ANNAMACHARYA UNIVERSITY
EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
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Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Lecture Notes
on
Strength of Materials

Prepared by:
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Civil Engineering

Title of the Course: **Strength of Materials**
Category: PCC
Semester: III Semester
Couse Code: 24ACIV31T
Branch(s): CE

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	-	-	3

Course Objectives:

1. To impart knowledge on the fundamental concepts of stress, strain, and elastic behavior of materials.
2. To enable students to understand and apply the concepts of flexural and shear stresses in beams and analyze the behavior of axially loaded compression members using Euler's theory
3. To develop the ability to calculate flexural and shear stresses in various structural members.
4. To provide methods for computing deflections in beams using standard techniques.
5. To introduce principal stresses and strain theories and failure theories.

Course Outcomes:

At the end of the course, the student will be able to

1. Analyze stresses and strains in materials under axial loading and determine elastic constants for various conditions.
2. Construct shear force and bending moment diagrams for beams under different loading scenarios.
3. Calculate flexural and shear stresses in various beam sections and evaluate the stability of columns using Euler's buckling theory under different end conditions.
4. Determine the slope and deflection of beams using analytical methods such as double integration and moment area method.
5. Apply theories of failure and principal stress analysis to evaluate materials and shells behaviour under complex stress conditions.

Unit 1 Simple Stresses and Strains 12

Concept of stress and strain- Principle-Stress and Strain Diagram - Elasticity and Plasticity–Types of stresses and strains – Hooke's law–stress –strain diagram for mild steel– Working stress –Factor of safety –Lateral strain, Poisson's ratio and volumetric strain –Elastic moduli and the relationship between them– Bars of varying section –composite bars– Temperature stresses. Strain energy –Resilience –Gradual, sudden and impact –simple applications.

Unit 2 Shear Force and Bending Moment 12

Definition and classification of beams– Concept of shear force and bending moment– S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, UDL, UVL and combination of loads– Point of contra flexure –Relation between S.F, B.M and rate of loading at a section of a beam.

Unit 3 Flexural Stresses and Shear Stresses 12

Flexural stresses: Theory of simple bending –Assumptions –Derivation , Neutral axis– Determination of bending stresses– section modulus of rectangular and circular sections (Solid and Hollow), I, T & C sections –Design of simple beam sections.

Shear stresses: Derivation of formula– Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T and C-Sections.

Basics of Columns: Axially loaded compression members – Euler's crippling load theory – Derivation of Euler's critical load formulae for various end conditions – Equivalent length – Slenderness ratio –Limitations of Euler's theory.

Unit 4 Deflection of Beams **12**

Bending in to a circular arc– slope, deflection and radius of curvature –and Determination of slope and deflection for cantilever and simply supported beams subjected to point loads, -UDL – Double Integration method and Moment area method.

Unit 5 Basics of Principal Stresses and Strains , Theories of Failures, Thin and Thick Shells **12**

Principal stresses and strains: Basics of Stresses on an inclined section of a bar under axial loading Normal and tangential stresses on an inclined plane for biaxial stresses–Mohr's circle of stresses– Principal stresses and strains – Analytical and graphical solutions.

Theories of failures: Various Theories of failures like Maximum Principal stress theory– Maximum Principal strain theory–Maximum shear stress theory– Maximum strain energy theory –Maximum shear strain energy theory.

Basics of Thin and Thick Shells- Longitudinal and circumferential stresses.

Prescribed Textbooks:

1. Mechanics of Solid, Ferdinand Beer and others – Tata McGraw-Hill Publications 2000
2. A Text book of Strength of materials by Dr. R. K.Bansal, 4th edition,Laxmi publications, 2010.

Reference Textbooks:

1. Strength of Materials by R. Subramaniyan, Oxford University Press, 2015.
2. Advanced Mechanics of Solids, L.S Srinath, McGraw Hill Education, , 3rd Edition, 2017.
3. Mechanics of Materials, Beer and Johnston, McGraw Hill India Pvt. Ltd. 8th Edition, 2020.
4. Mechanics of Solids — E P Popov, Prentice Hall, 2nd Edition, 2015.
5. Strength of Materials by S. S. Ratan Tata Mc Grill Publications 3rd Edition, 2016.

CO-PO Mapping:

Course Outcomes	Engineering Knowledge	Problem Analysis	Design/Development of solutions	Conduct investigations of complex problems	Engineering tool usage	The Engineer and the World	Ethics	Individual and collaborative Teamwork	Communication	Project management and finance	Life-long learning	**PSO1**	**PSO2**
24ACIV31T.1	3	3	1	2	-	2	-	-	-	-	1	3	2
24ACIV31T.2	3	3	1	2	-	1	-	-	-	-	1	3	2
24ACIV31T.3	3	3	1	2	-	2	-	-	-	-	1	3	2
24ACIV31T.4	3	3	1	2	-	2	-	-	-	-	1	3	2
24ACIV31T.5	3	3	1	2	-	1	-	-	-	-	1	3	2

CIVIL ENGINEERING

Strength of Materials

UNIT-1

Simple Stress and Strains

Strength: The strength of a material may be defined as the maximum resistance which a material can offer to the extremelly applied forces.

factors: * Type of loading, temperature, internal structures etc.

Stress: When some external forces are applied to a body, then the body offers internal resistance to these forces. The magnitude of the internal resisting force is numerically equal to the applied forces. This internal resisting force per unit area is called '**Stress**'.

$$\text{Stress} = \frac{\text{force}}{\text{Area}} \quad ; \quad \sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

Normal stress: The force ΔF can be resolved into components such that the one of them is along the outward drawn normal to the area ΔA and the other components lies in the plane of the area ΔA. Let ΔF_n be normal component then normal stress $\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$

This may be tensile or compressive depending upon the forces acting on the material to be either of the pull or push type respectively. Tensile and compressive stresses together are called direct stresses.

Shear Stress: The force ΔF may be resolved into infinite no. of components in the plane containing area ΔA, because there are infinite no. of directions in the plane containing area ΔA. which are perpendicular to unit normal $\bar{n}$.

$$\tau_x = \frac{dF_x}{dA}, \quad \tau_y = \frac{dF_y}{dA}$$

Conventional or Engineering Stress: It is defined as ratio of load P to the original area of cross-section A_0. $\left(\sigma = \frac{P}{A_0}\right)$

True Stress: It is defined as the ratio of load P to the instantaneous area of cross-section A. thus

$$\bar{\sigma} = \frac{P}{A} = \frac{P}{A_0} \times \frac{A_0}{A} = \sigma \frac{A_0}{A}$$

for volume constancy, $Al = A_0 l_0$, $\quad l = l_0(1+\epsilon)$; $\quad A = \frac{A_0}{1+\epsilon}$

$$\bar{\sigma} = \sigma(1+\epsilon)$$

Strain: It is defined as the change in length per unit length.

Conventional or engineering strain: It is the change in length per unit original length.

$$\epsilon = \frac{l - l_0}{l_0} = \int_{l_0}^{l} \frac{dl}{l_0} = \frac{1}{l_0}\int_{l_0}^{l} dl.$$

Natural strain: It is defined as the changed in length per unit instantaneous length.

$$\bar{\epsilon} = \int_{l_0}^{l} \frac{dl}{l} = \ln\frac{l}{l_0} = \ln(1+\epsilon)$$

true strain $\epsilon_t = \ln(1+\epsilon)$

Gauge length: It is that portion of the test specimen over which extension or deformation is measured. $5.65\sqrt{A}$

Poisson's ratio: When a material is subjected to longitudinal deformation then the lateral dimensions also change. The ratio of the lateral strain to longitudinal strain is a constant quantity called the poisson's ratio and is designated by μ or $\frac{1}{m}$

$$\mu = \frac{\text{lateral strain}}{\text{longitudinal strain}} \quad ? \quad \frac{\frac{\delta b}{b} \text{ (or) } \frac{\delta d}{d}}{\left(\frac{\delta L}{L}\right)}$$

Modulus of elasticity (E): within elastic limits the ratio of normal stress to normal strain is a constant quantity and is defined as the young's modulus of elasticity

$$E = \frac{\sigma}{\epsilon} = \frac{PL}{A\,dl}$$

Modulus of rigidity (G or C): It is defined as the ratio of shearing stress to shearing strain. i.e. $G = \frac{\tau}{\gamma}$

Bulk modulus (K): It is defined as the ratio of uniform stress intensity to volumetric strain, within the elastic limits and is denoted by K.

$$K = \frac{\sigma}{\epsilon_v} = \frac{\sigma}{-\left(\frac{\Delta V}{V}\right)}$$

Proof stress: It is the maximum stress which can be applied to a material without allowing the material to fail.

Factor of safety:

Because of uncertainities of loading conditions, we introduce a factor of safety, defined as the maximum stress to the allowable or working stress / permissible stress. The maximum stress is generally taken as the yield stress for ductile materials. This is also called the factor of ignorance.

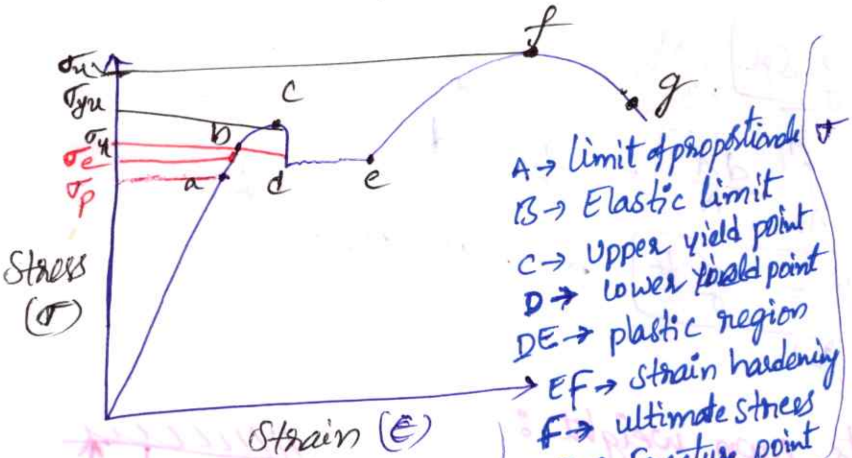
$$\text{factor of safety} = \frac{\text{Yield Stress}}{\text{working stress}} \quad \text{for ductile material.}$$

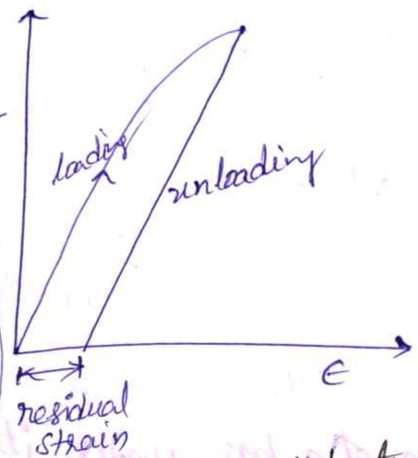
$$\text{factor of safety} = \frac{\text{ultimate Stress}}{\text{working stress}} \quad \text{for brittle material.}$$

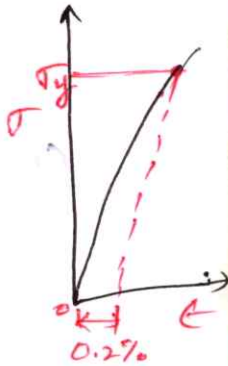
FOS → working stress method.
Concrete - 3
Steel - 1.78 - 1.79

FOS - limit state
Concrete - 1.5
Steel - 1.15

Stress-strain diagram



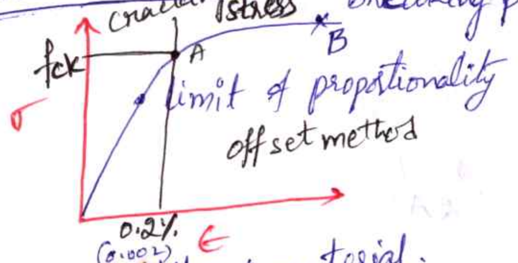




mild steel graph.

σ_p → proportionality limit stress
σ_e - elastic limit stress
σ_{yl} → lower yield stress
σ_{yu} → upper yield stress
σ_u → ultimate stress

Some times it is not possible to locate the yield point for such materials the yield point stress defined at some particular value of permanent set. The commonly used value of permanent set for the determining value of yield strength for mild steel is 0.2 % of the max strain.



brittle material.

→ fails suddenly [CI, glass, Concrete.
→ Strong in Compression
→ weak in tension

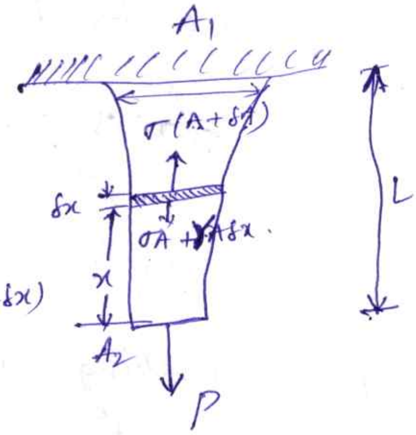
OA → Non linear elastic
AB → Strain hardening.

Strain at failure = 200 (strain at yield)

④ for square tapering bar, $\delta l = \frac{PL}{d_1 d_2 E}$

⑤ for circular tapering section $\delta l = \frac{4PL}{\pi d_1 d_2 E}$

Bar of uniform Strength :



Consider a bar which is acted upon by tensile load P. Consider elementary strip of the bar between cross-section x and $x+\delta x$ from lower end. let A be c/s area at x, $A+\delta A$ at $(x+\delta x)$

The force acting on strip upwards $= \sigma(A+\delta A)$

downwards $= \sigma A + \gamma \delta x . A$

($\gamma \to$ self weight of bar).

for equilibrium of the strip,

$$\sigma(A+\delta A) = \sigma A + \gamma A \delta x$$

$$\boxed{\frac{dA}{A} = \frac{\gamma}{\sigma}\delta x}$$

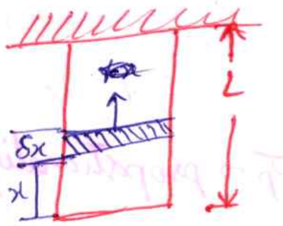
On Integration,
$$\int_{A_2}^{A_1} \frac{dA}{A} = \frac{\gamma}{\sigma}\int_0^L dx$$

$$\boxed{\ln \frac{A_1}{A_2} = \frac{\gamma}{\sigma} L} \qquad A_1 = A_2 e^{\frac{\gamma}{\sigma} L}$$

Then $A_1 = A_2 e^{\frac{\gamma}{\sigma} L}$

Extension of a bar under its own weight :

* Bar of uniform area :



Total stress in the strip, $\sigma = \frac{(\gamma A x)}{A} = \gamma x$

Strain $\epsilon = \frac{\sigma}{E} = \frac{\gamma x}{E}$

Extension of the strip $= \epsilon . dx = \frac{\gamma x}{E} dx$

Total elongation $(\delta L) = \int_0^L \frac{\gamma x}{E} dx = \frac{\gamma L^2}{2E}$

If total weight of the bar $W = \gamma A L$, $(\gamma = \frac{W}{AL})$

Total extension $= \frac{WL}{2AE}$

Bar of varying Cross-section:

Total Extension of bar = $\dfrac{\int_0^L \left(\gamma \int_0^x A\,dx\right) dx}{AE}$

Conical bar: $\delta = \dfrac{Wh}{2AE}$

UTM

Gauge length (GL) = $5.65\sqrt{A}$

GL - depends on 'A' (c/s area)

Independent on
1) length of bar
2) shape of c/s
3) Material used
4) Rate of loading.

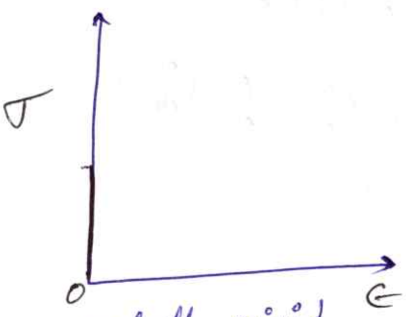
Nominal values ↓ Based on Initial parameters:

- Strain (ϵ) = $\dfrac{\delta(GL)}{GL}$
- stress = $\dfrac{R}{A}$

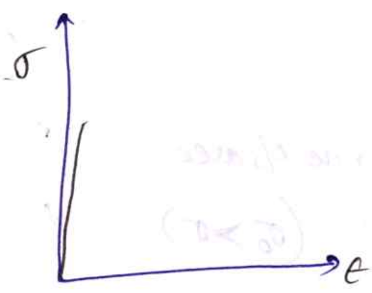
Margin of Safety = MS = $\dfrac{\text{ultimate stress} - \text{working stress}}{\text{working stress}}$ = FOS − 1

Idealised Stress-Strain Curves: used in Designs.

Simplified curves

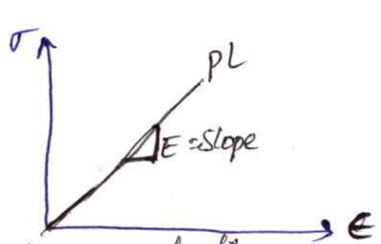


perfectly rigid
(not possible)



Nearly rigid
Eg: Diamond, Glass

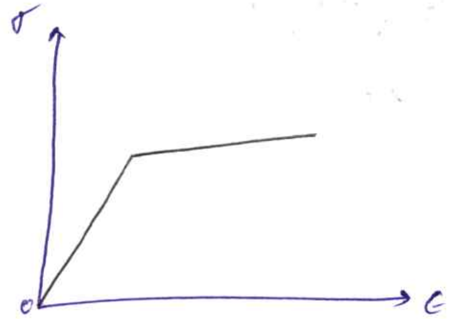
Ideal fluid
(nearly water)
It is incompressible ($\delta V = 0$)
density const.
Surface Tension = 0
viscosity = 0.



linear elastic
Hooke's law valid. working stress method.

Rigid-plastic
(Ideal plastic)

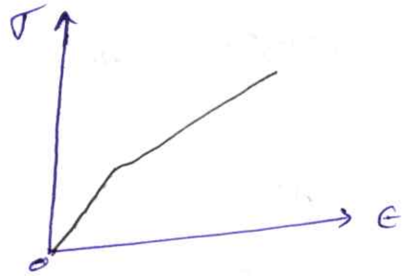
plasticity →



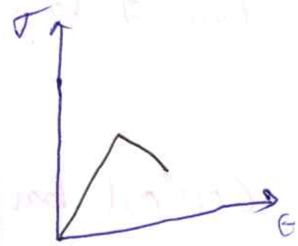
linear elastic - plastic

(or) Elasto - plastic

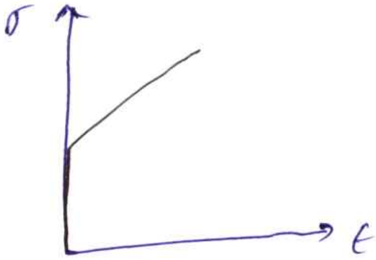
(or) visco - elastic



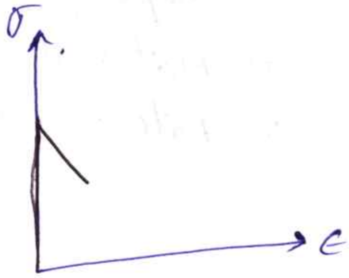
linear elastic - Strain hardening



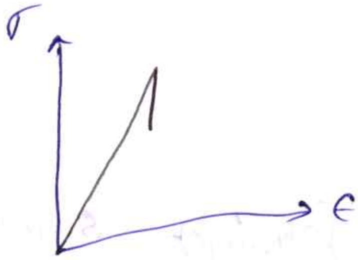
linear elastic - Strain softening



Rigid - Strain hardening



Rigid - Strain softening.



linear elastic - yielding

Nominal stress :

$$\sigma = \frac{R}{A} = \frac{P}{A}$$

A = Intial / nominal c/s Area.

True stress

$$\sigma_0 = \frac{R}{A_0} = \frac{P}{A_0}$$

A_0 = Instantaneous / True c/s aree

$\boxed{\sigma_0 = \sigma(1+\epsilon)}$ for Tensile. ($\sigma_0 > \sigma$)

$\boxed{\sigma_0 = \sigma(1-\epsilon)}$ for Compression ($\sigma_0 < \sigma$)

Intial volume = Instan volume

$$Al = A_0(l + \delta l)$$

$$\frac{A}{A_0} = \frac{l+\delta l}{l} = 1 + \frac{\delta l}{l}$$

$$A = A_0(1+\epsilon)$$

$$\sigma_0 = \frac{P}{A_0} \times \frac{A}{A} = \frac{P}{A}\left(\frac{A}{A_0}\right)$$

$$\sigma_0 = \sigma(1+\epsilon)$$

Material properties

Toughness → Resistance to impact load

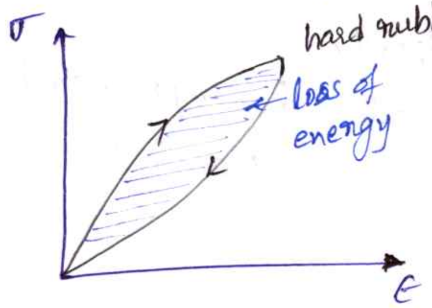
Malleability → Thin sheets / plates are made. Related to Compression.

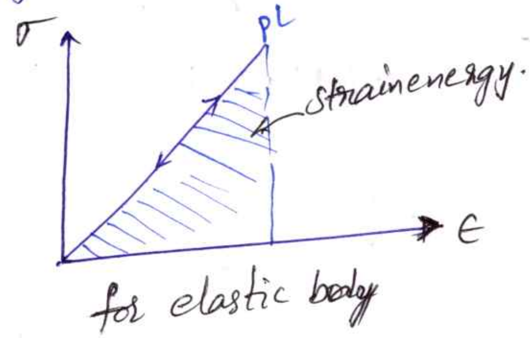
Ductility → Thin wire is made. related to Tension.

* All Ductile are Malleable.
* Strong in Tension
* weak in Shear

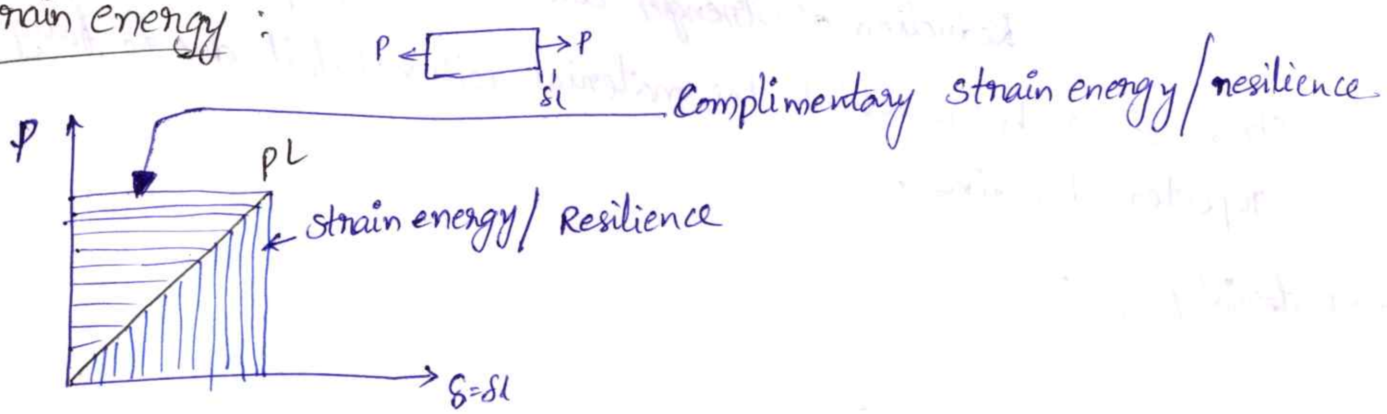
Creep: Deformation @ constant load / stress with prolonged time.
permanent deformation.

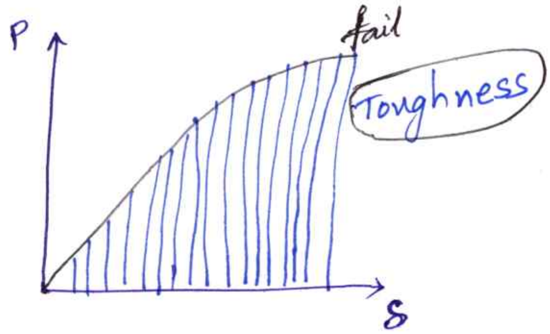
Hysteresis: → loss of energy due to cyclic load.





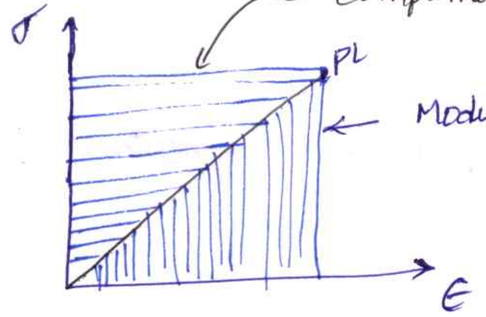
Strain energy:



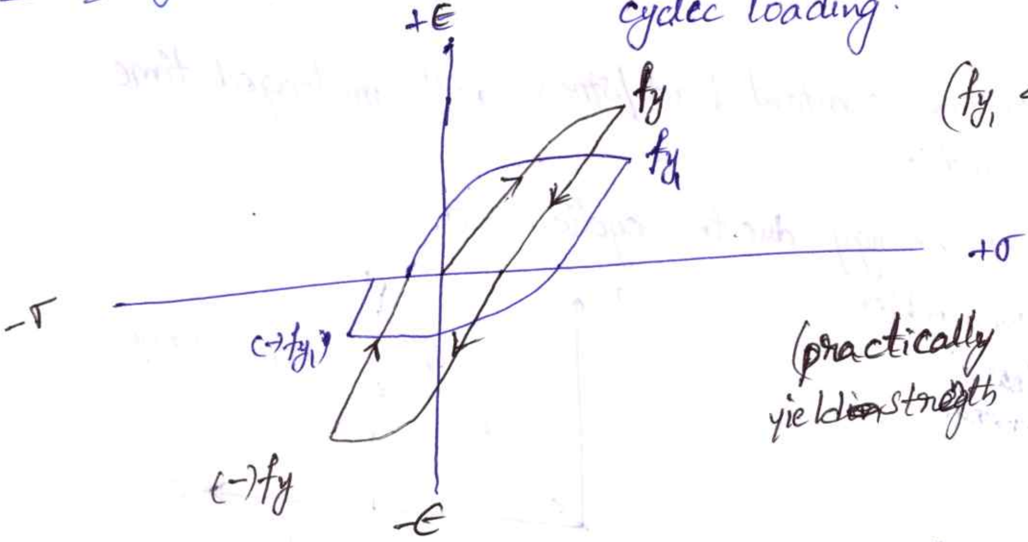


← Resistance to impact load.

* Max strain energy stored (upto failure) Toughness.



Bauschinger's effect : → Reduction of yield strength due to cyclic loading.



($f_{y_1} < f_y$)

(practically yield strength in compression is less than in tension)

Endurance limit / fatigue limit :

Reduction of strength due to cyclic/repeated loading.

↳ Stress @ & below which the material will not fail due to fatigue/repeated loading.

→ (Material property)

Elastic Constants :

① Modulus of elasticity [young's / elastic modulus]

$$E = \frac{\sigma}{\epsilon} \text{ (upto propotionality limit)}$$

(slope of stress-strain curve upto PL)

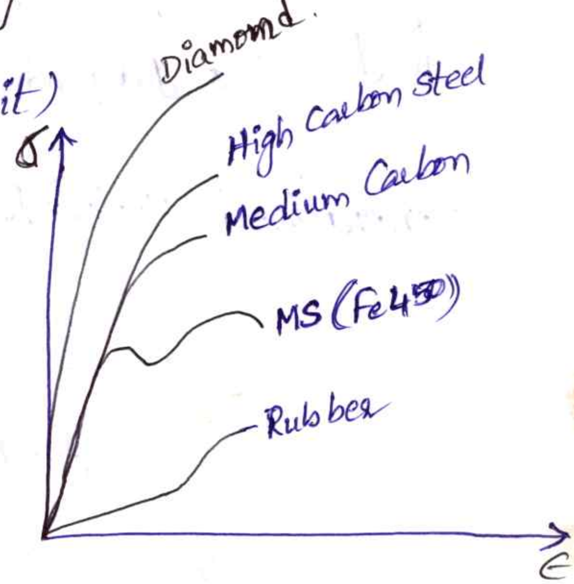
E (Steel) = 200 GPa.

E (Diamond) = 1200 GPa.

higher E = $\frac{\uparrow \sigma}{\downarrow \epsilon}$

(lesser strain), higher stress.

Capable of resisting higher loads.



2) Ridgidity Modulus : G (or) C (or) N = $\frac{\tau}{\phi}$ = $\frac{\text{Shear stress}}{\text{Shear strain}}$

③ Bulk Modulus : (Dialation constant) : which causes change in volume.

$$K = \frac{\sigma}{\epsilon_v}, \quad \epsilon_v = \frac{\delta V}{V},$$

σ → volumetric stress
Normal stress acting over all volume.

④ poisson's Ratio : (μ, ν) : $\mu = \frac{(-)\text{lateral strain}}{\text{linear strain / longitudinal strain}}$

Range : for General −ve to 0.5

μ is −ve for genetic material (or) Nano tubes (or) Auxetic Material.

* for Engg. material → 0 to 0.5

cork, μ = 0
↑
Bottle closures.

$$\mu = \frac{\text{lateral} \to 0}{\text{longitudinal}} = 0$$

μ = 0.5 → for Incompressible material [Eg: Ideal fluid, (water)

μ = 0.25 → Isotropic material. , μ = 0.3 steel (0.33)?
μ = 0.15 Concrete.

$\left.\begin{array}{l} E \\ G \\ K \\ \mu \end{array}\right\} \rightarrow$ for homogeneous + Isotropic

Total Elastic constants = ④

⊛ Independent Elastic const = ② [E & μ]

26/8/20

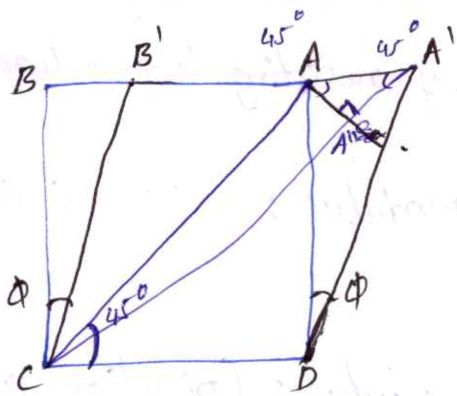
Generalised Hooke's Law:

$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

Relation between E, G, K, μ:



Shear Strain, $\phi = \frac{AA'}{AD}$

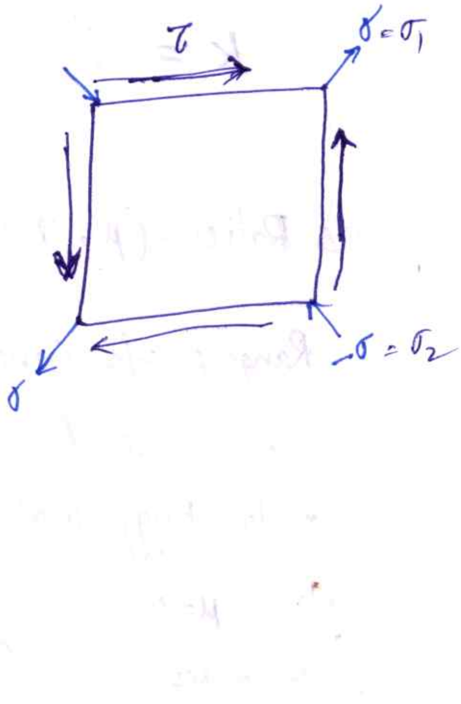
Normal strain, $\epsilon = \frac{\delta l}{l} = \frac{A'A''}{AC}$

$\cos 45° = \frac{A'A''}{AA'} \Rightarrow AA' = \sqrt{2}\, A'A''$

$\sin 45° = \frac{AD}{AC} \Rightarrow AD = \sqrt{2}/AC$

$\phi = \frac{AA'}{AD} = \frac{2 A'A''}{AC} = 2\frac{A'A''}{AC}$

$$\boxed{\phi = 2\epsilon}$$



$$\boxed{\sigma = -\sigma = \tau}$$

Generalised Hooke's law:

$$\epsilon_1 = \epsilon = \frac{\sigma_1}{E} - \frac{\mu \sigma_2}{E}$$

$$= \frac{\sigma}{E} - \frac{\mu(-\sigma)}{E}$$

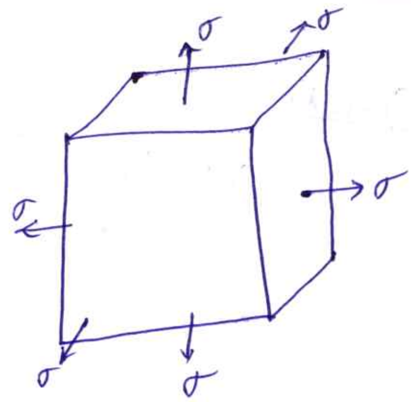
$$\epsilon = \frac{\sigma}{E}(1+\mu)$$

$$\frac{\phi}{2} = \frac{\sigma}{E}(1+\mu)$$

$$E = 2\frac{\sigma}{\phi}(1+\mu)$$

$$\boxed{E = 2\frac{\tau}{\phi}(1+\mu) = 2G(1+\mu)}$$

$$K = \frac{\sigma}{\epsilon_v}$$



$$\epsilon_x = \frac{\sigma}{E} - \mu\frac{\sigma}{E} - \mu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$$

$$\epsilon_y = \frac{\sigma}{E}(1-2\mu)$$

$$\epsilon_z = \frac{\sigma}{E}(1-2\mu)$$

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z = \frac{3\sigma}{E}(1-2\mu)$$

$$E = 3\frac{\sigma}{\epsilon_v}(1-2\mu) = 3K(1-2\mu)$$

$$E = \frac{9KG}{(3K+G)} \qquad \mu = \frac{3K-2G}{6K+2G}$$

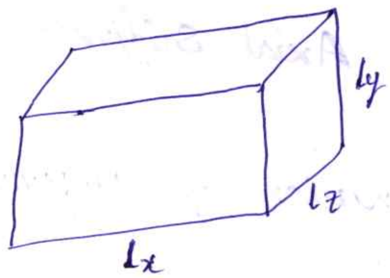
Independent Elastic constants :

*** Homogeneous + Isotropic = 2 [E & μ]

Homogeneous + orthotropic = 9

Homogeneous + Anisotropic = 21

volumetric strains:

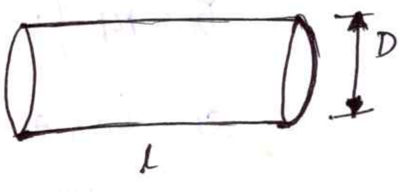


$$V = l_x l_y l_z$$

$$\delta V = \delta l_x l_y l_z + l_x \delta l_y l_z + l_x l_y \delta l_z$$

$$\epsilon_v = \frac{\delta V}{V} = \frac{\delta l_x}{l_x} + \frac{\delta l_y}{l_y} + \frac{\delta l_z}{l_z}$$

Volumetric strain of a cylinder :



$$V = \frac{\pi}{4}D^2 l$$

$$\delta V = \frac{\pi}{4} 2D\,\delta D \cdot l + \frac{\pi}{4}D^2 \delta l$$

$$\epsilon_v = \frac{\delta V}{V} = \frac{2\delta D}{D} + \frac{\delta l}{l} = 2(\epsilon_h) + \epsilon_l$$

$$\epsilon = 2\epsilon_h + \epsilon_l$$

$\epsilon_h \rightarrow$ hoop strain (or) circumferencial strain

$\epsilon_l \rightarrow$ longitudinal strain/axial stre

Sphere :

$$V = \frac{\pi}{6}\pi D^3$$

$$\delta V = \frac{\pi}{6}\pi 3D^2 \delta D$$

$$\epsilon_V = \frac{\delta V}{V} = \frac{3\,\delta D}{D} = 3\epsilon_h$$

$$\boxed{\epsilon_V = 3\epsilon_h}$$

Any member with uniform c/s

Elongation of a prismatic bar Subjected to axial force.

$$\sigma = \frac{R}{A} = \frac{P}{A}, \quad \epsilon = \frac{\delta l}{l}, \quad E = \frac{\sigma}{\epsilon} = \frac{P/A}{\delta l/l}$$

$$E = \frac{Pl}{A\delta l} \Rightarrow \boxed{\delta l = \frac{PL}{AE}}$$

l & A → Intial values.

