ANNAMACHARYA UNIVERSITY::RAJAMPET
(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)
HUMANITES AND SCIENCES

MATRIX THEORY AND CALCULUS (24AMAT11T)

I B.Tech. & I-Semester

Written by

Dr. M. Parvathi

ANNAMACHARYA UNIVERSITY

EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)

Title of the Course:	Matrix Theory and Calculus
Category:	BS
Semester:	I Semester
Course Code:	24AMAT11T
Branch/es:	Common to all branches

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	0	0	3

Course Objectives: This course aims to familiarize students with matrix theory and its practical applications, equipping them with essential tools for mathematical and engineering problem-solving. It also focuses on imparting knowledge of partial derivatives, mean value theorems, and multiple integrals to effectively address real-world engineering challenges. Additionally, students will develop proficiency in vector differentiation and integration to solve complex engineering problems.

Course Outcomes:

At the end of the course, the student will be able to

1. understand the methods for solving systems of linear equations
2. utilize matrix algebra techniques for engineering applications
3. analyze functions of several variables to optimization techniques
4. determine the area of solids using multiple integrals
5. apply vector integral theorems in evaluating double and triple integrals

Unit 1 Matrices 10

Rank of a matrix by echelon form, normal form, solving system of homogeneous and non-homogeneous linear equations, Cayley-Hamilton theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton theorem.

Unit 2 Eigen values and Eigen vectors 8

Eigen values and Eigen vectors and their properties, diagonalization of a matrix, quadratic forms and nature of the quadratic forms, reduction of quadratic form to canonical form by orthogonal transformation.

Unit 3 Mean Value Theorems & Multivariable calculus 10

Taylor's theorem and Maclaurin's theorem for one variable (without proofs) – simple problems.
Partial derivatives, total derivatives, Chain rule, change of variables, Jacobian, Maxima and Minima of functions of two variables, method of Lagrange multipliers for three variables.

Unit 4 Multiple Integrals 8

Double integrals, change of order of integration, change of variables (Cartesian to polar), areas enclosed by plane curves, evaluation of triple integrals.

Unit 5 Vector Calculus 10

Vector differentiation: Scalar and Vector point functions, vector operator Del, del applies to scalar point functions-Gradient, directional derivative, del applied to vector point functions-Divergence and Curl.
Vector integration: Line integral - work done, surface integral, Green's theorem in the plane (without proof), Stoke's theorem (without proof), volume integral, divergence theorem (without proof) and related problems.

Prescribed Textbooks:

1. E. Kreyszig. *Advanced Engineering Mathematics*. 10th Ed., John Wiley & Sons, 2011.
2. B. S. Grewal. *Higher Engineering Mathematics*, 44th Ed., Khanna Publishers, 2017.

Reference Books:

1. B. V. Ramana. *Higher Engineering Mathematics.* Mc Graw Hill Education
2. G. B. Thomas. *Maurice D. Weir and Joel* Hass. Thomas Calculus, 13th Ed., Pearson Publishers, 2013
3. R.L. Garg Nishu Gupta. *Engineering Mathematics Volumes-I &II.* Pearson Education
4. H. K. Das, Er. Rajnish Verma. *Higher Engineering Mathematics.* S. Chand

CO-PO Mapping:

Course Outcomes	Engineering Knowledge	Problem Analysis	Design/Development of solutions	Conduct investigations of complex problems	Engineering tool usage	The Engineer and world	Ethics	Individual and Collaborative teamwork	Communication	Project management and finance	Life-long learning
24AMAT11T.1	2	2	1	1	-	-	-	-	-	-	1
24AMAT11T.2	3	2	1	2	-	-	-	-	-	-	1
24AMAT11T.3	3	3	2	2	-	-	-	-	-	-	1
24AMAT11T.4	2	2	1	1	-	-	-	-	-	-	1
24AMAT11T.5	3	2	1	2	-	-	-	-	-	-	1

Title of the Course: Matrixe Theory and Calculus

Semester: I Semester

Course Code: 24AMAT11T

Branches: Common to all

Unit-1 Matrices

Matrix:- A matrix is a rectangular array of mn numbers arranged in 'm' rows and 'n' columns and bounded by brackets [ ] or ( ) or ‖ ‖. It is denoted by capital letter and the order of such matrix is $m\times n$

Thus $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n}$

where a_{ij} is the element in i^{th} row and j^{th} column.

