{EI} ds + HA \int \frac{xy ds}{EI} + M_A \int \frac{x ds}{EI} = 0$$

$$\Delta H = \Delta H_{A0} + \Delta H_{A1} + \Delta H_{A2} + \Delta H_{A3} = 0$$

$$\int \frac{M'y}{EI} ds + V_A \int \frac{xy}{EI} ds + H_A \int \frac{y^2}{EI} ds + M_A \int \frac{y}{EI} ds = 0$$

$$\theta A = \theta A_0 + \theta A_1 + \theta A_2 + \theta A_3$$

$$\int \frac{M' ds}{EI} + V_a \int \frac{x ds}{EI} + H_A \int \frac{y ds}{EI} + M_A \int \frac{ds}{EI} = 0$$

The above integrations had to be carried out to cover the entire arch which results in to 3- simultaneous equations V_A, H_A, M_A. The solutions of these equations gives the values of V_A, H_A & M_A once the end reactions at A are known, the B.M at any point can be found by using equations of static equilibrium

$$M = M' + M_A + (V_A x) + H_A y$$

Assignment - 1

(1)a Derive eddy's theorem

1(b) A semicircular arch of radius R is subjected to a UDL of w/unit length over the entire span. Assum. EI to be constant, determine Horizontal thrust

1(c) Derive the expressions for horizontal thrust due to effect of temperature and rib shortening

(2) (a) A three hinged arch ACB supported on of span 40m rise 8m carries a point load of 80kN at a distance of 10m from the left support find horizontal thrust at

each support. find also max B.M. and max +ve or Radial shear and normal thrust

2(b) A two hinged parabolic arch of span 25 m and rise 5 m carries UDL of 40 kN/m over the left half of the span and a concentrated load of 100 kN at crown. find horizontal thrust and maximum +ve moment.

2(c) A two hinged parabolic arch of span 18 m and rise 3.6 m carries 2 concentrated loads of 25 kN each at crown and at left quarter span section. find horizontal thrust at each support and BM at the loaded sections

3(a) A three hinged parabolic arch ACB of span 30 m

CIVIL ENGINEERING

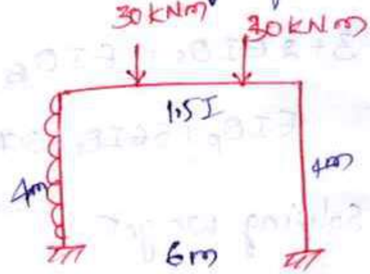
Advanced Structural Analysis

UNIT-2

Moment Distribution method

Prof. Hardy cross a professor in the University of Illinois

Fixed End moments



$M_{fAB} = \frac{-12 \times 4^2}{12} = +16 kNm$

$M_{fBA} = -16 kNm$

$M_{fBC} = -\left[\frac{30 \times 2 \times 4^2}{6^2} + \frac{30 \times 4 \times 2^2}{6^2}\right] = +40 kNm$

$M_{fCB} = -40 kNm$

$M_{fCD} = 16 kNm$

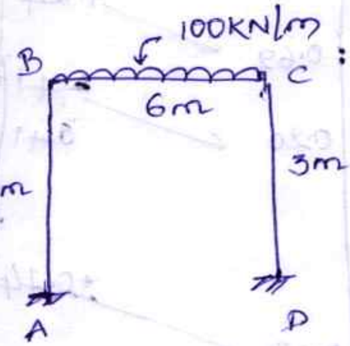
$M_{fDC} = -16 kNm$

Distribution factors

Joints	Members	k	Σk	Df
B	BA	$4EI/4$	$2EI$	0.5
	BC	$4E(1.5I)/6$		0.5
C	CB	$\frac{4E(1.5I)}{6} = EI$	$2EI$	0.5
	CD	$\frac{4EI}{4} = EI$		0.5

A	B (BA)	B (BC)	C (CB)	C (CD)	D
	0.5	0.5	0.5	0.5	
	-16	40	-40	16	-16
16	-12	-12	+12	12	
		6	-6		6
-6	-3	-3	+3	3	1.5
		1.5	-1.5		
-1.5	-0.75	-0.75	0.75	0.75	0.38
		0.38	-0.38		
-0.38	-0.19	-0.19	0.19	0.19	+0.1
		0.10	-0.10		
-0.10	-0.05	-0.05	0.05	0.05	
8.02	31.99	31.99	31.99	31.99	8.02

Column moments



$M_{fAB} = M_{fBA} = 0$

$M_{fBC} = \frac{100 \times 6^2}{12} = 300KNm$

$M_{fCB} = \frac{-100 \times 6^2}{12} = -300KNm$

$M_{fCD} = M_{fDC} = 0$

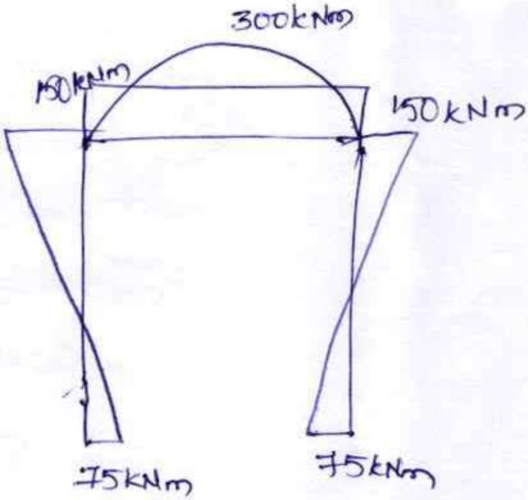
Joint	member	Relative stiffness	Total Relative Stiffness	Distribution factors
B	BA	$4EI/3$	$3I/3$	$1/3$
	BC	$4EI/6$		$2/3$
C	CB	$4EI/6$	$3I/3$	$2/3$
	CD	$4EI/3$		$1/3$

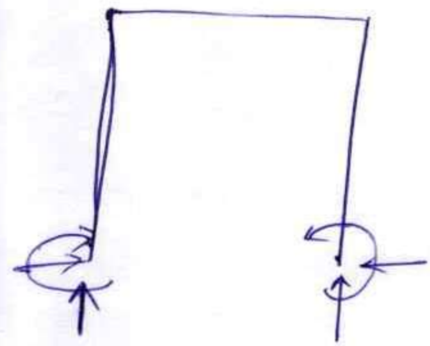
A	B (BA)	B (BC)	C (CB)	C (CD)	D
	1/3	2/3	2/3	1/3	
0	0	-300	300	0	0
	+100	+200	-200	-100	
50		-200	100		-50
	33.33	66.67	-66.67	-33.33	
16.67		-33.33	33.33		-16.67
	11.11	22.22	-22.23	-11.11	
5.56		-11.11	11.11		-5.56
	+3.70	7.41	-7.41	-3.70	
1.85		-3.71	3.71		-1.85
	1.24	2.47	-2.47	-1.24	
0.62		-1.23	1.23		-0.62
0.20	0.41	0.82	-0.82	-0.41	
		-0.41	+0.41		
	+0.14	+0.27	-0.27	-0.14	-0.07
0.07		-0.14	+0.14		
	0.05	0.09	-0.09	-0.05	-0.03
0.03		-0.05	+0.05		
	0.02	+0.03	-0.03	-0.02	
75	150	150	150	150	75

$$H_a = \frac{75+150}{3} = +75\text{ KN}$$

$$H_d = \frac{-150-75}{3} = -75\text{ KN}$$

Vertical reaction $= \frac{100 \times 6}{2} = 300\text{ KN}$


300kNm
150kNm
150kNm
75kNm
75kNm



Frames with side sway;

Step by step procedure to analyse frames with side sway.

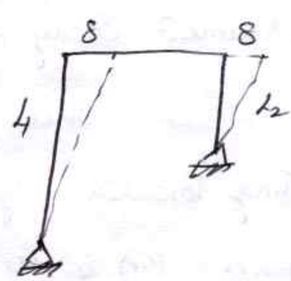
- → Fixed End moments
- → stiffness and Distribution factor
- → Moment distribution tabulation with out sway
- → Assumed sway moments
- → Moment distribution tabulation with assumed sway moments
- → Actual sway moments.

In case of continous beams or frames, the effect of yielding or settlement of support was taken in to account by introducing initial fixed end moments.

In case of portal frames, how ever the amount of sway or joint moment is not known and the analysis is done by assuming some arbitrary fixed moments. These assumed fixed reactions

Method Analysis

Case (i) Both ends are hinged



$$M_{AB} = \frac{3EI\delta}{L_1^2} \qquad M_{CD} = \frac{3EI_2\delta}{L_2^2}$$

$$\frac{M_{AB}}{M_{CD}} = \frac{3EI_1\delta/L_1^2}{3EI_2\delta/L_2^2} = \frac{I_1/L_1^2}{I_2/L_2^2}$$

Case (ii) Both ends are fixed

$$M_{AB} = \frac{6EI_1\delta}{L_1^2} \qquad M_{CD} = \frac{6EI_2\delta}{L_2^2}$$

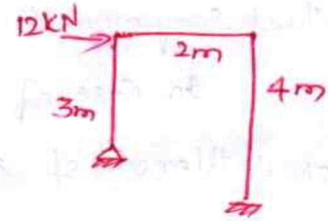
$$\frac{M_{AB}}{M_{CD}} = \frac{6EI_1\delta}{L_1^2} \times \frac{L_2^2}{6EI_2\delta} = \frac{I_1/L_1^2}{I_2/L_2^2}$$

Case (3) One end fixed other end hinged

$M_{AB} = \frac{6EI_1\delta}{L_1^2}$ $\quad M_{CD} = \frac{3EI_2\delta}{L_2^2}$

$$\frac{M_{AB}}{M_{CD}} = \frac{6EI_1\delta}{L_1^2} \times \frac{L_2^2}{3EI_2\delta} = \frac{2I_1L_2^2}{I_2L_1^2}$$

Analyse the given portal frame by using moment distribution method?



Fixed End moments

$M_{AB} = M_{BA} = 0$

$M_{CD} = M_{DC} = 0$

Stiffness & distribution factors

Joint	member	Stiffness	Σk	Distribution factor $k/\Sigma k$
B	BA	$\frac{3}{4}EI/3$	0.83	0.39
	BC	$EI/2$		0.602
C	CB	$EI/2$	0.75	0.67
	CD	$EI/4$		0.33

Assumed Sway moments

Since in the given condition the sway force is acting towards right so the moments at the joint B & C with span BA & CD will be in opposite direction i.e., towards left

$$\frac{M_{AB}}{M_{CD}} = \frac{3EI_1\delta/L_1^2}{6EI_2\delta/L_2^2} = \frac{I_1/L_1^2}{2I_2/L_2^2} = \frac{I/3^2}{2I/4^2}$$

$$\frac{I}{9} \times \frac{16 \cdot 8}{2I} = \frac{8}{9}$$

$M_{AB} = M_{BA} = 8 \text{ KN-m}$

$M_{CD} = M_{DC} = 9 \text{ KN m}$

Moment distribution tabulation with assumed sway moments

	A	B (BA)	B (BC)	C (CB)	C (CD)	D
		0.33	0.67	0.67	0.33	
	-8	-8			-9	-9
		2.64	5.36	6.03	2.97	
			3.015	2.68		1.485
		-1	-2.02	-1.795	-0.884	
			-0.89	-1.01		-0.442
		0.29	0.59	0.67	0.33	
			0.335	0.295		0.165
		-0.11	-0.22	-0.197	-0.097	
			-0.098	-0.11		-0.048
		0.032	0.065	0.07	0.03	
final mo		-6.148	6.135	6.64	-6.65	-7.74

Horizontal reaction at A

$$H_A = \frac{M_{AB} + M_{BA}}{L} = \frac{0 - 6.148}{3} = -2.04 \text{ kN}$$

$$H_D = \frac{M_{CD} + M_{DC}}{L_2} = \frac{-6.65 - 7.74}{4} = -3.59$$

let S be the sway force, by resolving the sway force horizontally we get

$$S = 2.04 + 3.59 = 5.63 \text{ KN}$$

Actual Sway moments

The actual sway moments will be determined as follows

	AB	BA	BC	CB	CD	DC
Assumed sway moments	0	-6.148	6.135	6.638	-6.659	-7.79
Actual sway moments $\frac{12}{5.63} \times$ Assumed sway moments	0	-13.10	13.07	14.14	-14.19	-16.49

Corrected horizontal reactions at A

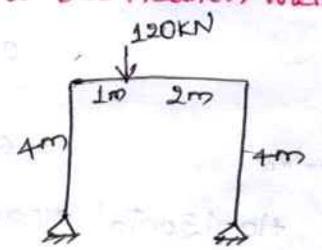
$$H_A = \frac{M_{AB} + M_{BA}}{L} = \frac{0 - 13.10}{3} = -4.36 \text{ kN}$$

Corrected horizontal reactions at D

$$H_D = \frac{M_{CD} + M_{DC}}{L_2} = \frac{-14.19 - 16.49}{4} = -7.67$$

$$H_A + H_D = -4.36 - 7.67 = -12 \text{ kN}$$

Analyse the given portal frame by using moment distribution method



Step 1: fixed End moments

$$M_{fAB} = M_{fBA} = 0$$

$$M_{fBC} = \frac{-wab^2}{l^2} = \frac{-120 \times 1 \times 2^2}{3^2} = -53.33 \text{ kNm}$$

$$M_{fCB} = \frac{wa^2 b}{l^2} = \frac{120 \times 1^2 \times 2}{3^2} = 26.66 \text{ kNm}$$

$$M_{fCD} = M_{fDC} = 0$$

Step 2:

Stiffness and Distribution factor

Joint	Member	Stiffness	ΣK	Df = K/ΣK
B	BA	3/4 EI/4	0.51 EI	0.35
	BC	EI/3		0.64
C	CB	EI/3	0.51 EI	0.64
	CD	3/4 EI/4		0.35

Moment distribution method

A: AB	B: BA	B: BC	C: CB	C: CD	D: DC
	0.35	0.64	0.64	0.35	0
		-53.33	26.66		
	18.66	34.13	-17.06	-9.33	
		-8.53	17.06		
	2.98	5.45	-10.91	-5.97	
		-5.45	2.72		
	1.90	3.48	-1.74	-0.95	
		-0.87	1.74		
	0.30	0.55	-1.11	0.60	
		-0.55	0.27		
	0.19	0.35	-0.17	-0.09	
		-0.08	0.17		
	0.02	0.05	-0.10	-0.05	
		-0.05	0.02		
	0.01	0.03	-0.01	-0.007	
	24.06	-24.82	17.06	-16.99	

$H_A = \dfrac{M_{AB} + M_{BA}}{L_1} = \dfrac{0 + 24.06}{4} = 6.015\,kN$

$H_D = \dfrac{M_{CD} + M_{DC}}{L_2} = \dfrac{-17+0}{4} = -4.25\,kN$

Sway force (S) = $6.015 - 4.25 = -1.765\,kN$

Assumed sway moment

$\dfrac{M_{BA}}{M_{CD}} = \dfrac{I_1/L_1^2}{I_2/L_2^2} = \dfrac{I/4^2}{I/4^2} = \dfrac{10}{10}$

$M_{BA} = -10\,kNm \qquad M_{CD} = -10\,kNm$

Moment distribution tabulation with assumed sway moments

A	B		C		D
	0.35	0.64	0.64	0.35	
	-10			-10	
	3.5	6.4	6.4	3.5	
		3.2	3.2		
	-1.12	-2.04	-2.04	-1.12	
		-1.02	-1.02		
	0.35	0.65	0.65	0.35	
		0.32	0.32		
	-0.11	-0.2	-0.2	-0.11	
		-0.1	-0.1		
	0.03	0.06	0.06	0.03	
		0.03	0.03		
	-0.01	-0.01	-0.01	-0.01	
	-7.3	7.3	7.3	-7.3	

final moments

$M_{AB} = 0, \quad M_{BA} = -7.3\,kNm, \quad M_{BC} = 7.3\,kNm \quad M_{CB} = 7.3\,KNm$

$M_{CD} = -7.36 kNm$ $M_{DC} = 0$

$$H_A = \frac{M_{AB} + M_{BA}}{4} = \frac{0 - 7.36}{4} = -1.84 kN$$

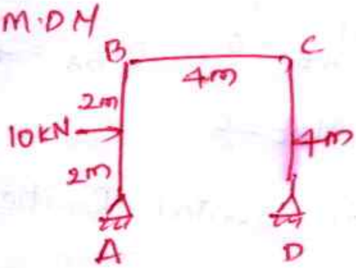
$$H_D = \frac{M_{CD} + M_{DC}}{L_2} = \frac{-7.36 + 0}{4} = -1.84 kN$$

Resolving Above forces hor Sway force = $-1.84 - 1.84 = 3.68$ kN

Actual Sway moments

Assumed sway moments	0	-7.36	7.29	7.29	-7.36	0
Actual sway moments $\frac{1.765}{3.68}$ × Assumed sway moments	0	-3.53	3.49	3.49	-3.53	0
Non sway moment	0	24.06	-24.82	17.06	-17	0
final moments	0	13.17	-14.04	27.84	-27.89	0

Analyse the given portal frame by using M.D.M



Fixed End moments

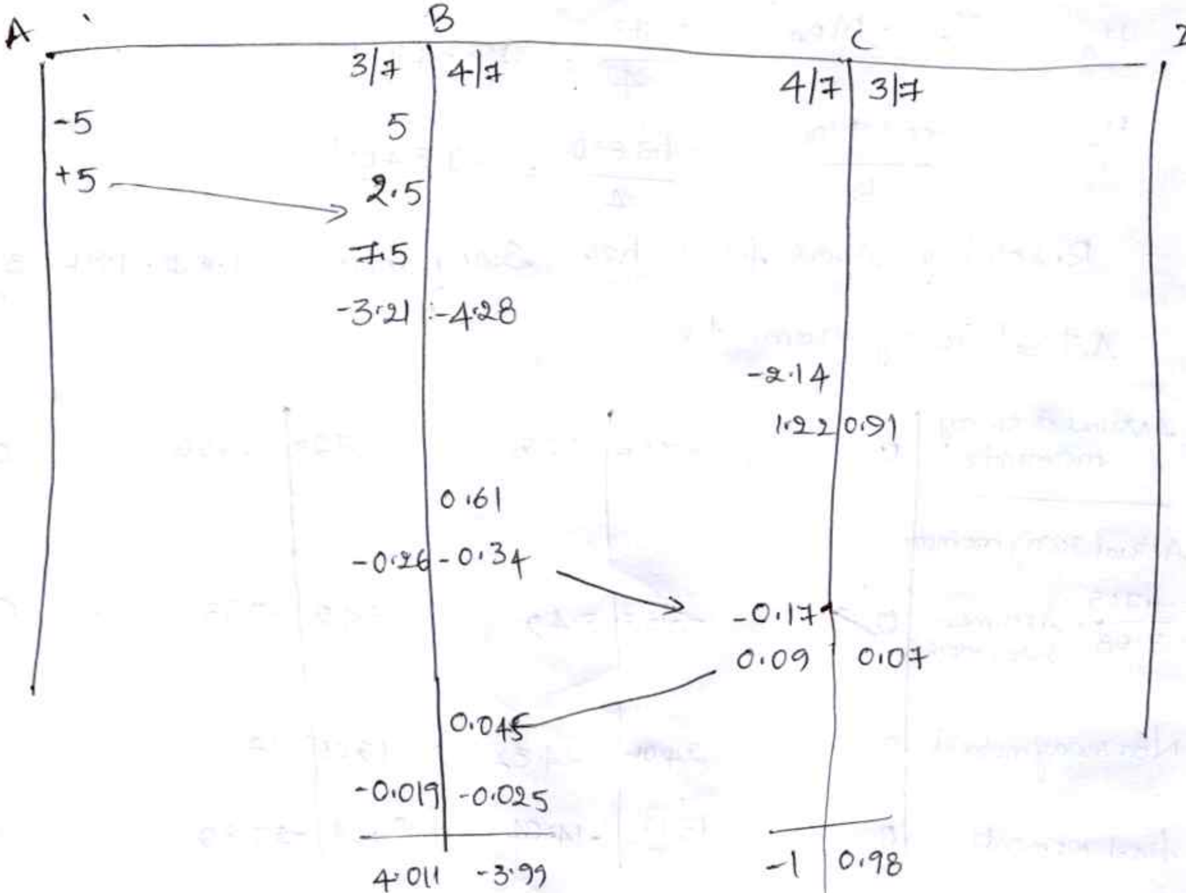
$$M_{fAB} = -\frac{WL}{8} = -\frac{10\times4}{8} = -5 kNm$$

$$M_{fBA} = \frac{WL}{8} = \frac{10\times4}{8} = 5 kNm$$

$$M_{fBC} = M_{fCB} = M_{fCD} = M_{fDC} = 0$$

B	BA	$\frac{3}{4}\frac{EI}{L} = \frac{3}{16}EI$	$7/16 EI$	3/7
	BC	$EI/4$		4/7
C	CB	$EI/4$	$7/16 EI$	4/7
	CD	$3/16 EI$		3/7

Moment Distribution method



$M_{AB} = 0$, $M_{BA} = 4.011$, $M_{BC} = -3.99$ $M_{CB} = -1$, $M_{CD} = 0.98$

$M_{DC} = 0$

Horizontal Reaction $H_A = \dfrac{M_{AB} + M_{BA} - WL}{L_1} = \dfrac{0 + 4.01 - 10\times 2}{4}$

$= -3.99\,kN \leftarrow$

$H_D = \dfrac{M_{CD} + M_{DC}}{L_2} = 1/4 = 0.25\,kN \rightarrow$

Force acting left to Right = 10 + 0.25 = 10.25 kN

force acting right to left = -3.99 kN

Sway force (S) = 10.25 - 3.99 = 6.26 kN

Step 4: Assumed sway moments

$$\frac{M_{BA}}{M_{CD}} = \frac{I_1/L_1^2}{I_2/L_2^2} = \frac{I/4^2}{I/4^2} = \frac{1}{1} \qquad I_1 = I_2 = I$$

Let us assume $\frac{M_{BA}}{M_{CD}} = \frac{6}{6}$

$M_{BA} = -6 kNm \qquad M_{CD} = -6 kNm$

Step 5;

AB	BA	BC	CB	CD	DC
	-6				
	2.57	3.42	3.42	2.57	
		1.71	1.71		
	-0.73	-0.97	-0.97	-0.73	
		-0.485	-0.485		
	0.207	0.277	0.277	0.207	
		0.138			
	-0.059	-0.078	-0.078	-0.059	
		-0.039	-0.039		
	0.016	0.022	0.022	0.016	
	-4	4	4	-4	

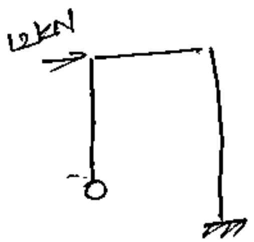
$$H_A = \frac{M_{AB} + M_{BA}}{4} = -4/4 = -1 kN$$

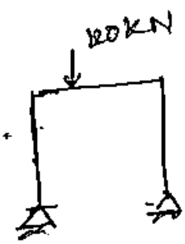
$$H_D = \frac{M_{CD} + M_{DC}}{L_2} = -4/4 = -1 kN$$

By resolving the above forces $S = 1 + 1 = 2 kN$

Actual sway moments

	AB	BA	BC	CB	CD	DC
Assumed sway moments	0	-4	4	4	-4	0
Actual sway moments $\frac{6.26}{2}$ × Actual sway	0	-12.52	12.52	12.52	-12.52	0
Non sway moments	0	4	-4	-1	1	0
final moments	0	-12.52	12.52	15.52	-15.52	0





6cycles

$\frac{m_{ba}}{m_{ca}} = \frac{8}{9}$

$M_{ba} = -8$ kNm

$M_{cd} = -9$ kNm

12/5.65 × column a

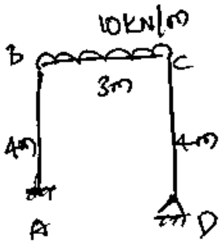
$\frac{M_{ba}}{M_{cd}} = 10/10$

-10, -10

column = $\frac{1.82}{3.635}$ × column

Previous moment after

Actual sway
Non sway
final moment



before 0.2275

Actual mo.
Non sway
final

$m_{ba}/m_{cd} = 10/10$

-10 -10

$\frac{0.2275}{5.16}$ × column

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CIVIL ENGINEERING

Advanced Structural Analysis

UNIT-3

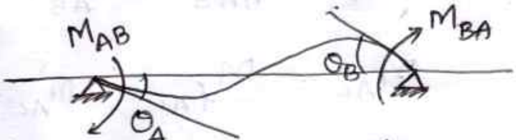
UNIT - III

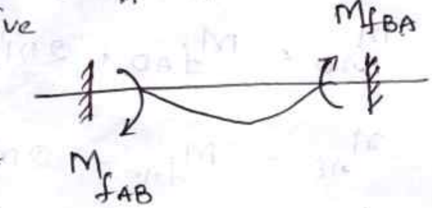
KANI'S METHOD OF ROTATION CONTRIBUTION

Gasper Kani, a German engineer, developed another distribution procedure based on slope deflection equation. Number of cycles it converges finally to the correct answer. Applicable to multistoried frames.

Analysis of structures with out Relative displacement at ends

Consider a member AB, shown in fig (1), is an intermediate member of a beam/frame, which has no relative displacements at the ends (ends A & B are at same level). let M_{AB} and M_{BA} be the final end moments. M_{AB} may consists of

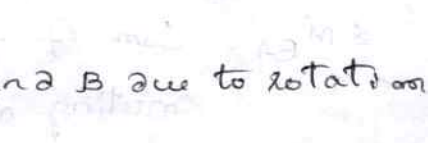




(i) fixed End moments ($\theta_A = \theta_B = 0$)(b) $\quad 2M'_{AB} = \frac{4EI}{L}\theta_A$

(ii) Moment due to rotation of end A only (c) $\quad 2M'_{BA} = \frac{4EI}{L}\theta_B$

(iii) Moment due to rotation of end B (d)



let the moment developed at A due to rotation θ_A only be $2M'_{AB}$

Naturally it is equal to $\frac{4EI}{L}\theta_A$.

Hence moment developed at ends A and B due to rotation θ_B only are M'_{BA} and $2M_{BA} = \frac{4EI}{L}\theta_B$

$$M_{AB} = M_{fAB} + 2M'_{AB} + M'_{BA}$$

$M_{BA} = M_{fBA} + M'_{AB} + 2M'_{BA}$

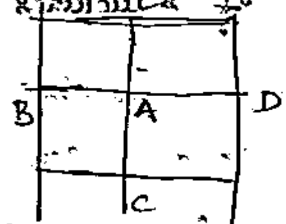
The moments M'_{AB} and M'_{BA} are called rotation contributions.