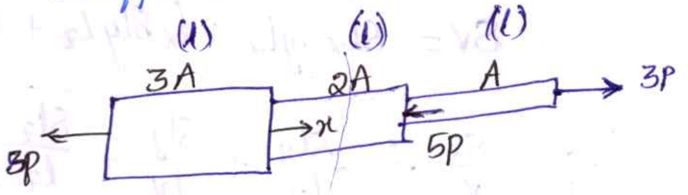
AE → axial rigidity → units = N

Axial stiffness → $k = \frac{P}{\delta l} = \frac{AE}{l}$ [force required to cause unit elongation of the bar

Elongation of Compound bars : (stepped bars)

Assume E = Constant

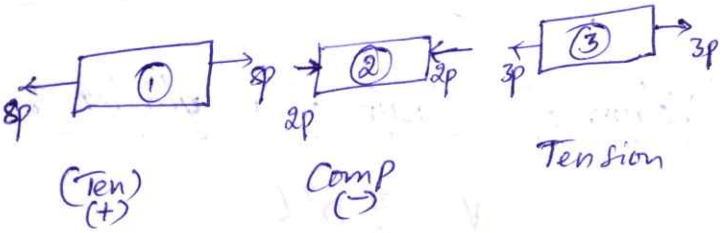
Calculate x = ?



$\Sigma F_x = 0 \quad (\rightarrow +)$

$3P - 5P + x - 8P = 0$

$\boxed{x = 10P}$



$$\delta l = (\delta l)_1 + (\delta l)_2 + (\delta l)_3$$

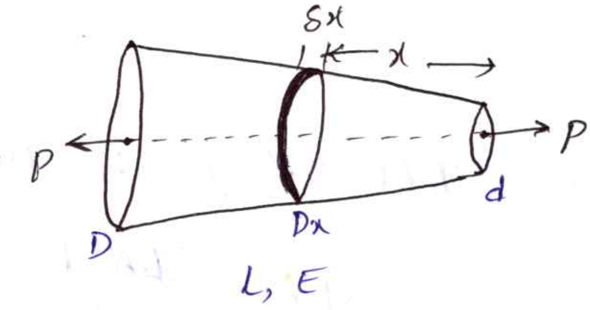
$$= \frac{8PL}{3AE} - \frac{2PL}{2AE} + \frac{3PL}{AE} = \frac{PL}{AE}\left(\frac{8}{3} - 1 + 3\right)$$

$$\boxed{\delta l = \frac{14}{3}\frac{PL}{AE}}$$

Elongation of tapering bars :

Circular tapering :

$$\delta l = \frac{P(dx)}{\frac{\pi}{4}(D_x)^2 E}$$



$l \rightarrow (D-d)$

$x \rightarrow (D_x - d)$

$$l(D_x - d) = x(D-d)$$

~~$D_x l = xD - xd + ld$~~

~~$D_x = x + d(l-x)$~~

$$\boxed{D_x = \frac{x}{l}(D-d) + d} = \boxed{D_x = d + kx}$$

stress is linearly var[illegible]

$$\delta l = \frac{4P\,dx}{\pi(d+kx)^2 E}$$

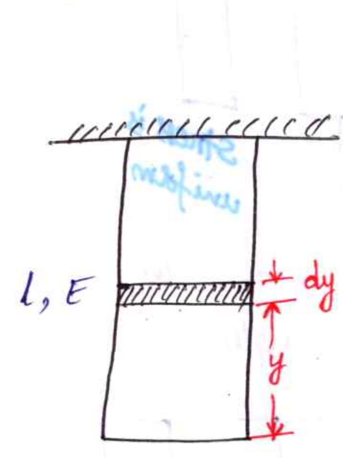
Total change in length of entire body

$$\delta l = \int_0^l \frac{4P\,dx}{\pi(d+kx)^2 E} = \boxed{\frac{4PL}{\pi E D d} = \delta l}$$

Square Tapering : $\boxed{\delta l = \frac{PL}{abE}}$

Tapering bar of uniform thickness : $\delta l = \frac{Pl}{t(B-b)E} \ln\left(\frac{B}{b}\right)$

Deformation due to Self weight :



prismatic bar

weight of bar acting below element.

$$dW = \gamma A y$$

$$\delta l\ (\text{element}) = \frac{(\gamma A y)\,dy}{AE}$$

$$\delta l\ (\text{Total bar}) = \int_0^l \left(\frac{\gamma}{E}\right) y\,dy = \frac{\gamma}{E}\left(\frac{y^2}{2}\right)_0^l = \frac{\gamma L^2}{2E}$$

$$\boxed{\delta l = \frac{\gamma L^2}{2E}}$$

Self weight Deformation → is directly proportional to l^2

→ Independent of area, shape of c/s

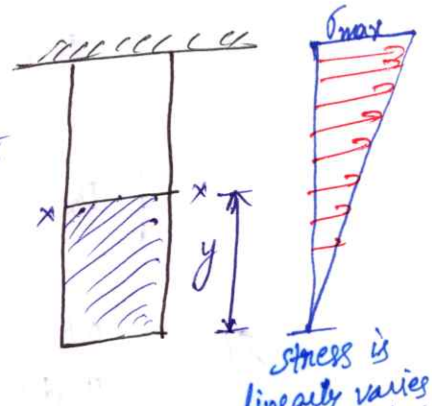
Total Selfweight $W = \gamma l A$

$$\delta l = \frac{(\gamma l A)\, l}{2AE} = \frac{WL}{2AE}$$

Stress due to self weight :

$$\sigma_x = \frac{P}{A} = \frac{\gamma A y}{A}$$

$$\boxed{\sigma_x = \gamma y}$$



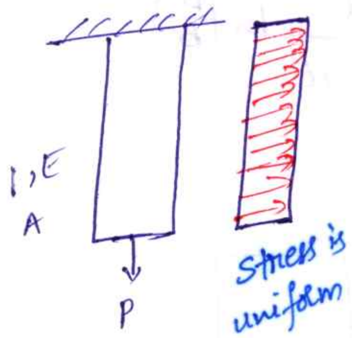
Stress @ free end, i.e. $y=0$; $\sigma_x = 0$

@ fixed end, $y = l$, $\sigma_x = \gamma L = \sigma_{x(max)}$

Stress due to selfweight → is independent of area of c/s, shape of c/s.

→ directly proportional to L

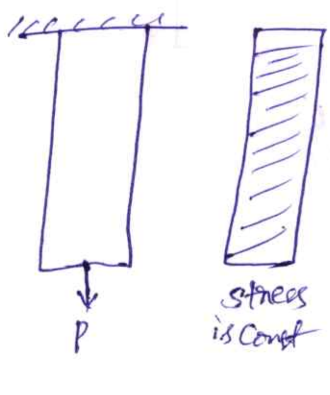
Stress due to external load on a wt less bar :



$\sigma = \frac{P}{A}$ → Independent of length, shape of c/s

→ ~~depend~~ Inversely proportional to A

Bar of uniform strength : Stess developed is constant lengths at the length.

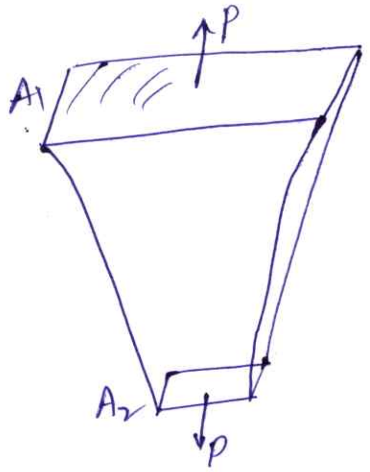


$\sigma = \frac{P}{A}$

Bar of uniform strength

(impossible in practice)

Eg :



$$\frac{A_1}{A_2} = e^{\left(\frac{\gamma l}{\sigma}\right)}$$

$$\ln\left(\frac{A_1}{A_2}\right) = \frac{\gamma l}{\sigma}$$

28/8/20

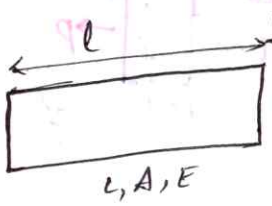
Thermal Stresses [Temperature Stresses]

→ secondary stress / Indirect stress

α = Coeff. of thermal/Linear Expansion (units : $/^\circ C$)
- → Material property
- → Strain developed in the material for unit change in temp.

$\alpha = 12 \times 10^{-6} /^\circ C$ → for Concrete & Steel.

Expansion of a prismatic bar due to Change in Temp.



free expansion of length of bar

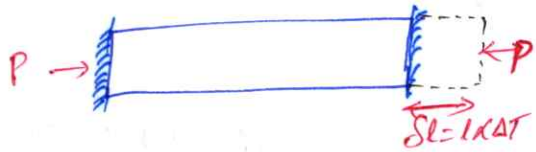
$$\delta l = \alpha (\Delta T) l$$

strain in bar due to temp = $\epsilon = \frac{\delta l}{l}$

free expansion of width of bar = $\delta b = \alpha \Delta T\, b$

thickness = $\delta t = \alpha \Delta T\, t$

prismatic bar expansion of length is prevented :



free expansion = Expansion prevented.

$$l \alpha \Delta T = \frac{Pl}{AE}$$

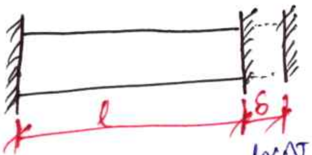
$$\sigma = \frac{P}{A} = (\alpha \Delta T) E = \epsilon E$$

$$\boxed{\sigma = \alpha \Delta T E}$$ – along length direction.

↑ temp	→ Compression
↓ temp	→ Tension

prismatic bar with yielding Support : [Support moves freely by a little. (δ)]

①



Case ① → free exp. $\delta l = l \alpha \Delta T$

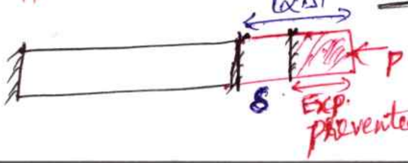
$l \alpha \Delta T \le \delta$

σ = Zero.

Case② free exp. $\delta = l \alpha \Delta T$

$l \alpha \Delta T > \delta$

②



Exp. prevented. ⇒ $\frac{Pl}{AE} = (l \alpha \Delta T - \delta)$

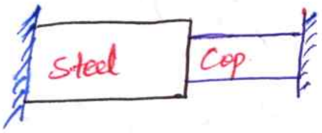
$$\sigma = \alpha \Delta T E - \frac{\delta E}{l}$$

Composite Bars : made of different material

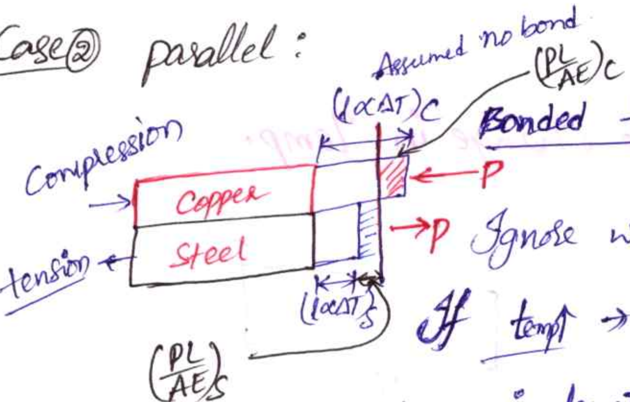
Case ① Series :

Sum of change in length = 0

$$(\delta l)_c + (\delta l)_s = 0$$



Case ② parallel :



Bonded together $\alpha_c > \alpha_s$

Ignore warping & bending.

If temp↑ → ΔT

Net change in length of steel = net change in length of copper.

(Compatability eq.) $(\delta l)_s = (\delta l)_c$

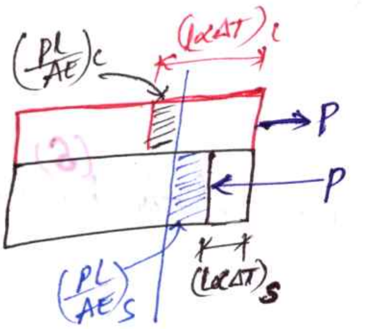
$$(l\alpha\Delta T)_s + \left(\frac{PL}{AE}\right)_s = (l\alpha\Delta T)_c - \left(\frac{PL}{AE}\right)_c$$

$$\boxed{P_s = P_c = P}$$

Stresses $\sigma_s = \frac{P_s}{A_s} = \frac{P}{A_s}$, $\sigma_c = \frac{P_c}{A_c} = \frac{P}{A_c}$

parallel (Bonded)

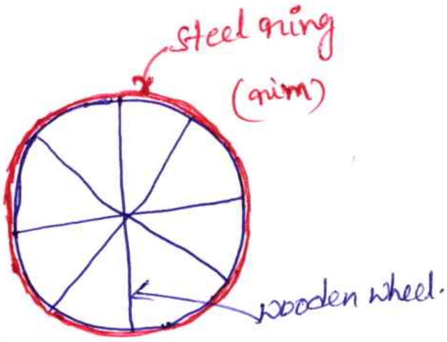
$\downarrow t = \Delta T$ (If there is no bond → no Stress.



$$(\delta l)_c = (\delta l)_s$$

$$(l\alpha\Delta T)_c - \left(\frac{PL}{AE}\right)_c = (l\alpha\Delta T)_s + \left(\frac{PL}{AE}\right)_s$$

Hoop Stress / Circumferencial Stress (due to temp)



Dia. of rigid wooden wheel = D

Intial diameter of steel ring = d [$d < D$]

final dia of steel ring after fixing = D

Hoop Strain in steel ring, $\epsilon_h = \frac{\pi D - \pi d}{\pi d} = \left(\frac{D-d}{d}\right)$

Hoop Stress, $\sigma_h = \epsilon_h E = \left(\frac{D-d}{d}\right) E$ [Tension] in steel ring

Compression in wooden ring.

Min ↑ in temp. of steel ring to fit it over wooden wheel.

$\epsilon_h = \epsilon_t$?

$$\boxed{\left(\frac{D-d}{d}\right) = (\alpha \Delta T)}$$

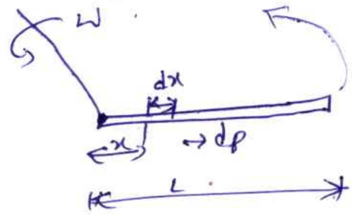
Brinell's Hardness number [BHN] = $\dfrac{P}{\frac{\pi D}{2}\left[D - \sqrt{D^2 - d^2}\right]}$

Where D = Dia of the ball, mm

d = Avg diameter value of the indentation, mm,

P = Test force in N.

(Q) uniform slender rod of length 'L' and c/s area A is rotating in horizontal plane about a vertical axis through one end. The unit mass of rod is 'ρ'. it is rotating with const. angular velocity ω. Total elongation.



$\delta = \frac{PL}{AE}$

$d\delta = \frac{dP \cdot x}{AE}$

$dp = dM \cdot \omega^2 x = (\rho A dx)\, \omega^2 x^2$

$d\delta = \frac{\rho A \omega^2 x^2 dx}{AE}$

$\delta = \int_0^L \frac{\rho \omega^2 x^2}{E} dx = \frac{\rho \omega^2 L^3}{3E}$??

$F = mr\omega^2$

CIVIL ENGINEERING

Strength of Materials

UNIT-2

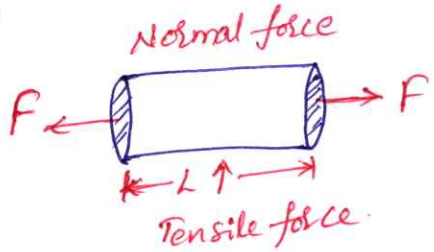
30/8/20

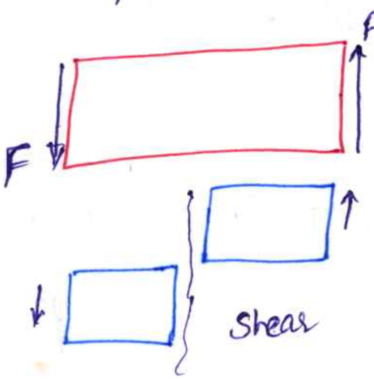
Shear force & Bending Moment :

Contents :

What is Shear Force :

Force acts parallel to the Surface.

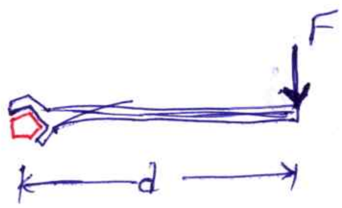




What is Bending Moment :

Moment → It gives turning effect of a force

- Rotate
- Twist
- Bend.

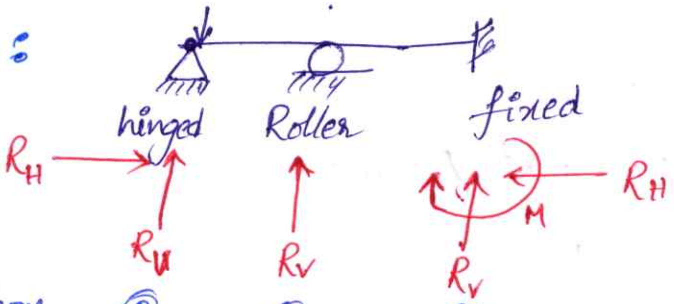


→ It is the moment of force which tends to Bend the object.

Beam : To carry Transverse load :

Beams subject to lateral load & s/o flexural load.

Types of Supports :



No. of Reactions ② ① ③

Types of Beams :—

Statically determined

((statics)

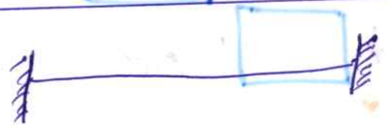
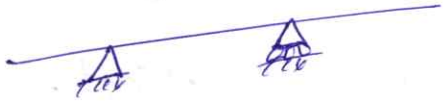
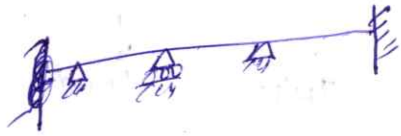
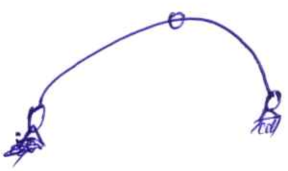
equation of equilibrium

$\Sigma F_x = 0$, $\Sigma F_y = 0$ $\Sigma F_z = 0$

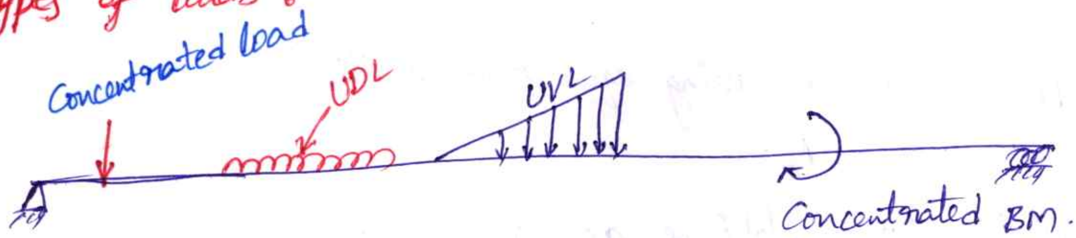
$\Sigma M_z = 0$

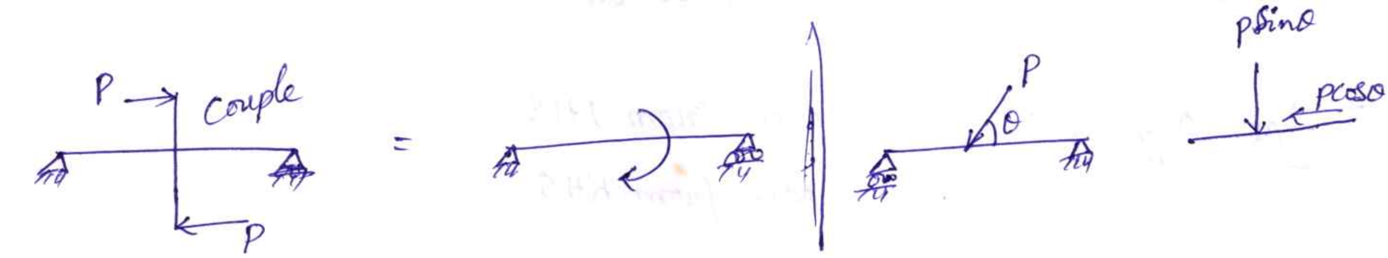
Statically Indetermined

we cannot determine unknown by using statics alone only. for this extra eq. are req. " compatable equation.

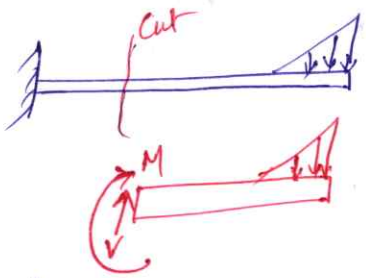
S.D. (eq. equilibrium eq)	S.ID (Eq + Comp.)
Cantilever 	
	
	
	

Types of loads :





Concept of SF & BM related to Beams :



Internal Resistance developed at any c/s of the beam SF & BM

why-? ① V → τ > τ_allow

② if M → σ > σ_allow

for a design

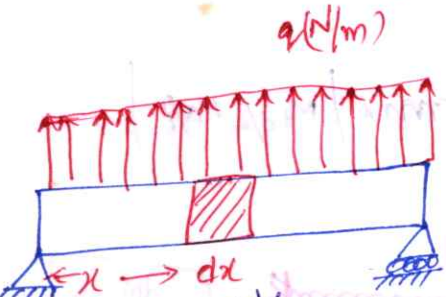
V → τ }
M → σ } ≤ allowable.

Then only beam is safe.

Shearforce
↳ V → dominates in Shorter beam.

M → dominates in longer beam.

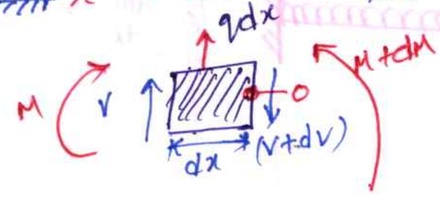
Relation b/w intensity of load, Shear force and Bending Moment.



① q - applied load

② V → SF

③ M → BM.



Equilibrium eq.

① $\Sigma F_y = 0$

$V + q\,dx = V + dV$

$$\frac{dV}{dx} = q \quad \text{—— ①}$$

Moment about O.

$\Sigma M = 0 \Rightarrow \quad V dx + (q\,dx)\frac{dx}{2} + M = M + dM$

$$V = \frac{dM}{dx} \quad \text{—— ②}$$

Procedure to draw SFD and BMD.

① findout support reactions using equilibrium equations.

$$\Sigma F_x = \Sigma F_y = \Sigma M_z = 0$$

② Ignore Axial force / Horizontal force acting on beam.

③ V at any section = Σ force from LHS

= − Σ force from RHS

④ M at any section = Σ Moment of force from LHS (or) RHS.

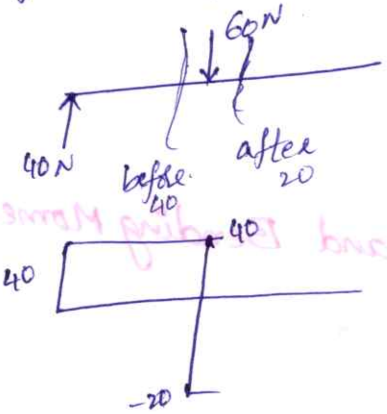
↑ +ve ↓ −ve (−ve) +ve

Rules to follow:

① 

Always find V, M from free end

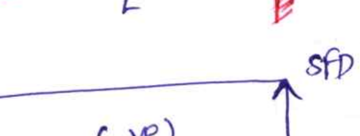
② find SF from LHS —

③ 

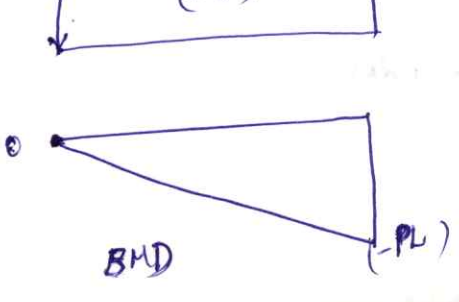
If you want to find load at concentrated load

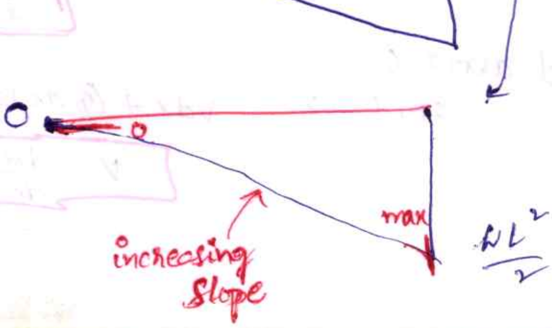
for Design $V_c = \text{Max} \left| V_{before}, V_{after} \right|$

= 40

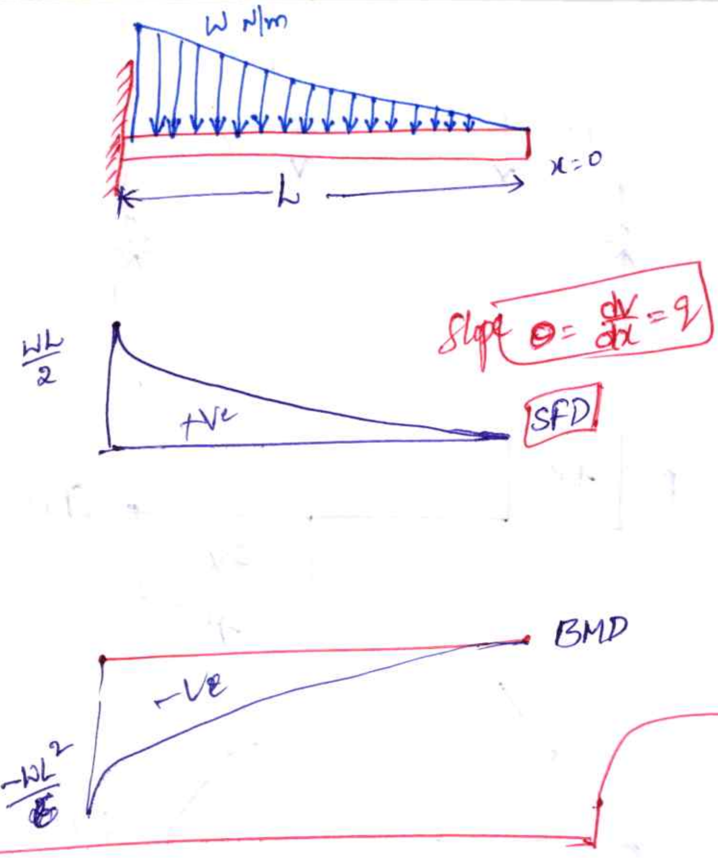
④ 

for Design $M_c = \max \left| M_{before}, M_{after} \right|$





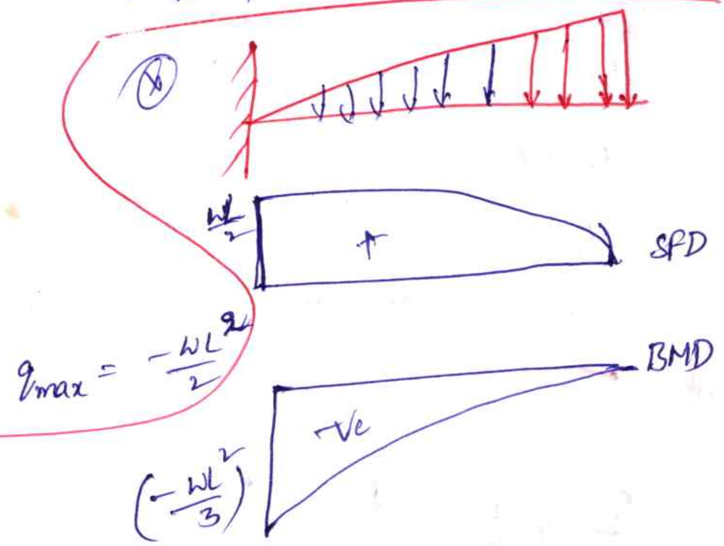
$$\theta = \frac{dM}{dx} = V$$

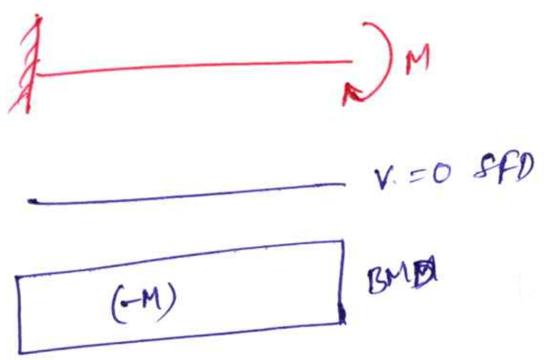


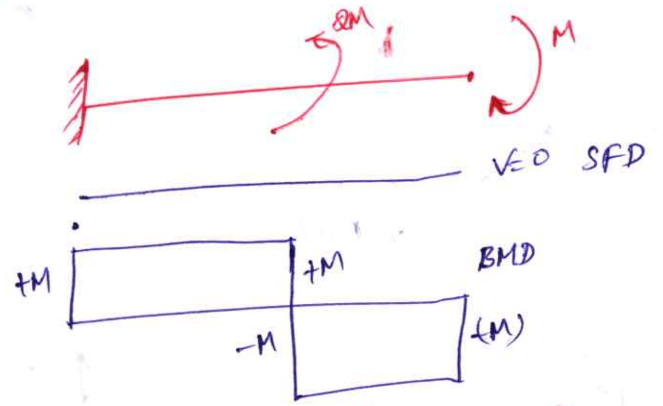
at $x = L$.

$$V = -\left[-\frac{1}{2} w \times L\right] = \frac{wL}{2}$$

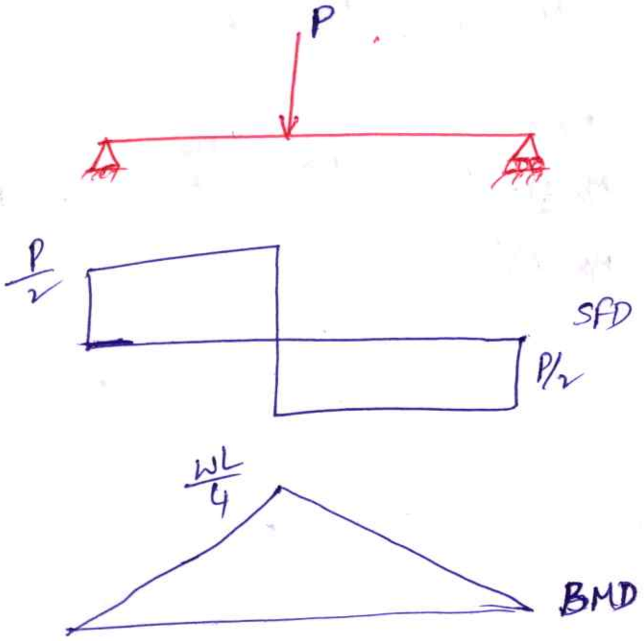
at $x=0$, $V=0$.