Types of matrices

Row matrix:- A matrix having a single row is called a row matrix. Its order is always $1\times n$

Eg:- $A=[1\ 2\ 3]_{1\times 3}$

Column matrix:- A matrix having a single column is called a column matrix. Its order is always $m\times 1$

Eg:- $A=\begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times 1}$

Square matrix:- A matrix having equal number of rows and columns is called a square matrix

Eg:- $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}_{2\times 2}$

Diagonal matrix:- Diagonal matrix is a square matrix whose elements are all zeros except the principal diagonal

Eg:- $D=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

Scalar matrix:- Scalar matrix is a diagonal matrix whose all leading diagonal elements are equal.

Eg:- $A=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$

Unit matrix or Identity matrix:- Unit matrix is a scalar matrix whose all principal diagonal elements are one's & it is denoted by I or I_n

Eg:- I_3 or I $=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Null matrix or Zero matrix:- If all the elements of a matrix are zero then it is called a null matrix & it is denoted by O.

Eg:- $O=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$

Triangular matrix :- A matrix which is either upper triangular matrix or lower triangular matrix is called Triangular matrix.

Upper triangular matrix :- A square matrix in which all the elements below the principal diagonal are zeros are known as upper triangular matrix.

Eg:- $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Lower triangular matrix:- A square matrix in which all the elements above the principal diagonal are zeros are known as lower triangular matrix.

Eg:- $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

Idempotent matrix :- If A is a square matrix such that $A^2 = A$ then A is called Idempotent matrix

Nilpotent matrix :- If A is a square matrix such that $A^m = 0$, where 'm' is a +ve integer then A is called Nilpotent matrix.

Involutory matrix :- If A is a square matrix such that $A^2 = I$ then A is called Involutory matrix.

Transpose of a matrix :- The matrix obtained by interchanging its rows and columns is called the transpose of a matrix. It is denoted by A' or A^T

Eg:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Inverse of a matrix :- Inverse of a 'n' squared matrix A is denoted by A^{-1} & it is defined as $AA^{-1} = A^{-1}A = I$

Note :- 1) Inverse of A exists only if A is non-singular i.e., $|A| \neq 0$.

2.) A is singular means $|A| = 0$

Elementary transformations of a matrix :-

1.) The interchange of any two rows (columns) Eg:- $R_i \leftrightarrow R_j$

2.) The multiplication of any row (column) by a non-zero number

Eg:- $R_i \rightarrow kR_i$

3) The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Eg:- $R_i \rightarrow R_i + kR_j$

Echelon form:- A matrix is said to be in Echelon form, if

i.) zero rows, if any, are below any non-zero row

ii.) the first non zero entry in each non-zero row is equal to '1'.

iii.) the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note:- In Echelon form second condition is optional.

The Rank of a matrix in Echelon form is the number of non-zero rows of the matrix.

1.) Reduce the matrix $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$ into echelon form and hence find its rank.

Sol:- Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{pmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ is the echelon form

$\therefore$ Rank = no. of non-zero rows = 3

2) Reduce matrix $A = \begin{pmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix}$ into echelon form and hence find its rank

Sol:- Let $A = \begin{pmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix}$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{pmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$R_4 \to 6R_4 + R_3$

$$\sim \begin{pmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ is in Echelon form}$$

$\therefore$ Rank $= 4$

3) find the rank of the matrix $A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$ by reducing to Echelon form

Sol:- Given $A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$

$R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{pmatrix}$$

$R_3 \to R_3 - 5R_1$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{pmatrix}$$

$R_3 \to R_3/4$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -1 \end{pmatrix} \text{ is in Echelon form}$$

$\therefore$ Rank = no. of nonzero rows in Echelon form $= 3$

Normal form:- A matrix is said to be in normal form, if it is reduced to the form $[I_r]$ or $[I_r \; 0]$ or $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a sequence of elementary transformations.

Rank of a matrix in normal form = no. of non-zero rows or order of the matrix I.

1.) Reduce the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{pmatrix}$ to normal form and hence find its rank.