Final moment = fixed end moment + 2(Rotation contribution of near end) + Rotation contribution of far end.

Now consider the moments at joint A in the frame

A/c to EV

$M_{AB} = M_{fAB} + 2M'_{AB} + M'_{BA}$



$M_{AC} = M_{fAC} + 2M'_{AC} + M'_{CA}$

$M_{AD} = M_{fAD} + 2M'_{AD} + M'_{DA}$

$M_{AE} = M_{fAE} + 2M'_{AE} + M'_{EA}$

$\Sigma M_{AB} = \Sigma M_{fAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA}$

ΣM_{AB} = Sum of near end moments in all the members meeting at joint A

ΣM_{fAB} = Sum of fixed end moments in all the members at joint A

$\Sigma M'_{AB}$ = Sum of near end rotation contributions of all the members meeting at joint A

$\Sigma M'_{BA}$ = Sum of far end rotation contributions of all the members meeting at joint A

from the joint Equilibrium condition, we know

$\Sigma M_{AB} = 0$

$\Sigma M_{fAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA} = 0$

$$\Sigma M'_{AB} = -\frac{1}{2}(\Sigma M_{fAB} + \Sigma M'_{BA}) \longrightarrow (a)$$

for each member meeting at joint A

$$2M'_{AB} = \left(\frac{4EI}{L}\right)\theta_A = K_{AB}\theta_A$$

$$M'_{AB} = \frac{1}{2} K_{AB}\theta_A$$

$$\Sigma M'_{AB} = \frac{1}{2}\Sigma K_{AB}\theta_A$$

$$= \frac{1}{2}\theta_A \Sigma K_{AB}$$

Since θ_A is the same for all the members meeting at joint A

$$\frac{M'_{AB}}{\Sigma M'_{AB}} = \frac{K_{AB}}{\Sigma K_{AB}}$$

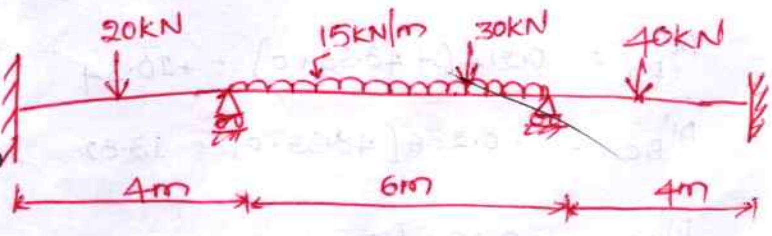
$$M'_{AB} = \left[\frac{K_{AB}}{\Sigma K_{AB}}\right] \Sigma M'_{AB} \longrightarrow (b)$$

substituting Eq (a) in Eq(b) we get

$$M'_{AB} = -\frac{1}{2}\left[\frac{K_{AB}}{\Sigma K_{AB}}\right](\Sigma M_{fAB} + \Sigma M'_{AB})$$

The expression $-\frac{1}{2}\left[\frac{K_{AB}}{\Sigma K_{AB}}\right]$ is called Rotation factor for member AB at joint A.

Analyse the Continous beam



Step1: fixed End moments

$$M_{fAB} = \frac{Wl}{8} = \frac{20\times4}{8} = 10 KNm$$

$$M_{fBA} = -10 KN$$

$M_{fBC} = \frac{wl^2}{12} = \frac{15\times6^2}{12} + \frac{30\times4\times2^2}{6^2} = -58.33\,kNm$

$M_{fCB} = \frac{-15\times6^2}{12} - \frac{30\times4^2\times2}{6^2} = -71.67\,kNm$

$M_{fCD} = \frac{+40\times4}{8} = +20\,kNm$ $\quad M_{fDC} = -20\,kNm$

Rotation factor $= -\frac{1}{2}\left[\frac{K_{AB}}{\Sigma K}\right]$

Joint	Member	k	Σk	Rf
B	BA	$\frac{4EI}{4} = EI$	2.33EI	−0.214
	BC	$\frac{4E(2I)}{6} = 4/3\,EI$		−0.286
C	CB	$\frac{4E(2I)}{6} = 4/3\,EI$	2.33EI	−0.286
	CD	$\frac{4EI}{4} = EI$		−0.214

10 −10 | −0.214 [48.33] −0.286 | 58.33 −71.67 | −0.286 [51.67] −0.214 | 20 −20

−10.34	−13.84	18.73	14.01
−14.35	−19.18	20.26	15.16
−14.68	−19.62	20.39	15.26
−14.70	19.65	20.40	15.26

$M'_{BA} = -0.214[+48.33+0] = +10.34$

$M'_{BC} = -0.286[48.33+0] = 13.82$

$M'_{CB} = -0.286[-51.67+13.82] = -18.73$

~~ΣM~~ ~~AB~~ $M'_{CD} = -0.214[-51.67 + 13.82] = 14.01$

$M'_{BA} = -0.214[48.33 + 18.73] = -14.35$

$M'_{BC} = -0.286[48.33 + 18.73] = -19.18$

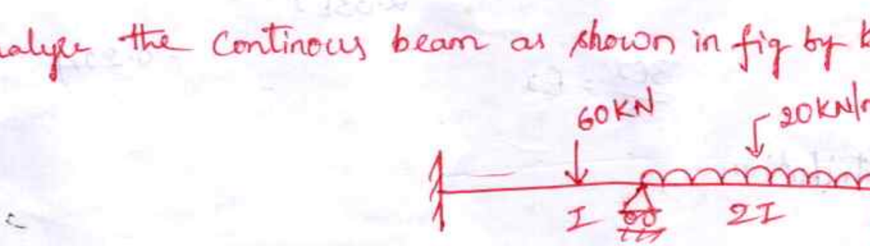
Consider joint C

$M'_{CB} = -0.286[-51.67 + -19.18] = +20.26$

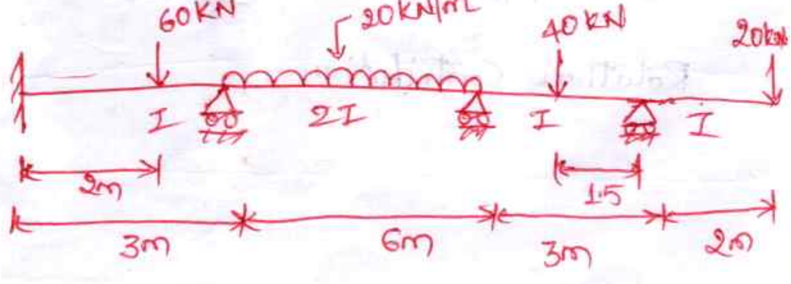
$M'_{CD} = -0.214[-51.67 + -19.18] = +15.16$

Final moment = fixed end moment + 2(Near end rotation contribution) + far end contribution

	A		B	C		D
FEM	-10	+10	-58.33	+71.67	-20	+20
Near end contribution	2×0	2×14.7	2×19.65	-2×20.40	2×15.16	2×0
Far end contribution	14.7	0	-20.40	19.65	0	15.26
	4.7	39.4	-39.4	50.52	-50.52	4.74



Analyse the continous beam as shown in fig by kani's method?



Fixed end moments

$M_{fAB} = \frac{+60\times2\times1^2}{3^2} = 13.33$; $M_{fBA} = \frac{-60\times2^2\times1}{3^2} = 26.67$

$M_{fBC} = \frac{20\times6^2}{12} = 60kNm$; $M_{fBC} = -60kNm$

$M_{fCD} = \frac{40\times3}{8} = 15kNm$; $M_{fDC} = -15kNm$

$M_{fDE} = -2\times20 = -40kNm$

Modification in fem for rotation D

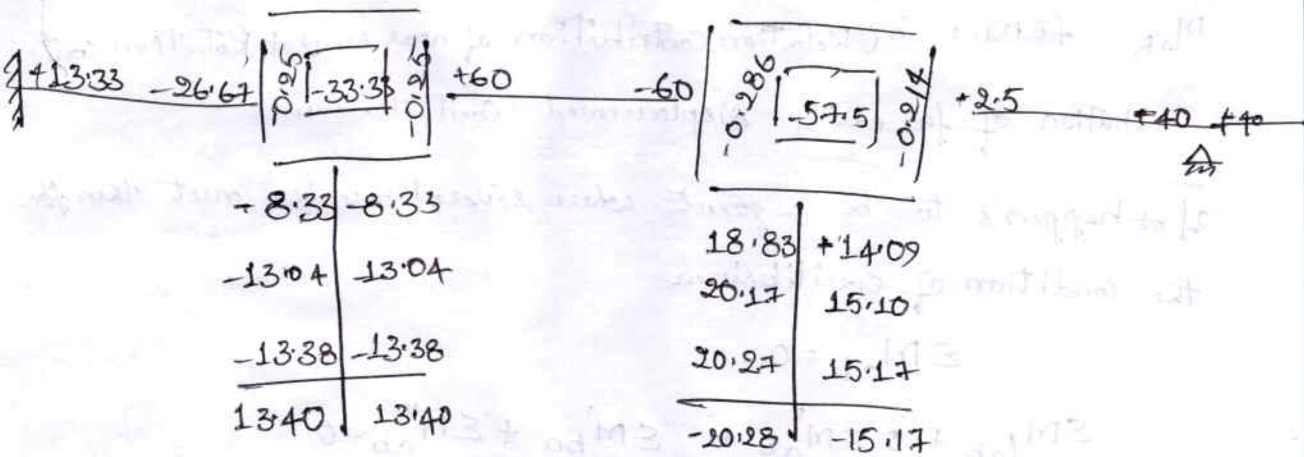
$M_{fDC} = -40$

$M'_{fCD} = -15 - 0.5(15-40) = -2.5kNm$

Rotation factor $= -1/2\left(\frac{k}{\Sigma k}\right)$

Joint	Members	K	ΣK	Rf
B	BA	$4EI/3$	$8/3EI$	-0.25
	BC	$4/6E(2I) = 4/3EI$		-0.25
C	CB	$\frac{4E(2I)}{6} = 4/3EI$	2.33EI	-0.286
	CD	$\frac{3EI}{3} = EI$		-0.214

Rotation Contribution

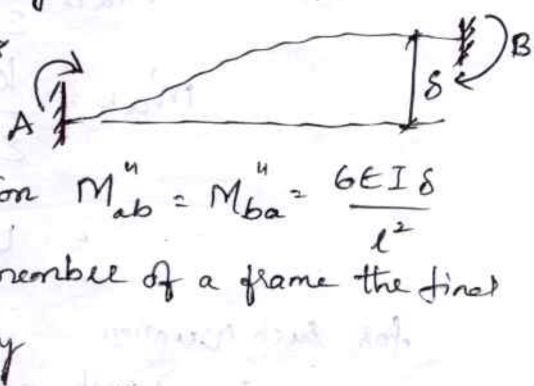


Final moment calculation

	A	B		C		D		E
FEM	+13.33	-26.67	60	-60	2.5	-40	40	-
Near end moment	2×0	2(13.40)	2(-13.40)	2(20.28)	(2×15.17	0	-	—
far end moment	13.40	0	20.28	13.40	0	-	-	—
	0.07	-53.47	+53.47	-32.84	32.84	-40	+40	-40

Members with relative lateral displacement

A beam AB whose ends have undergone lateral displacement so the relative displacement b/n ends equals δ



The FEM due to this condition $M''_{ab} = M''_{ba} = \frac{6EI\delta}{l^2}$

when such displacements occur for a member of a frame the final moment at A and B are given by

$$M_{ab} = M_{fab} + 2M'_{ab} + M'_{ba} + M''_{ab}$$

$$M_{ba} = M_{fba} + 2M'_{ba} + M'_{ab} + M''_{ba}$$

where $M''_{ab} = M''_{ba}$ called as displacement contribution of the member AB

M_{AB} = fem + 2(Rotation contribution of near end) + Rotation contribution of far end + displacement contribution.