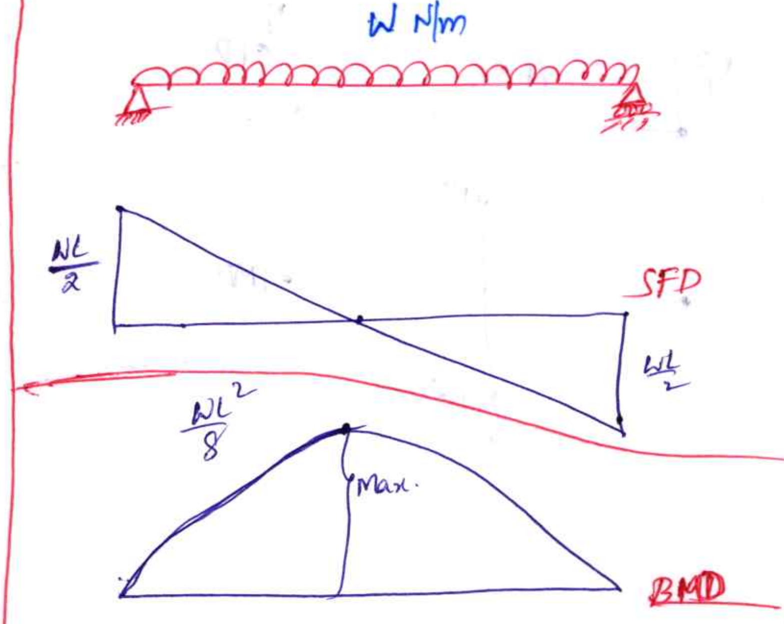






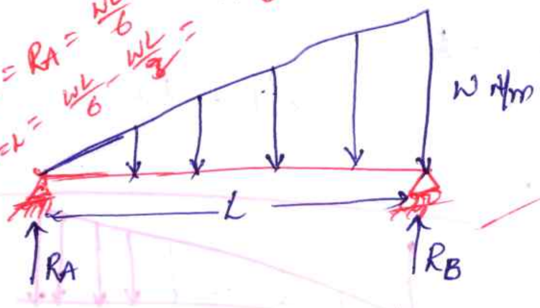
Simply Supported Beams:





$V_{x=0} = R_A = \frac{WL}{6}$

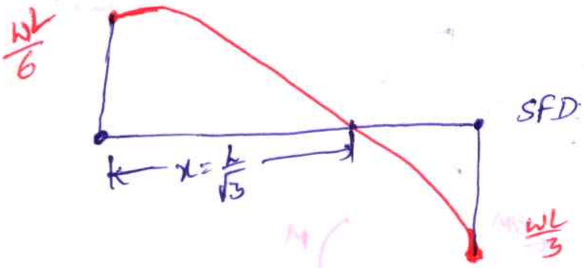
$V_{x=L} = \frac{WL}{6} - \frac{WL}{3} = -\frac{WL}{3}$

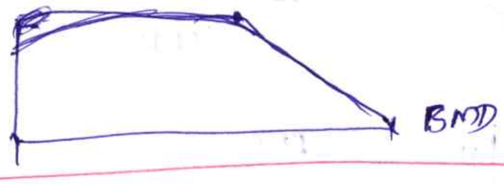


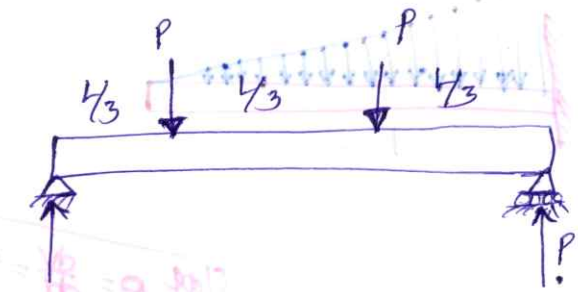
$R_A + R_B = \frac{1}{2} WL$

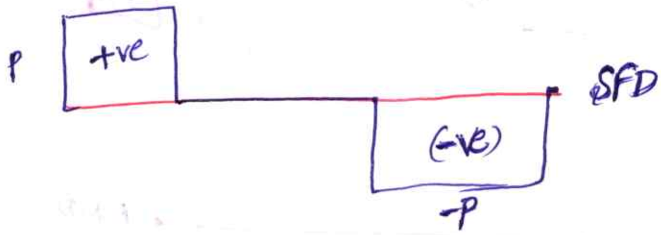
$R_A \times L = \frac{1}{2} WL \times \frac{L}{3}$

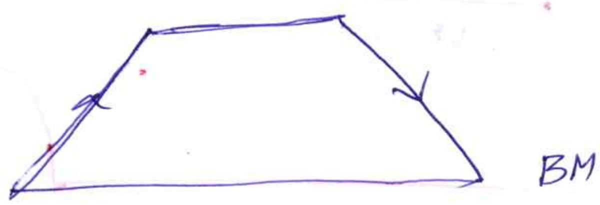
$R_A = \frac{WL}{6}$, $R_B = \frac{WL}{3}$



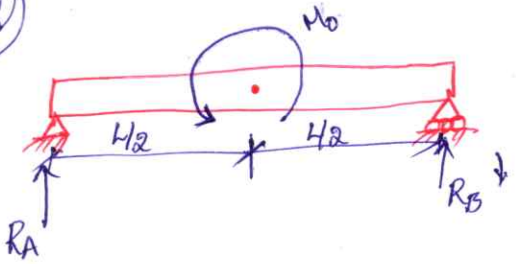


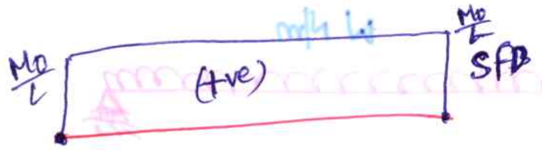


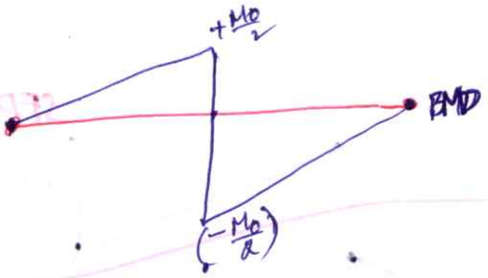




(Q)







$\uparrow \; = \; \downarrow$

$R_A + R_B = 0$

$\Sigma M_B = 0 \Rightarrow$ ↻ = ↺

$R_A \times L = M_0$

$R_A = \frac{M_0}{L}$, $R_B = \frac{-M_0}{L}$

$M_{x=0} = 0$

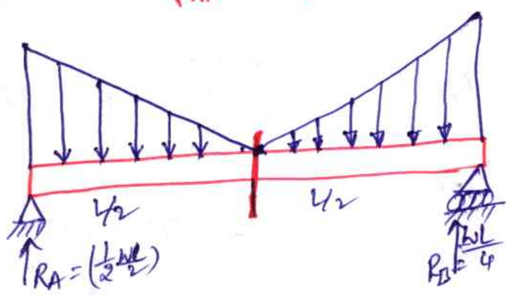
$M_{(x=\frac{L}{2})} = \frac{M_0}{L} \times \frac{L}{2} - M_0 = -\frac{M_0}{2}$

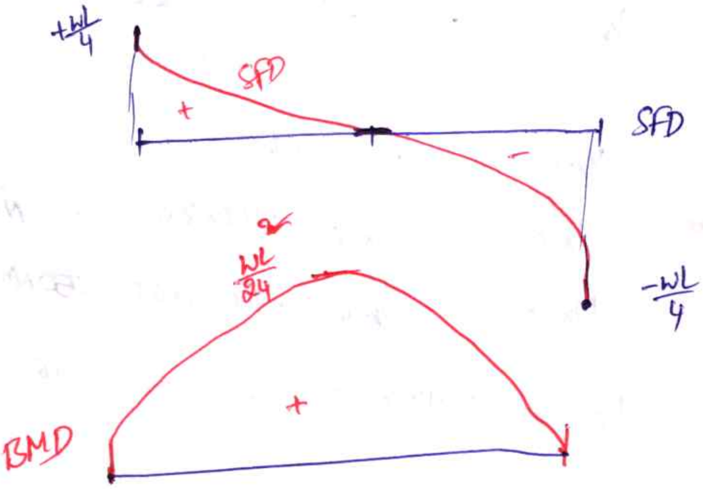
$M_{(x=L)} =$

$V_{(x=0)} = \frac{WL}{4}$

$V_{(x=\frac{L}{2})} = \frac{WL}{4} - \frac{WL}{4} = 0$

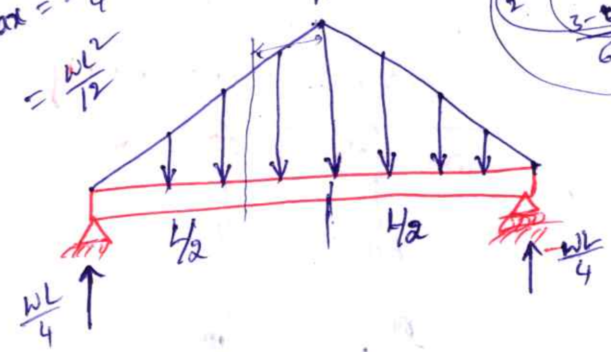
$M_{max} = \frac{WL}{4}\cdot\frac{L}{2} - \frac{WL}{4}\cdot\frac{L}{3} = \frac{WL^2}{8} - \frac{WL^2}{12} = \frac{WL^2}{24}$

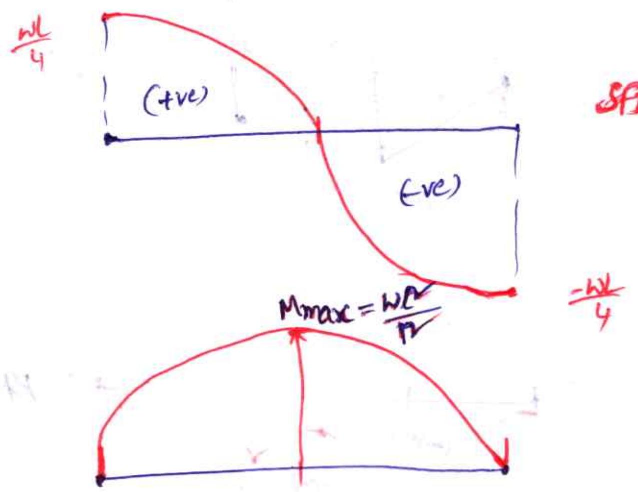




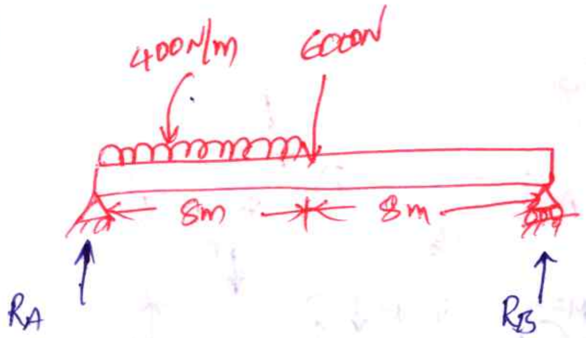
$M_{max} = \frac{WL}{4}\times\frac{L}{2} - \frac{WL}{4}\times\frac{L}{6}$

$= \frac{WL^2}{12}$





②



$\Rightarrow R_A + R_B = 400\times 8 + 6000$

$= 9200\ N$

$\Sigma M_B = 0$

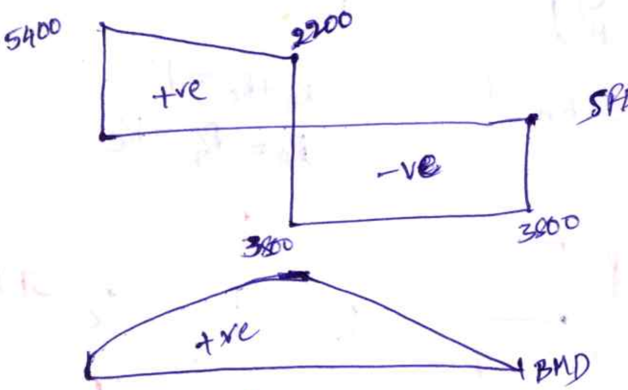
$R_A \times 16 = 400\times 8\times 12 + 6000\times 8$

$R_A = 5400\ N$

$R_B = 3800\ N$

$V_{(x=8)} = 5400 - 3200 - 6000 = -3800$

$= 2200 - 6000$



$M_{(x=8)} = 5400\times 8 - 400\times 8\times 4$

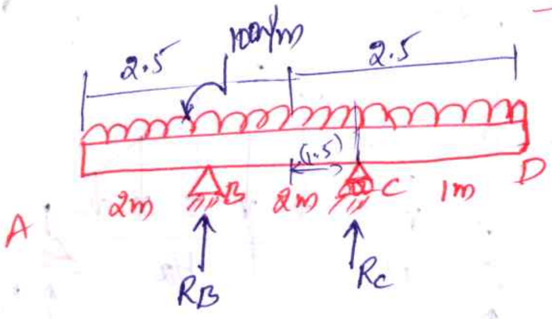
$= 30400\ N\text{-}m.$

* note: ① whenever point load is there then vertical step is there in SFD.

② whenever point Moment is there then vertical step is there in BMD.

(8)

Overhanging Beam



$R_B + R_c = 100 \times 5 = 500$

$\Sigma M_c = 0$

$R_B \times 2 = 500 \times 1.5$

$R_B = 375, \; R_c = 125$

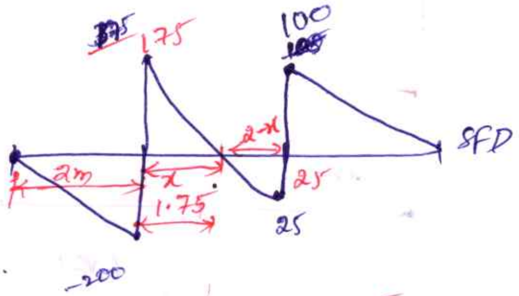
$V_{x=0} = 0$

$V_{(x=2)} = -200 + 375 = 175$

$V_{(x=4)} = (-400 + 375) + 125 = 100$

$= -25 + 125 = 100$

$V_{x=5} = -500 + 375 + 125 = 0$

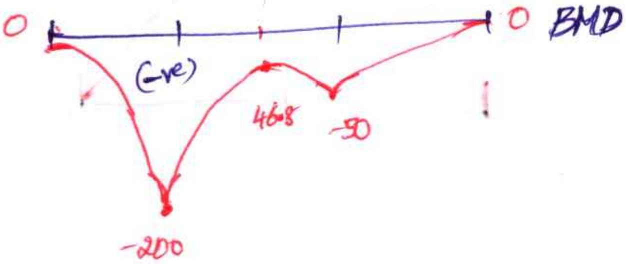


$\frac{175}{x} = \frac{25}{2-x} \Rightarrow x = 1.75m$

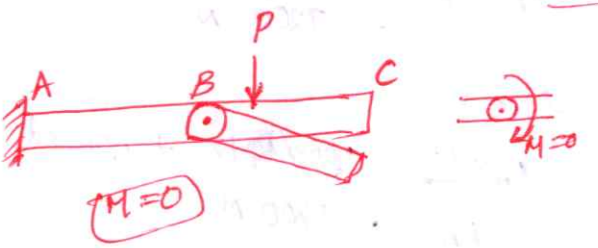
$M_{x=0} = 0, \quad M_{(x=2)} = -100 \times 2 \times 1 = -200 \text{ Nm}$

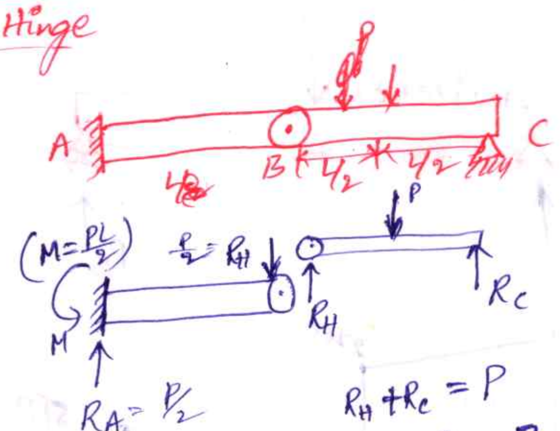
$M_{(x=5)} = 0, \quad M_{(x=4)} = -100 \times 1 \times 0.5 = -50 \text{ Nm}$

$M_{(x=3.75)} = -100 \times 3.75 \times \frac{3.75}{2} + 375 \times 2 = -46.8$



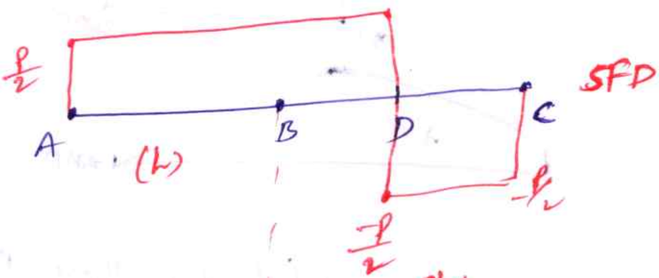
Internal Hinge

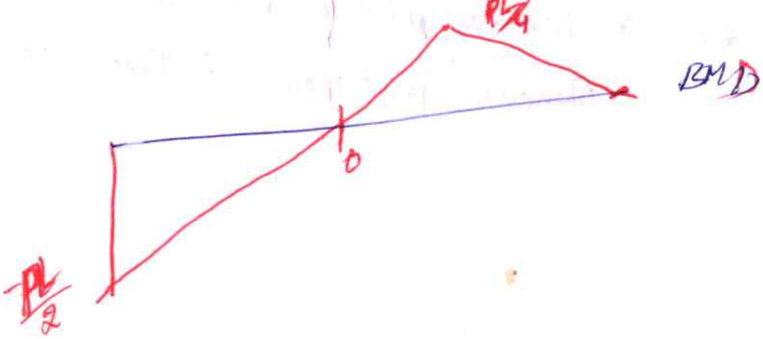




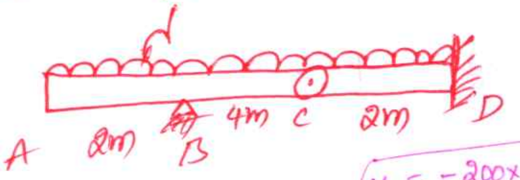
$R_H + R_c = P$

$R_H = P/2 = R_C$





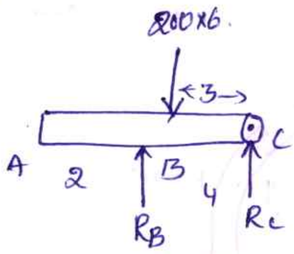
Internal Hinge



$V_C = -200\times6 + 900$

$V_D = -[700]$

$V_D = -200\times8 + 900 = -700$

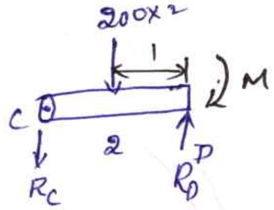


$R_B + R_C = 1200$

$R_B \times 4 = 200\times6\times3$

$R_B = 900$

$R_C = 300$

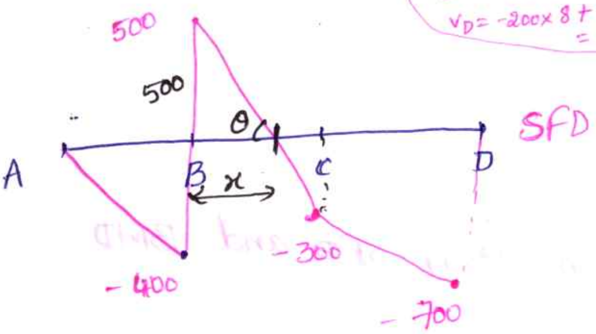


$R_C\times2 + 400\times1 = M$

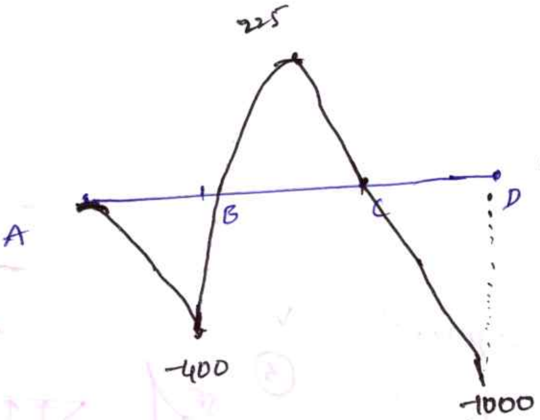
$M = 300\times2 + 400$

$M = 1000\ N\text{-}m$

$R_D = 300 + 400 = 700$



$\frac{500}{x} = \frac{300}{4-x} \Rightarrow x = 2.5\,m.$



BMD

$M_{(x=0)} = 0,$

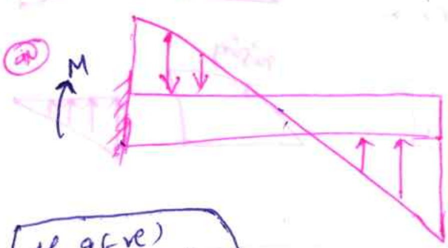
$M_{(x=2)} = -200\times2\times1 = -400\ N\text{-}m.$

$M_{(x=4.5)} = -200\times4.5\times\frac{4.5}{2} + 900\times2.5$

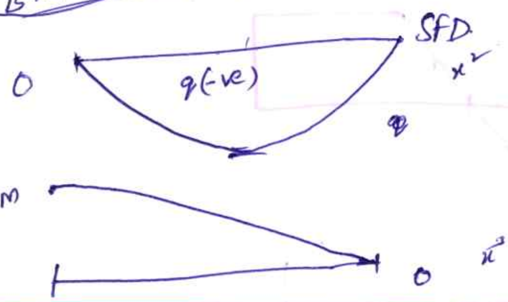
$= 225\ N\text{-}m.$

$M_{x=6} = 0 = -200\times6\times\frac{6}{2} + 900\times4 = 0$

$M_{x=8} = -1000$



If q(-ve)
BMD → falling



Q) Load intensity on a beam segment is given as $\sqrt{x}$ N/m. Bending moment would be a function of

$q = x^{1/2}$

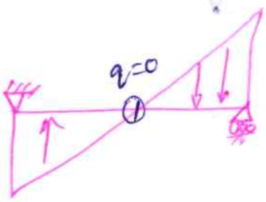
$\frac{dV}{dx} = x^n \qquad q = x^n$

$\int dV = \int x^n dx \Rightarrow V = \frac{x^{n+1}}{n+1}$

$\frac{dM}{dx} = kx^{n+1} \Rightarrow M = \frac{kx^{n+2}}{n+2}$

$q \propto \sqrt{x}$

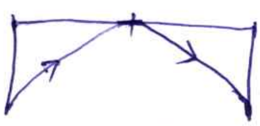
$M \propto x^{\frac{1}{2}+2} \Rightarrow M \propto M^{5/2}$



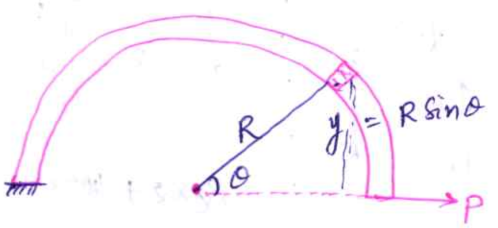
$\frac{dV}{dx} = q = 0 \Rightarrow$ slope of SF — flat slope

(↑) q is +ve → SF — ↗

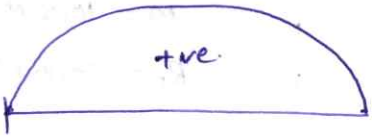
(↓) q is −ve → SF → ↘



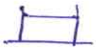
Ⓠ Choose the correct BMD for the beam

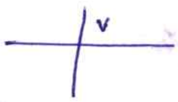


$m = P \times y = PR \sin\theta$

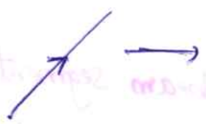


⇒ Procedure to draw loading diagram from SFD and BMD.

① Identify segment, $v = c$ → load = 0 

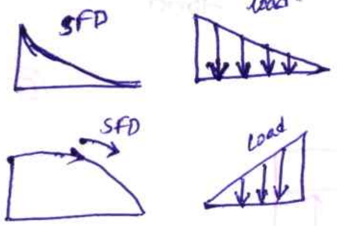
② Sudden jump of SF  ⇒ point load.

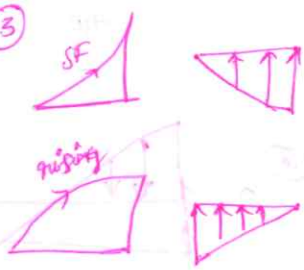
left hand side If jump ↑ → load ↑
jump ↓ → load ↓

③  ⟶ upward UDL

 UDL

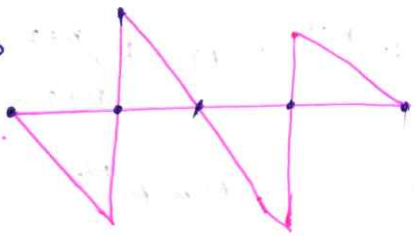
④ If SFD → parabola → UVL

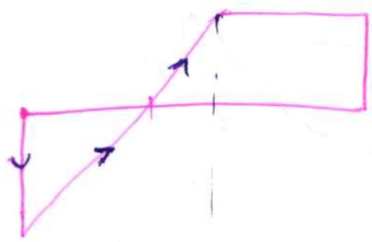


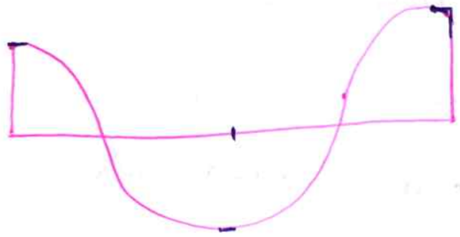


⑤ 

SFD →



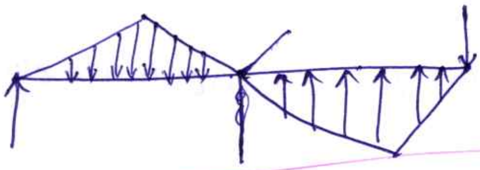




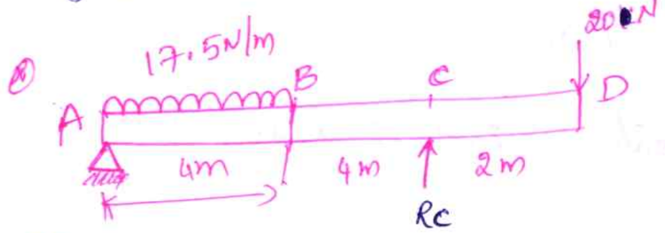
Point of contraflexure:

POC → ⊛ curvature changing from saging to haging [hog → sag]

⊛ The point at which BM changing +ve to -ve (or) -ve to +ve.



Significance: ① It is useful to where to Reinforce



$\Sigma M_A = 0$

$20 \times 10 + 17.5 \times 4 \times 2 = R_C \times 8$

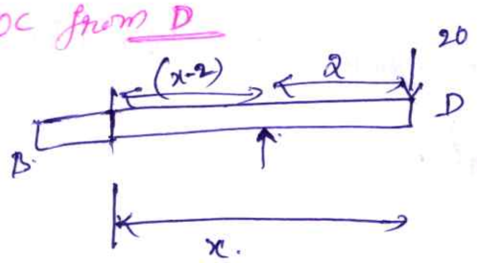
$R_C = 42.5 N$

find POC from D

$M_D = 0$,

$M_C = -20 \times 2 = -40$

$M_B = -20 \times 2 + 42.5 \times 4 = +90$



85

$M_x = 0 \Rightarrow -20 \times x + 42.5 \times (x-2) = 0$

$x = 3.78 m$

CIVIL ENGINEERING

Strength of Materials

UNIT-3

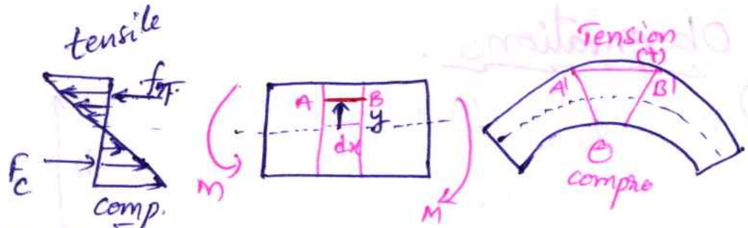
Theory of Simple Bending

Assumptions

1. Beam is homogeneous & Isotropic and obey's hook's law.
2. C/s area symmetric wrt any axis. plane of symmetry.
3. Applied load lies on plane of symmetry.
4. Radius of curvature is large compared to C/s dimensions.
5. plane sections remains plane before and after bending. [V=0]

pure bending $V=0$, $\frac{dm}{dx} = V = 0 \Rightarrow M = C$.

Derivation of flexure formula



$\sigma \propto \epsilon$

$\sigma = E \times \frac{y}{R}$

at any c/s $\frac{E}{R}$ = constant $F_{net} = 0$

$\sigma \propto y \Rightarrow$ $\boxed{\sigma = cy}$

$dF = (\sigma_{(y)}) dA$

$dF = c \cdot y \cdot dA$

$F_{net} = \int dF = \int c \cdot y \cdot dA$

$F_{net} = c \cdot \int y dA = 0$.

$\epsilon_{AB} = \frac{A'B' - AB}{AB}$.

$A'B' = (R+y)d\theta$, $AB = dx = R \times d\theta$

$\epsilon_y = \frac{(R+y)d\theta - Rd\theta}{Rd\theta} = \frac{y}{R}$

* Moment of any area about an axis passing through centroid also zero.

Compatability eq. ① $\epsilon_{(y)} = \frac{y}{R}$ ② $\sigma_{(y)} = E \cdot \frac{y}{R} \Rightarrow \boxed{\frac{\sigma}{y} = \frac{E}{R}}$ — ②

③ $F_{net} = 0$, $\Rightarrow$ N.A $\rightarrow$ centroid. ④ M & σ.

$dF = \sigma \cdot dA = \frac{E}{R} y \, dA$, $dM = (ky \, dA) y$.

$M = \int k y^2 dA$

$M = k \times I$

$M = \frac{E}{R} \cdot I \Rightarrow \boxed{\frac{M}{I} = \frac{E}{R}}$ — ③

$$\boxed{\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}}$$

$$\frac{E}{R} = \frac{M}{I} = \frac{\sigma_b(f)}{y}$$

E = C (young's modulus of beams)

R → Radius of curvature of NA

$K = \frac{1}{R}$ = curvature

M = Bending moment at c/s

I = Area Moment of Inertia of c/s

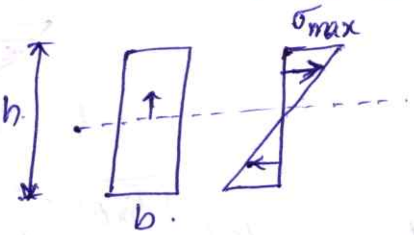
σ_b = Normal stress / flexural stress due to bending

y = distance from N.A.

Observations :

(I) ① $\sigma_b = \frac{M}{I} \cdot y = C \cdot y$

$\sigma \propto y$



$y \to 0, \sigma = 0$

$y = \pm \frac{h}{2}, \sigma = \sigma_{max}$, $\frac{\sigma}{\sigma_{max}} = \frac{y}{y_{max}}$ ⇒ $\sigma_{max} = \frac{M}{I} y_{max} = \frac{M \frac{h}{2}}{\frac{bh^3}{12}} = \frac{6M}{bh^2}$

② for circular c/s. $\sigma_{max} = \frac{M \times \frac{d}{2}}{\frac{\pi d^4}{64}} = \frac{32M}{\pi d^3}$

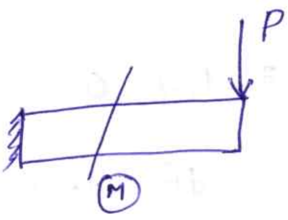
(II) ⇒ $\epsilon = \frac{\sigma}{E} = \frac{y}{R}$ ⇒ $\boxed{\epsilon \propto y}$

(3) EI = flexural Rigidity

Significance : $\frac{E}{R} = \frac{M}{I}$ ⇒ $K = \frac{1}{R} = \frac{M}{EI}$ Resistance of a beam to bend.

EI ↑ → K ↓ → difficult to bend.