Sol:- Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{pmatrix}$

$R_2 \to R_2 - R_1$

$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}$

$R_3 \to R_3 + R_2$

$\sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$C_2 \to C_2 - C_1, \; C_3 \to C_3 - 2C_1$

$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$C_3 \to C_3 - C_2$

$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Rank = 2

2.) Find the rank of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{pmatrix}$ by reducing to normal form

Sol:- $A = \begin{pmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{pmatrix}$

$C_1 \leftrightarrow C_2$

$\sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{pmatrix}$

$R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{pmatrix}$

$R_3 \to 2R_3 - R_2$

$\sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R_2 \to R_2/2$

$$\sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_3 \to C_3 - 2C_1 \ , \ C_4 \to C_4 + 2C_1$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_2 \to C_2/2 \ , \ C_4 \to C_4/3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_3 \to C_3 - C_2 \ , \ C_4 \to C_4 - C_2$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$ is the normal form

$\therefore$ Rank $= 2$

3.) By reducing the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{pmatrix}$ into normal form, find its rank.

Sol $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{pmatrix}$

$R_2 \to R_2 - 2R_1 , \ R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{pmatrix}$$

$R_3 \to R_3/-12$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_2 \to C_2 - 2C_1 , \ C_3 \to C_3 - 3C_1 , \ C_4 \to C_4 - 4C_1$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_2 \to C_2/-3 , \ C_3 \to C_3/-2$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_3 \to C_3 - C_2 \ , \ C_4 \to C_4 + 5C_2$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_3 \leftrightarrow C_4$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} I_3 & 0 \end{pmatrix}$$

$\therefore$ Rank $= 3$

Inverse of a matrix by elementary transformations: (Gauss Jordan method)

By using elementary row operations on a non-singular matrix, we can find the inverse of a non-singular matrix. This method is known as Gauss-Jordan Method.

Working rule to find the inverse of a matrix:

Suppose A is a non-singular matrix of order n.

1) Write $A = I_n A$.
2) Apply elementary row operations only to a matrix A and the prefactor I_n of R.H.S.
3) Repeat the above process until we get $I_n = BA$, where B is the required inverse of a matrix A.

1) Find the inverse of a matrix A using elementary row operations for $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Sol: Write $A = I_4 A$, since the given matrix is of order 4.

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 2R_1 - 3R_2$, $R_3 \to 2R_3 - 11R_2$, $R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_4$, $R_3 \to R_3 + 6R_4$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -1 & -3 & 0 & -2 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3$, $R_2 \to R_2 + R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -2 & -2 & 2 & 4 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1/-2$, $R_2 \to R_2/-2$, $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$$I_4 = BA$$

$$\therefore A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}.$$

2) Find the inverse of a matrix by elementary row operations for a matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Ans $A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$.

3) Given $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find its inverse using elementary transformations.

Ans $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$..

8

Linear system of equations:

Consider a system of 'm' linear equations in 'n' unknowns say $x_1, x_2, ---, x_n$ as given below.

$$a_{11}x_1 + a_{12}x_2 + --- + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + --- + a_{2n}x_n = b_2$$
$$-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-$$
$$a_{m1}x_1 + a_{m2}x_2 + --- + a_{mn}x_n = b_m$$
, where a's & b's are constants.

The above equations in matrix form can be written as

$$\begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Rightarrow AX = B$, where A is called the co-efficient matrix,
B is the constant matrix.

The matrix $[A,B]$ is called the Augumented matrix and is given by,

$$[A,B] = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} & b_1 \\ a_{21} & a_{22} & --- & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & --- & a_{mn} & b_m \end{bmatrix}$$

The given system of equations are said to be consistent, if the system possesses one or more solutions. Otherwise the system is said to be inconsistent.

Gauss - Elimination method:

Consider the system of equation in-for matrix form as $AX=B$. Now, reduce the augumented matrix $[A,B]$ to an upper-triangular matrix. By back substitution, we get the values to the unknowns $x_1, x_2, --, x_n$ for the given linear system of equations.

1.) Express the following system of equations in matrix form and solve them by Gauss elimination method.