If A happens to be a joint where several members meet then for the condition of equilibrium

$$\Sigma M_{ab} = 0$$

$$\Sigma M_{fab} + 2\Sigma M'_{ab} + \Sigma M'_{ba} + \Sigma M''_{ab} = 0$$

$$\Sigma M'_{ab} = \left(-\tfrac{1}{2}\right)\left\{\Sigma \bar{M}_{ab} + \Sigma M'_{ba} + \Sigma M''_{ab}\right\}$$

$$M'_{ab} = 2E\, k_{ab}\, \theta_a$$

$$\Sigma M'_{ab} = 2E\theta_a \Sigma k_{ab}$$

$$\frac{M'_{ab}}{\Sigma M'_{ab}} = \frac{k_{ab}}{\Sigma k_{ab}}$$

$$M'_{ab} = \frac{k}{\Sigma k_{ab}} \Sigma M'_{ab}$$

$$M'_{ab} = \frac{k_{ab}}{\Sigma k_{ab}}\left[-\tfrac{1}{2}\right]\left[\Sigma \bar{M}_{fab} + \Sigma M'_{ba} + \Sigma M''_{ab}\right]$$

$$= \upsilon_{ab}\left(\Sigma M_{fab} + \Sigma M'_{ba} + \Sigma M''_{ab}\right)$$

for such equation

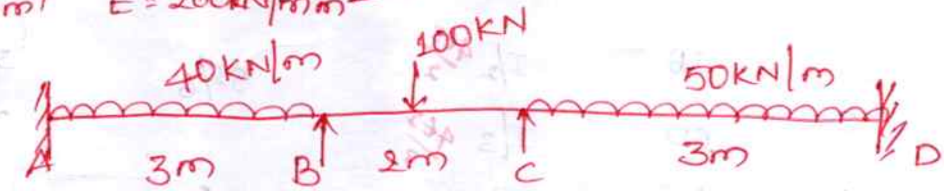
Here at end each of a member the resultant fixed end moment can be computed

for instance for a span AB resultant fixed end moment at A

$$= M_{fab} + M''_{ab}$$

Ily Resultant fixed End moment = $M_{fba} + M_{ab}$"

Determine the support moments for the Continous girder if the support B sinks by 2.5mm. for all members take $I = 3.5 \times 10^7 mm^4$ $E = 200 kN/mm^2$



Fixed End moments

$$M_{Rfab} = M_{fab} + \frac{6EI\delta}{L_1^2}$$

$$= \frac{40 \times 3^2}{12} + \frac{6 \times 200 \times 3.5 \times 10^7 \times 2.5}{3^2 \times 10^9} = ~~18.33~~ \; 41.67 \; kNm$$

$$M_{Rba} = -30 + 11.67 = -~~41.67~~ \; -18.33 \; kNm$$

Span BC

Resultant fixed End moment at B

$$M_{fbc} + \frac{6EI\delta}{l^2} = \frac{100 \times 2}{8} + \frac{6 \times 200 \times 3.5 \times 10^7 \times 2.5}{2^2 \times 10^9}$$

$$= 25 + 26.25 = ~~51.25~~ \; 1.25$$

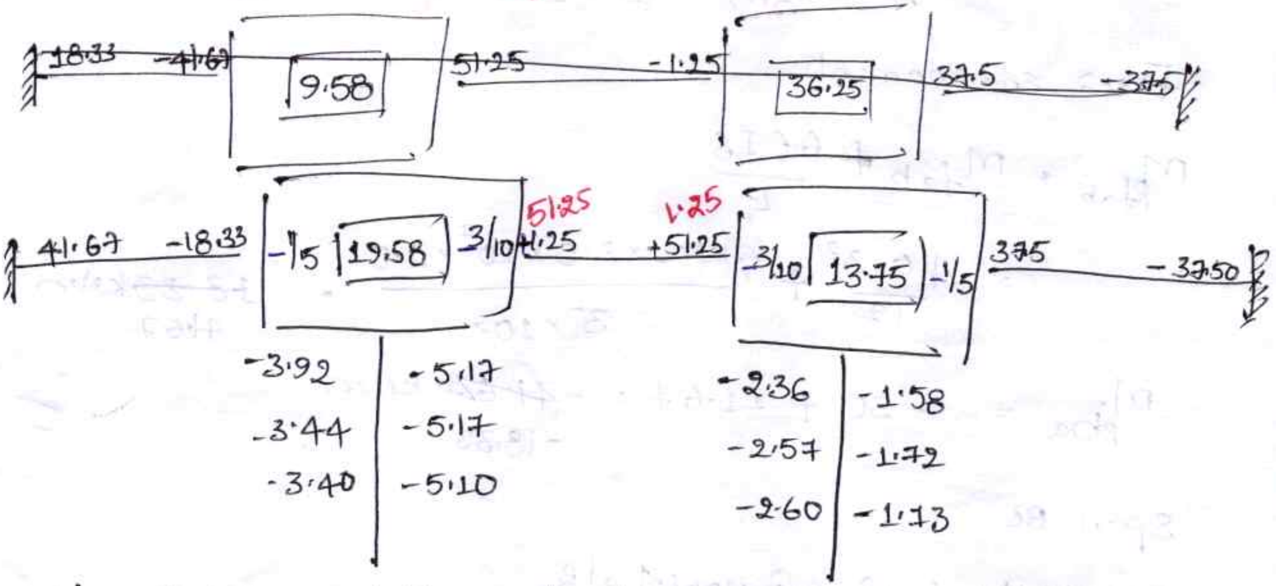
$$M_{fBC} = +25 + 26.25 = 51.25$$

At CD $M_{fCB} = -25 + 26.25 = 1.25$

$$M_{cd} = +\frac{50 \times 3^2}{12} = +37.5 \qquad M_{fdc} = -37.5 kNm$$

$M_{DC} =$

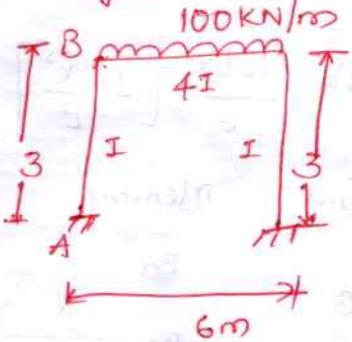
Joint	member	Relative stiffness k	Total relative stiffness Σk	Rotation factor $-\frac{1}{2}\,k/\Sigma k$
B	BA	$I/3 = 2I/6$ $\frac{4EI}{3} = 1.33$	$5I/6$	$2/5(-\frac{1}{2}) = -1/5 = 0.2$
	BC	$I/2 = 3I/6$ $\frac{4EI}{2} = 2$	333	$3/5[-\frac{1}{2}] = -3/10 = 0.6$
C	CB	$I/2$ $\frac{4EI}{2}$	$5I/6$	$3/5(-\frac{1}{2}) = -3/10 =$
	CD	$I/3$ $\frac{4EI}{3}$		$2/5(-\frac{1}{2}) = -1/5$



Now that the rotation contributions have been calculated we can find the end moments. See table below

A		B	C		D
41.67	-18.3	41.25	-51.25	37.5	-37.5
0	2×-3.4	2×5.10	2×2.6	2×1.73	0
3.4	0	2.6	5.10	0	1.73
-45.07	11.53	11.55	+40.96	-40.96	35.77

Determine the moments ABCD for the portal frame



$M_{fab} = M_{fba} = M_{fcd} = M_{fdc} = 0$

$M_{fbc} = \frac{100 \times 6^2}{12} = 300 \text{ KN/m}$

$M_{fcb} = -300 \text{ KN/m}$

Joint	Member	Relative Stiffness	Total Relative Stiffness	Rotation factor $-\frac{1}{2}[K/\Sigma K]$
B	BA	$I/3$	$3I/3$	$-\frac{1}{2}[\frac{1}{3}] = -\frac{1}{6}$
	BC	$4I/6 = \frac{2}{3}I$		$-\frac{1}{3}$
C	CB	$4I/6 = 2I/3$	$3I/3$	$-\frac{1}{3}$
	CD	$I/3$		$-\frac{1}{6}$

Joint B: [-300], $-\frac{1}{3}$, $-\frac{1}{6}$

Member BC at B: 300, -100, -144.4, -149.38, -149.93, -150.00

Member BA at B: -50, -72.22, -74.69, -74.96, -75.00

Joint C: [ ], $-\frac{1}{3}$, $-\frac{1}{6}$

Member CB at C: -300, 133.33, 148.15, 149.79, 149.98, 150.00

Member CD at C: 66.67, 74.07, 74.80, 74.99, 75.00

final moments

At B (member BC): 300, 150, 150, 150

At B (member BA): 0, 75, 75, +75

At A: +75, 0

At C (member CB): -300, 150, 150, +150

At C (member CD): 0, 75, 75, -75, -75

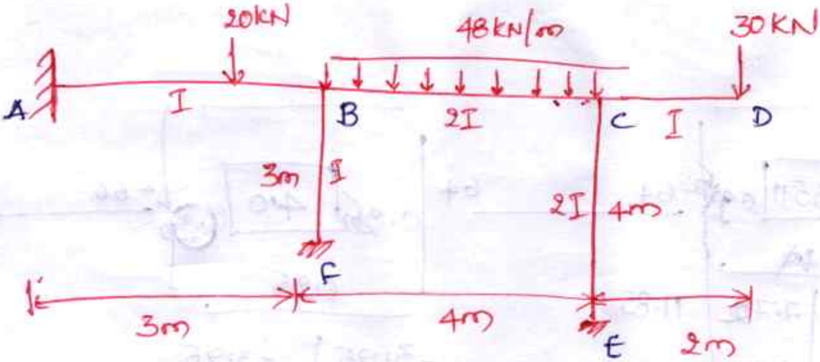
At D: 0, 0

$$M = \frac{WLx}{6}\left[1 - \frac{x^2}{L^2}\right]$$

Joint	Member	K	ΣK	$-\frac{1}{2} \cdot \frac{K}{\Sigma K}$	
B	BA	EI/3	0.996	$-\frac{1}{2} \times \frac{0.33}{0.996}$	0.16
	BC	4EI/6		$-\frac{1}{2} \times \frac{0.66}{0.996}$	0.34
C	CB	4EI/6	0.996	0.34	
	CD	EI/3		0.16	

B: -0.16	B: 0.34	C: 0.34	C: 0.16
	300	-300	
-48	-102	+136.68	+64.32
-69.86	-148.34	152.44	71.73
72.39	153.8	154.29	72.61
72.68	154.45	154.5	72.71
72.72	154.43	154.44	72.68
72.71	154.5	154.46	72.68

A	B	B	C	C	D
0	0	300	-300	0	0
2×0	2×72.7	2×-154.5	2×154.5	2×72.7	0
72.7	0	154.46	154.5	0 72.68	72.7
	145.42	145.46	145.46	145.42	72.2



Step 1; fixed End Moments

$M_{fAB} = \frac{-20\times2\times1^2}{3^2} = -4.44\,kNm$ $\quad M_{fBA} = \frac{-20\times2^2\times1}{3^2} = 8.89\,kNm$

$M_{fBC} = \frac{-48\times4^2}{12} = -64\,kNm$ $\quad M_{fCB} = 64\,kNm$

$M_{fCD} = -30\times2 = -60\,kNm$

$M_{fBF} = M_{fFB} = M_{fCE} = M_{fEC} = 0$

Joint	Member	k	Σk	$-\frac{1}{2}\frac{k}{\Sigma k}$
B	BA	EI/3	1.16	-0.14
	BF	EI/3		-0.14
	BC	2EI/4		-0.215
C	CB	2EI/4	~~1.375~~ 1	-0.25 (~~0.36~~ 0.41)
	CD	~~3EI/4~~ 0		~~0.272~~ 0
	CE	2EI/4		-0.25 (~~0.36~~ ~~0.5~~)

-4.44 8.89 | 0.44 | -55.11 | 0.56 | -64 64 | 0.36 | 4.0 | 0.64 | -60

-0.28 0.25

7.72	7.72	11.85	-3.95	-3.95
8.27	8.27	12.69	-4.17	-4.17
8.29	8.29	12.70	-4.17	-4.17

Member	A	B	C	D
Near end moment				
far end moment				
final moment				

$M_{AB} = -4.44 + 2\times 0 + 8.48 = 4.04$

$BA = 8.89 + 2\times 8.48 + 0 = 25.85$

$BF = 0 + 2\times 8.48 = 16.96$

$BC = -64 + 2\times 12.68 - 4.17 = -42.81$

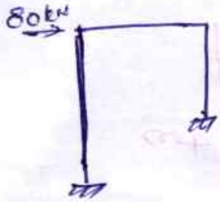
$CB = 64 + 2(-4.17) + 12.68 = 68.34$

$CD = -60$

$CE = 2(-4.17)$

$EC = -4.17$

$FB = 8.48$



1 Problem 2.12 Pg-73[1] - Bhavikatti SA-II
2 Ex 2.11 Pg-69[2]
3 Ex-2.13 Pg-75[3]
4 Ex-1.11 Pg-31[4]
1 Ex-1.10 Pg-27[5]

MDM
2 Pr-75 Pg-393[6]
3 Pr-76 Pg-396[1]
4 Pr-78 Pg-401[2]
1 Pr-79 Pg-401[3]
2 Pr-59 Pg-356[4]
3 Pr-54 Pg-350[5]
4 Pr-81 Pg-408[6]

Ramamrutham SDM
1 Pr-49 Pg-636[1]
2 Pr-50 Pg-636[2]
3 Pr-51 Pg 637[3]
4 Pr-53 Pg-639[4]
1 Pr-55 Pg-641[5]
2 Pr-62 Pg-649[6]
3 Pr-76 Pg-667[1]
4 Pr-77 Pg-669[2]

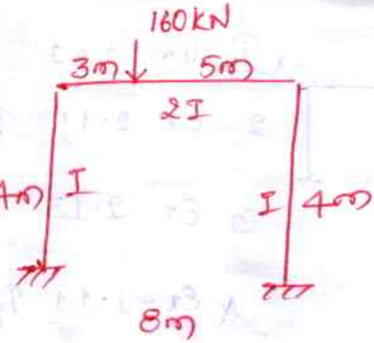
RCM
Pr2 - 723 Pg[3]
Pr4 - pg 725[4]
Pr7 - Pg-728[5]
Pr11 - Pg 732[6]
Pr 14 - Pg-735[1]

MDM Pandit & Gupta
Ex-9.6.1 - Pg 563[2]
Ex-9.7.6 - Pg 575[3]
Prakash Rao
Pr-VII - Pg 168[4]

SDM
Ex-20.8 Pg 20.23[5] Devadas Menon
Ex-20.9 Pg 20.26[6]