④ **Strength of a section :** $\sigma = \frac{My}{I}$

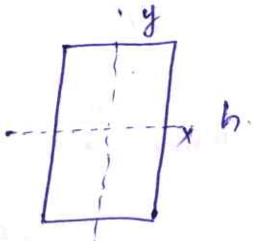


$M_{max} = \sigma_{all} \times \frac{I}{y_{max}}$

Bending strength → $M_{max} = \sigma_{all} \times Z$

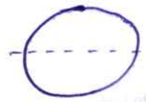
Significance of 'Z' : used to compare various beam c/s made with same material.

Section modulus for various beam c/s : $Z = \frac{I}{y_{max}}$

① 

$$Z_1 = \frac{\frac{bh^3}{12}}{\frac{h}{2}} = \frac{bh^2}{6}$$

$$Z_2 = \frac{hb^2}{6}$$

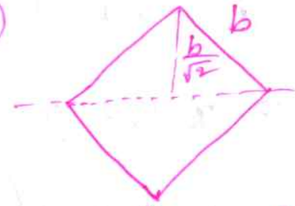
② 

$$Z = \frac{\frac{\pi d^4}{64}}{\frac{d}{2}}$$

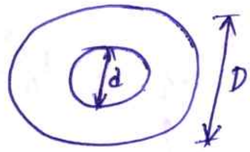
$$Z = \frac{\pi d^3}{32}$$

③ 

$$Z = \frac{\frac{bh^3}{36}}{\frac{2h}{3}} = \frac{bh^2}{24}$$

④ 

$$Z = \frac{\frac{b^4}{12}}{\frac{b}{\sqrt{2}}} = \frac{b^3}{6\sqrt{2}}$$

⑤ 

$$Z = \frac{\frac{\pi}{64}(D^4 - d^4)}{\frac{D}{2}}$$

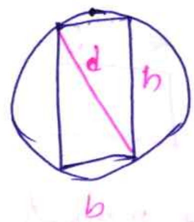
ⓠ A square beam laid flat is then rotated in such a way that one of its diagonal becomes horizontal. How is its moment capacity affect

$M \propto Z$

$$\frac{M_{\square}}{M_{\diamond}} = \frac{Z_{\square}}{Z_{\diamond}} = \frac{\frac{a^3}{6}}{\frac{a^3}{6\sqrt{2}}} = \sqrt{2} = 1.414 \Rightarrow \boxed{\frac{M_{\square}}{M_{\diamond}} = 1.414}$$

$$\%\text{ affected } z = \frac{Z_{\diamond} - Z_{\square}}{Z_{\square}} = -29.27\%$$

ⓠ find the dimensions of strongest rectangle to be cut from a circular log of wood of dia 'd'.



$$Z = \frac{bh^2}{6} \; ; \qquad d^2 = b^2 + h^2$$

$$h^2 = d^2 - b^2$$

$$Z = \frac{b(d^2 - b^2)}{6} = \frac{1}{6}(bd^2 - b^3)$$

$$\frac{dZ}{db} = 0 \Rightarrow \quad d^2 - 3b^2 = 0 \Rightarrow \boxed{d = \sqrt{3}\, b} \Rightarrow \boxed{\frac{d}{b} = \sqrt{3}}$$

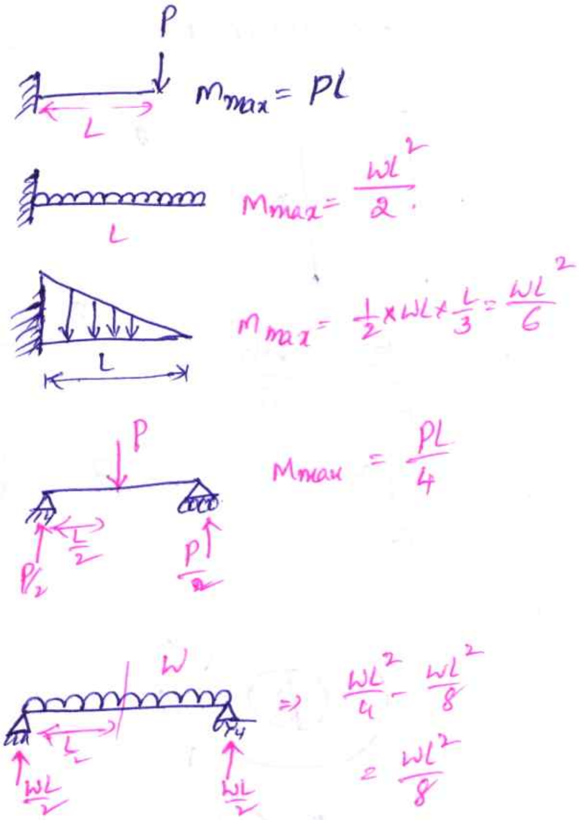
$$h^2 = 3b^2 - b^2 \Rightarrow \boxed{\frac{h}{b} = \sqrt{2}}$$

procedure to solve flexure problems.

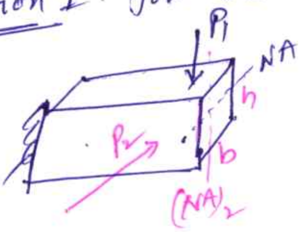
① find M_{max}

② To find $y \rightarrow$ N.A location $\rightarrow$ centroid

③ $I \rightarrow$ moment of inertia.



Confusion I for beam.



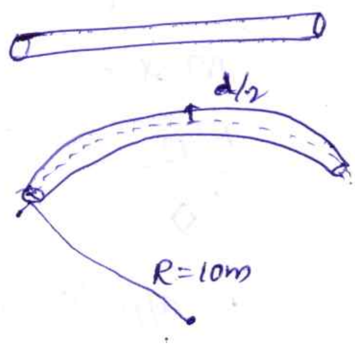
$P_1 \Rightarrow I = \frac{bh^3}{12}$

$P_2 \Rightarrow I = \frac{hb^3}{12}$

$$I = \frac{\text{Dim} \parallel^{l} \text{ to NA} \times (\text{Dim} \perp^{r} \text{ to NA})^3}{12}$$

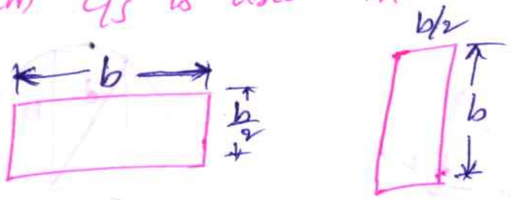
Q) A steel wire of 20mm dia is bent into a circular shape of 10m radius. If the modulus of elasticity is E is 2×10^5 N/mm², Then the maximum stress induced in the wire is.

$$\sigma_b = \frac{E \times y}{R} = \frac{2 \times 10^5 \times \left(\frac{20}{2}\right)}{10 \times 10^3}$$



$\sigma_b = 2 \times 10^2$ MPa.

Q A beam c/s is used in two different orientations as shown in fig.



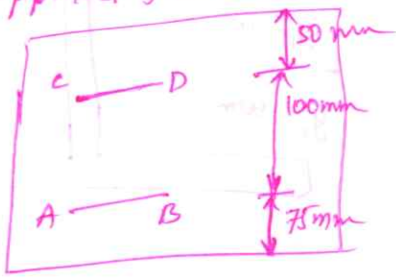
Bending moments applied to the beam in both cases are same. The max. bending stresses induced in cases (A) and (B) are related as

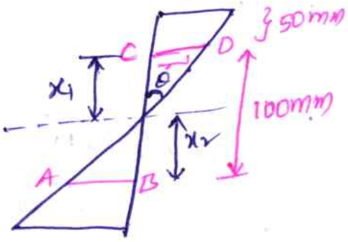
$M_A = M_B$, $\frac{\sigma_A}{\sigma_B} = ?$

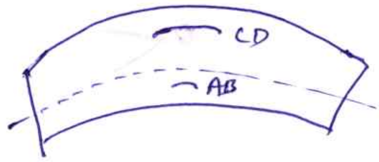
$$\sigma_A Z_A = \sigma_B Z_B \Rightarrow \frac{\sigma_A}{\sigma_B} = \frac{\frac{b}{2} \times \frac{b^2}{6}}{\frac{b \times \left(\frac{b}{2}\right)^2}{6}} = 2$$

Ⓐ A test is conducted on a beam loaded by end couples. The fibres at layer CD are found to lengthen by 0.03 mm and fibers at AB shorten by 0.09 mm for 20mm gauge length as shown in fig. Taking $E = 2\times10^5$ N/mm². the flexural stress at toppmost fibre would be

ⓐ 900 MPa (tensile) ⓑ 1000 MPa (tensile)
ⓒ 1200 MPa (tensile) ⓓ 1200 MPa (Compressive)







$x_1 + x_2 = 100$ mm

$\epsilon_{CD} = \frac{0.03}{20}$ $\quad \tan\theta = \frac{(0.03/20)}{x_1} = \frac{(0.09/20)}{x_2}$

$\epsilon_{AB} = \frac{0.09}{20}$, $\quad x_1 = 25, \quad x_2 = 75$

$\frac{\epsilon_{fiber}}{(50+25)} = \frac{(0.03/20)}{25} \Rightarrow \boxed{\epsilon_{fiber} = 0.0045}$

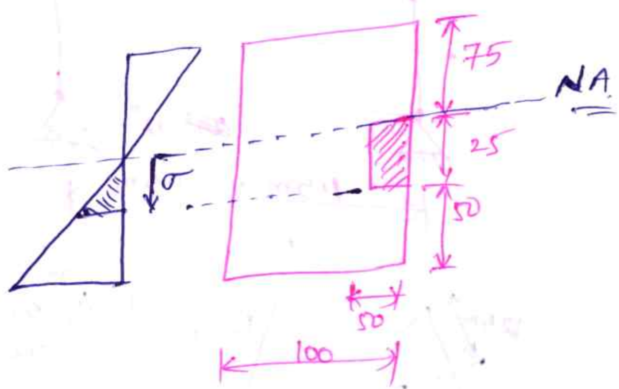
$\sigma_{fiber} = E(\epsilon_{fiber}) = 2\times10^5 \times 0.0045 = 900$ MPa (tensile).

Q) A beam with the c/s given below is subjected to a positive bending moment (causing compression at the top) of 16 KN-m acting around the horizontal axis. The tensile force acting on the hatched area of c/s is.

$M = 16$ KN-m.

$F = \frac{1}{2}(0+\sigma)\, 50\times25$

$F = \sigma(625)$

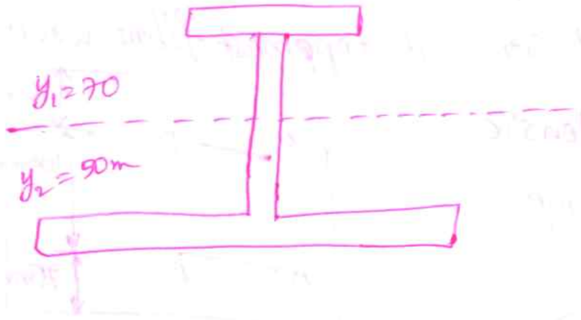


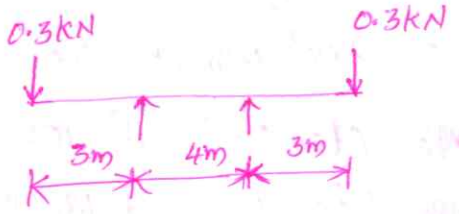
$\sigma = \frac{My}{I}$

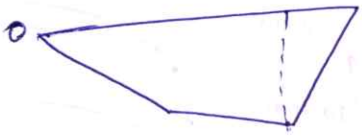
$= \frac{16\times10^3\times10^3 \times 25}{\frac{100\times(150)^3}{12}} = 14.22$ MPa

$F = 14.22\times10^6 \times 625\times10^{-6} = 8.9$ KN

(Q) The c/s of a beam is shown in fig-I. Its I_{xx} is equal to $3\times10^6 mm^4$. It is subjected to a load as shown in fig-II. The maximum tensile stress in the beam would be







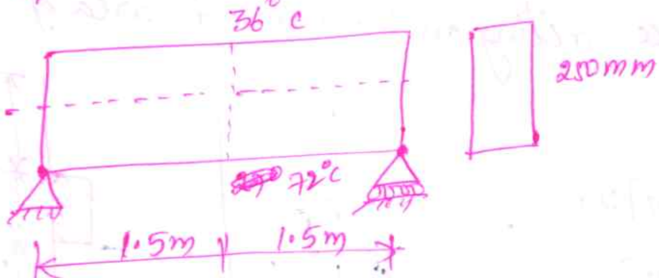
$$\sigma_{max} = \frac{M_{max} \times y_{max}}{I}$$

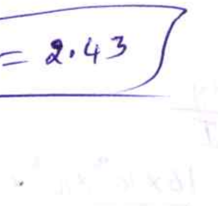
$$M_{max} = 0.3\times10^3 \times 3\times10^3 = 0.9\times10^6 \ N\text{-}mm.$$

$$= \frac{0.9\times10^6 \times 70}{3\times10^6}$$

$$\boxed{\sigma_{max} = 21 \ MPa}$$

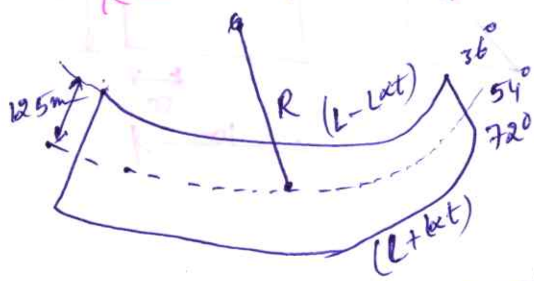
(Q) The beam of an overall depth 250 mm (shown below) is used in a building subjected to two different thermal environments. The temperatures at the top and bottom surfaces of the beam are 36°c and 72°c respectively. Considering coefficient of thermal expansion 'α' as 1.50×10^{-5} /°c, the vertical deflection of the beam (in mm) at its mid span due to temp gradient ——





$$\delta = R - d = R - \sqrt{R^2 - 1.5^2}$$

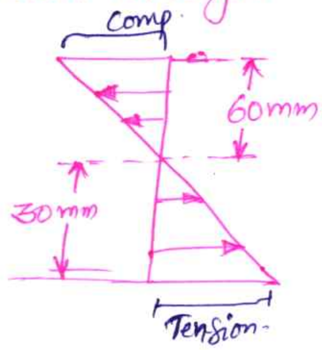
$$\boxed{\delta = 2.43}$$

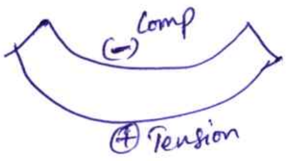


$$\epsilon = \frac{y}{R} = \alpha t \Rightarrow \frac{125\times10^{-3}}{R} = 1.5\times10^{-5} \times (54-36)$$

$$\boxed{R = 462.9 m}$$

ⓐ The bending stress distribution in a beam subjected to sagging shown in fig. The permissible stress in Tension and Compression are 70 MPa and 120 MPa Respectively. $I = 6 \times 10^5 \, mm^4$. Determine Beam strength.





$\sigma_T = 70 \, MPa$

$M_T = \sigma_T \dfrac{I}{y_{max}}$

$= \dfrac{70 \times 6 \times 10^5 \times 10^{-3}}{30}$

$M_T = 1400 \, N\text{-}m$

$\sigma_c = 120 \, MPa$

$M_c = 120 \times \dfrac{I}{y_{max}}$

$M_c = 120 \times \dfrac{6 \times 10^5}{60} \times 10^{-3}$

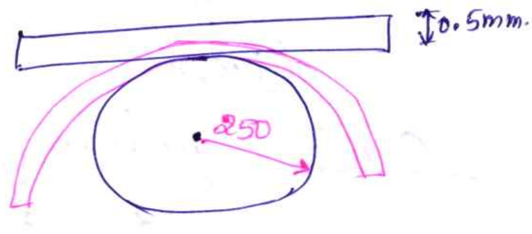
$M_c = 1200 \, Nm$

Safe ⇒ min $[M_T, M_c]$

Strength = 1200 Nm.

ⓐ A flat ribbon of steel 3 mm wide and 0.5 mm thick is wound round a cylinder 500 mm in diameter. The max. stress in the steel ribbon is N/mm^2 is.

ⓐ 100 ⓑ 200 ⓒ 400 ⓓ None



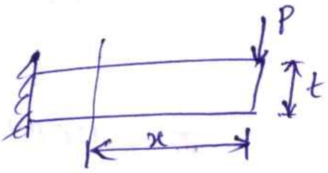
$\sigma_b = E \times \dfrac{y}{R} = \dfrac{200 \times 10^3 \times \left(\frac{0.5}{2}\right)}{\left(250 + \frac{0.5}{2}\right)}$

$\boxed{\sigma_b = 200 \, MPa}$

Beam of Uniform Strength

BOUS → $\sigma_{max} = C$ at any c/s.

Beam strength / section strength → (M) moment carrying strength



$M = Px$.

$\sigma_b = \frac{My}{I} = \frac{6M}{bt^2} = C$

$\frac{6(Px)}{bt^2} = C$

Case ① ⇒ $t = const$.

$\frac{C_1 x}{b} = C$

$b \propto x$

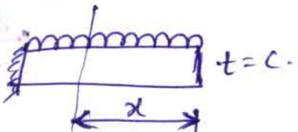
Case ② $b = Const$.

$\frac{6Px}{bt^2} = Const$

$x \propto t^2$

$t \propto \sqrt{x}$

Case ③



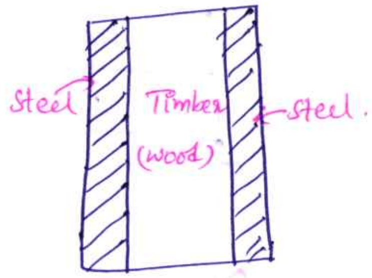
$M = \frac{wx^2}{2}$

$\sigma_b = \frac{My}{I} = \frac{wx^2}{2} \times \frac{6}{bt^2} = C$

$b \propto x^2$

Flitched Beams / Composite beams.

Flitched Beam Analysis:



$$\epsilon_{st} = \epsilon_T$$

$$\sigma = E\epsilon$$

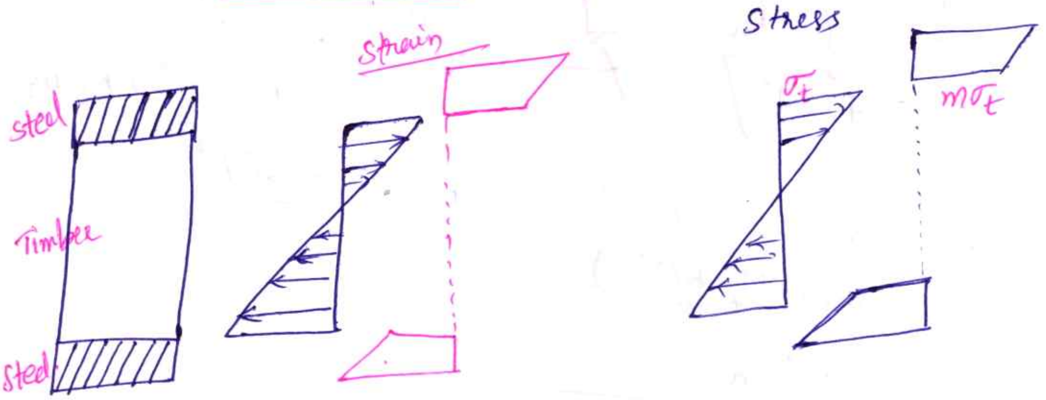
$$\frac{\sigma_S}{E_S} = \frac{\sigma_T}{E_T} \Rightarrow \frac{\sigma_S}{\sigma_T} = \frac{E_S}{E_T}$$

$$\frac{\sigma_S}{\sigma_T} = m \quad \text{[modular ratio.}$$

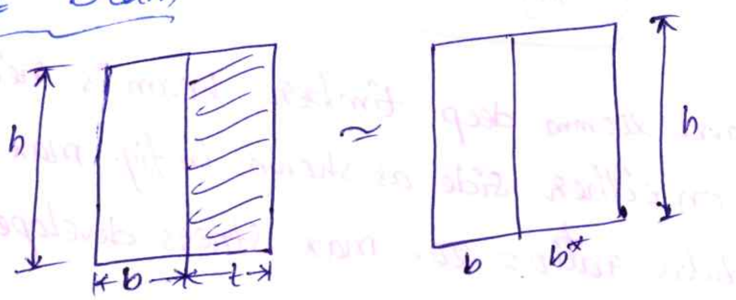
$$\boxed{\sigma_S = m\,\sigma_T}$$

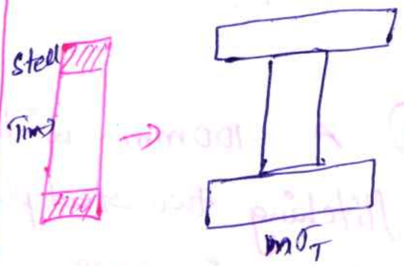
$$M = M_T + M_{St}$$

$$= (\sigma Z)_T + (\sigma Z)_{st}$$



Equivalent Beam





$$M_C = M_{ET}$$

$$M_T + M_S = M_{equi}$$

$$\sigma_T Z_T + \sigma_S Z_S = \sigma_T \times Z_{eq}$$

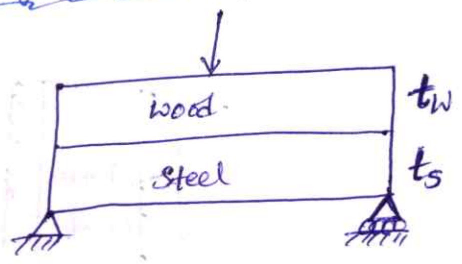
$$\sigma_T \frac{bh^2}{6} + m\sigma_T \times \frac{th^2}{6} = \sigma_T \frac{(b+b^*)h^2}{6}$$

$$b + mt = b + b^*$$

$$\boxed{b^* = m\cdot t}$$

→ Always changes the dimension of equivalent beam $\parallel^{h}$ to NA.

Not rigidly fixed :

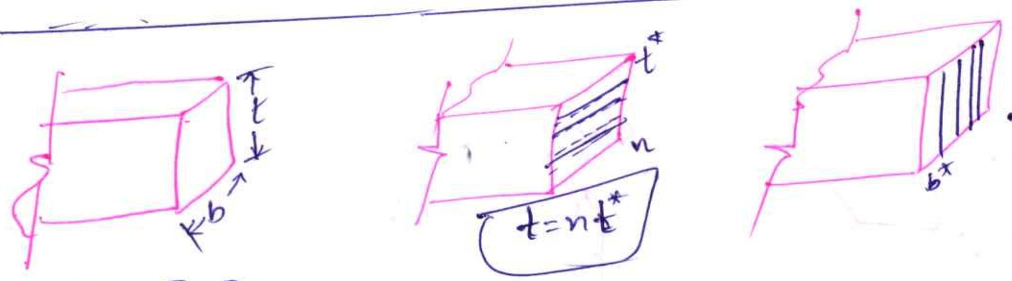


$R_s = R_w$

$R = \frac{EIy}{\sigma}$

$\frac{E_s\left(\frac{t_s}{2}\right)}{\sigma_s} = \frac{E_w\left(\frac{t_w}{2}\right)}{\sigma_w}$

$\frac{\sigma_s}{\sigma_w} = \frac{E_s t_s}{E_w t_w}$



$b = nb^*$

$M_3 = \min(M_1, M_2 \ldots)$

$M_1 = \sigma \times Z$

$= \sigma \times \frac{bt^2}{6}$

$M_2 = (M_1 + M_2 + \cdots)$

$= \sigma\left(\frac{bt^{*2}}{6} + \frac{bt^{*2}}{6} \ldots\right)$

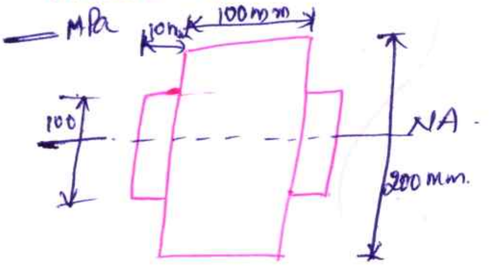
$= \sigma \frac{bt^{*2}}{6} \times n.$

$M_3 = \sigma \frac{b^* t^2}{6}$

$\frac{M_1}{M_2} = \left(\frac{t}{t^*}\right)^2 \frac{1}{n} = n,$

$\frac{M_1}{M_3} = n$

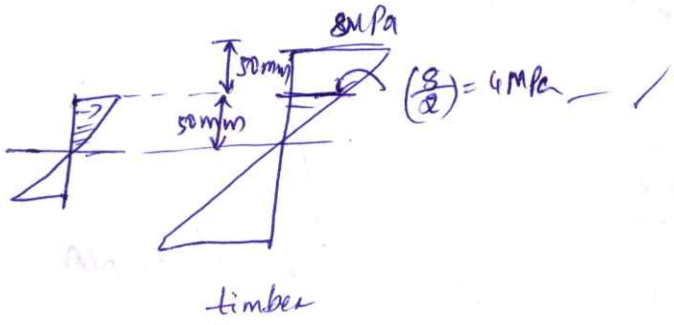
Q) A 100mm wide and 200mm deep timber beam is reinforced by flitching two steel plates on either side as shown in fig. max. stress in timber is 8MPa, modular ratio = 20. max stress developed in steel is — MPa



$\sigma_{Tim} = 8\,MPa$

$\sigma_s = ?$

$\sigma_s = m\sigma_T = 20 \times 4 = 80\,MPa$

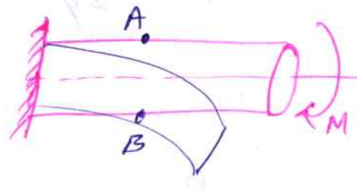


Stress due to Combined loads:

CL → combination of Axial load + Bending load

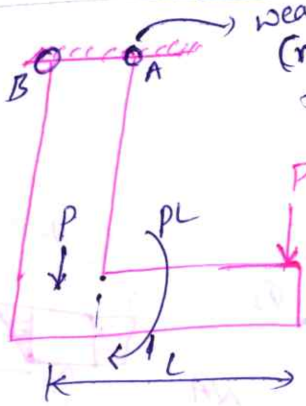
direct normal stress σ_d $\left(\frac{P}{A}\right)$

Indirect ~~direct~~ normal stress $\sigma_b = f = \frac{My}{I}$



$\sigma_A = +\sigma_d + \sigma_b = \frac{P}{A} + \frac{My}{I}$

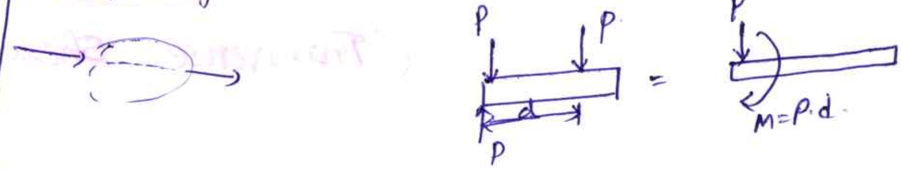
$\sigma_B = +\sigma_d - \sigma_b = \frac{P}{A} - \frac{My}{I}$



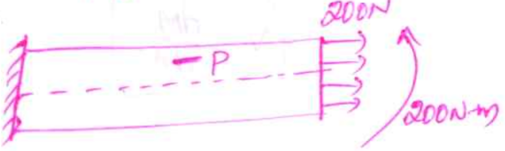
$\sigma_A = \frac{P}{A} + \frac{My}{I}$

$\sigma_B = \frac{P}{A} - \frac{My}{I}$

Transferening force.



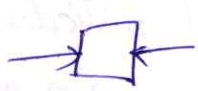
Q) A Cantilever beam having c/s area as $0.1 m^2$ and moment of inertia $1.33 \times 10^{-3} m^4$ as shown fig. Subjected to uniform tension of 200N and a Couple of 200N-m at the free end. The state of stress at point P [20mm above the neutral axis) is ______

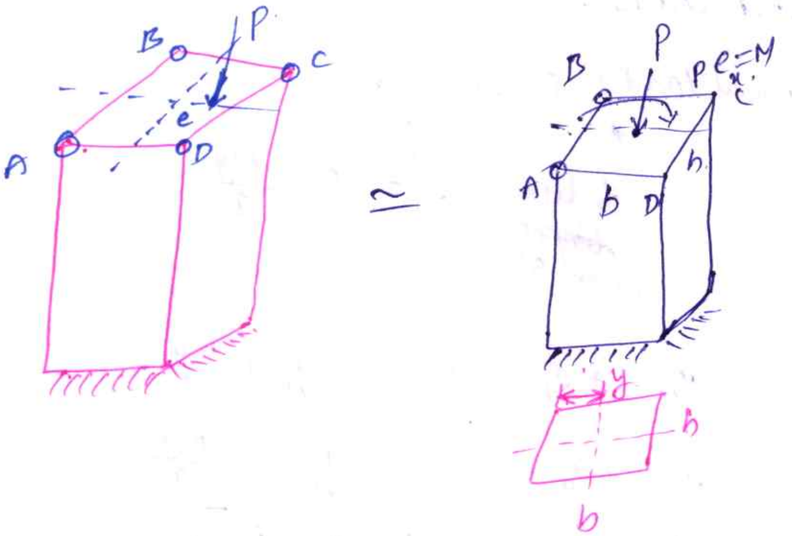


$\sigma_P = \sigma_d \cdot \frac{P}{A} - \frac{My}{I}$

$\sigma_P = \frac{200}{0.1} - \frac{2000 \times 20 \times 10^{-3}}{1.33 \times 10^{-3}} = -1007.5$

(−) → compressive





$$\sigma_A = \frac{-P}{A} + \frac{M}{Z_y}$$

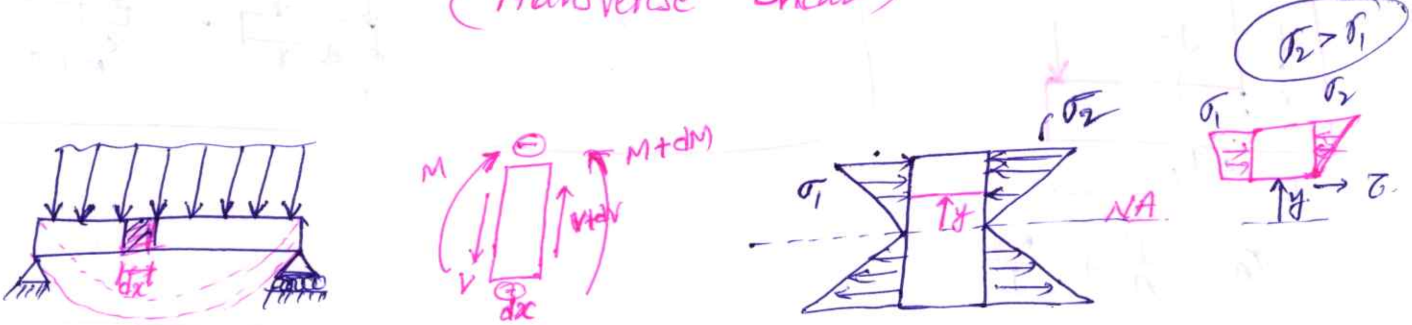
$$\sigma_B = \sigma_A$$

$$\sigma_C = \sigma_D = \frac{-P}{A} - \frac{M}{Z_y}$$

$$A = bh$$

$$Z = \frac{hb^2}{6}$$

Shear Stress due to Bending
(Transverse Shear)



$$\Sigma F = 0$$

$$\int_A \sigma_1 dA + \tau dx \cdot b = \int_A \sigma_2 dA$$

$$\int_A \frac{My}{I} dA + \tau dx b = \int_A \frac{(M+dM)y}{I} dA$$

$$\tau dx b = \int_A \frac{dM}{I} y dA$$

$$\tau = \frac{dM}{dx} \frac{1}{Ib} \int y dA$$

$$V = \frac{dM}{dx}$$

$$\tau = \frac{V}{Ib} \int_A y dA = \frac{VQ}{Ib}$$

$V \to$ Shear force at particular section.

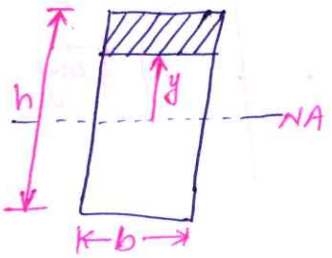
$Q = A\bar{y}$, $A \to$ area concerned for τ.

$\bar{y} \to$ distance b/w centroid of A to N.A of c/s.

$I \to$ Area moment of Inertia of entire c/s area.

$b \to$ width of the c/s at which shear stress is to be calculated.

Shear formula - practical observations



$$\tau = \frac{VQ}{Ib}$$

$$Q = A\bar{y}$$

$$Q = b\left(\frac{h}{2} - y\right) \times \left(y + \frac{\frac{h}{2} - y}{2}\right)$$

$$\tau = C \times \frac{Q}{b}$$

$$Q = \frac{b}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$

$$\tau = \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$

observations :

① $\tau \propto y^2$

② at $y = \pm\frac{h}{2}$ ⇒ $\tau = 0$ [Topmost & Bottommost fiber have zero stress]

③ $\tau \propto \frac{VQ}{Ib}$; If $b = C$ ⇒ $\tau \propto Q \propto A\bar{y}$

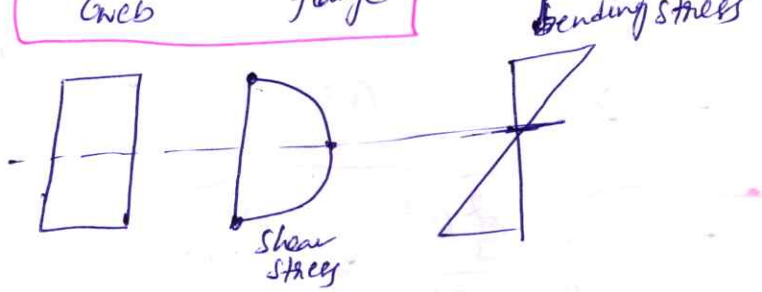
τ_{max} @ NA.

④ If $A\bar{y}$ = Constant , at junction. $Q = C$, $A\bar{y} = C$

$\tau \propto \frac{1}{b}$

$$\frac{\tau_{flange}}{\tau_{web}} = \frac{b_{web}}{b_{flange}}$$

⑤



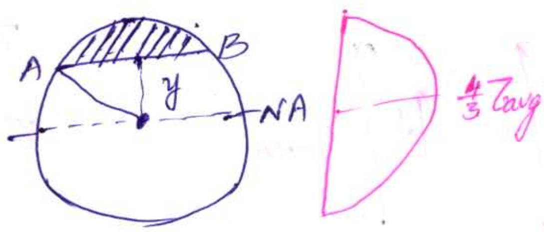
Rectangular c/s section:

$$\tau = \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$, at $y = \pm\frac{h}{2}$ ⇒ $\tau = 0$

τ at $y = 0$, $\tau_{max} = \frac{V}{2\left(\frac{bh^3}{12}\right)} \times \frac{h^2}{4}$ ⇒ $\frac{3}{2}\frac{V}{bh}$

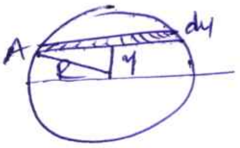
$\tau_{avg} = \frac{V}{bh}$, ⇒ $\tau_{max} = \frac{3}{2}\tau_{avg}$

Shear Stress Distribution in Circular C/S section.



$$\tau = \frac{VQ}{Ib}$$

$$Q = A\bar{y}, \quad I = \frac{\pi d^4}{64} = \frac{\pi}{4}R^4$$



$$dA = 2\,dy\sqrt{R^2-y^2}$$

$$y\,dA = 2y\sqrt{R^2-y^2}\,dy$$

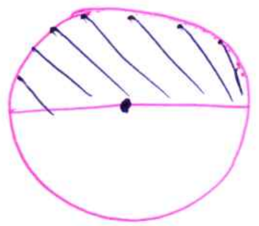
$$Q = A\bar{y} = \int_y^R 2y\sqrt{R^2-y^2}\,dy$$

$$Q = \frac{2}{3}(R^2-y^2)^{3/2}$$

$$\tau = \frac{V\frac{2}{3}(R^2-y^2)^{3/2}}{\frac{\pi}{4}R^4\,2\sqrt{R^2-y^2}} = \frac{4V}{3\pi R^4}[R^2-y^2]$$

$$\tau_{@y=R} = 0, \quad \tau_{@R=0} = \frac{4}{3}\frac{V}{\pi R^4}R^2 = \frac{4}{3}\cdot\left(\frac{V}{\pi R^2}\right) = \tau_{max}$$

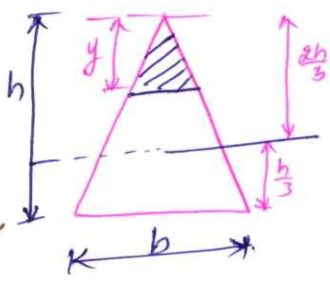
$$\boxed{\tau_{max} = \frac{4}{3}\tau_{avg}} \qquad \tau_{avg} = \frac{V}{\pi R^2}$$



$$\tau = \frac{VQ}{Ib}, \qquad Q = A\bar{y} = \frac{\pi R^2}{2}\frac{4R}{3\pi}$$

$$\boxed{\tau = \frac{V\,\frac{2}{3}R^3}{\frac{\pi}{4}R^4\cdot 2R} = \frac{4V}{3\pi R^2} = 1.33\,\tau_{avg}}$$

Shear stress distribution in triangular C/S section:



$$Q = A\bar{y} = \frac{1}{2}b'y\left(\frac{2}{3}(h-y)\right) = \frac{b'y}{3}(h-y)$$

$$\tau = \frac{VQ}{Iy} = \frac{V\,\frac{b'y}{3}(h-y)}{\frac{bh^3}{36}\times b'} = \frac{12V}{bh^3}y(h-y)$$

$$\tau_{@\,y=0,\,h=0} = 0$$

$$\tau_{@(y=\frac{2h}{3})} = \frac{8}{3}\frac{V}{bh}$$

$$\tau_{@\,y=\frac{h}{2}} = \frac{3V}{bh}$$

$$\tau_{avg} = \frac{V}{\frac{1}{2}bh} = \frac{2V}{bh}$$

$$\boxed{\tau_{@NA} = \frac{4}{3}\tau_{avg}}$$

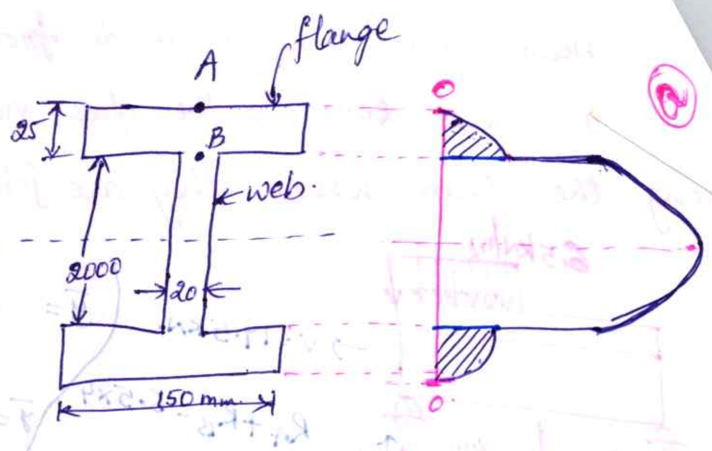
$$\boxed{\tau_{@\frac{h}{2}} = \frac{3}{2}\tau_{avg}}$$

I - Cross section.

$V = 50\,kN$, $\tau_{@A} = ?$

$\tau_{@B\,flange} =$,

$\tau_{@B\,web}$



$$I = \frac{BH^3}{12} - \frac{2bh^3}{12} = 108.6\times10^6\ mm^4$$

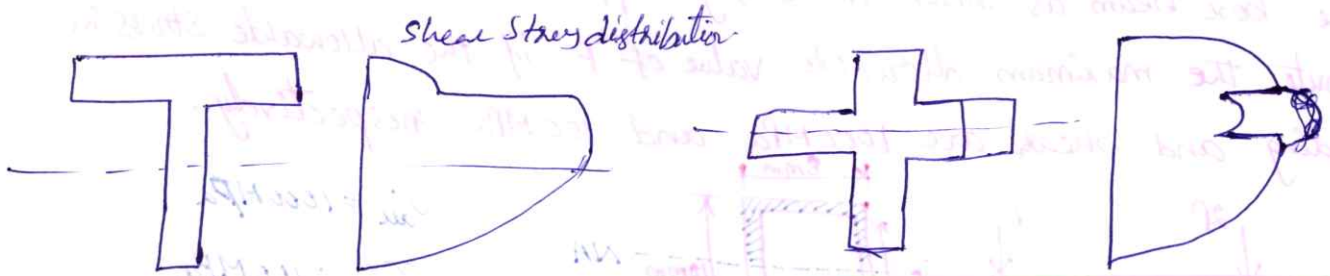
$$\tau_{@A} = 0$$

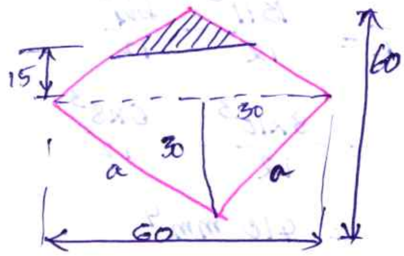
$$\tau_{@B\,flange} = \frac{50\times10^3\times(150\times25)\times112.5}{108.6\times10^6\times150} = 1.29\ MPa$$

$$\tau_{@B\,on\,web} = \frac{50\times10^3\times150\times25\times112.5}{108.6\times10^6\times20} = 9.7\ MPa$$

$$\tau_{max@C} = \frac{50\times10^3\left(150\times25\times112.5+20\times100\times50\right)}{108.6\times10^6\times20} = 12.01\ MPa$$

Shear Stress distribution





$V = 5\,kN$

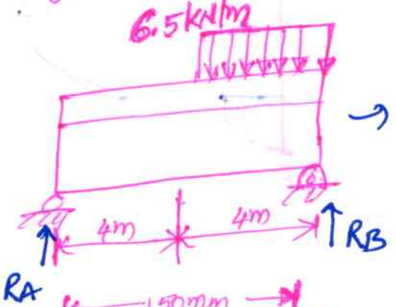
$\tau_{@AB} = ?$

$$\tau = \frac{VQ}{Ib} \Rightarrow \quad I = \frac{a^4}{12} = \frac{(30\sqrt{2})^4}{12}$$

$$Q = A\bar{y} = \tfrac{1}{2}b\times15\times20$$

$$\tau = \frac{5\times10^3\times\tfrac{1}{2}b\times15\times20}{\frac{(30\sqrt{2})^4}{12}\times b}$$

The beam shown in fig is made from two boards. Determine the maximum shear stress in the glue necessary to hold the boards together along the seam where they are joined.



$\rightarrow V = 19.5\,kN.$

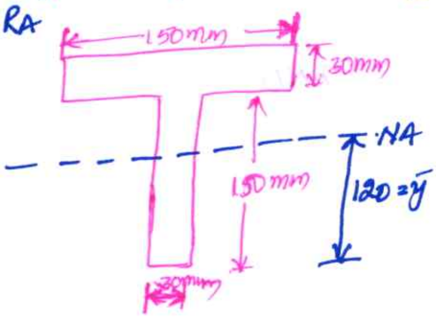
$R_A + R_B = 6.5\times4$

$\Sigma M_B = 0$

$R_A\times8 = 6.5\times4\times2$

$R_A = 6.5$

$R_B = 19.5$



$$\bar{y} = \frac{A_1y_1 - 2(A_2y_2)}{A_1 - 2A_2}$$

$$\bar{y} = \frac{(150\times180)\,90 - 2(60\times150\times75)}{150\times180 - 2(60\times150)}$$

$\bar{y} = 120\,mm$

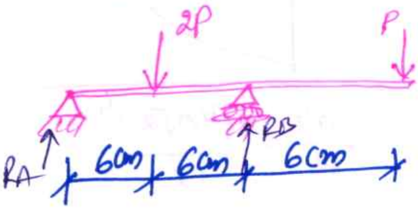
$$I = \frac{150\times30^3}{12} + (150\times30)\,45^2 + \frac{30\times150^3}{12} + 30\times150\times15^2$$

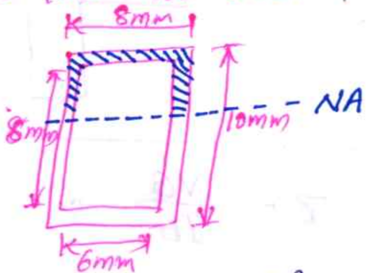
$I = 2.7\times10^6\ mm^4$

$$\tau = \frac{VQ}{Ib} = \frac{19.5\times10^3\times(150\times30\times45)}{2.7\times10^6\times30}$$

$\tau = 4.8\,MPa.$

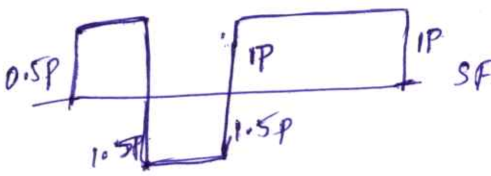
(Q) The box beam as shown in the fig. support the concentrated loads 2P and P. Compute the maximum allowable value of 'P' if the allowable stress in bending and shear are 1000 MPa and 100 MPa respectively.

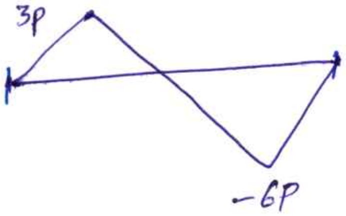




$\sigma_{all} = 1000\,MPa$

$\tau_{all} = 100\,MPa.$





$R_A + R_B = 3P$

$R_B\times12 = 2P\times6 + P(18)$

$R_B = 2.5P$

$R_A = 0.5P$

$V_{max} = 1.5P$

$M_{max} = 6P\ N\text{-}cm$

$= 60P\ N\text{-}mm$

$$I = \frac{BH^3}{12} - \frac{bh^3}{12}$$

$$= \frac{8\times10^3}{12} - \frac{6\times8^3}{12}$$

$= 410\ mm^4$

$$\sigma_{all} = \frac{My}{I} = 1000$$

$$1000 = \frac{60P\times5}{410}$$

$P_1 = 1369\,N$

$Q = (8\times5\times2.5 - 6\times4\times2)$

$$\tau_{all} = \frac{VQ}{Ib} = 100 = \frac{1.5P\times(Q)}{410\times2}$$

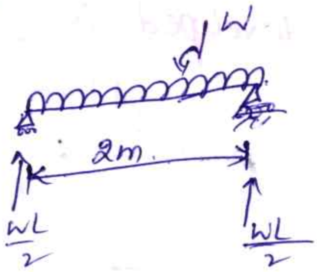
$P_2 = 1053\,N$

$P_{safe} = \min(P_1, P_2) \Rightarrow 1053\,N.$

Q) A simply supported beam (span 2m) of rectangular crosssection is subjected to UDL throughout its length. If the ratio of max normal stress to max. transverse shear stress in the beam is 10. the depth of the c/s is

① 100mm ② 200mm ③ 300mm ④ None of the above

$$\frac{\sigma_{max}}{\tau_{max}} = 10$$



$$\frac{\frac{6M}{bh^2}}{1.5\times\frac{V}{bh}} = 10 \Rightarrow \frac{6M}{1.5Vh} = 10$$

$V_{max} = \frac{wL}{2} =$

$M_{max} = \frac{wL^2}{8}$

$$\frac{6\times\frac{wL^2}{8}}{1.5\left(\frac{wL}{2}\right)h} = 10$$

$$\frac{12\times 2}{1.5h\times 8} = 10 \Rightarrow h = 200mm$$

Q) A rectangular beam of rectangular c/s 10cm wide, is subjected to a maximum shear force of 5000N, the corresponding maximum shear stress is being 3 N/mm². The depth of the beam is

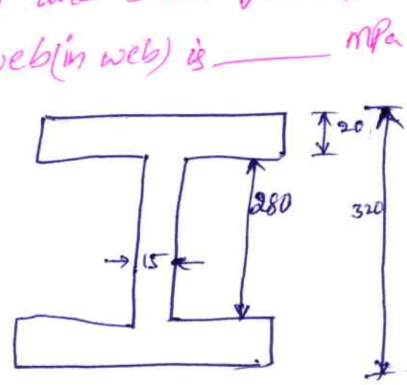
ⓐ 25cm ⓑ 22cm ⓒ 16.67cm ⓓ 30cm.

$$\tau = \frac{VQ}{Ib}$$

$$\tau_{max} = \frac{3}{2}\frac{V}{bh} \Rightarrow 3 = \frac{3}{2}\times\frac{50\times10^3}{100\times h} \Rightarrow h = 250mm$$

Q) A symmetrical I-section consists of web thickness 15mm and total depth of I section 320mm, flanges 160mm wide and 20mm thickness is subjected to bending moment of 100kN-m and shear force of 200kN. Shear stress at the junction of flange and web (in web) is ______ MPa.

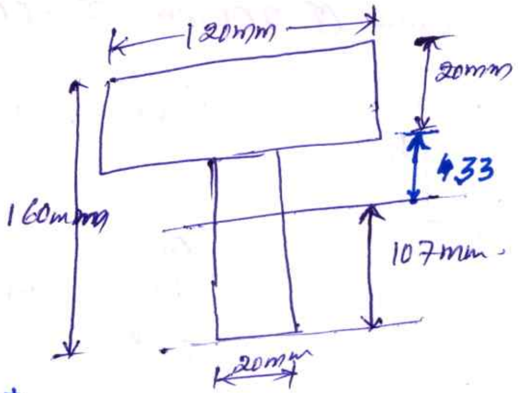
$t_{web} = 15mm$, $t_{flange} = 20$,

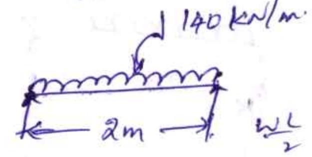


$$(\tau) = \frac{VQ}{Ib} \Rightarrow \quad I = \frac{BH^3}{12} - \frac{bh^3}{12} = 171.65\times10^6$$

$$\tau = \frac{200\times10^3\times160\times20\times150}{171.65\times10^6\times15} = 37.28 MPa$$

Q) A simply supported beam of 2m span carried a udl of 140 kN/m over the whole span. Cross-section of beam is shown in fig. Neutral axis is 107 mm from base. $I_{NA} = 13 \times 10^6$ mm⁴. Max. shear stress developed is ____ MPa.





$\frac{wL}{2} = \frac{140 \times 2}{2} = 140$ kN $= V$.

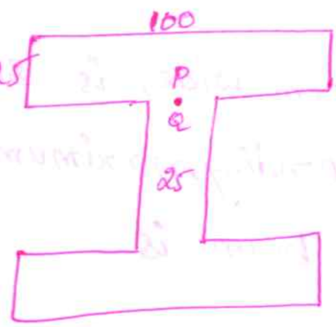
$\tau_{max} = \frac{V \cdot Q}{Ib} =$

$Q = A_1 y_1 + A_2 y_2 = (120 \times 20 \times 43) + 33 \times 20 \times 16.5$

$Q = 103200 + 10890 = 114090 \text{ mm}^3$

$\tau_{max} = \frac{140 \times 10^3 \times 114{,}090}{13 \times 10^6 \times 20} = 61.43 \text{MPa}.$

Q)



Points P and Q lies very close to the junction of flange and web respectively.

find $\frac{\tau_P}{\tau_Q} = ?$

$\frac{\tau_{flange}}{\tau_{web}} = \frac{b_{web}}{b_{flange}} = \frac{25}{100} = \frac{1}{4}$

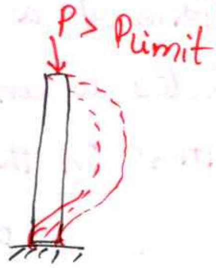
$\boxed{\tau_Q = 4 \tau_P}$

Buckling of Columns

Buckling :- Due to Sudden loss of Stiffness of column

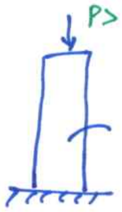
⇒ In case of buckling column crushing never takes before buckling.

$P > P_{limit}$



Types of Columns. (Based on failure mechanism.

Short Column

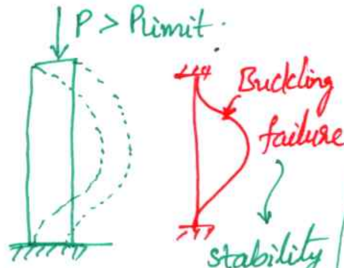


$P_B >>> P_{cr}$.

$L \leq 8$ least lateral Dimension (LLD)

→ Sudden failure.

Long Columns.



$P_{cr} >>> P_B$

$L > 30$ LLD

$P_{cr} >>> P_B$

Medium Columns.

Crushing + Buckling.

↓

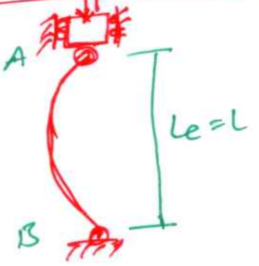
Analysis → Difficult.

→ Emperical formula.

Introduction to Euler's formula : (critical load)

⇒ Applicable for long columns fails due to Buckling under axial compressive loads.

Both ends pinned :



$$P_B = \frac{\pi^2 E I_{min}}{L_e^2}$$

↷ derived using differential equation elastic curve.

$$EI \frac{d^2y}{dx^2} = M$$

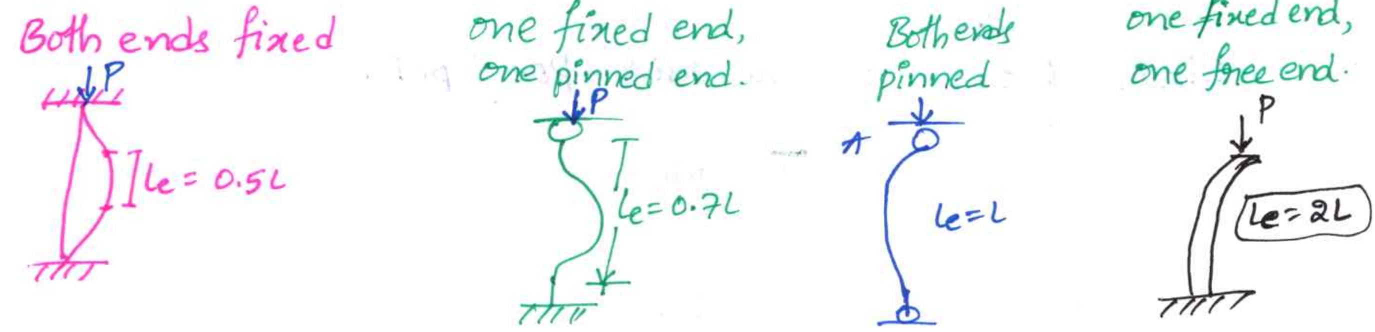
Assumptions : ① Column material is Homogeneous & isotropic.

② Self weight of columns neglected.

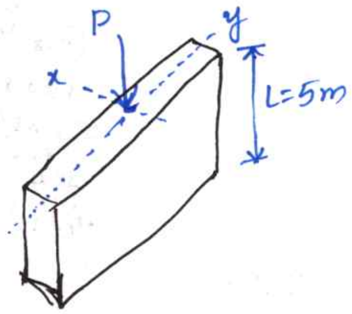
③ length of column is very large as compared to the C/S of column.

④ Column is initially straight and applied axial compressive loads only

⑤ Shortening of column due to axial compression is neglected.



practical observations of Euler formula :



$$(P_B) = \frac{\pi^2 E I_{min}}{(l_e)^2}$$

let

$$(P_B)_y = \frac{\pi^2 E I_{yy}}{(l_e)^2} \quad , \quad (P_B)_x = \frac{\pi^2 E I_{xx}}{(l_e)^2}$$

$$\frac{(P_B)_y}{(P_B)_x} = \frac{I_{yy}}{I_{xx}} , \qquad (P_B)_x > (P_B)_y .$$

$$P < \min \{ (P_B)_x \text{ (or) } (P_B)_y \}$$

① $\uparrow P_B \propto \uparrow I_{min}$ $\qquad P_B \propto AK^2 \qquad (K\uparrow)$ radius of gyration.

② $P_B \propto \frac{1}{l_e^2}$

$$(P_B)_{fix-fix} > (P_B)_{fix-hinge} > (P_B)_{Hinge\ Hinge} > (P_B)_{fixe\ free}$$

$$\left\{ \begin{aligned} (P_B)_{fix-fix} &= 2 (P_B)_{fix-hinge} \\ &= 4 (P_B)_{H-H} \\ &= 16 (P_B)_{fix-free} . \end{aligned} \right\}$$

Effective length and Significance

⇒ It is the length of equivalent pinned - pinned end Column having same load Carrying capacity as the given Column with the given Conditions.

(*) Distance between two Successive zero moment points.

(or)

Distance between two Successive POC / POI.

Significance:

$$P_B = \frac{\pi^2 E I_{min}}{l_e^2} \qquad \boxed{P_B \propto \frac{1}{l_e^2}}$$

as $l_e \downarrow$ and $P_B \uparrow$.

(Q) If dia of long Column is reduced by 20% the percentage reduction in Euler Buckling load is.

(a) 4 (b) 36 (c) 49 (d) 59.

$(0.8)^4$

0.8×0.8

$.64 \times 64$

256

384

4/16

Sol:

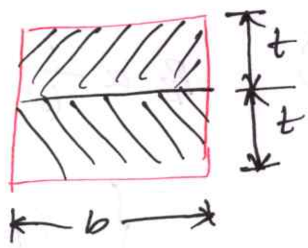
$P \propto I \Rightarrow$

$$\frac{P_2}{P_1} = \frac{d_2^4}{d_1^4} = (0.8)^4 = 0.41$$

$\boxed{P_2 = 0.41 P_1}$ → Reduction by 59%.

(P=)

(Q) C/s of a Column Consisting of two steel strips, each of thickness 't' and width b is shown in fig. The critical loads of the column with perfect bond and without bond b/w the strips are P & P_0. Then the ratio $\boxed{\frac{P}{P_0} = 4}$



$$P_B = \frac{\pi^2 E I_{min}}{(l_e)^2}$$

with bond, I

without bond I_0

$P_B \propto I_{min}$,

$$\frac{P}{P_0} = \frac{I}{I_0} = \frac{\frac{b(2t)^3}{12}}{2\left(\frac{bt^3}{12}\right)} = \frac{8}{2} = 4$$

Rankine formula

Limitation of Euler formula: → Applicable for Long Columns only.

L.C → fails by bucklings.

S.C → fails by crushing.

⇒ Rankine's formula is applicable for all types of column.
(Long Column, Short, Intermediate)

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_B}$$

P_R → Critical load given by Rankine.

P_C → Crushing load = $\sigma_C \cdot A$.

P_B → Euler Buckling load.

for S.C: $L\downarrow : P_B\uparrow : \frac{1}{P_B}\downarrow$ $\frac{1}{P_B}=0$

$$\frac{1}{P_R} = \frac{1}{P_C} + 0 \Rightarrow P_R = P_C$$

for Long Columns: $\uparrow L : \downarrow P_B : \frac{1}{P_B}\uparrow$ ($\frac{1}{P_B}$ is more.)

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_B} \simeq \frac{1}{P_B}$$

$$P_R = \frac{P_C P_B}{P_C + P_B} = \frac{P_C}{\left(1 + \frac{P_B}{P_C}\right)}$$

$P_C = \sigma_C A$

$P_B = \frac{\pi^2 E A K^2}{L_e^2}$

$$P_R = \frac{\sigma_C A}{1 + \frac{\sigma_C A}{\frac{\pi^2 E A K^2}{L_e^2}}} = \frac{\sigma_C A}{1 + \left(\frac{\sigma_C}{\pi^2 E}\right)\lambda^2} = \frac{\sigma_C A}{1 + a\lambda^2}$$

↖ Rankine's constant.

$$P_R = \frac{\sigma_C A}{1 + a\lambda^2}$$

$$\lambda = \frac{L_e}{K}$$

$a = \frac{1}{7500}$ → mild steel

$a = \frac{1}{1600}$ – Cast Iron

$a = \frac{1}{750}$ – wood.

Core (or) kernel of section

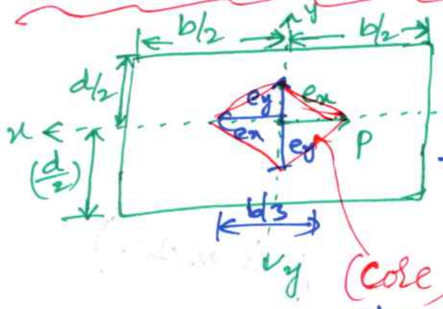
Core: It is part of colum c/s with in the load is placed so that no tension present in the column c/s. [i.e c/s column is fully compression]

$$-\sigma_{axial} + \sigma_{bending} \le 0$$

$$\sigma_{bending} \le \sigma_{axial} \Rightarrow \frac{M}{Z} \le \frac{P}{A}$$

$$\frac{Pe}{Z} \le \frac{P}{A} \Rightarrow \boxed{e \le \frac{Z}{A}}$$

Core for Different sections:



Shape: Rhombus.

Rectangular

$M_y = P \cdot e_x$. $Z_y = \frac{db^2}{12}$

To avoid Tension, $\sigma_{min} \le 0$

$$-\sigma_{axial} + \sigma_{bending} \le 0$$

$$\sigma_{bending} \le \sigma_{axial}$$

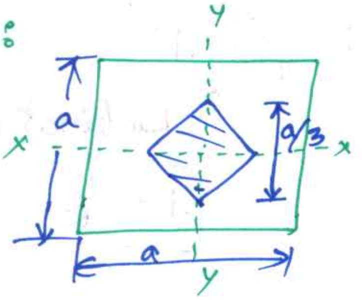
$$\frac{M_y}{Z_y} \le \frac{P}{A}$$

$$\frac{P e_x}{\frac{db^2}{6}} \le \frac{P}{db}$$

$$\boxed{e_x \le \frac{b}{6}}$$

$$A_{core} = 2\frac{1}{2}\left(\frac{b}{3}\right)\left(\frac{d}{6}\right) = \frac{bd}{18} = \frac{1}{18} A_{gross}.$$

Square:

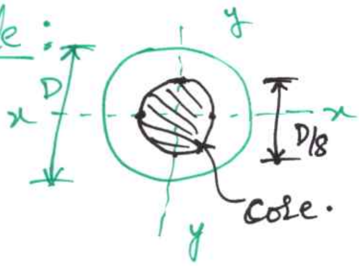


$$\sigma_b \le \sigma_{axial}.$$

$$\frac{P \cdot e_x}{\frac{a^3}{6}} \le \frac{P}{a^2}$$

$$e_x \le \frac{a}{6} \quad ||^{ly} \quad e_y \le \frac{a}{6}$$

Circle:



To avoid Tension.

$$\sigma_{min} \le 0 \qquad \frac{M_y}{Z_y} \le \frac{P}{A}$$

$$\frac{P e_x}{\frac{\pi}{32} D^3} \le \frac{P}{\frac{\pi}{4} D^2}$$

Middle fourth zone. $\left(e_x \le \frac{D}{8}\right)$

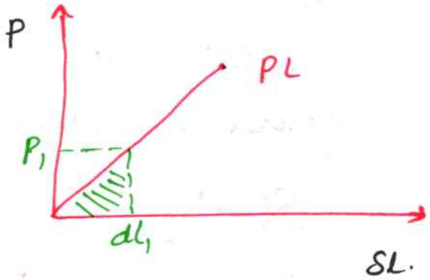
$$A_{core} = \frac{\pi}{4}\left(\frac{D}{4}\right)^2 = \frac{1}{16} A_{gross}.$$

Strain Energy

Strain Energy: The energy absorbed by a member when it is strained due to external load.

⇒ Stored energy due deformation

Resilience: SE stored in the member within the Elastic Limit.



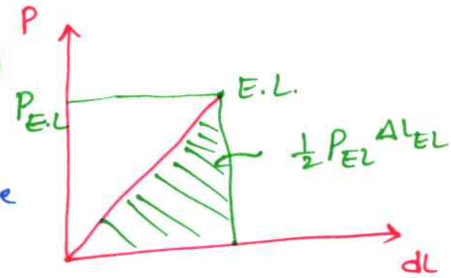
workdone by load $P_1 = \frac{1}{2} P_1 \, dl_1$

= Resilience corresponding load P_1

⇒ Area under P Vs δl curve within EL

⇒ Elastic SE / Recoverable SE. — Resilience.

Proof Resilience: max. SE stored in the member within elastic region.



proof Resilience = Area under P vs. δL curve upto E.L.

$$\text{proof Resilience} = \frac{1}{2} P_{EL} \cdot dL_{EL} = \frac{1}{2} \sigma_{EL} A \cdot (\epsilon_{EL} L)$$

$$= \frac{1}{2} \sigma_{EL} \epsilon_{EL} (AL) = \frac{1}{2} \sigma_{EL} \epsilon_{EL} (\text{Volume})$$

$$= \frac{(\sigma_{EL})^2}{2E} (\text{Volume})$$

Conclusion: PR → function of { material, volume }.

↑ PR : ↑ σ_{EL}

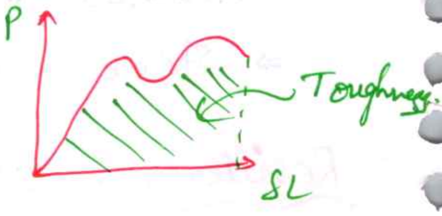
: ↓ E

: ↑ Volume.

Toughness : SE stored in the member upto fracture.

→ Area under P vs dL curve upto fracture.

Toughness : ability to absorb energy in plastic range.

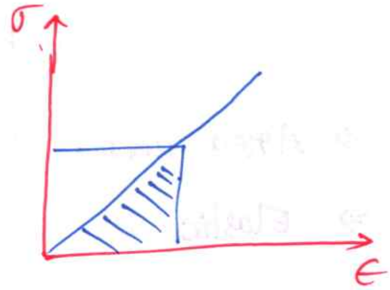


⇒ useful when we are designing members subjected to shock loads.

Eg : Shock absorbers,

Toughness : Strength + ductility.

Strain Energy density



Strain Energy density (u) = $\frac{S.E}{volume}$

$$u = \frac{U}{vol.} = \frac{N\text{-}m}{m^3} \rightarrow \frac{J}{m^3}$$

$$u = \frac{\frac{1}{2}(P)\,dL}{A.L} = \frac{1}{2}\sigma.\epsilon$$

Strain energy density

→ Modulus of Resilience

⇒ Max SE stored per unit volume under elastic limit.

$$MOR = \frac{\text{proof Resilience}}{\text{volume}}$$

$$MOR = \frac{\frac{1}{2}P\,dL}{AL} = \frac{1}{2}\sigma_{EL}\epsilon_{EL} = \left(\frac{\sigma_{EL}^2}{2E}\right)$$

Area under σ vs ε curve upto EL.