MDM
Ex-21.6 Pg 21.20
Ex-21.7 Pg-21.22
Ex-21.11 Pg-21.30

$M_{fab} = M_{fBA} = M_{fCD} = M_{fDC} = 0$



$M_{fBC} = \frac{16 \times 3 \times 5^2}{8^2} = 187.5 \text{ KNm}$

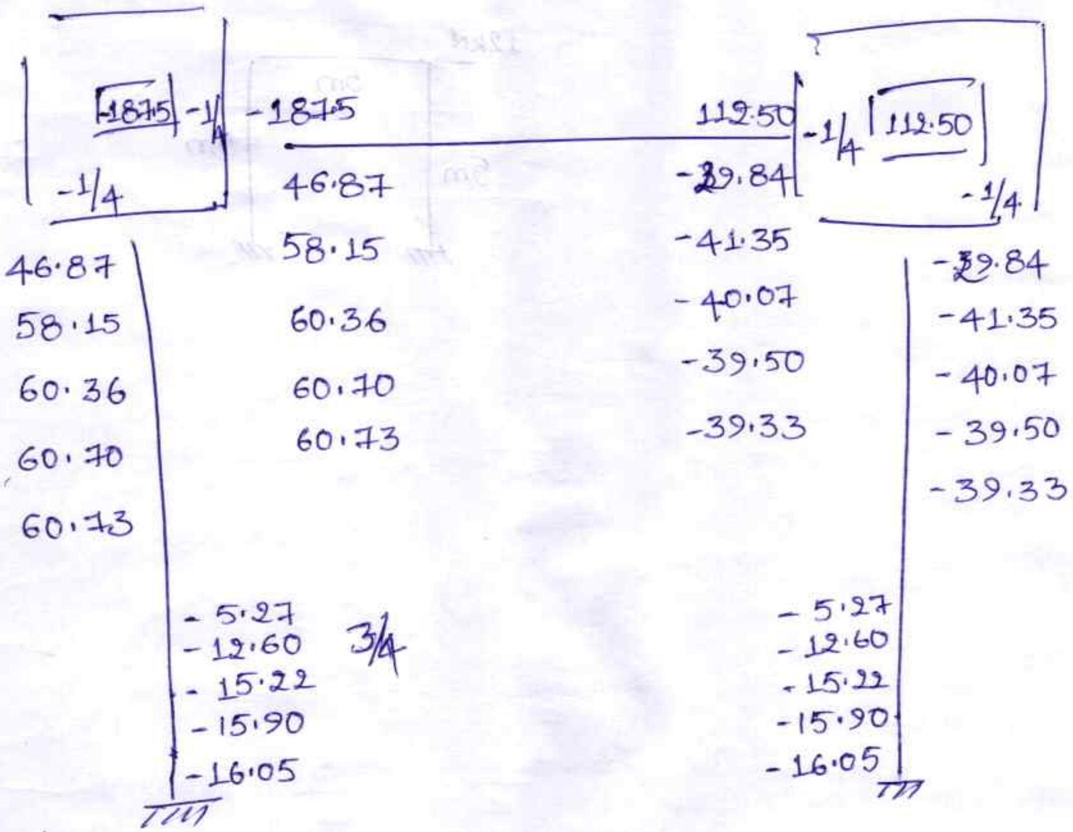
$M_{fCB} = 112.5 \text{ kNm}$

Joint	Member	Relative Stiffness	Total Relative Stiffness	d_f
B	BA	I/4	2I/4	-1/4
	BC	2I/8		-1/4
C	CB	2I/8	2I/4	-1/4
	CD	I/4		-1/4

Displacement factor $= -\frac{K}{\Sigma k} \, 3/2$

Vertical member	Relative Stiffness k	Total Relative Stiffness Σk	Displacement factor $\frac{k}{\Sigma k}\left[-3/2\right]$
AB	I/4	2I/4	$1/2\left[-3/2\right] = -3/4$
CD	I/4		$1/2\left[-3/2\right] = -3/4$

$60.73 - 39.33 = 21.40 \quad -3/4(21.40) = -16.05$



Storey one

Rotation contribution at top of column AB = + 60.73

Rotation contribution at top of colum CD = -39.33

Rotation contribution at bottom of columns = 0

21.40

A B C D

$M'_{ab} = -3/4\,(21.40) = -16.05\,kNm$

$M'_{cD} = -3/4\,(21.40) = -16.05\,kNm$

A	B		C		D
		-187.50	-112.50		
	2× 60.73	2× 60.73	+39.33	2×-39.33	-39.05
60.73	-16.05	-39.33	-39.33	-16.05	-16.50
-16.05			+ 60.73		
44.68	105.41	-105.37	94.61	-94.71	-55.38

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CIVIL ENGINEERING

Advanced Structural Analysis

UNIT-4

UNIT - IV

Flexibility & Stiffness method

Flexibility method

The flexibility method is a matrix approach method which is used to analyse indeterminate structure, also called force method (or) Compatability method. In this method the basic unknowns are redundant forces. The number of redundant forces is equal to degree of static indeterminacy. The number of equations required over and above the equations of static equilibrium for the analysis of structure is known as degree of static indeterminancy or degree of redundancy.

Step-by-step Procedure of Analysis;

Determine degree of static indeterminancy.

Choose the redundants.

Assign the coordinates to the redundant force direction.

Remove restraints to redundant forces and get basic determinate structure.

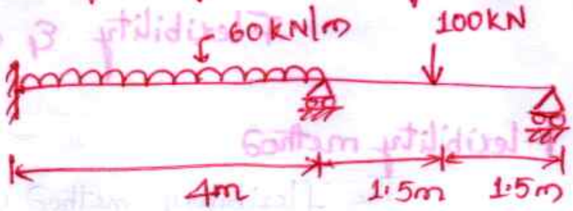
Determine the deflections in the coordinate directions due to given loading condition in the basic determinate structure

Determine flexibility matrix

$$\text{where } P = \delta^{-1}[\Delta - \Delta']$$

Analyse the given Continous beam by using flexibility method?

Degree of static indeterminancy



$D_s = 9 - 3$

Fixed support = 3

Roller - 1

Hinged - 2

Number of Equilibrium equation = 3

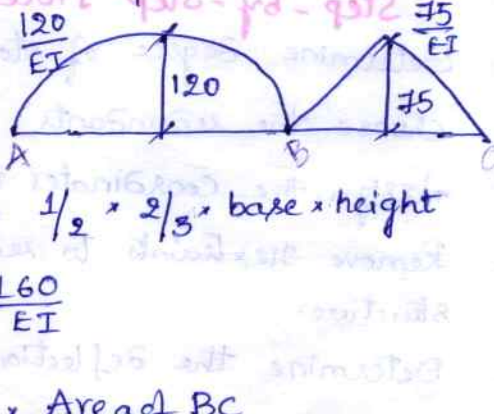
Total No of Reactions = 5 $\quad D_s = 5 - 3 = 2$

Consider the given as a simply supported & assign the coordinates.

for Span AB, maximum B.M = $\frac{wl^2}{8} = \frac{60 \times 4^2}{8} = 120\,KNm$

for Span BC, maximum B.M = $\frac{wl}{4} = \frac{100 \times 3}{4} = 75\,KNm$

formation of 'Δ' matrix



$$\Delta' = \begin{bmatrix} \Delta_1' \\ \Delta_2' \end{bmatrix}$$

$\Delta_1' = 1/2 \times$ Area of portion AB $= 1/2 \times 2/3 \times$ base $\times$ height

$$= 1/2 \times 2/3 \times 4 \times \frac{120}{EI} = \frac{160}{EI}$$

$\Delta_2' = 1/2 \times$ Area of AB $+ 1/2 \times$ Area of BC

$$= \frac{160}{EI} + 1/2 \times 1/2 \times \text{base} \times \text{height}$$

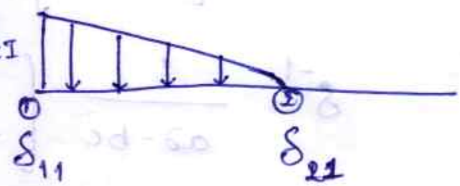
$$= \frac{160}{EI} + 1/2 \times 1/2 \times 3 \times \frac{75}{EI} = \frac{160}{EI} + \frac{56.25}{EI}$$

$$= \frac{216.25}{EI}$$

$$\Delta' = \begin{bmatrix} 160/EI \\ \\ \dfrac{216.25}{EI} \end{bmatrix}$$

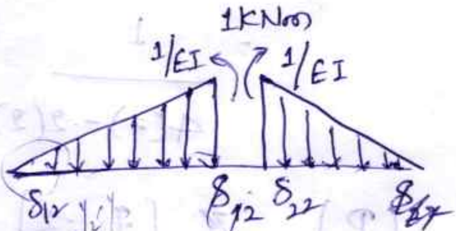
Formation of δ matrix ;

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$



$$\delta_{11} = 1/2 \times 2/3 \times 4 \times 1/EI = 4/3EI$$

$$\delta_{21} = 1/2 \times 1/3 \times 4 \times 1/EI = \frac{2}{3EI}$$



$$\delta_{12} = 1/2 \times 1/3 \times 4 \times 1/EI = \frac{2}{3EI}$$

$$\delta_{22} = 1/2 \times 2/3 \times 4 \times 1/EI + 1/2 \times 2/3 \times 3 \times 1/EI$$

$$= \frac{7}{3EI}$$

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{4}{3EI} & \frac{2}{3EI} \\ \frac{2}{3EI} & \frac{7}{3EI} \end{bmatrix}$$

Flexibility Equation

$$[P] = \delta^{-1}[\Delta - \Delta']$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}^{-1} \left[\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} - \begin{bmatrix} \Delta_1' \\ \Delta_2' \end{bmatrix} \right]$$

$$= \begin{bmatrix} \frac{4}{3EI} & \frac{2}{3EI} \\ \frac{2}{3EI} & \frac{7}{3EI} \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{160}{EI} \\ \frac{216.25}{EI} \end{bmatrix} \right]$$

$$= \frac{3EI}{1} \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}^{-1} \frac{1}{EI} \begin{bmatrix} -160 \\ -216.25 \end{bmatrix}$$

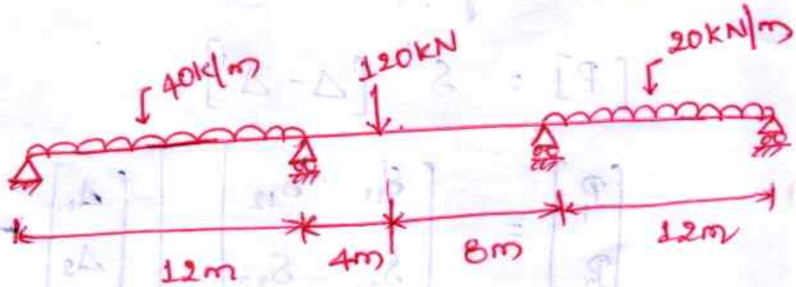
$$\delta^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{4(7) - 2(2)} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{3}{24} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -160 \\ -216.25 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 7 \times -160 + -2 \times -216.25 \\ -2 \times -160 + 4 \times -216.25 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -1120 + 432.5 \end{bmatrix} = \begin{bmatrix} -85.93 \\ -68.12 \end{bmatrix}$$

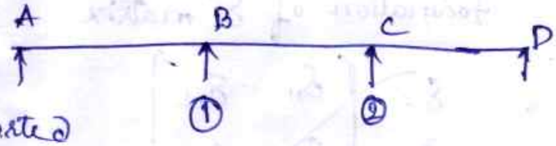


Degree of Indeterminacy

$D_s = r - 3$

Hinged = 2
Roller = 3×1 } = 5

$D_s = 7 - 3 = 5 - 3 = 2$



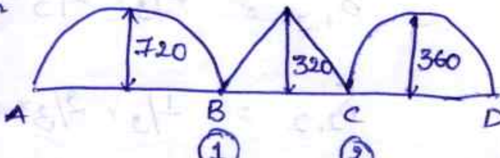
Assuming the beam as simply supported

$BM\ AB = \frac{WL^2}{8} = \frac{40 \times 12^2}{8} = 720\,kNm$

$BC = \frac{Wab}{L} = \frac{120 \times 4 \times 8}{12} = 320\,kNm$

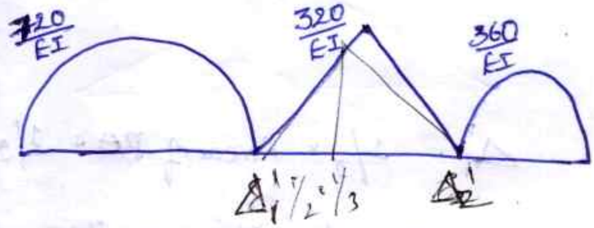
$CD = \frac{WL^2}{8} = \frac{20 \times 12^2}{8} = 360\,kNm$

Now, Draw the BMD & also $\frac{M}{EI}$ diagram



formation of 'Δ'' matrix;

$\Delta' = \begin{bmatrix} \Delta'_1 \\ \Delta'_2 \end{bmatrix}$



$\Delta'_1 = \frac{1}{2} \times \text{Area of } AB$

$= \frac{1}{2} \times \frac{2}{3} \times 4 \times \frac{720}{EI} = \frac{160}{EI}$

$\Delta'_2 = \frac{1}{2} \times \text{Area of } AB + \frac{1}{2} \times \text{Area of } BC$

$= \frac{160}{EI} + \frac{1}{2} \times \frac{1}{2} \times \text{base} \times \text{height}$

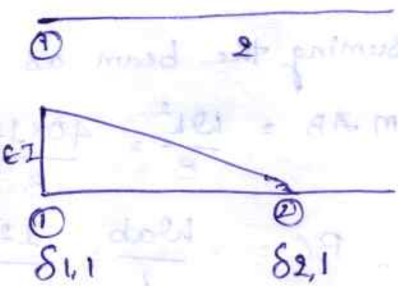
$= \frac{160}{EI} + 56 \cdot \frac{1}{2} \times \frac{1}{2} \times 3 \times \frac{45}{EI} = \frac{160}{EI} + \frac{56.25}{EI}$