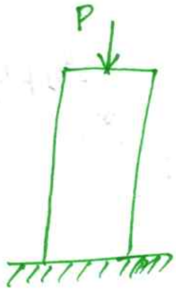
→ Modulus of Toughness.

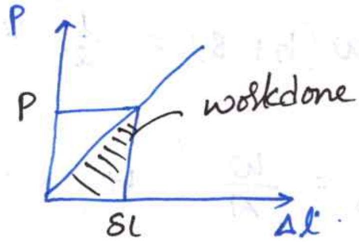
⇒ max SE stored per unit volume upto failure point

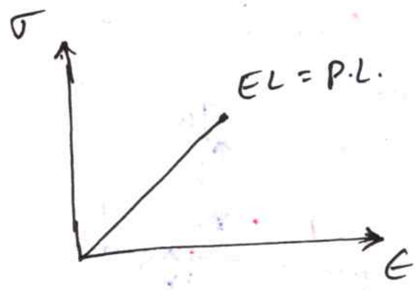
Area under σ vs ε curve upto failure

Stress due to different types of axial loading :

Gradual applied load :







Resistance (gradually)

workdone by load $P = \frac{1}{2} P \cdot \Delta L$

Energy stored with in EL $= \frac{1}{2} \sigma \epsilon \times$ Volume.

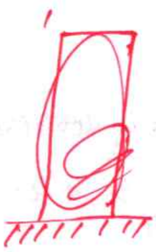
WD = Energy stored with in EL

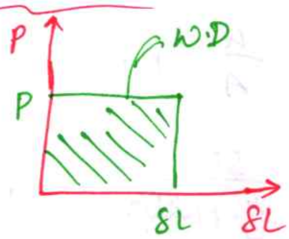
$$\frac{1}{2} P(\Delta L) = \frac{1}{2} \sigma \epsilon \times AL$$

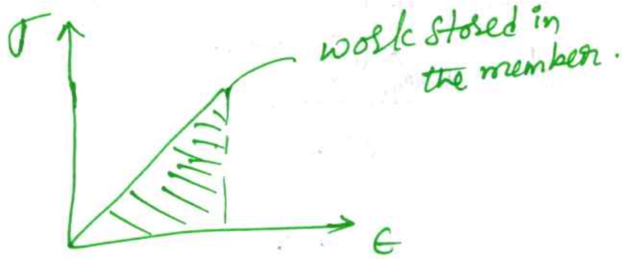
$$\frac{1}{2} P\left(\frac{\sigma L}{E}\right) = \frac{1}{2} \sigma \cdot \left(\frac{\sigma}{E}\right) \times AL$$

$$\boxed{\sigma_{GAL} = \frac{P}{A}}$$

Suddenly applied load :







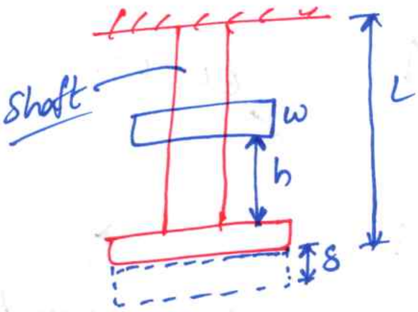
WD = work stored (EL)

$$P(\delta L) = \frac{\sigma^2}{2E} AL \Rightarrow P\left(\frac{\sigma L}{E}\right) = \frac{\sigma^2 AL}{2E}$$

$$\boxed{\sigma_{SAL} = \frac{2P}{A}} \Rightarrow \boxed{\sigma_{SAL} = 2\sigma_{GAL}}$$

$$\boxed{U_{SAL} = 4 U_{GAL}}$$

Impact load



$$WD = \text{Energy Stored}$$

$$W(h+\delta) = \frac{\sigma^2}{2E} \times AL$$

$$\sigma_{imp} = \frac{W}{A}\left[1+\sqrt{1+\frac{2AEh}{WL}}\right]$$

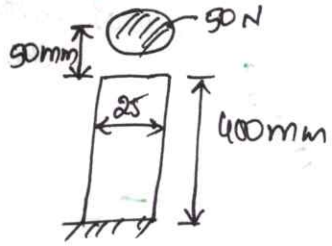
(the bracketed term: Impact factor, I.F.)

$$\sigma_{imp} = \sigma_{GAL} \times IF.$$

$$\delta_{imp} = \frac{\sigma_{imp} \cdot L}{E}$$

Q. A 50N load falls through a height of 90mm onto a steel bar of a dia of 25mm and a lengths of 400mm, supported on the ground. What is the instantaneous stress developed in the bar and what is the change in its length? (Take E = 200 GPa)

Sol.



$$\sigma = \frac{W}{A}\left[1+\sqrt{1+\frac{2AEh}{WL}}\right]$$

$$= \frac{50}{\frac{\pi}{4}(25)^2}\left[1+\sqrt{1+\frac{2\times\frac{\pi}{4}(25)^2 \, 200\times10^3\times 90}{50\times400}}\right]$$

$$= \frac{200}{625\times\pi}\left[1+\sqrt{1+\frac{625\times\pi\times25\times200\times10^3}{50\times400}}\right]$$

$$\sigma_{inst} = 71.467 \text{ MPa}.$$

$$\delta L_{inst} = \frac{\sigma_{inst} L}{E} = \frac{71.467\times400}{200\times10^3} = 0.14 \text{ mm}.$$

ANNAMACHARYA UNIVERSITY

EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY

ESTD, UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)

Rajampet, Annamayya District, A.P – 516126, INDIA

CIVIL ENGINEERING

Strength of Materials

UNIT-4

Deflection of Beams

Design
- Strength
  - $\sigma \le \sigma_{all}$
  - $\tau \le \tau_{all}$
  - safe.
- Stiffness
  - $k \downarrow \rightarrow \delta \uparrow$
  - $\delta \le \delta_{all}$.
  - $k = \frac{F}{\delta}$

Std building code

$$\delta_{max} = \frac{1}{360} L_{span}$$

methods of determing slope and Deflection.

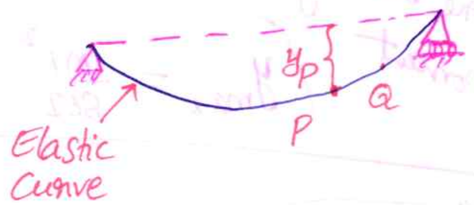
→ Double integration method.
→ Macaulay's method
⇒ Moment area method
→ Conjugate beam method
⇒ Method of Superposition
→ Strain Energy method

Differential Equation of Elastic Curve:

→ Relation b/w P, θ, δ

Assumptions: ① Curvature is small [slope & deflections is very small, if stress are with in elastic limit]

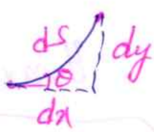
② Hook's law is valid. $\sigma \propto \epsilon$
③ Material is homogeneous & isotropic

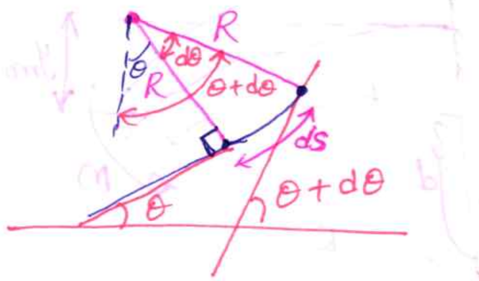


→ let $y \rightarrow$ deflection

$y' \rightarrow \frac{dy}{dx}$ [slope − θ]

$\tan\theta = \frac{dy}{dx}$





If θ is small → $\theta = \frac{dy}{dx}$

θ is very small → $ds = dx$

$R\,d\theta = ds$

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx}$$

$$\frac{1}{R} = \frac{d}{dx}\left[\frac{dy}{dx}\right] \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{R}$$

$$\frac{E}{R} = \frac{M}{I} = \frac{f}{y} \Rightarrow \frac{E}{R} = \frac{M}{I} \Rightarrow \boxed{\frac{1}{R} = \frac{M}{EI}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{R} = \frac{M}{EI} = K} = \text{Curvature.}$$

$\Rightarrow$ $EI\,y'' = M$

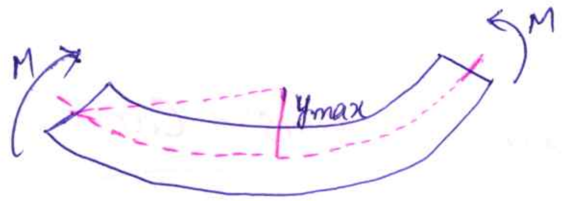
$\Rightarrow$ $EI\,y' = \int M\,dx + C_1$ → Slope

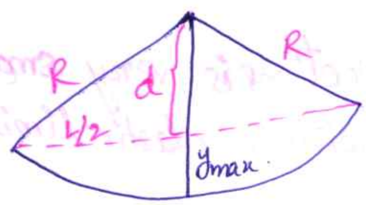
$\Rightarrow$ $EI\,y = \iint M\,dx\,dx + C_1x + C_2$ → deflection

$\Rightarrow$ $EI\,y''' = \frac{dM}{dx} = V$

$\Rightarrow$ $EI\,y^{IV} = \frac{dV}{dx} = q$

Beams Subjected to pure Bending Moment





$\boxed{d = R - y_{max}}$

$$R^2 = \left(\frac{L}{2}\right)^2 + d^2$$

$$R^2 = \left(\frac{L}{2}\right)^2 + R^2 + y_{max}^2 - 2R\,y_{max}$$

$$y_{max}\left[2R - y_{max}\right] = \frac{L^2}{4}$$

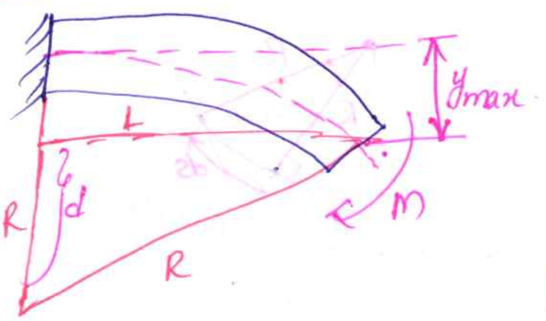
$$\frac{E}{R} = \frac{M}{I} \Rightarrow \frac{1}{R} = \frac{M}{EI}$$

$2R >> y_{max} \Rightarrow \boxed{2R - y_{max} = 2R}$

$$y_{max}(2R) = \frac{L^2}{4} \Rightarrow \boxed{y_{max} = \frac{L^2}{8R}}$$

pure bending moment

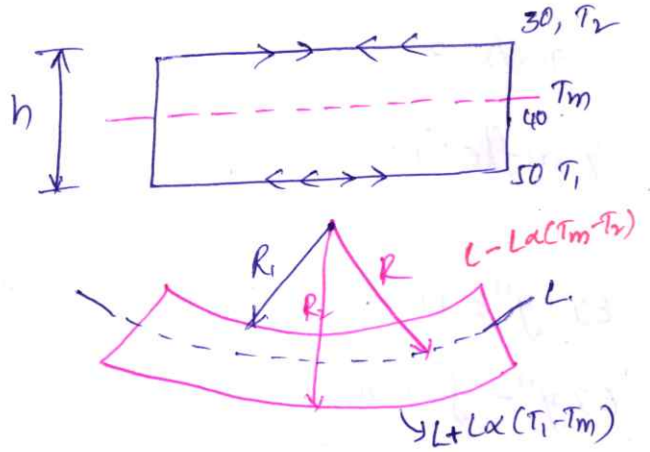
$$\boxed{y_{max} = \frac{ML^2}{8EI}}$$



$d = R - y_{max}$; $R^2 = L^2 + (R - y_m)^2$

$$y_{max} = \frac{L^2}{2R}$$

$$\boxed{y_{max} = \frac{ML^2}{2EI}}$$



$$R_1\theta = L - L\alpha(T_m - T_2)$$

$$\left(R - \frac{h}{2}\right)\theta = L - L\alpha(T_m - T_2) \quad -(1)$$

$$\left(R + \frac{h}{2}\right)\theta = L + L\alpha(T_1 - T_m) \quad -(2)$$

$$R\theta = L$$

$$(2) - (1) \Rightarrow \quad h\theta = L\alpha(T_1 - T_2)$$

$$h\left(\frac{L}{R}\right) = L\alpha(T_1 - T_2)$$

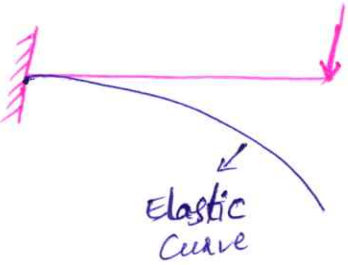
$$\boxed{\frac{1}{R} = \frac{\alpha \Delta T}{h}} \rightarrow \text{Temp difference.}$$

Q) A simply supported reinforced concrete beam of length 10 m sags while undergoing shrinkage. Assuming a uniform curvature of 0.004 m^{-1} along the span, the max deflection (in m) of the beam at mid-span is.

$$y_{max} = \frac{ML^2}{8EI} = \frac{L^2}{8R} = \frac{100 \times 0.004}{8} = 50\text{ mm}$$

$$k = \frac{1}{R} = 0.004\ m^{-1}$$

Double Integration method



$$EI\, y'' = M$$

y - deflection

$y' \rightarrow$ Slope

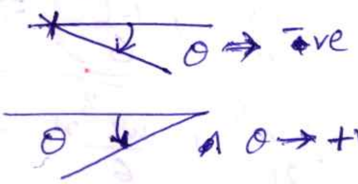
$M \rightarrow$ BM in the segment

$EI \rightarrow$ flexural Rigidity

Sign Convention.

y →	
	↑ +ve
	↓ −ve

θ = Slope



$$\int EIy'' = \int M;$$

$$EIy' = \int M dx + C_1 \rightarrow \text{Slope Equation.}$$

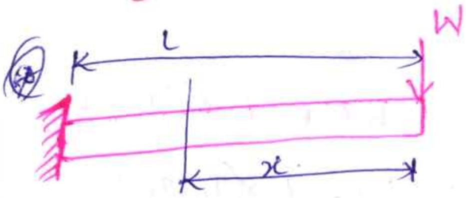
$$EIy = \iint M dx \cdot dx + C_1 x + C_2 \rightarrow \text{deflection eq}$$

C_1 & C_2 → determined from boundary Conditions.

① Hinge/Roller Support : → It restricts only deflection. $\delta = y = 0$

② fixed : Both slope & deflection $\theta = y' = 0$

Ⓐ



$$EI\,y'' = M \Rightarrow EI\,y'' = -Wx.$$

$$\int EIy'' = \int -Wx$$

$$EI\,y' = -\frac{Wx^2}{2} + C_1$$

BC → @ $x = L \Rightarrow y' = 0$

$\Rightarrow y = 0$

$$EI\,y = -\frac{Wx^3}{6} + C_1 x + C_2$$

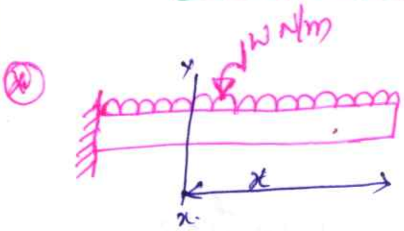
$$\Rightarrow EI(0) = -\frac{WL^2}{2} + C_1 \Rightarrow C_1 = \frac{WL^2}{2},$$

$$0 = \frac{-WL^3}{6} + \frac{WL^3}{2} + C_2 \Rightarrow C_2 = \frac{-2WL^3}{6} = \frac{-WL^3}{3}$$

$$EIy = -\frac{Wx^3}{6} + \frac{WL^2}{2}x - \frac{WL^3}{3} \Rightarrow @\ x = 0,\ |y_{max}| = \left|\frac{-WL^3}{3EI}\right| = \frac{WL^3}{3EI}$$

$$EIy' = \frac{-Wx^2}{2} + \frac{WL^2}{2} \Rightarrow y'_{max} = \frac{WL^2}{2EI}$$

Ⓑ



$$M_x = (wx)\frac{x}{2} = \frac{-wx^2}{2}$$

$$EI\,y'' = M_x = \frac{-wx^2}{2}$$

$$EIy' = -\frac{wx^3}{6} + C_1$$

$$EIy = -\frac{wx^4}{24} + C_1 x + C_2$$

@ $x = L,\ y' = 0,\ y = 0$

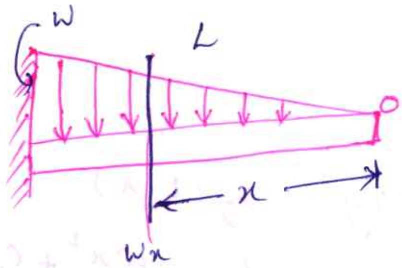
$$C_1 = \frac{wL^3}{6}$$

$$\frac{wL^4}{24} - \frac{wL^4}{6} = C_2 = \frac{-3wL^4}{24} = \frac{-wL^4}{8}$$

$$EIy' = -\frac{wx^3}{6} + \frac{wL^3}{6} \Rightarrow y'_{max} = \frac{wL^3}{6EI}$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^3}{6}x - \frac{wL^4}{8} \Rightarrow y_{max} = \frac{-wL^4}{8EI}$$

max displacement $= |y_{max}| = \frac{wL^4}{8EI}$



$$\frac{w}{L} = \frac{w_x}{x} \Rightarrow w_x = \frac{wx}{L}$$

$$M_x = -\frac{1}{2}\left(\frac{wx}{L}\right)x \cdot \frac{x}{3} = \frac{-wx^3}{6L}$$

$$EIy'' = \frac{-wx^3}{6L}$$

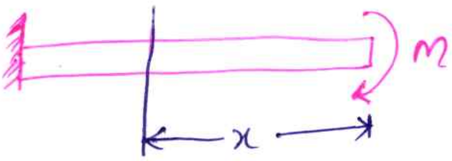
$$EIy' = -\frac{wx^4}{24L} + C_1 \quad @x=0,\ y=L \quad C_1 = \frac{wL^3}{24}$$

$$EIy = \frac{-wx^5}{120L} + C_1 x + C_2$$

$$@\ x=L,\ y=0, \Rightarrow \quad \frac{wL^4}{120} - \frac{wL^4}{24} = C_2 = \frac{-wL^4}{30}$$

$$EIy = \frac{-wx^5}{120L} + \frac{wL^3}{24}x - \frac{wL^4}{30} \Rightarrow @\ x=0,\ y_{max} = \frac{-wL^4}{30EI}$$

$$EIy' = \frac{-wx^4}{24L} + \frac{wL^3}{24} \rightarrow @\ x=0,\ y'_{max} = \frac{wL^3}{24EI}$$



$$M_x = -M$$

$$EIy'' = -M$$

$$EIy' = -Mx + C_1$$

$$EIy = -\frac{Mx^2}{2} + C_1 x + C_2$$

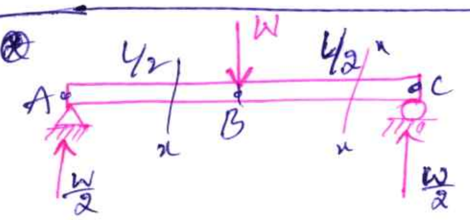
$@\ x=L,\ y'=0,\ y=0$

$C_1 = ML,\ C_2 = \frac{-ML^2}{2}$

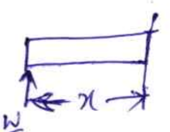
$$EIy' = -Mx + ML \quad @x=0,\ y'_{max} = \frac{ML}{EI}$$

$$EIy = -\frac{Mx^2}{2} + MLx - \frac{ML^2}{2} \quad @x=0,\ y_{max} = \frac{-ML^2}{2}$$

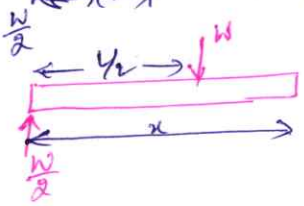
$$\delta = \frac{ML^2}{2EI}$$



If loading is discontinues on the beam, then beam must be divided into segments at each each discontinuity and write separate moment equation for each segment.



$M_x = \frac{W}{2}x$ (AB)



$$M_x = \frac{W}{2}x - W\left(x - \frac{L}{2}\right)$$

$$M_x = \frac{W}{2}(L - x)$$

$M_x = \frac{w}{2}x, \quad (0 \le x \le \frac{L}{2})$

$= \frac{w}{2}(L-x) \quad [\frac{L}{2} \le x \le L]$

(AB) $\Rightarrow$

$EIy'' = \frac{w}{2}x$

$EIy' = \frac{wx^2}{4} + C_1$

$EIy = \frac{wx^3}{12} + C_1 x + C_2$

@ $x=0,\ y=0$ $\quad C_2 = 0;$ $\quad \frac{-wL^2}{16} = C_1$

$x = \frac{L}{2},\ y' = 0,$

$EIy' = \frac{wx^2}{4} - \frac{wL^2}{16}$

$EIy = \frac{wx^3}{12} - \frac{wL^2}{16}x.$

(BC)

$EIy'' = \frac{w}{2}(L-x)$

$EIy' = \frac{wL}{2}x - \frac{wx^2}{4} + C_1$

$EIy = \frac{wLx^2}{4} - \frac{wx^3}{12} + C_1 x + C_2$

@ $x = \frac{L}{2},\ y' = 0$

$x = L,\ y = 0$

$C_1 = \frac{-3wL^2}{16}$

$0 = \frac{wL^3}{4} - \frac{wL^3}{12} - \frac{3wL^3}{16} + C_2$

$C_2 = \frac{12-4-9}{48}(wL^3) = \frac{-wL^3}{48}$

Macaulay's method.

Rules :

① BM equation to be written for the last segment of beam.

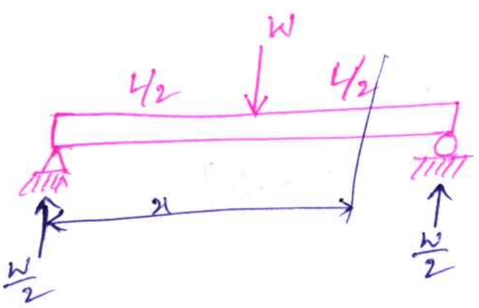
② If load is acting only part of the section write the distance in Special Brackets. " $\langle x-a \rangle$ ".

③ If -ve term comes in special bracket ignore the entire term.

④ If couple is present in part of the beam, it is to be multiplied with a distance raised to power zero.

⑤ If distribute load is present on part of the beam, it must be extended till last segment and must be compensated by introducing equal and opposite load.

⑥ Quantities with in a special bracket must be integrated as whole.



$$M_x = \frac{W}{2}x - W.\langle x-\tfrac{L}{2}\rangle$$

$$EIy'' = M_x = \frac{W}{2}x - W\langle x-\tfrac{L}{2}\rangle$$

$$EIy' = \frac{Wx^2}{4} - W\frac{\langle x-\frac{L}{2}\rangle^2}{2} + C_1 \quad -①$$

$$EIy = \frac{Wx^3}{12} - \frac{W\langle x-\frac{L}{2}\rangle^3}{6} + C_1x + C_2 \quad -②$$

@ $x=0$, $y=0$, $\Rightarrow$ $0 = 0-0+0+C_2 = 0 \Rightarrow C_2 = 0$

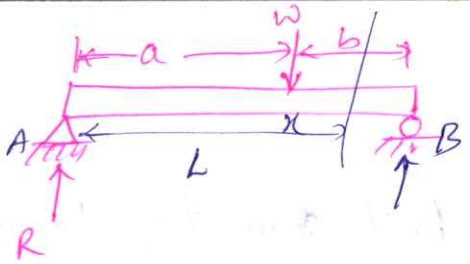
@ $x=\frac{L}{2}$, $y'=0$ $\Rightarrow$ $0 = \frac{WL^2}{16} - \frac{W(0)}{2} + C_1 \Rightarrow \boxed{C_1 = \frac{-WL^2}{16}}$

$$\boxed{EIy' = \frac{Wx^2}{4} - \frac{W\langle x-\frac{L}{2}\rangle^2}{2} - \frac{WL^2}{16}}$$

@ $x=0$, $y'_{max} = \frac{-WL^2}{16EI}$

$$\boxed{EIy = \frac{Wx^3}{12} - \frac{W\langle x-\frac{L}{2}\rangle^3}{6} - \frac{WL^2}{16}x}$$

@ $x=\frac{L}{2}$, $\boxed{y_{max} = \frac{-WL^3}{48EI}}$



$M_x = Rx - W\langle x-a\rangle$

$EIy'' = Rx - W\langle x-a\rangle$

$EIy' = \frac{Rx^2}{2} - W\frac{\langle x-a\rangle^2}{2} + C_1$

$\Sigma M_B = 0$

$R \times L = W \times b$

$R = \frac{Wb}{L}$

$EIy = \frac{Rx^3}{6} - \frac{W\langle x-a\rangle^3}{6} + C_1x + C_2$

@ $x=0,\ y=0$ | $0 = C_2 = 0$

$x=L,\ y=0$ | $0 = \frac{Wb}{L}\left(\frac{L^3}{6}\right) - \frac{W\langle L-a\rangle^3}{6} + C_1L$

$\left(\frac{-WbL^2}{6L} + \frac{W\langle L-a\rangle^3}{6L}\right) = C_1$

$C_1 = \frac{Wb}{6L}(b^2 - L^2)$

$EIy = \frac{Wb}{6L}x^3 - \frac{W\langle x-a\rangle^3}{6} + \frac{Wb}{6L}(b^2-L^2)x$

@ $x=a$

$EIy = \frac{Wba^3}{6L} + \frac{Wb}{6L}(b^2-L^2)a$

$y = \frac{Wba}{6EIL}[a^2+b^2-L^2] = \frac{Wab}{6EIL}[-2ab]$

$$y_{x=a} = -\frac{Wa^2b^2}{3LEI}$$

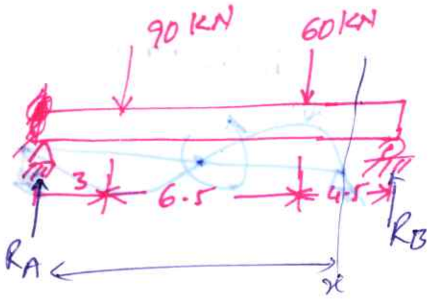
@ δ_{max}, $y'=0$

$0 = \frac{Wb}{2L}x^2 + \frac{Wb}{6L}(b^2-L^2)$

$0 = \frac{Wb}{L}\left[\frac{x^2}{2} + \frac{b^2-L^2}{6}\right] = 0$

$x^2 = \frac{(L^2-b^2)}{3} \Rightarrow x = \sqrt{\frac{L^2-b^2}{3}}$

$y_{max} = \frac{Wb[L^2-b^2]^{3/2}}{9\sqrt{3}\,L\,EI}$



Take $E = 210\ GPa$, $I = 64 \times 10^{-4}\ m^4$, $EI = 1344 \times 10^6\ N\text{-}m^2$

$R_A \times 14 = 990 + 270 \Rightarrow R_A = \frac{1260}{14} = 90\ kN$

$M_x = 90 \times x - 90\langle x - 3\rangle - 60\langle x - 9.5\rangle$

$EIy'' = 90x - 90\langle x-3\rangle - 60\langle x - 9.5\rangle$

$EIy' = \frac{90x^2}{2} - 90\frac{\langle x-3\rangle^2}{2} - 60\frac{\langle x-9.5\rangle^2}{2} + C_1$

$EIy = \frac{90x^3}{6} - 90\frac{\langle x-3\rangle^3}{6} - 60\frac{\langle x-9.5\rangle^3}{6} + C_1x + C_2$

@ $x=0$, $y=0 \Rightarrow$ $C_2 = 0$

@ $x = L$, $y=0 \Rightarrow$ $0 = 41160 - 19965 - 911.25$

$x = 14$ $C_1 = 1448.84$

$EIy = (15x^3 - 15\langle x-3\rangle^3 - 10\langle x - 9.5\rangle^3 + 1448.84x)\ 10^3$

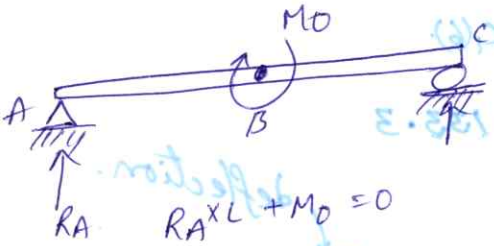
@ $x = 3$, $y_c = 2.52\ mm \downarrow$

@ $x = 9.5$ $y_D = 3.737\ mm \downarrow$

δ_{max} | $y' = 0$, $\Rightarrow$ b/w C & D. $x < 9.5$

$0 = 45x^2 - 45\langle x-3\rangle^2 + 1448.$

$x =$



$R_A \times L + M_0 = 0$

$R_A = \frac{-M_0}{L}$

$M_x = R_A x + M_0$

$= -\frac{M_0}{L}x + M_0\langle x - \frac{L}{2}\rangle^0$

$EIy'' = \frac{-M_0}{L}x + M_0\langle x - \frac{L}{2}\rangle^0$

$EIy' = \frac{-M_0}{L}\frac{x^2}{2} + M_0\langle x - \frac{L}{2}\rangle + C_1$

$EIy = -\frac{M_0}{L}\frac{x^3}{6} + M_0\frac{\langle x - \frac{L}{2}\rangle^2}{2} + C_1x + C_2$

@ $x=0$, $y=0$ $\Rightarrow C_2 = 0$

@ $x=L$, $y=0$ $\Rightarrow$ $0 = \frac{-M_0L^2}{6} + \frac{M_0L^2}{8} + C_1\frac{L}{2}$

$C_1 = \frac{M_0L}{24}$

$$EIy = \frac{-M_0 x^3}{6L} + \frac{M_0 \langle x - \frac{L}{2} \rangle^2}{2} + \frac{M_0 L}{24} x$$

@ $x=0$, $y' = \theta_A = \frac{M_0 L}{24EI}$

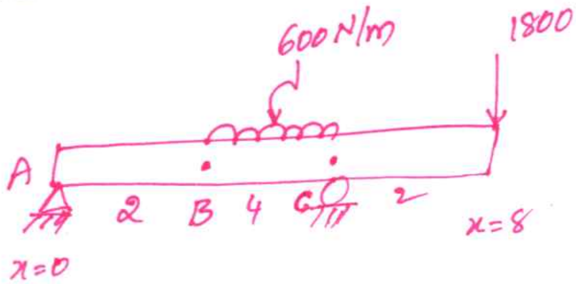
@ $x = \frac{L}{2}$, $y_B = \delta_B = 0$

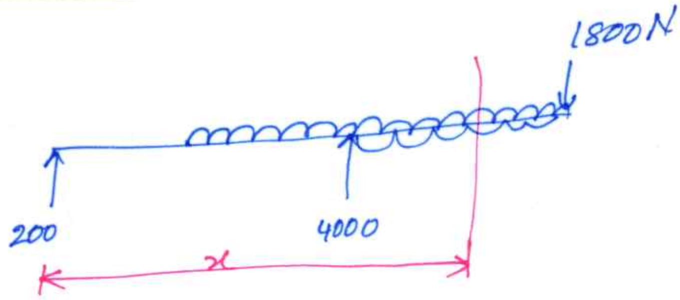
Assuming δ_{max} in AB : $x < \frac{L}{2}$

$$EIy' = 0 = -\frac{M_0 x^2}{2L} + \frac{M_0 L}{24} \Rightarrow x^2 = \frac{L^2}{12} \Rightarrow x = \frac{L}{\sqrt{12}}$$

@ $x = \frac{L}{\sqrt{12}}$

$$y_{max} = \frac{1}{EI}\left[\frac{-M_0 L^3}{6L \times 12 \times \sqrt{12}} + \frac{M_0 L}{24} \times \frac{L}{\sqrt{12}} \right] = \frac{+M_0 L^2}{36\sqrt{12}\, EI}$$





$R_A + R_C = 4200$ N

$R_A \times 6 + 1800 \times 2 = 600 \times 4 \times 2$

$R_A = 200$ N

$R_C = 4000$ N

$$M_x = 200x - 4000\langle x-6 \rangle - 600\frac{\langle x-2 \rangle^2}{2} + 600\frac{\langle x-6 \rangle^2}{2}$$

$$EIy'' = 200x - 4000\langle x-6 \rangle - 600\frac{\langle x-2 \rangle^2}{2} + 600\frac{\langle x-6 \rangle^2}{2}$$

$$EIy' = 100x^2 - 2000\langle x-6 \rangle^2 - 100\langle x-2 \rangle^3 + 100\langle x-6 \rangle^3 + C_1$$

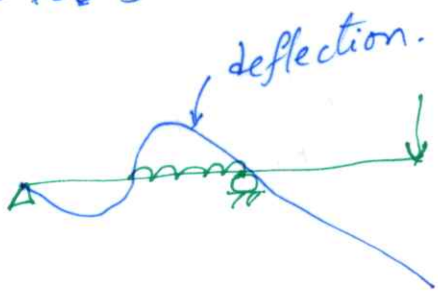
$$EIy = \frac{100}{3}x^3 - \frac{2000}{3}\langle x-6 \rangle^3 - 25\langle x-2 \rangle^4 + 25\langle x-6 \rangle^4 + C_1 x + C_2$$

@ $x=0$, $y=0$ $\Rightarrow$ $C_2 = 0$

@ $x=6$, $y=0$ $\Rightarrow$ $0 = \frac{100}{3}6^3 - 25(4)^4 + C_1(6)$

~~7200~~ $C_1 = -133.3$

@ $x=0$, $y' = \frac{1}{EI}[133.3] = \frac{133.3}{EI}$



@ $x=2$, $y=0$

@ $x=8$ $\Rightarrow$ $y = -2.962$ mm

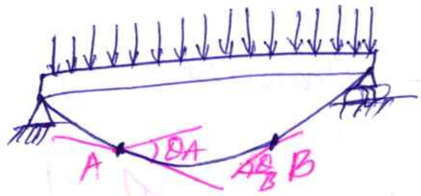
Moment Area Method :

Moment area method : useful for finding θ, δ at Specified Locations.