$= \frac{216.25}{EI}$

$\Delta' = \begin{bmatrix} \Delta'_1 \\ \Delta'_2 \end{bmatrix} = \begin{bmatrix} 160/EI \\ 216.25/EI \end{bmatrix}$

formation of 'δ' matrix

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

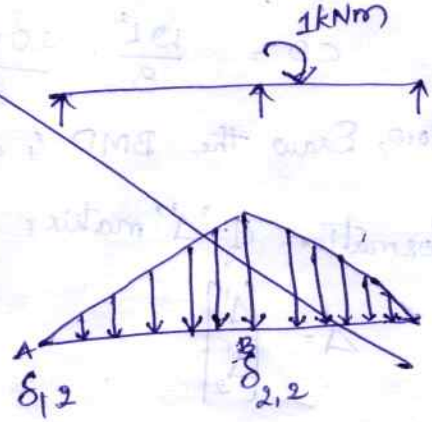


$$\delta_{11} = 1/2 \times 2/3 \times 4 \times 1/EI = \frac{4}{3EI}$$

$$\delta_{21} = 1/2 \times 1/3 \times 4 \times 1/EI = \frac{2}{3EI}$$

$$\delta_{1,2} = 1/2 \times 1/3 \times 4 \times 1/EI = \frac{2}{3EI}$$

$$\delta_{2,2} = 1/2 \times 2/3 \times 4 \times 1/EI +$$
$$1/2 \times 2/3 \times 3 \times$$



$$\Delta_1' = 1/2 \times \text{Area of BA} + 1/2 \times \text{Area of BC}$$

$$= 1/2 \times 2/3 \times 12 \times \frac{720}{EI} + \left[1/2 \times 1/3 \times 4 \times \frac{320}{EI}\right] + \left[\frac{1}{2} \times 2/3 \times 8 \times \frac{320}{EI}\right]$$

$$= \frac{2880}{EI} + \frac{213.3}{EI} + \frac{853.3}{EI}$$

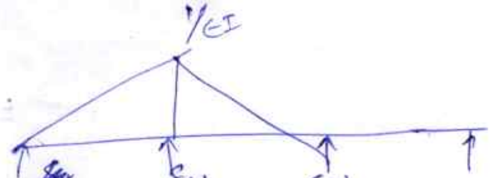
$$= \frac{2880}{EI} + \frac{1066.63}{EI} = \frac{3946.63}{EI}$$

$$\Delta_2' = 1/2 \times \text{Area of CB} + 1/2 \times \text{Area of CD}$$

$$= \left[1/2 \times 2/3 \times 4 \times \frac{320}{EI}\right] + \left[1/2 \times 1/3 \times 8 \times \frac{320}{EI}\right] + \left[1/2 \times 2/3 \times 12 \times \frac{360}{EI}\right]$$

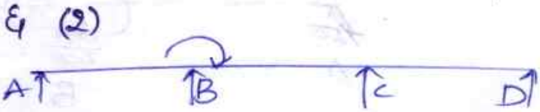
$$= \frac{426.66}{EI} + \frac{426.66}{EI} + \frac{1440}{EI} = \frac{2293.33}{EI}$$

$$\begin{bmatrix} \Delta_1' \\ \Delta_2' \end{bmatrix} = \begin{bmatrix} 3946.66/EI \\ 2293.33/EI \end{bmatrix}$$



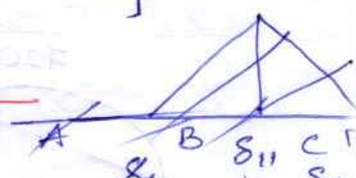
Formation of 'δ' matrix;

In the given beam remove the applied load and apply an unit load at the coordinates (1) & (2)



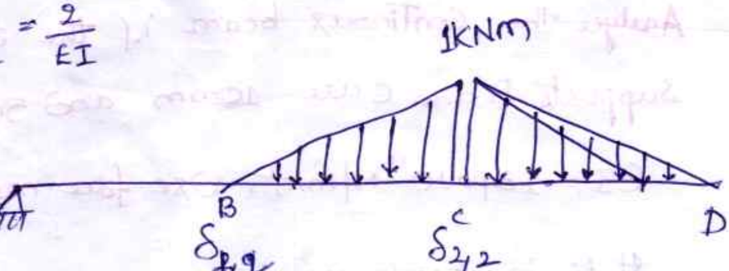
$$\delta_{11} = \left[\frac{1}{2} \times \text{Area of span AB} + \frac{1}{2} \times \text{Area of span BC}\right]$$

$$= \left[\frac{1}{2} \times \frac{2}{3} \times 12 \times \frac{1}{EI}\right] + \left[\frac{1}{2} \times \frac{2}{3} \times 12 \times \frac{1}{EI}\right]$$



$$= \frac{4}{EI} + \frac{4}{EI} = \frac{8}{EI}$$

$$\delta_{21} = 1/3 \times 1/2 \times 12 \times 1/EI = \frac{2}{EI}$$



$$\delta_{12} = 1/3 \times 1/2 \times 12 \times 1/EI$$

$$= 2/EI$$

$$\delta_{2,2} = 2/3 \times 1/2 \times 12 \times 1/EI + 2/3 \times 1/2 \times 12 \times 1/EI = \frac{8}{EI}$$

The final displacements at (1) and (2) are zero

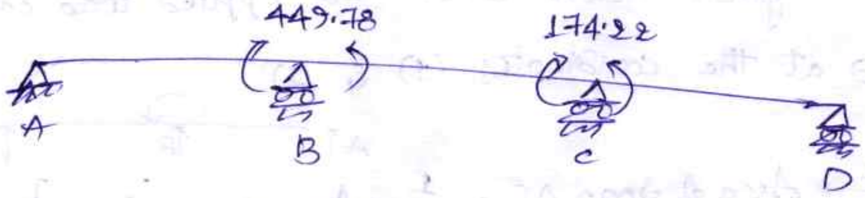
$$[\delta]\,[P] = [\Delta] - [\Delta_L]$$

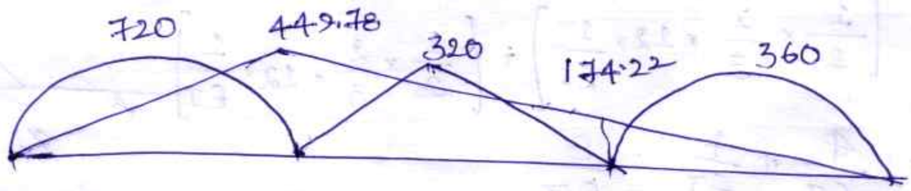
$$P = \delta^{-1}\left[\Delta - \Delta_L\right]$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 8/EI & 2/EI \\ 2/EI & 8/EI \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3946.6/EI \\ 22933/EI \end{bmatrix}\right]$$

$$\frac{1}{EI}\begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\frac{1}{EI}\begin{bmatrix} 3946.67 \\ 2293.33 \end{bmatrix}$$

$$= \begin{bmatrix} -449.78 \\ -174.22 \end{bmatrix}$$





Analyse the Continous beam if the downward settlement of supports B and C are 10mm and 5mm, respectively. Take $EI = 184 \times 10^{11}\ N/mm^2$. Use force method.

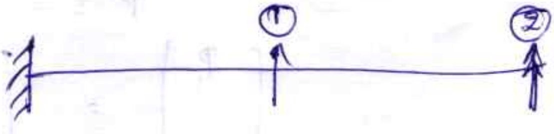
Static indeterminancy = 2

Simple Bending moment

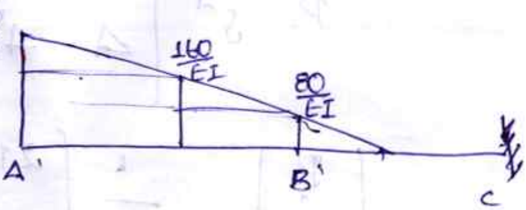
$M_{AB} = \frac{wl}{4} = \frac{80\times4}{4} = 80KNm$

$M_{BC} = \frac{wl}{4} = \frac{40\times4}{4} = 40KNm$

Assigning coordinates



$\Delta_{1L} = \left[\frac{160}{EI} \times 2 \times 3 + \frac{1}{2} \times 2 \times \frac{(400-160)}{EI} \times \left[2 + \frac{4}{3}\right] + 2 \times \frac{80}{EI} \times 1 + \frac{1}{2} \times 2 \times \frac{80}{EI} \times \frac{4}{3}\right]$



$= \frac{-2026.67}{EI}$

$EI = 184 \times 10^{11}\ N\text{-}mm^2 = 18400\ KNm^2$

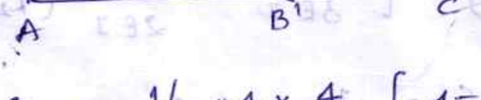
$\Delta_{1L} = \frac{2026.67}{18400} = -0.110\,m$

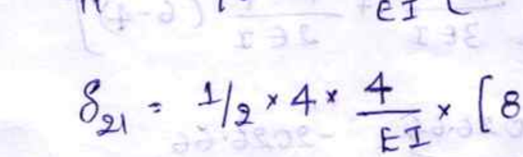
Δ_{22} = moment at 'C'

$= -\left[\frac{160}{EI} \times 2 \times 7 + \frac{1}{2} \times 2\,(400-160) \times 7.333 + \frac{1}{2} \times 4 \times 160 \times 5.333\right]$

$= \frac{-5706.667}{EI} = -0.310\,m$

Formation δ matrix - Apply unit force at coordinate ①





$\delta_{11} = \frac{1}{2} \times 4 \times \frac{4}{EI}\left[4 - 4/3\right] = \frac{21.33}{EI}$

$\delta_{21} = \frac{1}{2} \times 4 \times \frac{4}{EI} \times \left[8 - 4/3\right] = \frac{53.33}{EI}$

$\delta_{12} = \frac{1}{2} \times \frac{4}{EI} \times 4 \times 8/3 + \frac{4}{EI} \times 4 \times 2 = \frac{53.33}{EI}$

$\delta_{22} = \frac{1}{2} \times \frac{8}{EI} \times 8 \times 16/3 = \frac{170.67}{EI}$

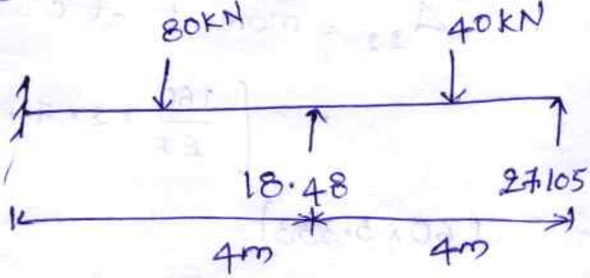
$$P = \delta^{-1}\left[\Delta - \Delta'\right]$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 21.33 & 51.33 \\ 51.33 & 170.67 \end{bmatrix}^{-1}\left(\begin{bmatrix} -0.01 \\ -0.005 \end{bmatrix} - \begin{bmatrix} -0.110 \\ -0.310 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 18.485 \\ 27.105 \end{bmatrix}$$

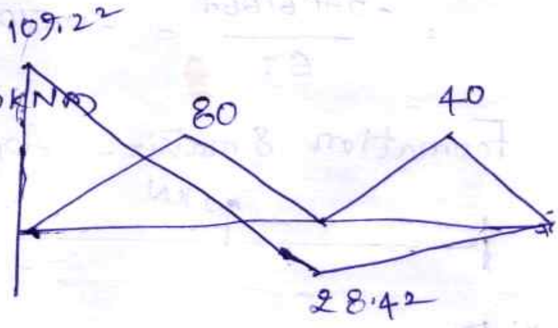
$M_B = 27.105 \times 4 - 40 \times 2 = 28.42\,kNm$

$M_A = 27.105 \times 8 - 40 \times 6 + 18.485 \times 4$
-80×2
$= -109.22\,kNm$



Simple moment $M_{AB} = \frac{40 \times 4}{4} = 40\,kNm$

$M_{BC} = \frac{80 \times 4}{4} = 80\,kNm$



$$\Delta_1' = -\left[\left(\frac{w_1 a_1^3}{3EI} + \frac{w_1 a_1^2}{2EI}(L_1 - a_1)\right) + \left[\frac{w_2 a_2^3}{3EI} + \frac{w_2 a_2^2}{2EI}(L_2 - a_2)\right]\right]$$

$$= \left[\left[\frac{80 \times 2^3}{3EI} + \frac{80 \times 2^2}{2EI}(4-2)\right] + \left[\frac{40 \times 4^3}{3EI} + \frac{40 \times 4^2}{2EI}(6-4)\right]\right]$$

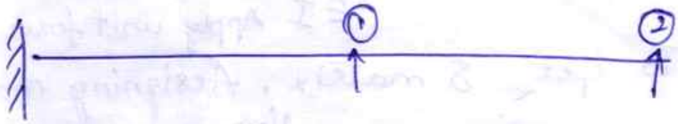
$$= -\left(\frac{1600}{3EI} + \frac{4480}{3EI}\right)\Bigg] = \frac{2026.66}{EI} = \frac{-2026.66}{18400} = -0.11m$$

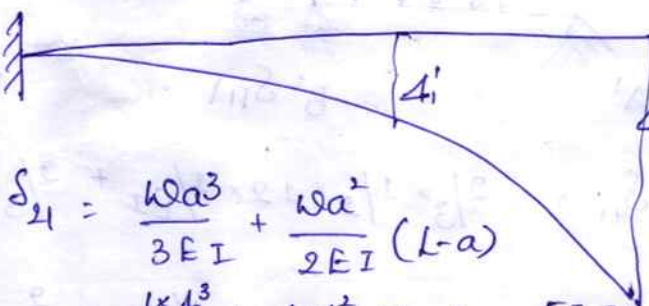
Stiffness matrix method;

$$\Delta_2' = \left[\left[\frac{w_1 a_1^3}{3EI} + \frac{w_1 a_1^2}{2EI}(L-a_1)\right] + \left[\frac{w_2 a_2^3}{3EI} + \frac{w_2 a_2^2}{2EI}(L_2-a_2)\right]\right]$$

$$= \left[\frac{80\times 2^3}{3EI} + \frac{80\times 2^2}{2EI}(8-2)\right] + \left[\frac{40\times 6^3}{3EI} + \frac{40\times 6^2}{2EI}(8-2)\right]$$

$$= \frac{3520}{3EI} + \frac{4320}{EI} = -\frac{5493.33}{EI} = -\frac{5493.33}{18400} = -0.298 \simeq -0.3\ m$$





$$\delta_{11} = \frac{wa^3}{3EI} = \frac{1\times 4^3}{3EI} \qquad \delta_{21} = \frac{wa^3}{3EI} + \frac{wa^2}{2EI}(L-a)$$

$$\frac{1\times 4^3}{3EI} + \frac{1\times 4^2}{2EI}(8-4) = \frac{53.33}{EI}$$

$$\delta_{12} = \frac{53.33}{EI} \qquad \delta_{22} = \frac{wL^3}{3EI} = \frac{1\times 8^3}{3EI}$$

(2) Problem

$$A_{1L} = \frac{1}{2} \times \frac{2}{3} \times \frac{720}{EI} \times 12 + \frac{1}{2} \times 12 \times \frac{320}{EI} \times \frac{(12+8)}{3} \times \frac{1}{12}$$

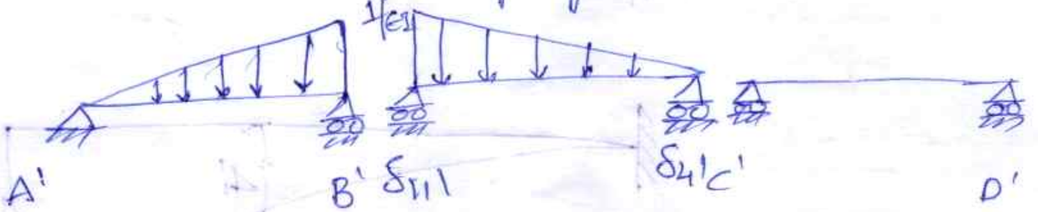
$$= \frac{3946.67}{EI}$$

$$A_{2L} = \frac{1}{2} \times \frac{320}{EI} \times 12 \times \left[\frac{12+4}{3}\right] \times \frac{1}{12} + \frac{1}{2} \times \frac{2}{3} \times 12 \times \frac{360}{EI}$$

$$= \frac{2293.33}{EI}$$

Apply unit force at coordinate direction 1.

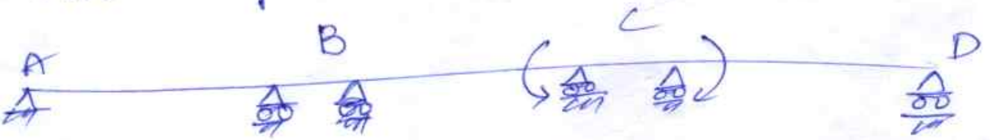
To get δ matrix, Assigning coordinates



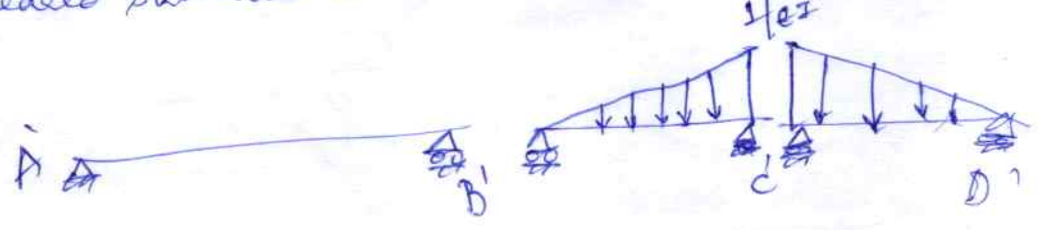
$$\delta_{11} = \frac{2}{3} \times \frac{1}{2} \times 12 \times \frac{1}{EI} + \frac{2}{3} \times \frac{1}{2} \times 12 \times \frac{1}{EI} = \frac{8}{EI}$$

$$\delta_{21} = \frac{1}{3} \times \frac{1}{2} \times 12 \times \frac{1}{EI} = \frac{2}{EI}$$

(b) Unit force is applied in coordinate direction 2. The conjugate beam and loading on it



Released structured with unit load in coordinate direction:



$$\delta_{1,2} = 1/3 \times 1/2 \times 12 \times 1/EI = 2/EI$$

$$\delta_{2,2} = 1/2 \times 12 \times 1/EI \times 2/3 + 1/2 \times 12 \times 1/EI \times 2/3 = \frac{8}{EI}$$

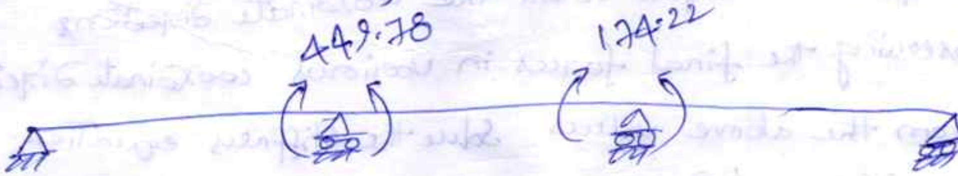
$$[\delta]\,[P] = [\Delta] - [\Delta_L]$$

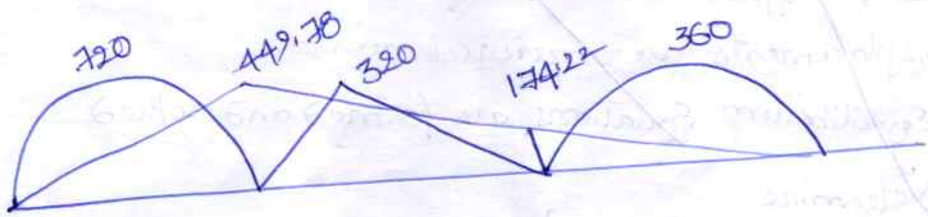
$$[P] = \delta^{-1}\left[[\Delta] - [\Delta_L]\right]$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}^{-1}\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3946.67/EI \\ 2293.33/EI \end{bmatrix}\right]$$

$$= \frac{1}{(64-4)}\begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 3946.67/EI \\ 2293.33/EI \end{bmatrix}$$

$$= \begin{bmatrix} -449.78 \\ -174.22 \end{bmatrix}$$





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CIVIL ENGINEERING

Advanced Structural Analysis

UNIT-5

Stiffness matrix

In this method, the basic unknowns to be determined in the analysis are the displacement components of various joints. Hence the degree of kinematic indeterminacy is determined. This method is also called as displacement method or equilibrium method

→ Determine the degree of kinematic indeterminancy / Degree of freedom
→ Assign coordinate members to the unknown displacements
→ Imposed restrains in all coordinate directions
→ Determine the forces developed in each of the coordinate directions.
→ Determine the stiffness matrix by giving unit displacement to the restrained structure in each of the coordinate directions and find the forces developed in all the coordinate directions
→ Observing the final forces in various coordinate directions
→ from the above values solve the stiffness equation.

Step by step procedure

1) Since, stiffness matrix is used
2) Displacements are basic unknowns
3) Equilibrium Equations are formed and solved

→ Determine

$$[K][\Delta] = [P - P^{1}]$$

$$\Delta = K^{-1}[P - P^{1}]$$

→ Calculate the member forces using these joint displacements

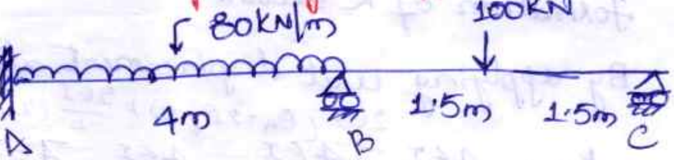
Degree of kinematic indeterminancy (or) Degree of freedom

A structure is said to be kinematically indeterminate if the displacement components of the joints cannot be determined by compactability equations alone. For these structures, additional eq's based on equilibrium conditions must be formulated to obtain the number of eq's necessary for determining all the unknown displacement components.

The number of equilibrium conditions needed to find the displacement components of all joints of the structure are known as degree of kinematic indeterminancy of the structure.

Analyse the given continous beam by using stiffness method.

Degree of kinematic indeterminancy

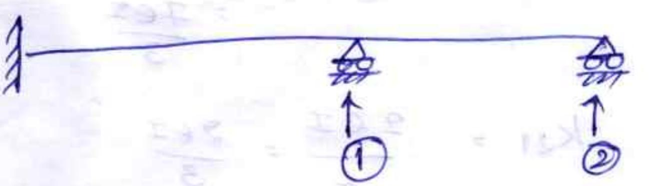


$D_k = 3j - (r+m)$

j = number of joints

r = number of reactions

$m = 2$



$D_k = 3(3) - (5+2)$

$= 9 - 7 = 2$

$D_k = 2$

fixed End moments

$M_{fAB} = \frac{wl^2}{12} = \frac{80 \times 4^2}{12} = 106.6 \text{ KNm}$

$M_{fBA} = -106.6 \text{ kNm}$

$M_{fBC} = \frac{WL}{8} = \frac{100 \times 3}{8} = 37.5 \text{ KNm}$ $M_{fCB} = -37.5 \text{ KNm}$

formation of P' matrix

$$P' = \begin{bmatrix} P_1' \\ P_2' \end{bmatrix}$$

At coordinate ①. The values of P_1' & P_2' can be determined by considering the coordinates ① & (2) are at equilibrium

$$P_1' = M_{fBA} + M_{fBC}$$

$$= -106.6 + 37.5 = -69.1 \text{ KNm}$$

At coordinate (2) $P_2' = M_{fCB} = -37.5$

$$\begin{bmatrix} P_1' \\ P_2' \end{bmatrix} = \begin{bmatrix} -69.1 \\ -37.5 \end{bmatrix}$$

formation of K matrix

By applying unit displacement at coordinate 1, we get

$$\frac{2EI}{L}(\theta_A + 2\theta_B - 0) + \frac{2EI}{L}(2\theta_B + \theta_C - 0)$$

$$K_{11} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{4EI}{4} + \frac{4EI}{3}$$

$$= \frac{7EI}{3} \qquad \frac{2EI}{L}\left[2\theta_B + \theta_A\right] \quad \theta_A = \theta_C = 0$$

$$\theta_B = 1$$

$$K_{21} = \frac{2EI}{L} = \frac{2EI}{3} \qquad \frac{2EI}{L}(\theta_B + 2\theta_C)$$

By applying unit displacement at 2nd coordinate

$$k_{12} = \frac{2EI}{L} = \frac{2EI}{3} \qquad (2\theta_B + \theta_C)\frac{2EI}{L} \quad \theta_B = 0 \; \theta_C = 1$$

$$k_{22} = \frac{4EI}{L} = \frac{4EI}{3} \qquad (\theta_B + 2\theta_C)$$

$$K = \begin{bmatrix} k_{11} & K_{21} \\ k_{12} & k_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7EI}{3} & \frac{2EI}{3} \\ \frac{2EI}{3} & \frac{4EI}{3} \end{bmatrix}$$

Stiffness Equation $\Delta = K^{-1}[P - P']$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{bmatrix}^{-1} \left[\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - \begin{bmatrix} P_1' \\ P_2' \end{bmatrix} \right]$$

$$= \begin{bmatrix} 7EI/3 & 2EI/3 \\ 2EI/3 & 4EI/3 \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -69.1 \\ -37.5 \end{bmatrix} \right]$$

$$K^{-1} = \frac{EI}{3} \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}^{-1}$$

$$K = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{28 - 4} \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix}$$

$$\frac{EI}{3} \times \frac{3}{EI} \times \frac{1}{24} \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} +69.11 \\ -37.5 \end{bmatrix}$$

$$\frac{1}{8EI} \begin{bmatrix} 4 \times 69.11 + -2 \times -37.5 \\ -2 \times 69.11 + 7 \times -37.5 \end{bmatrix} = \frac{1}{8EI} \begin{bmatrix} 201.4 \\ -124.28 \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} \frac{25.17}{EI} \\ -\frac{15.53}{EI} \end{bmatrix}$$

final moments

$$M_{AB} = M_{fAB} + \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\delta}{L}\right]$$

$$= -106.6 + \frac{2EI}{4}\left(2(0) + \frac{-25.7}{EI}\right)$$

$$= 106.6 - 12.85 = 93.75$$

$$M_{BA} = M_{fBA} + \frac{2EI}{L}\left[2\theta_B + \theta_A - \frac{3\delta}{L}\right]$$

$$= -106.6 + \frac{2EI}{4}\left[2\left[\frac{-25.17}{EI}\right]\right]$$

$$= -106.6 + 25.17 = 81.43\,kNm$$

$$M_{BC} = M_{fBC} + \frac{2EI}{L}\left[2\theta_B + \theta_C - \frac{3\delta}{L}\right]$$

$$= -37.5 + \frac{2EI}{3}\left[2\left[\frac{-25.17}{EI}\right] + \left(\frac{-15.53}{EI}\right)\right]$$

$$= -81.43\,kNm$$

$$M_{CB} = M_{fCB} + \frac{2EI}{L}\left[2\theta_C + \theta_B - \frac{3\delta}{L}\right]$$

$$= -37.5 + \frac{2EI}{3}\left[2\left[\frac{-15.53}{EI}\right) + \left[\frac{-25.17}{EI}\right)\right]$$

$$= -37.5 + \frac{2EI}{3}\left[\frac{-56.23}{EI}\right] = 0.013$$

Analyse the given Continous beam

Degree of kinematic indeterminacy

$$D_K = 3j - (r + m)$$

$j = 3 \quad r = 5 \quad m = 2$

$D_k = 3(3) - (5+2) = 2.$

fixed End moments

$M_{fAB} = -\frac{wab^2}{l^2} = \frac{-90 \times 2 \times 4^2}{6^2} = -80 \text{KNm}$