Theorem-1 : Area of curvature diagram b/w two points is equal to change in slope. $k = \frac{1}{R} = \frac{M}{EI}$

(gives slope)

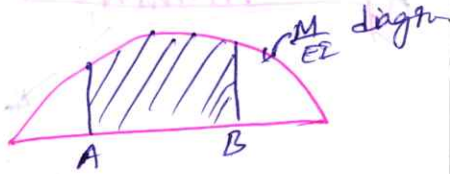
$$\theta_B - \theta_A = \int_A^B \frac{M}{EI}\, dx$$



$EI\, y'' = M$, $\quad y' = \frac{dy}{dx} = \theta$

$y'' = \frac{d\theta}{dx} = \frac{M}{EI}$ $\quad y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\theta}{dx}$

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}\, dx$$



$\theta_B - \theta_A = $ (Area under $\frac{M}{EI}$ curve AB)

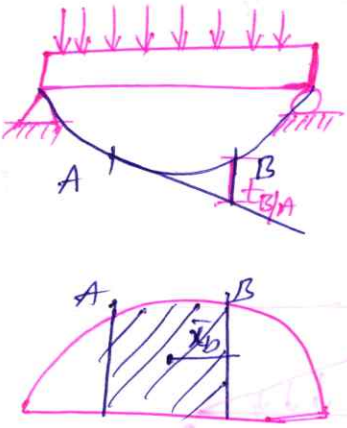
Theorem-2 : (deals with deflection δ)

→ In elastic curve AB, The vertical distance of point 'B' from the tangent to the elastic curve at A ($t_{B/A}$) is equal to 1st moment of $\frac{M}{EI}$ diagram b/w A & B taken moment about B.

$t_{x/y}$ → (vertical distance x)
→ tangent @ y
→ (Area b/w x & y
→ Moment about x)

$$t_{B/A} = \left[\int_A^B \frac{M}{EI}\, dx\right] \bar{x}_B$$

$$t_{A/B} = \left(\int_A^B \frac{M}{EI}\, dx\right) \bar{x}_A$$



$t_{A/B}$ → Tangential deviation of A w.r.to B

parabolic segment

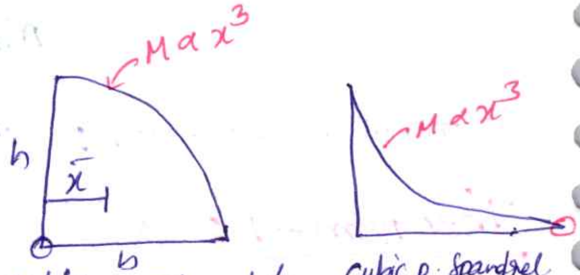
$A = \frac{2}{3}bh$

$\bar{x} = \frac{3}{8}b$

parabolic spandrel

$A = \frac{1}{3}bh$

$\bar{x} = \frac{3}{4}b$



cubic parabolic

$A = \frac{3}{4}bh$

$\bar{x} = \frac{2}{5}b$

Cubic p. spandrel

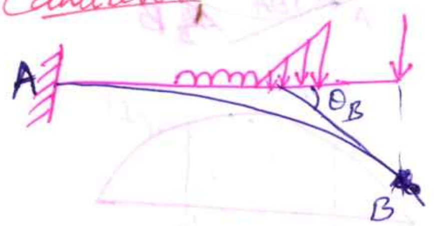
$A = \frac{1}{4}bh$

$\bar{x} = \frac{4}{5}b$

→ Draw BMD by parts [each part represents BMD of one load]

Cantilever

(* If zero slope location is known to us use Moment Area Method.

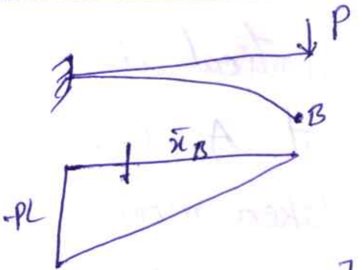


$$\theta_B - \theta_A = \left[\text{Area } \frac{M}{EI}\right]_A^B$$

0

$$\theta = \frac{ML}{EI}$$

$$\delta_{max} = \frac{ML^2}{2EI}$$

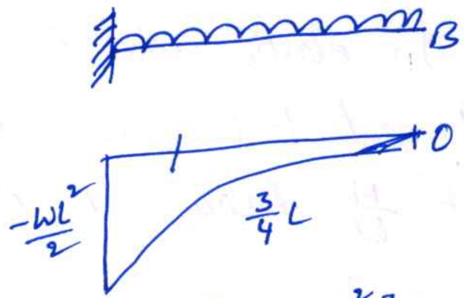


$$\theta_B - \theta_A = \frac{1}{EI}\left[\frac{1}{2} \times L \times PL\right]$$

$$\theta_B = \frac{PL^2}{2EI}$$

$$\delta_B = t_{B/A} = \left(\frac{PL^2}{2EI}\right)\frac{2}{3}L$$

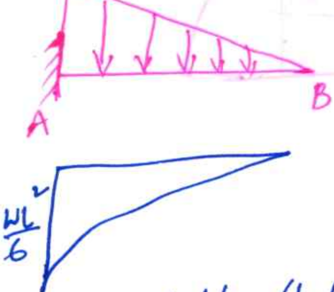
$$\delta_B = \frac{PL^3}{3EI}$$



$$\theta_B = \left(\frac{1}{3} \times L \times \frac{WL^2}{2}\right)\frac{1}{EI}$$

$$= \frac{WL^3}{6EI}$$

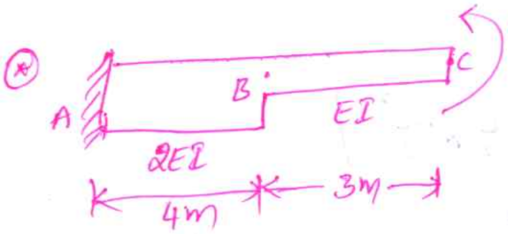
$$t_{B/A} = \delta_B = \frac{WL^3}{6EI} \times \frac{3}{4} \times L = \frac{WL^4}{8EI}$$



$$\theta_B = \frac{1}{4}bh = \left(\frac{1}{4}L \times \frac{WL^2}{6}\right)\frac{1}{EI} = \frac{WL^3}{24EI}$$

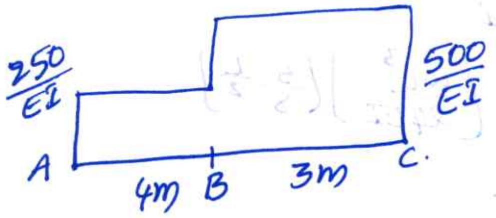
$$\boxed{\delta_B = \frac{WL^3}{24EI} \times \frac{4}{5}(L) = \frac{WL^4}{30EI}}$$

EI → stepped variation.



$E = 200\ GPa$, $I = 4 \times 10^{-6}\ m^4$

determine, $\delta_B = ?$, $\delta_C = ?$



$$\delta_B = t_{B/A} = \left[\frac{M}{EI}\ \text{area}\right]_A^B \times \bar{x}_B$$

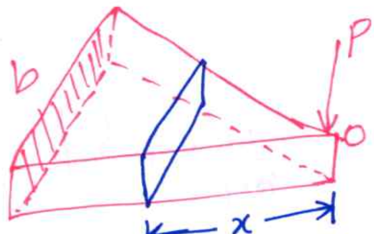
$$\delta_b = \left[\frac{250 \times 4}{EI}\right] \times 2 = 2.5\,mm$$

$$\delta_c = t_{C/A} = [A_1 x_{1C}] + A_2 x_{2C}$$

$$= \left[\frac{1000}{EI} \times 5\right] + \left[\frac{500}{EI} \times 3\right] \times 1.5$$

$$\delta_c = 9.06\,mm$$

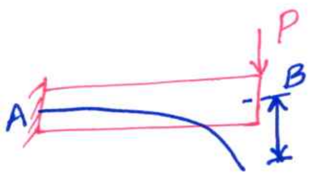
EI varies



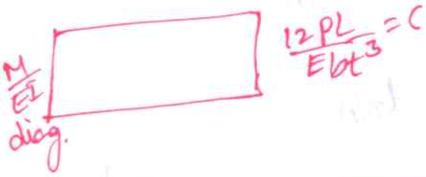
$$b_x = \frac{bx}{L}$$

$$I = \frac{bx}{L} \cdot \frac{t^3}{12}$$

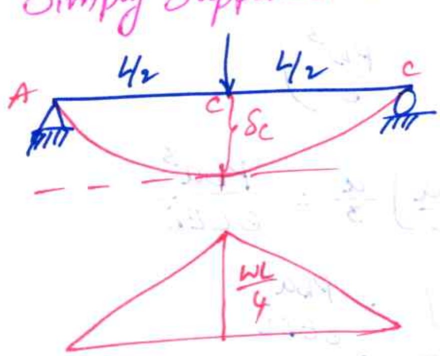
$$\frac{M}{EI} = \frac{Px}{E\left(\frac{bx\,t^3}{12L}\right)} = \frac{12PL}{Ebt^3} = \text{constant}$$



$$\delta_B = t_{B/A} = \left[\frac{12PL}{Ebt^3}\right] \times L \times \frac{L}{2} = \frac{12PL^3}{Ebt^3(2)} = \frac{6PL^3}{Ebt^3}$$

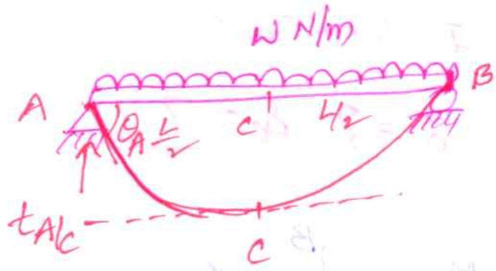


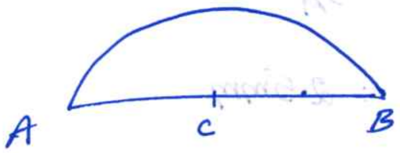
Simply Supported beams [Symmetrical Loading]



$$\theta_A = \frac{1}{EI}\left[\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4}\right] = \frac{WL^2}{16EI}$$

$$\delta_c = \left(\frac{WL^2}{16EI}\right)\left(\frac{2}{3} \times \frac{L}{2}\right) = \frac{WL^3}{48EI}$$





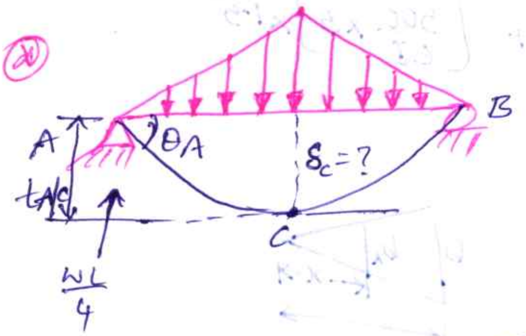
$$\theta_A = \int_A^C \frac{M}{EI}\,dx$$

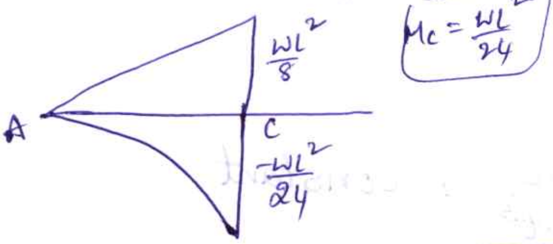
$$\theta_A = \frac{1}{EI}\left[\frac{2}{3}\ \frac{L}{2} \times \frac{wL^2}{8}\right]$$

$$\theta_A = \frac{wL^3}{24EI}$$

$$\delta_C = t_{A/C} = \left[\frac{wL^3}{24EI}\right]\left[\frac{5}{8}\frac{L}{2}\right]$$

$$\delta_C = \frac{5\,wL^4}{384\,EI}$$





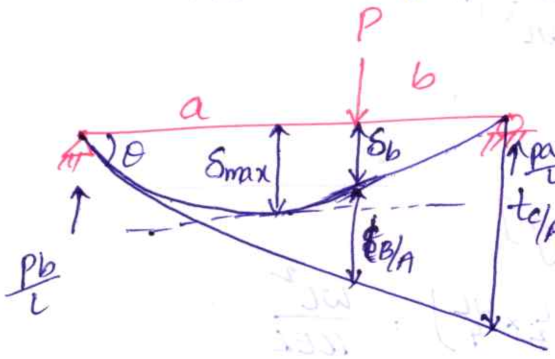
$$\theta_A - \overset{0}{\theta_C} = \int_A^C \frac{M}{EI}\,dx$$

$$\theta_A = \frac{1}{EI}\left[\frac{1}{2} \times \frac{L}{2} \times \frac{wL^2}{8}\right] - \frac{1}{EI}\left[\frac{1}{4} \times \frac{L}{2} \times \frac{wL^2}{24}\right]$$

$$= \frac{5wL^3}{192EI}$$

$$\delta_C = \frac{1}{EI}\left[\frac{wL^3}{32}\ \frac{L}{3}\right] - \frac{1}{EI}\left[\frac{wL^3}{48 \times 4} \times \frac{4}{5}\ \frac{L}{2}\right]$$

Simple supported beams [Non-Symmetrical loading]

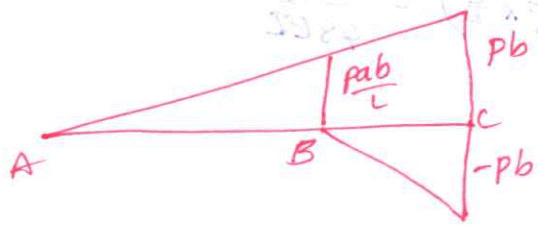


$$\tan\theta = \frac{t_{B/A} + \delta_b}{a} = \frac{t_{C/A}}{L}$$

$$\delta_b = \frac{a}{L} t_{C/A} - t_{B/A}$$

$$t_{C/A} = \frac{1}{EI}\left[\frac{1}{2} Pb \times L\right]\frac{L}{3} - \frac{1}{EI}\left[\frac{1}{2} b \times Pb \times \frac{b}{3}\right]$$

$$t_{C/A} = \frac{1}{EI}\left[\frac{PbL^2}{6} - \frac{Pb^3}{6}\right]$$

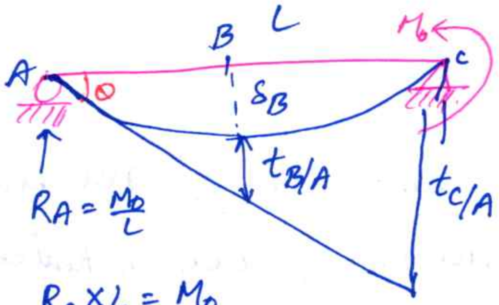


$$t_{B/A} = \frac{1}{EI}\left[\frac{1}{2} \times a\ \frac{Pba}{L}\right]\frac{a}{3} = \frac{Pba^3}{6LEI}$$

$$\delta_b = \frac{a}{L}\left[\frac{Pb}{6EI}\right]\left[L^2 - b^2\right] - \frac{Pba^3}{6EIL}$$

$$\boxed{\delta_b = \frac{Pba}{6EIL}\left[L^2 - a^2 - b^2\right]}$$

$$\delta_b = \frac{Pa^2b^2}{3EIL}$$



$R_A = \frac{M_0}{L}$

$R_A \times L = M_0$

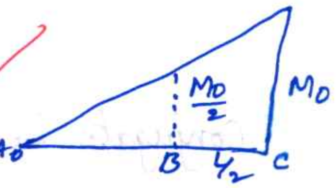
$R_A = \frac{M_0}{L}$

$R_A + R_B = 0$

$R_A = -R_B$

$\tan\theta = \frac{T_{C/A}}{L} = \frac{\delta_B + t_{B/A}}{L/2}$

$\delta_B = \frac{1}{2} t_{C/A} - t_{B/A}$



$t_{C/A} = \frac{1}{2} L \times M_0 \times \frac{L}{3}$

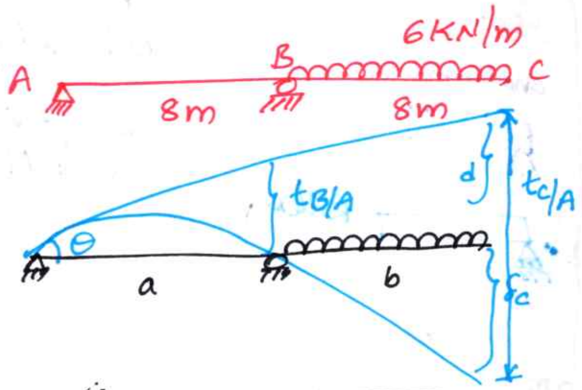
$= \frac{M_0 L^2}{6EI}$

$t_{B/A} = \left(\frac{1}{2} \times \left(\frac{L}{2}\right) \cdot \frac{M_0}{2} \times\right) \frac{L}{3\times 2}$

$= \frac{M_0 L^2}{48EI}$

$\delta_B = \frac{1}{2} \frac{M_0 L^2}{6EI} - \frac{M_0 L^2}{48EI} = \frac{3 M_0 L^2}{48EI} = \frac{M_0 L^2}{16EI}$

Q



$E = 200\,GPa$, $I = 250 \times 10^{-6} m^4$ find $\delta_C = ?$

Sol

$d = t_{C/A} - \delta_C$

$\tan\theta = \frac{d}{L} = \frac{t_{B/A}}{a}$

$\frac{t_{C/A} - \delta_C}{16} = \frac{t_{B/A}}{8}$

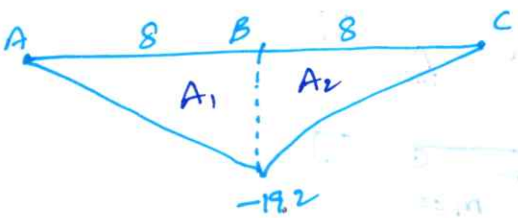
$6 \times 8 \times \frac{8}{2}$

$\boxed{\delta_C = t_{C/A} - 2t_{B/A}}$

$\delta_C = \frac{1}{EI}\left[A_1 x_{1c} + A_2 x_2\right] - 2\, t_{B/A}$

$= \frac{1}{EI}\left[\frac{1}{2} \times 192 \times 8 \times \left(8 + \frac{8}{3}\right) + \frac{1}{3}\, 192 \times 8 \times \frac{3}{4}(8)\right]$

$- \frac{1}{EI}\left(2 \times \frac{1}{2}\, 192 \times 8 \times \frac{8}{3}\right)$



$\boxed{\delta_C = 0.143 m}$

Conjugate Beam method

Conjugate beam: Imaginary beam with same length of real beam but load on the "Conjugate beam is $\frac{M}{EI}$ diagram of loads on Real beam.

→ To find slope and deflection.

- Slope at any section of R.B = S.F at that section on C.B.
- Deflection at any section of R.B = Moment at that section on C.B.

R.B	C.B
L_R =	L_C
Load →	$\frac{M}{EI}$
θ →	V } @ section
δ →	M } @ section

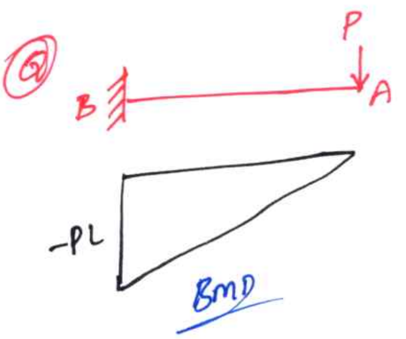
R.B	C.B
$\theta = 0$	$V = 0$
$\delta = 0$	$M = 0$
δ_{max}	M_{max}

Support conditions.

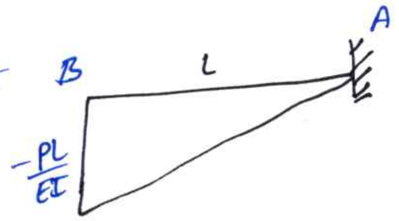
Real beam	Conjugate beam
end hinge. $\delta = 0, \theta \neq 0$	end hinge.
Roller ($\delta = 0, \theta \neq 0$)	Roller
fixed end. $\delta = 0, \theta = 0$	free end. → $M = 0$ → $V = 0$
free end $\delta \neq 0$ $\theta \neq 0$	fixed
Internal hinge	
Intermediate hinge $\delta = 0$	$M = 0$

1. H/R → H/R
2. fixed → free
3. free – fixed
4. Internal hinge – Intermediate hinge

Q



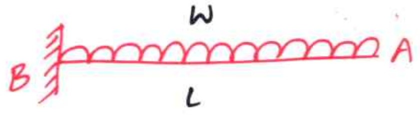
Sol.

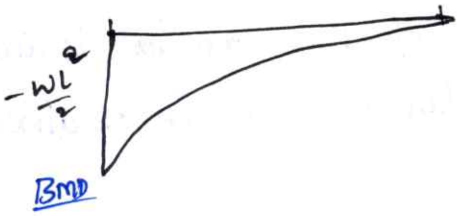


$$\theta_A = V_A = \frac{1}{2} \times L \times \frac{-PL}{EI} = \frac{PL^2}{2EI}$$

$$\delta_A = M_A = \frac{PL^2}{2EI} \times \frac{2L}{3} = \frac{PL^3}{3EI}$$

Q



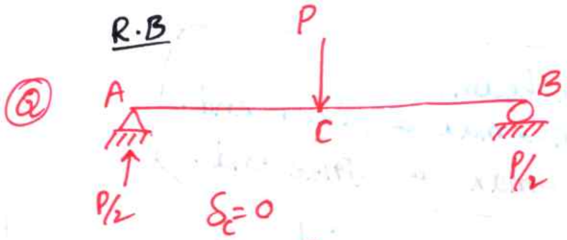


Conjugate beam.

$$\delta_A = M_A = \frac{1}{3} L \times \frac{WL^2}{2EI} \times \frac{3}{4} L$$

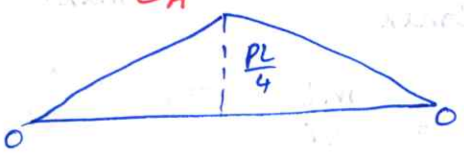
$$\delta_A = \frac{WL^4}{8EI}$$

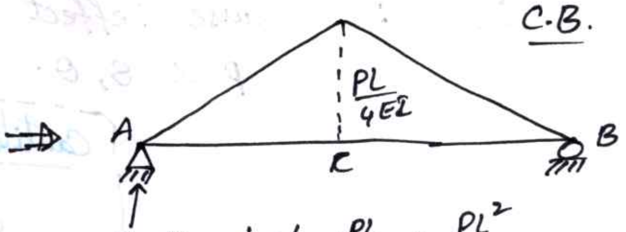
Q



$\delta_c = 0$

$\theta_A = ?$





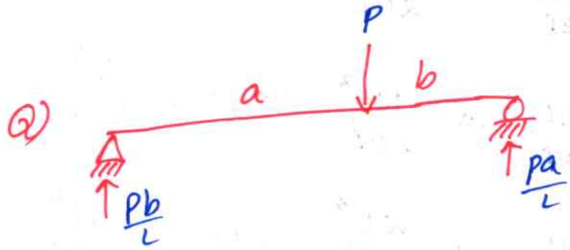
$$R_A = V_A = \frac{1}{2} \times \frac{L}{2} \times \frac{PL}{4EI} = \frac{PL^2}{16EI}$$

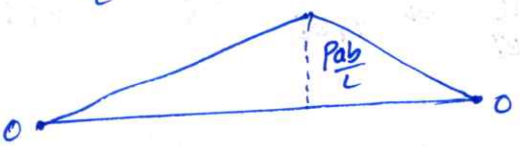
$$V_A = \frac{PL^2}{16EI}$$

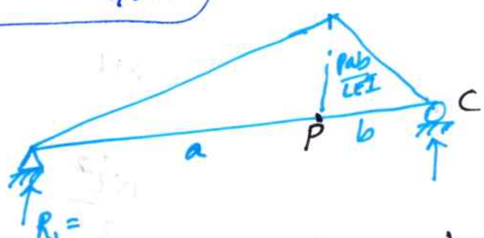
$$M_c = R_A \cdot \frac{L}{2} = \frac{PL^2}{16EI} \times \frac{L}{2} - \frac{PL^2}{16EI}\left(\frac{L}{6}\right)$$

$$\boxed{\delta_c = M_c = \frac{PL^3}{48EI}}$$

Q







$$\delta_p = M_p = R_1 \cdot a - \frac{1}{2} a \cdot \frac{Pab}{LEI} \frac{a}{3}$$

$$\Sigma M_c = 0$$

$$R_1 L = \frac{1}{2} a \cdot \frac{Pab}{LEI}\left(b + \frac{a}{3}\right) + \frac{1}{2} b \cdot \frac{Pab}{LEI}\left(\frac{2b}{3}\right)$$

$$R_1 = \frac{Pab}{6L^2EI}\left(3ab + a^2 + 2b^2\right)$$

$$\delta_p = \frac{Pa^2b}{6L^2EI}\left[3ab + a^2 + 2b^2\right] - \frac{Pa^3b}{6LEI} = \frac{Pa^2b}{6EIL^2}\left[3ab + a^2 + 2b^2 - a \cdot L\right]$$

$$\delta_p = \frac{Pa^2b}{6EIL^2}\left[2ab + 2b^2\right] = \frac{Pa^2b^2}{3EIL^2}\left[a + b\right] = \frac{Pa^2b^2}{3EIL}$$

Method of Superposition [Cantilever Beams]

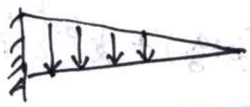
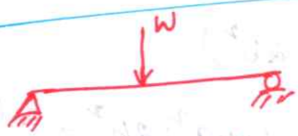
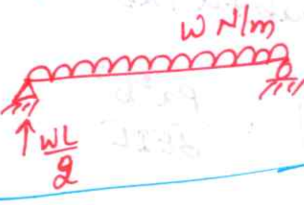
principle of Superposition : If the response of the structure is linear then effect of several loads acting simaltaneously can be obtained by adding the effect of Individual loads.

* Response → Linear : Cause ∝ effect

$P \propto \delta, \theta$.

Remember δ, θ → Simple loads
↓ obtained
δ, θ for complicated loads.

Cantilever beams
M → max @ fixed end.
δ, θ - max @ free end.

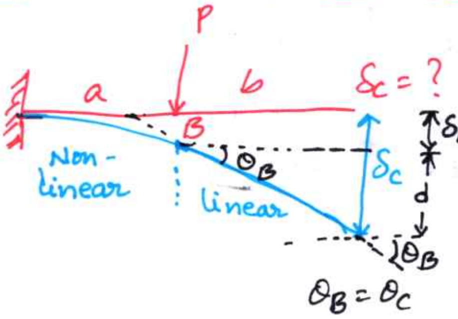
Loading	M_{max}	θ_{max}	δ_{max}
	M	$\frac{ML}{EI} = \frac{ML}{EI}$	$\theta \times \frac{1}{2}L = \frac{ML^2}{2EI}$
	WL	$\frac{ML}{2EI} = \frac{WL^2}{2EI}$	$\theta \times \frac{2}{3}L = \frac{WL^3}{3EI}$
	$\frac{WL^2}{2}$	$\frac{ML}{3EI} \Rightarrow \frac{WL^3}{6EI}$	$\theta \times \frac{3}{4}L = \frac{WL^4}{8EI}$
	$\frac{WL^2}{6}$	$\frac{ML}{4EI} \Rightarrow \frac{WL^3}{24EI}$	$\theta \times \frac{4}{5}L = \frac{WL^4}{30EI}$
	$\frac{WL}{4}$	→ $\frac{WL^2}{16EI}$	$\frac{WL^3}{48EI}$
	$\frac{WL^2}{8}$	→ $\frac{WL^3}{24EI}$	$\frac{5}{384}\frac{WL^4}{EI}$

Cantilever beams : [Complicated Loading]

Eg: ①

$$\delta_{free\ end} = \frac{wL^4}{8EI} + \frac{PL^3}{3EI}$$

Eg ②:



$$\delta_c = \delta_B + d$$

$$\tan\theta_B = \frac{d}{b} \Rightarrow \theta_B = \frac{d}{b}$$

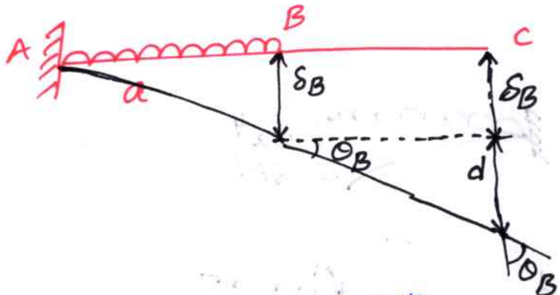
$$\delta_c = \delta_B + b\theta_B$$

$$\delta_B = \frac{Pa^3}{3EI}, \quad \theta_B = \frac{Pa^2}{2EI}$$

$$\delta_c = \frac{Pa^3}{3EI} + \frac{Pa^2 b}{2EI}$$

$$\theta_c = \theta_B = \frac{Pa^2}{2EI}$$

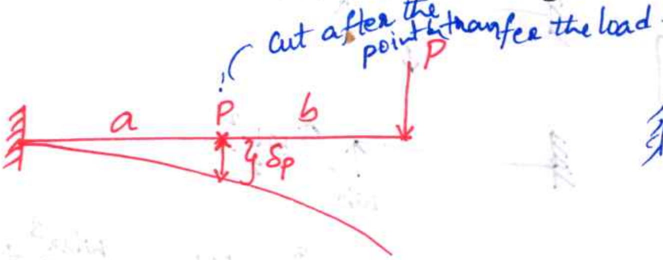
Eg -③:

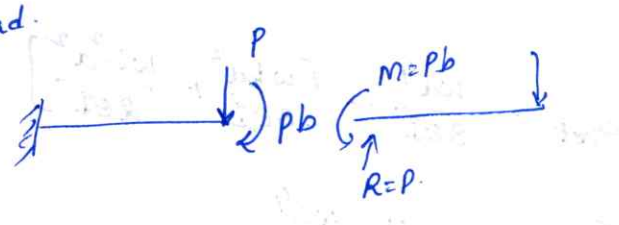


$$\delta_c = \delta_B + b\theta_B, \quad \tan\theta_B = \frac{d}{b}, \quad \theta_B = \frac{d}{b}$$

$$\delta_c = \frac{wa^4}{8EI} + \frac{wa^3 b}{6EI}$$

Eg ④

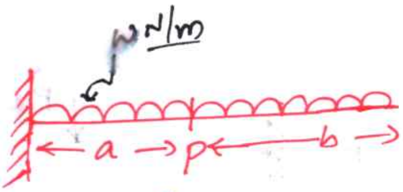


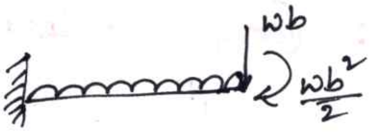


$$\delta_P = \frac{Pa^3}{3EI} + \frac{ML^2}{2EI} \Rightarrow \frac{Pa^3}{3EI} + \frac{Pba^2}{2EI}$$

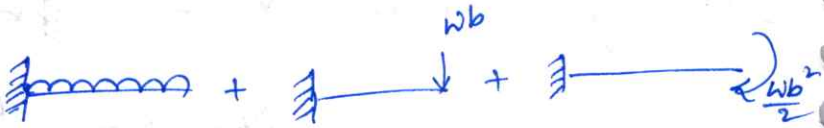
$$\theta_P = \frac{Pa^2}{2EI} + \frac{Pab}{EI}$$

Eg: 5





(δ_p)

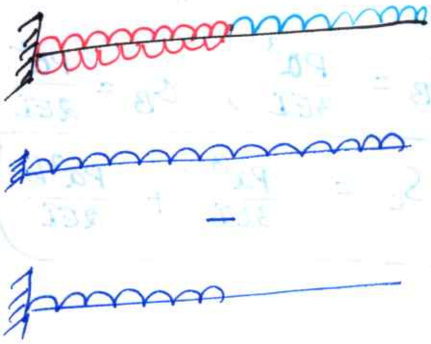


$$\delta_p = \frac{wa^4}{8EI} + \frac{wba^3}{3EI} + \frac{wb^2a^2}{4EI}$$

$$\theta_p = \frac{wa^3}{6EI} + \frac{wba^2}{2EI} + \frac{wb^2a}{2EI}$$

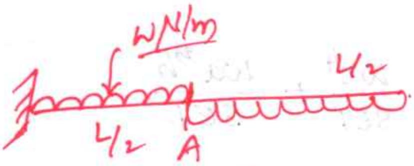
Eg 6

⇒



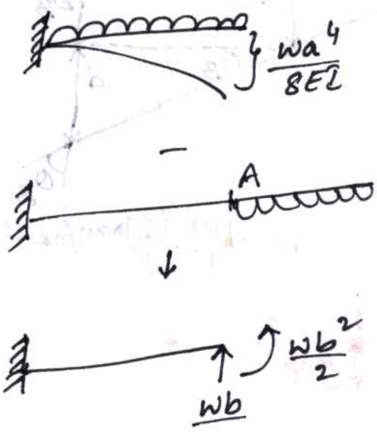
$$\delta_c = \frac{wL^4}{8EI} - \left[\frac{wa^4}{8EI} + \frac{wa^3b}{6EI}\right] \Rightarrow$$

Eg 7



find - $\delta_A = ?$

=

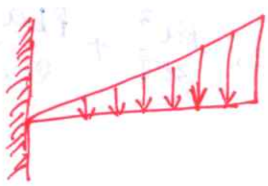


$$\delta_{net} = \frac{wa^4}{8EI} - \left[\frac{wba^3}{3EI} + \frac{wb^2a^2}{2EI}\right]$$

$$\delta_{net} = -\frac{11}{384}\frac{wL^4}{EI}$$

$$\delta_p = \frac{wba^3}{3EI} + \frac{ML}{2EI} = \frac{wba^3}{3EI} + \frac{wb^2a^2}{4EI}$$

Eg 8



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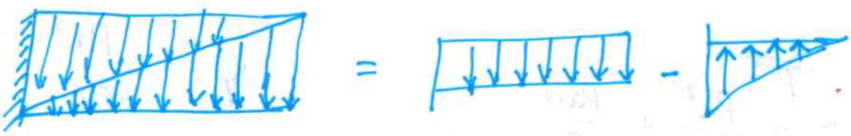


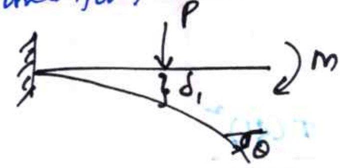
$$= \frac{wL^4}{8EI} - \frac{wL^4}{30EI}$$

$$\delta_{net} = \frac{11}{120}\frac{wL^4}{EI}$$

$$\theta = \frac{wL^3}{6EI} - \frac{wL^3}{24EI} = \frac{1\,wL^3}{8\,EI}$$

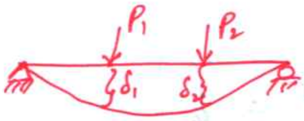
Strain Energy Method. [~~Stis~~

Castigliano's Theorem: (It is used for frames)



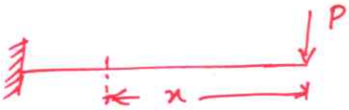
$\frac{\partial U}{\partial P_i} = \delta_i \qquad \frac{\partial U}{\partial m_i} = \theta$

$$U = \int_0^L \frac{m^2}{2EI}\, dx.$$



$\frac{\partial U}{\partial P_1} = \delta_1 \qquad \frac{\partial U}{\partial P_2} = \delta_2$

(a)



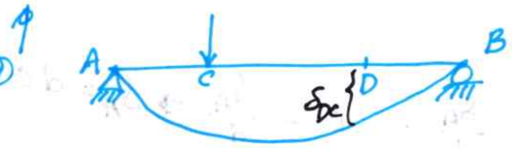
$m = -Px$

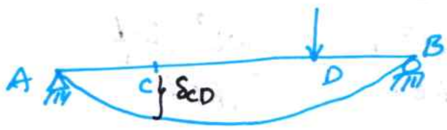
$$U = \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$\delta = \frac{\partial U}{\partial P} = \int_0^L \frac{2Px^2}{2EI} dx = \frac{Px^3}{3EI}\Big|_0^L = \frac{PL^3}{3EI}.$$

Maxwell's Law of Reciprocal deflection.