$M_{fBA} = \frac{wa^2b}{l^2} = 40 \text{kNm}$

$M_{fBC} = \frac{wl^2}{12} = \frac{80 \times 4^2}{12} = 106.6 \text{KNm}$

$P_1' = M_{fBA} + M_{fBC} = 40 - 106.6 = -66.67$

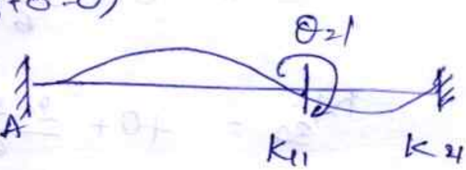
$P_2' \quad M_{fCB} = 106.67 \quad 120 - 106.67 = 13.33$

stiffness matrix

unit displacement at coordinate 1

$K_{11} = \frac{2E(I)}{L}(\theta_A + 2\theta_B - 0) + \frac{2EI}{L}(2\theta_B + 0 - 0)$

$= \frac{4EI}{6}[0 + 2 \times 1 - 0) + \frac{2EI}{4}(2 \times 1 + 0 - 0)$



$= \frac{8EI}{6} + EI = 7/3 EI$

$K_{21} = \frac{2EI}{L}\left[2\theta_c + \theta_B - 0\right] = 0.5EI$

Unit displacement at coordinate 2

$K_{21} = \frac{2EI}{4} = 0.5EI \quad K_{22} = \frac{4EI}{4} = EI$

There fore stiffness matrix

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{pmatrix} 66.67 \\ 120 - 106.67 \end{pmatrix}$$

$$\frac{EI}{3}\begin{bmatrix} 7 & 1.5 \\ 1.5 & 3 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \frac{3}{EI}\begin{bmatrix} 7 & 1.5 \\ 1.5 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$= \frac{3}{EI}\left[\frac{1}{7\times 3 - 1.5^2}\right]\begin{bmatrix} 3 & -1.5 \\ -1.5 & 7 \end{bmatrix}\begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$= \frac{3}{EI}\times\frac{1}{18.75}\begin{bmatrix} 180.015 \\ -66.95 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 28.802 \\ -1.070 \end{bmatrix}$$

$$\theta_B = \frac{28.802}{EI} \qquad \theta_C = \frac{-1.070}{EI}$$

$$M_{AB} = -80 \times \frac{2E2I}{6}\left(2\theta_A + \theta_B - 0\right)$$

$$= -80 + \frac{4}{6}EI\left[\frac{28.802}{EI}\right] = -60.80\,KNm$$

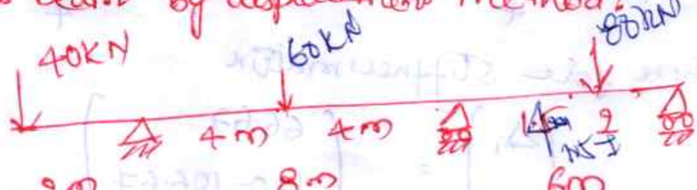
$$M_{BA} = 40 + \frac{2E2I}{6}\left[0 + 2\times\frac{28.802}{EI} - 0\right] = 78.403$$

$$M_{BC} = -106.67 + \frac{2EI}{4}\left[\frac{2\times 28.802}{EI} - \frac{1.071}{EI}\right] = -78.403$$

$$M_{CB} = 106.67 + \frac{2EI}{4}\left[\theta_B + 2\theta_C - 0\right]$$

$$= 106.67 + \frac{2EI}{4}\left[\frac{28.802}{EI} - \frac{2\times 1.071}{EI}\right] = 120\,KNm$$

Analyse the Continous beam by displacement method?



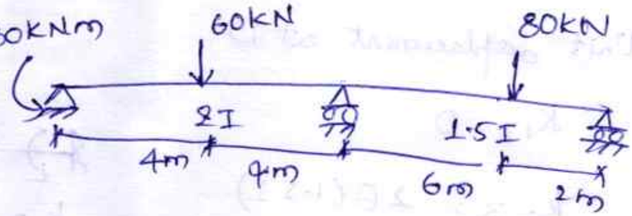
$3j - m + r$

$3\times 3 - (3+3)$

$9 - 6 = 3$

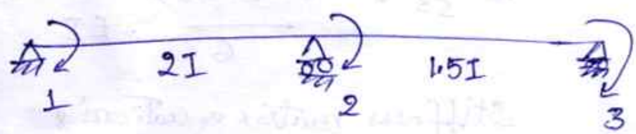
Final force

$$[P] = \begin{bmatrix} -80 \\ 0 \\ 0 \end{bmatrix}$$

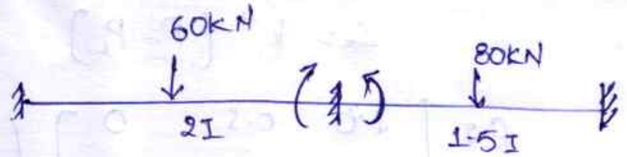


$$M_{fBC} = -60 \times \frac{8}{8} = -60KNm$$

$$M_{fCB} = 60\ KNm$$



$$M_{fCD} = \frac{-80 \times 4 \times 2^2}{6^2} = -35.55KNm$$



$$M_{fDC} = \frac{80 \times 4^2 \times 2}{6^2} = 71.11\ KNm$$

$$P_L = \begin{bmatrix} -60.0 \\ 60-35.55 \\ 71.11 \end{bmatrix}$$

$3j - m + r$

$3 \times 3 - 3 + 3$

$9 - 6 = 3$

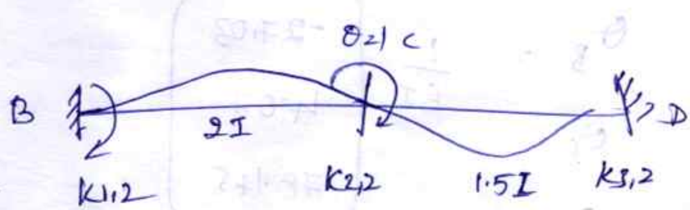
Stiffness matrix unit displace at B

$$k_{11} = \frac{4E \cdot (2I)}{8} = EI$$

$$k_{21} = \frac{2EI(2I)}{8}$$

$$k_{31} = 0$$

unit displacement at C

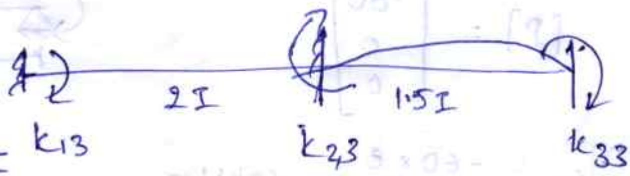


$$K_{1,2} = \frac{2EI}{8}[2\theta + \theta_C] = 0.5EI$$

$$K_{2,2} = \frac{2EI}{8}[2\theta_B + \theta_C] + \frac{2E1.5I}{6} = 2EI$$

$$K_{32} = \frac{2EI}{6}(2\theta_C + \theta_D) = \frac{2E(1.5I)}{6} = 0.5EI$$

Unit displacement at D



$k_{13} = 0$

$k_{23} = \frac{2E(1.5I)}{6} = 0.5EI$

$k_{33} = \frac{4E(1.5I)}{6} = EI$

Stiffness matrix equation is

$$\Delta = k^{-1}[P - P_L]$$

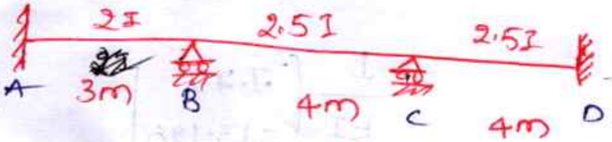
$$EI \begin{bmatrix} 1.0 & 0.5 & 0 \\ 0.5 & 2.0 & 0.5 \\ 0 & 0.5 & 1.0 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} -80+60 \\ 0-24.45 \\ 0-71.11 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -20 \\ -24.45 \\ -71.11 \end{bmatrix}$$

$$= \frac{1}{EI}\left[\frac{1}{2\times1 - 0.5^2 - 0.5(0.5-0)}\right] \times \begin{bmatrix} 1.75 & -0.5 & 0.25 \\ -0.5 & 1 & 0.5 \\ 0.25 & -0.5 & 1.75 \end{bmatrix} \begin{bmatrix} -20 \\ -24.25 \\ -71.11 \end{bmatrix}$$

$$\begin{matrix} \theta_B \\ \theta_C \\ \theta_D \end{matrix} = \frac{1}{EI} \begin{bmatrix} -27.03 \\ 14.07 \\ -78.145 \end{bmatrix}$$

Analyse the continous beam shown in fig. If the support B sinks by 10mm. Use displacement method. Take EI = 6000kNm²

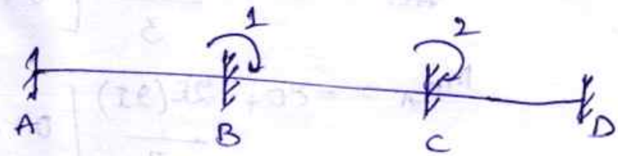


$$M_{fAB} = \frac{-6E(2I)0.01}{3^2} = -80kNm$$

$$M_{fBA} = -80kNm$$

$$M_{fBC} = \frac{6E(2.5I)0.01}{4^2} = 56.25kNm$$

$$M_{fCB} = 56.25kNm$$



$$P_{1L} = -80 + 56.25 = -23.27$$

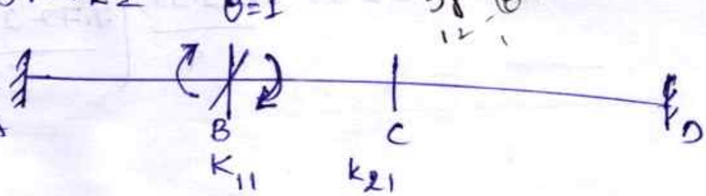
$$P_{2L} = 56.25 = 56.25$$

Unit displacement in coordinate direction 1

$3j - m + r$
$3 \times 4 - (3+8)$
$12 - 11$
$3j - r + m$
$12 - (8+3)$
$12 - 1$

$$k_{11} = \frac{4E(2I)}{3} + \frac{4E(2.5I)}{4} = 5.167EI$$

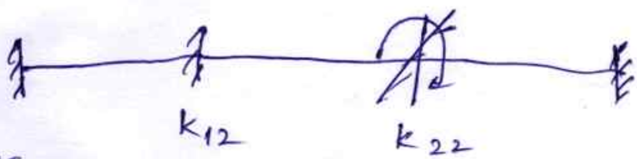
$$k_{21} = \frac{2E\,2.5I}{4} = 1.25EI$$



Unit displacement in coordinate direction 2

$$k_{12} = \frac{2E\,2.5I}{4} = 1.25EI$$

$$k_{22} = \frac{4E\,2.5I}{4} + \frac{4E\,2.5I}{4} = 5EI$$



$$EI\begin{bmatrix} 5.167 & 1.25 \\ 1.25 & 5 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 0-(-23.75) \\ 0-56.25 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 5.167 & 1.25 \\ 1.25 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 23.75 \\ -56.25 \end{bmatrix}$$

$$= \frac{1}{EI}\left[\frac{1}{24.27}\right]\begin{bmatrix}5 & -1.25\\ -1.25 & 5.167\end{bmatrix}^{-1}\begin{bmatrix}23.75\\ -56.25\end{bmatrix}$$

$$\frac{1}{EI}\begin{bmatrix}7.79\\ -13.198\end{bmatrix}$$

$\theta_B = 7.79/EI \quad \theta_C = -13.198/EI$

$$M_{AB} = -80 + \frac{2E(2I)}{3}\left[0 + \frac{7.79}{EI} - 0\right] = -69.613\ kNm$$

$$M_{BA} = -80 + \frac{2E(2I)}{3}\left[0 + \frac{2 \times 7.79}{EI} - 0\right] = -59.227\ kNm$$

$$M_{BC} = 56.25 + \frac{2E(2.5I)}{4}\left[\frac{2 \times 7.79 - 13.198}{EI}\right] = 59.227\ kNm$$

$$M_{CB} = 56.25 + \frac{2E(2.5I)}{4}\left[\frac{7.79\ 2 \times 13.198}{EI}\right] = 33\ kNm$$