(a)

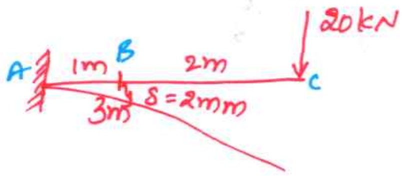




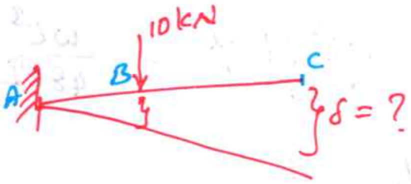
* Deflection at point D due to applied load at C (δ_{DC}) is equal to deflection at point C due to applied Load at D. (δ_{CD}).

$$\boxed{\delta_{DC} = \delta_{CD}}$$

(a)



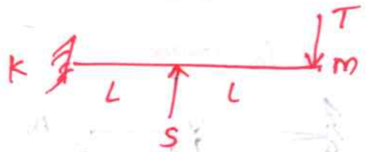
Then



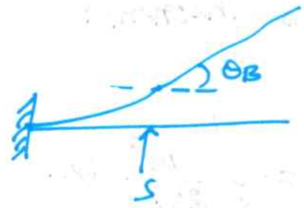
$P \propto \delta$

$\delta_c = 1mm.$

Q) for the beam-system as show, if the Slope at m is zero, then ratio of $\frac{T}{S}$ is



$\theta_c = \frac{T(2L)^2}{2EI}$



$\theta_B = \frac{SL^2}{2EI}$

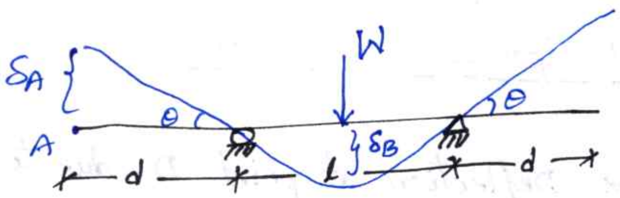
Slope @ m = 0

i.e. $\theta_C = \theta_B$

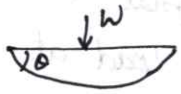
$\frac{4TL^2}{2EI} = \frac{SL^2}{2EI} \Rightarrow \boxed{\frac{T}{S} = \frac{1}{4}}$

Q) A uniform beam of length 'L' is simply supported and symmetrically supported on a span l the ratio $\frac{L}{l}$ so that the upward deflection at each end equals the downward deflection at mid span due to central point load of 'W' is ......

Sol



$\delta_A = \delta_B$



$\delta_B = \frac{Wl^3}{48EI}$

$\theta = \frac{Wl^2}{16EI}$

$\tan\theta = \frac{\delta_A}{d} \Rightarrow \theta = \frac{\delta_A}{d} \Rightarrow \delta_A = d\theta$

$\delta_A = \frac{Wl^2 d}{16EI} = \frac{Wl^2\left(\frac{L-l}{2}\right)}{16EI}$

$\frac{Wl^2\left(\frac{L-l}{2}\right)}{16EI} = \frac{Wl^3}{48EI} \Rightarrow \frac{L-l}{2} = \frac{l}{3}$

$\boxed{\frac{L}{l} = \frac{5}{3}}$

Q) A simply supported RC beam of length 10m sags while undergoing shrinkage. Assuming a uniform curvature of $0.004\ m^{-1}$ along the span, the max deflection (in m) of the beam at mid-span is ——.

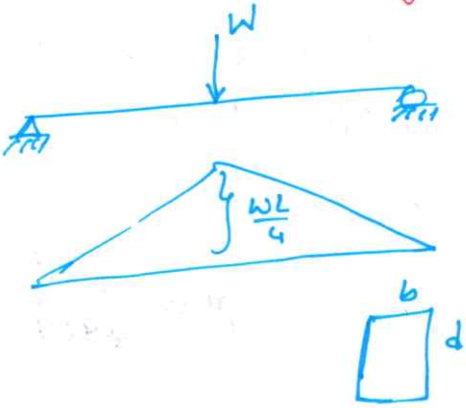
$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx} = 0.004\ m^{-1}$

for SSB $\quad y_{max} = \frac{L^2}{8R} = \frac{100(0.004)}{8} = 0.05\,m$

Q) A SS rectangular beam of span L and depth d carries a central load W. The ratio of max deflection to max. bending stress.

$$\sigma_{max} = \frac{M_{max}}{Z}$$

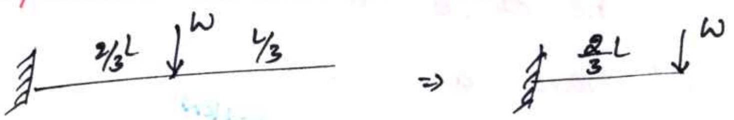
$$= \frac{WL}{4\left(\frac{bd^2}{6}\right)} = \frac{3}{2}\frac{WL}{bd^2}$$



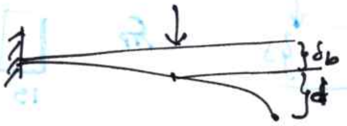
$$\delta_{max} = \frac{WL^3}{48EI} = \frac{WL^3}{48E\left(\frac{bd^3}{12}\right)} = \frac{WL^3}{4Ebd^3}$$

$$\circledast \quad \frac{\delta_{max}}{\sigma_{max}} = \frac{\frac{WL^3}{4Ebd^3}}{\frac{3}{2}\frac{WL}{bd^2}} = \frac{L^2}{6Ed}$$

Q) In a cantilever of span 'L' subjected to a concentrated load of 'W' acting at a distance of $\frac{1}{3}L$ from the free end, the deflection under load will be ______

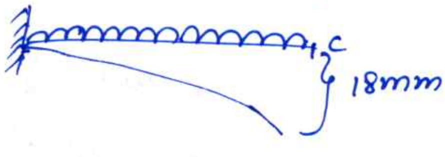


$$\frac{W\left(\frac{2L}{3}\right)^3}{3EI} = \frac{8WL^3}{81EI}$$



$$\delta_{free\ end} = \frac{Pa^3}{3EI} + \frac{Pa^2b}{2EI} \quad \left(\because a = \frac{2L}{3},\ b = \frac{L}{3}\right.$$

Q) If the deflection at the free end of a uniformly loaded cantilever beam is 18mm and the slope of the deflection curve at the free end is 0.02 radians then the length of the beam is —



$$\delta = 18mm = \frac{wL^4}{8EI}$$

$$\theta = \frac{wL^3}{6EI} = 0.02$$

$$\frac{\delta}{\theta} = \frac{6L}{8} = \frac{18\times10^{-3}}{0.02}$$

$$\boxed{L = 1.2\ m}$$

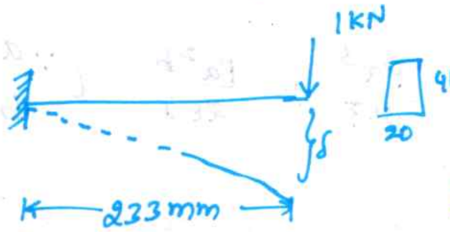
Q) A simply supported beam of width b and depth d is ~~supp~~ subjected to a point load 'W' at its centre causing deflection of y at that point. If the beam be turned such that its width becomes 'd' and depth is 'b' and it is subjected to same load at the same point then the central deflection would be ___?

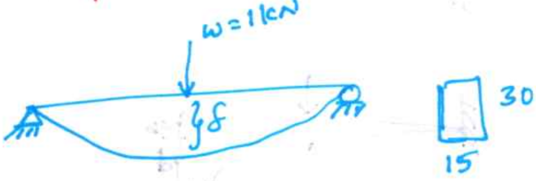
$$y = \frac{WL^3}{48EI} \Rightarrow y \propto \frac{1}{I}$$

$$\frac{y_2}{y_1} = \frac{I_1}{I_2} = \frac{\frac{bd^3}{12}}{\frac{db^3}{12}} = \frac{d^2}{b^2}$$

$$y_2 = \left(\frac{d}{b}\right)^2 y$$

Q) A cantilever beam of c/s (b×h) = 20×40 mm and length 233 mm is supporting a load of 1 kN at the free end. A simply supported beam made of same material and having a c/s (b×h) 15×30 mm with identical load at centre of its span and identical max deflection at the centre of span will have a span of ___?





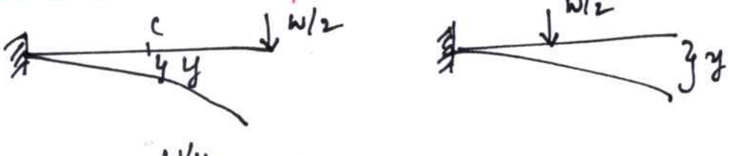
$$\delta_{cant} = \delta_{SSB}$$

$$\frac{WL_1^3}{3EI_1} = \frac{WL_2^3}{48EI_2}$$

$$\frac{L_1^3}{I_1} = \frac{L_2^3}{16I_2} \Rightarrow \frac{(233)^3}{\frac{20\times(40)^3}{12}} = \frac{L_2^3}{\frac{15\times 30^3}{12}}$$

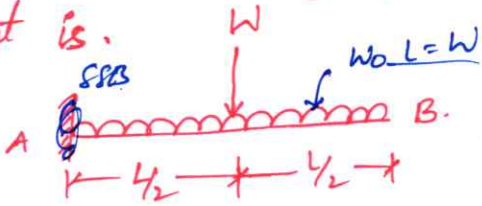
$$\boxed{L_2 = 400\ mm}$$

Q) A cantilever beam AB fixed at 'A' and carrying a load $\frac{W}{2}$ at the free end B is found to deflect by y at midpoint of AB. The deflection of B due to load $\frac{W}{4}$ at the mid point will be. $\left(\frac{y}{2}\right)$



Ans

Q) A simply supported beam of span L shown in the above fig. is subjected to a concentrated load W at its mid-span and also to a udl equivalent to W. It has a flexural rigidity of EI. The total deflection at its mid-point is.



$$\delta = \frac{5}{384}\frac{w_0 L^4}{EI} + \frac{WL^3}{48EI}$$

$$= \frac{WL^3}{EI}\left[\frac{5}{384} + \frac{1}{48}\right] = \frac{13}{384}\frac{WL^3}{EI}.$$

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EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY
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CIVIL ENGINEERING

Strength of Materials

UNIT-5

Principal Stresses and Principal plane

⇒ The planes at which the stress vector coincides with the normal of the plane are principal planes and stresses are known to be principal stresses.

let $\hat{n}$ be the principal plane with direction cosines n_x, n_y, n_z on which the stress is wholly normal.

$$\overset{n}{\vec{T}} = \sigma \vec{n}$$

$$\overset{n}{T}_x = \sigma n_x, \quad \overset{n}{T}_y = \sigma n_y, \quad \overset{n}{T}_z = \sigma n_z$$

Principal Stresses and Principal plane

from Cauchy's formula

$$\begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \\ \overset{n}{T}_z \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{Bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix},$$

on principal plane.

$$\begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \\ \overset{n}{T}_z \end{Bmatrix} = \sigma \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

Re-arranging

$$\begin{Bmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma \end{Bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = 0$$

for non-trivial solution

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma \end{vmatrix} = 0 \Rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0.$$

$$\sigma_1 > \sigma_2 > \sigma_3 > 0$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{xz} & \sigma_{zz} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{vmatrix} = \left(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} \pm \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2\right)$$

for 2-D. $\sigma^3 - \sigma^2(\sigma_{xx}+\sigma_{yy}) + \sigma(\sigma_{xx}\sigma_{yy} - \tau_{xy}^2) = 0$

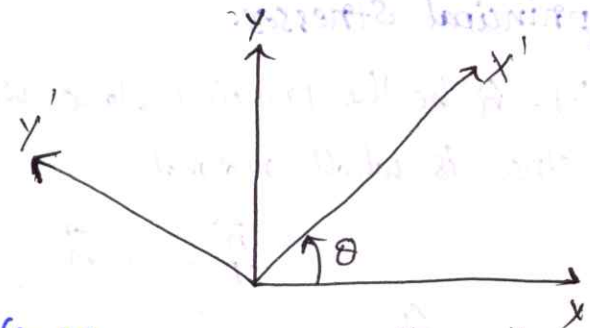
$$\sigma^2 - \sigma(\sigma_{xx}+\sigma_{yy}) + (\sigma_{xx}\sigma_{yy} - \tau_{xy})^2 = 0.$$

$$\sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\frac{(\sigma_{xx}+\sigma_{yy})^2 - 4(\sigma_{xx}\sigma_{yy} - \tau_{xy}^2)}{4}}$$

$$= \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Orientation of plane

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{pmatrix} \qquad \begin{pmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'y'} \end{pmatrix}$$



$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau_{xy}\sin 2\theta$$

$$\begin{array}{c|cc} & x & y \\ \hline x' & \cos\theta & \sin\theta \\ y' & -\sin\theta & \cos\theta \end{array}$$

$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \tau_{xy}\sin 2\theta.$$

On principal plane Shear is zero.

$\tau_{x'y'} = 0 \Rightarrow$

$$\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta = \tau_{xy}\cos 2\theta$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}\right)$$

Utility of Invariants

$\Rightarrow$ helps to Identify the nature of stress state such as uniaxial or pure shear

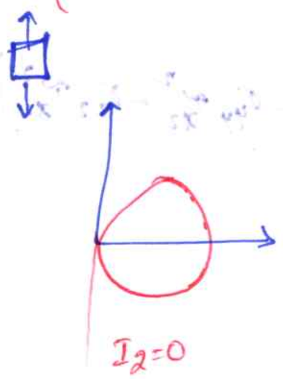
$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{pmatrix} \qquad I_1 = \sigma_{xx} + \sigma_{yy} \qquad I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix}$$

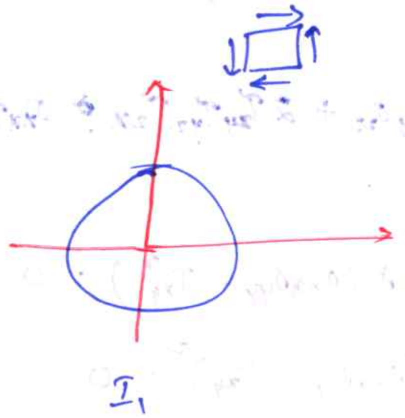
Uniaxial Stress

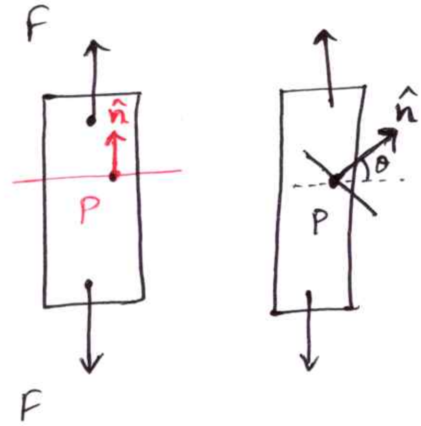
$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma_{yy} \end{pmatrix}$$

State of Pure Shear

$$\begin{pmatrix} 0 & \tau_{xy} \\ \tau_{yx} & 0 \end{pmatrix}$$







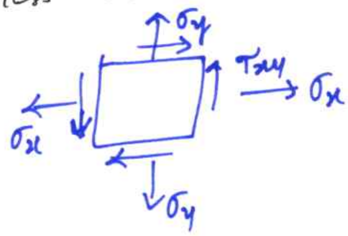
$$[\sigma] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{F}{A} \end{bmatrix}$$

$$\sigma_n = \overset{n}{T} \cdot \hat{n} = \frac{F}{A} \sin^2\theta$$

$$\overset{n}{\tau} = \sqrt{\overset{n}{T}{}^2 - \overset{n}{\sigma}{}^2} = \frac{F}{A} \sin\theta \cos\theta$$

Construction of Mohr Circle :

1. locate x which is the point representing the stress condition on the x-plane of the element ($\theta = 0^\circ$)



2. for this one has $\sigma = \sigma_x$, $\tau = \tau_{xy}$

   Sign : +ve shear on x-plane plot downwards

3. locate point y, representing the stress condition on the y face of element ($\theta = 90^\circ$)

   for this co-ordinates are, $\sigma = \sigma_y$, $\tau = \tau_{xy}$

4. Joint points x and y. This locates the center C of the circle.

   It's co-ordinates are $\sigma = \sigma_{avg}$ and $\tau = 0$

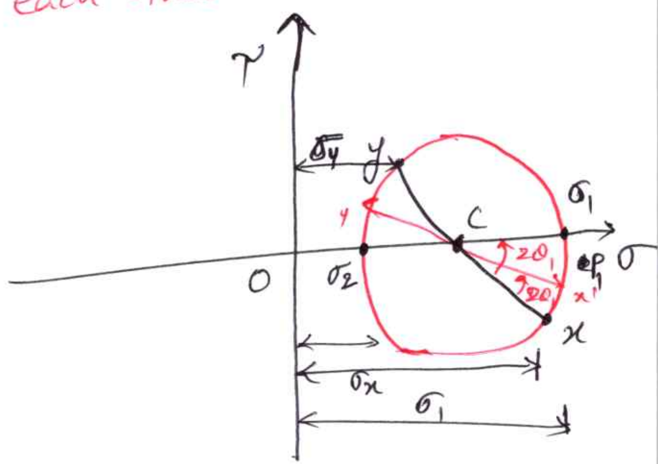
5. Draw the circle through points x and y using the centre at C.

⇒ points x and y represent planes at 90° to each other.

⇒ These are 180° apart on the circle.

$$\sigma_1 = OC + CP_1 = \frac{\sigma_x + \sigma_y}{2} + R$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Mohr's Circle for Plane Stress:

- ⇒ Given by Christian Otto Mohr.
- → It is basically developed for 2-D
- → Graphical method.
- → Used for analysis of stress, strain & inertia.
- ⇒ The transformation equations for plane stress can be represented in graphical form by a plot known as Mohr's Circle.
- ⇒ Less Accurate.

Construction of Mohr's Circle:

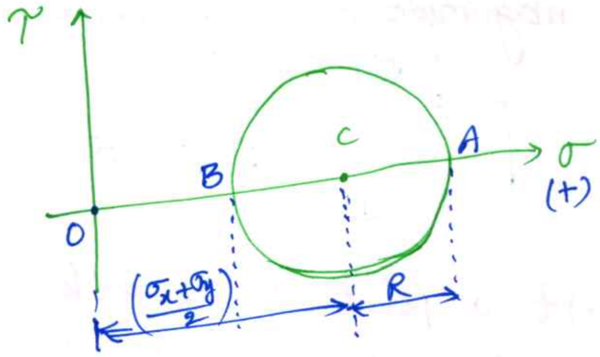
1. Distance from origin to center of Mohr's Circle $= OC = \left(\frac{\sigma_x + \sigma_y}{2}\right)$

2. Radius $= R = AC = (\tau_{max})_{In\ plane} = \left|\frac{\sigma_1 - \sigma_2}{2}\right|$ (or) $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$\sigma_1 = OA = \frac{\sigma_x + \sigma_y}{2} + R =$$

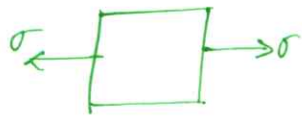
$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy}^2)}$$

$$\sigma_2 = OB = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Special Cases:

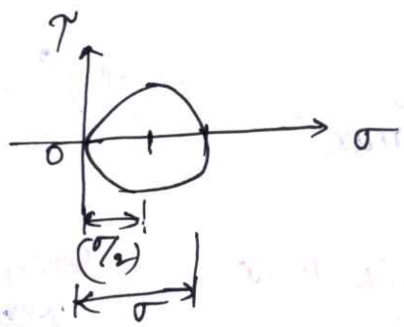
Uniaxial Stress Condition:



$\sigma_x = \sigma$ $\quad OC = \frac{\sigma}{2}$

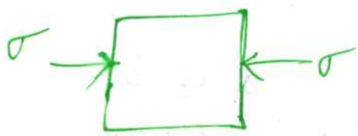
$\sigma_y = 0$ $\quad R = \frac{\sigma}{2}$

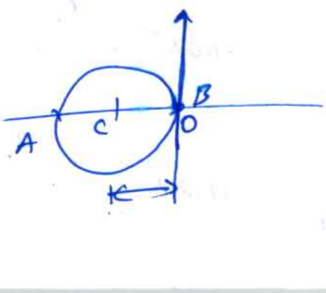
$\tau_{xy} = 0$



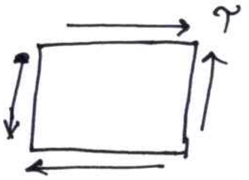
$\sigma_1 = \sigma$, $\sigma_3 = 0$, $\tau_{max} = \frac{\sigma}{2}$

Normal Stress on τ_{max} plane $\boxed{OC = \frac{\sigma}{2}}$



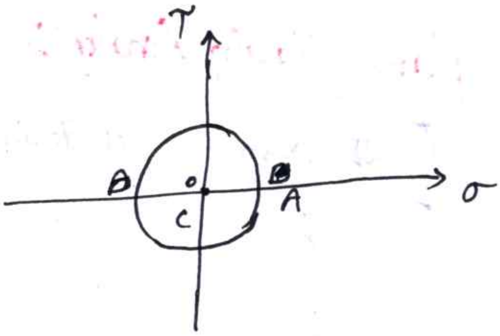


pure Shear Stress Condition



$\sigma_x = \sigma_y = 0$

$\tau_{xy} = \tau$

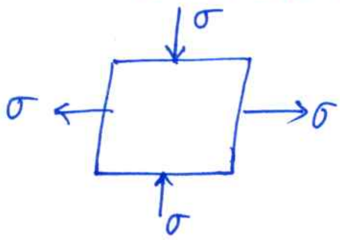


① $OC = \frac{\sigma_x + \sigma_y}{2} = 0$

② $R = \tau_{max} = \tau$

$OA = \sigma_1 = \tau$, $\sigma_2 = OB = -\tau$

pure Shear Stress Condition : [Equal & opposite normal stresses]



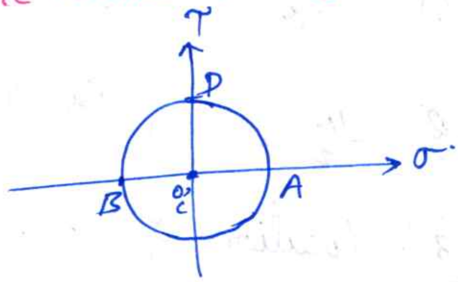
$\sigma_x = \sigma$

$\sigma_y = -\sigma$

$\tau_{xy} = 0.$

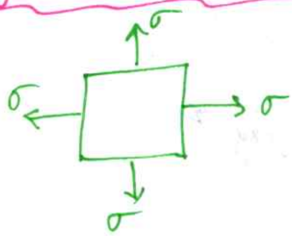
$OC = \frac{\sigma_x + \sigma_y}{2} = 0.$

$R = \tau$



$\sigma_1 = OA = \sigma$, $\sigma_2 = OB = -\sigma$

Hydrostatic Stress Condition :



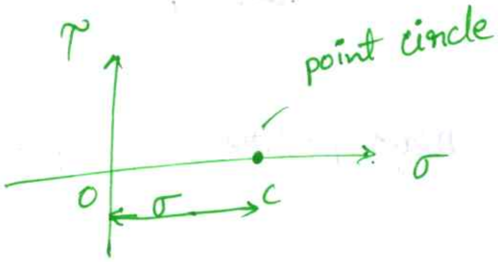
$\sigma_x = \sigma$

$\sigma_y = \sigma$

$\tau_{xy} = 0.$

$OC = \frac{\sigma + \sigma}{2} = \sigma$

$R = 0$



Q. Mohr's circle for state of Stress defined by $\begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix}$ MPa is circle with ______

Centre $= \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right)$,

$\sigma_x = 30$, $\sigma_y = 30$. $\tau_{xy} = 0$.

$OC = \frac{\sigma_x + \sigma_y}{2} = 30$,

$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 0$

Centre $= (30, 0)$

Radius = zero.

plane Strain Analysis

2D Stress	2D Strain
σ_x	ϵ_x
σ_y	ϵ_y
τ_{xy}	$\frac{\gamma_{xy}}{2}$

Strain Transformation Equations.

① $\epsilon_\theta = \frac{\epsilon_x + \epsilon_y}{2} + \left(\frac{\epsilon_x - \epsilon_y}{2}\right)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$

② $\frac{\gamma_\theta}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$

③ Location of P-1 in terms of strains.

$$\tan 2\theta_p = \frac{2\left(\frac{\gamma_{xy}}{2}\right)}{\epsilon_x - \epsilon_y} = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}.$$

④ Major p-strain = $\epsilon_1 = \frac{\epsilon_x + \epsilon_y}{2} + \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$

⑤ Minor p-strain → $\epsilon_2 = \frac{\epsilon_x + \epsilon_y}{2} - \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$

⑥ Max shear strain plane

$$\tan 2\theta_s = -\left(\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}\right)$$

⑦ $\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$, $\gamma_{max} = |\epsilon_1 - \epsilon_2|$

⑧ Normal strain, on γ_{max} plane $\left(\epsilon_{avg} = \frac{\epsilon_x + \epsilon_y}{2}\right)$

Q. An element of a certain material in plane strain has normal strains in x and y directions are 800×10^{-6}, 400×10^{-6} and shear strain in xy plane is 300×10^{-6}. What is the max shearing strain?

$\epsilon_x = 800 \times 10^{-6}$, $\epsilon_y = 400 \times 10^{-6}$, $\gamma_{xy} = 300 \times 10^{-6}$.

$$\gamma_{max} = \text{~~(800 - (400)) } 10^{-6} = 400 \times 10^{-6}\text{~~}$$

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 500 \times 10^{-6}.$$

Relation b/w principal strain & p. stress.

① $\sigma_1 = \frac{E}{1-\mu^2}(\epsilon_1 + \mu \epsilon_2)$

② $\sigma_2 = \frac{E}{1-\mu^2}(\epsilon_2 + \mu \epsilon_1)$

Theories of failure

Ductile materials → failure occurs at the onset of plastic deformation (yielding)

Brittle materials → failure occurs at fracture.
↳ $\sigma_{ultimate}$

failure theories

Max. principal stress theory —— Rankine theory

Max. Principal strain theory → St. Venant's.

Max. Shear stress theory → Tresca's & Guest Theory

Max. Strain Energy theory → Haigh & Beltrami

Max. Distortion energy theory → Von mises and Henky Theory.

① Rankine's Theory : material subjected to complex stresses will fail when max Principal stress induced is equal to yield stress in uni axial loading.

Limitations :

→ It neglect the eff. of minor & Intermediate p. stress

→ Not suitable for ductile materials.

⇒ when material subjected to pure shear, results for ductile materials are unsafe.

⇒ Results of this theory are not suitable during hydrostatic loading.

Max. Principal Strain theory (St. Venants). (MPSt.T)

⇒ failure occurs when max. strain in complex stress system equal to the yield strain under uni axial loading.

3-D $(\epsilon_1)_{3D} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$ $(\epsilon_1)_{3D} \simeq (\epsilon_{yield})_{1-D} \rightsquigarrow$ failure

$$(\epsilon_1)_{3D} = (\epsilon_{yield})_{1-D} = \frac{\sigma_{yield}}{E} \rightsquigarrow \text{failure}.$$

$$\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} = \frac{\sigma_{yield}}{E}$$

$$\sigma_1 - \mu\sigma_2 - \mu\sigma_3 = \sigma_{yield} \rightsquigarrow \text{for 3D}.$$

$$\sigma_1 - \mu\sigma_2 = \sigma_{yield} \rightsquigarrow 2D.$$

Limitation: ① In hydrostatic loading the results are not accurate.

② In Case of pure shear results are still unsafe for ductile materials. but better than MPST).

Max. Shear Stress theory (MSST): (Tresca theory).

failure occurs when max shear stress in complex stress system is equal to the shear stress at yield point under uniaxial loading.

$$(\tau_{max})_{3-D} = (\tau_{sy})_{1-D} \sim \text{failure}$$

$$(\tau_{max})_{3-D} = max\left\{\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_1-\sigma_3}{2}\right|\right\}$$

$$(\tau_{max})_{2D} = max\left\{\frac{\sigma_1-\sigma_2}{2}, \frac{\sigma_2}{2}, \frac{\sigma_1}{2}\right\}$$

$$(\tau_{sy})_{1-D} = \frac{\sigma_{yield}}{2}$$

$$\left(\frac{\sigma_1-\sigma_2}{2}\right) = \frac{\sigma_{yield}}{2}$$

$$\boxed{\sigma_1 - \sigma_2 = \sigma_{yield}} \sim \text{failure}$$

$$(\sigma_1 - \sigma_2) = \frac{\sigma_{yield}}{FOS.}$$

⇒ Recommended for ductile materials.

⇒ Not applicable for brittle materials.

Max. Strain Energy Theory (MSET) - Beltrami & Haighs theory.

failure occurs when max SE per unit volume in complex stress system equals to strain energy developed per unit volume at yielding in uniaxial loading.

$$\left(\frac{U}{V}\right)_{3D} = \frac{1}{2E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right\}$$

1-D : $\sigma_2 = 0,\ \sigma_3 = 0,\ \sigma_1 = \sigma_{yield}$.

$$\left(\frac{U}{V}\right)_{1D} = \frac{\sigma_{yield}^2}{2E}$$

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) = \sigma_{yield}^2$$

Shape - (ellipse)

2D ⇒ $\boxed{\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2 = \sigma_{yield}^2}$ ~ failure

Limitations: ⊗ Cannot be applied for brittle materials.

⊗ In case of pure shear, result are still unsafe. ductile.

Max. Distortion Energy Theory (vonmises theory):

⇒ failure occurs when the max distortion energy per unit vol. in complex stress system is equal to distortion energy per unit volume at yielding under uniaxial loading.

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2 = 2\sigma_{yield}^2$$

2D $\quad \sigma_1^2+\sigma_2^2-\sigma_1\sigma_2 = \sigma_{yield}^2$ ~ failure

⊗ can be applied to ductile

⊗ Cannot be applied to brittle

⊗ Cannot be applied for materials under hydrostatic pressure.