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10.$$

(6) Sol: The matrix form for the given system of linear eqs is $AX=B$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

Now, the augumented matrix $[A,B] = \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$

$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_2 \to R_2/9$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_3 \to R_3 + R_2,\ R_4 \to R_4 + R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -3 & 1 & 6 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right]$$

Then the matrix equivalent equation is $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -11 \\ 39 \end{bmatrix}$

$\Rightarrow 2x_1 + x_2 + 2x_3 + x_4 = 6$ — (1)

$-x_2 + x_4 = 2$ — (2)

$-x_3 - 4x_4 = -11$ — (3)

$13x_4 = 39$ — (4)

(4) $\Rightarrow x_4 = 39/13 = 3$

Put $x_4 = 3$ in (3), then (3) $\Rightarrow -x_3 - 4(3) = -11$

$\Rightarrow -x_3 = 1$

$\Rightarrow x_3 = -1$

Put $x_4 = 3$ in (2), then (2) $\Rightarrow -x_2 + 3 = 2$

$\Rightarrow x_2 = 1$

Put $x_2 = 1, x_3 = -1, x_4 = 3$ in (1), then (1) $\Rightarrow 2x_1 + 1 + 2(-1) + 3 = 6$

$\Rightarrow 2x_1 = 6 - 1 + 2 - 3 = 4$

$\Rightarrow x_1 = 2$

$\therefore$ Required $X = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

2) Solve $6x-y+z=13$; $x+y+z=9$, $10x+y-z=19$ by Gauss Elimination method.

Ans $x=2$, $y=3$, $z=4$.

3) Solve $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ by Gauss Elimination method. Ans: $x=1, y=2, z=-1$

4) Solve $x+2y+3z=1$, $2x+3y+8z=2$, $x+y+z=3$ by Gauss Elimination method Ans: $x=9/2, y=-1, z=-1/2$

5) Solve

Consistency Linear system of non-homogeneous equations:

Consider 'm' linear non-homogeneous equations with 'n' unknowns.

Let the matrix form for the linear equations be $AX=B$.

The system of linear equations are consistent iff $\rho(A)=\rho([AB])$

To find the rank of A and [A/B], reduce the augmented matrix [AB] to an upper triangular matrix Echelon form by elementary row transformations then automatically matrix A reduced to Echelon form.

Nature of the solution ($m \neq n$): The system of equations $AX=B$ are said to be

i.) Consistent, if $\rho(A)=\rho([A/B])$.

ii) Consistent and an unique solution, if $\rho(A)=\rho([A/B])=n$.

iii.) Consistent and an infinite number of solutions, if $\rho(A)=\rho([A/B])$ i.e., $r<n$

In this case, we have to give arbitrary values to '$n-r$' variables and the remaining variables can be expressed in terms of these arbitrary values.

iv.) inconsistent, if $\rho(A) \neq \rho([A/B])$

Nature of the solution when $m=n$:

Consistent and possesses solution, if A be a n-rowed non-singular matrix and $\rho(A)=\rho([AB])=n$.

1.) Discuss for what values of λ, μ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have i.) no solution ii.) a unique solution iii.) an infinite number of solutions.

Sol: The matrix form for the given system of equations is $AX=B$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Then Augumented matrix is $[A/B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$$R_2 \to R_2-R_1,\ R_3 \to R_3-R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$$R_3 \to R_3-R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i.) If $\lambda \neq 3$, then $\rho(A)=\rho([A/B])=$ number of unknowns $=3$, then system is consistent and has an unique solution ii.) If $\lambda=3, \mu=10$, then $\rho(A)=\rho([A|B])=2<3$ (number of unknowns), then the system has infinitely many solutions.

iii.) If $\lambda=3, \mu\neq 10$, then $\rho(A)\neq\rho([A|B])$ i.e., $2<3$ i.e., $2\neq 3$.
Then the system of equations are said to be inconsistent i.e., it has no solution.

2) Find whether the following system of equations are consistent. If so solve them
$x+2y+2z=2$, $3x-2y-z=5$, $2x-5y+3z=-4$, $x+4y+6z=0$.

Sol: Given eqs can be written in the matrix form as $AX=B$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

Then augumented matrix $[A|B]=\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{array}\right]$

$R_2\to R_2-3R_1$, $R_3\to R_3-2R_1$, $R_4\to R_4-R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{array}\right]$$

$R_3\to 8R_3-9R_2$, $R_4\to 4R_4+R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{array}\right]$$

$R_3\to R_3/55$, $R_4\to R_4/9$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$R_4\to R_4-R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho(A)=\rho([A|B])=3$

$\therefore$ The given system of equations is consistent and it has solution, which is unique.

The system of equations is equivalent to $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$$\Rightarrow x+2y+2z=2 \Rightarrow x=2-2y-2z=2-\cancel{2}+2$$
$$-8y-7z=-1 \Rightarrow y=\frac{-1-7}{-8}=1$$
$$+z=-1 \Rightarrow z=-1$$

$\therefore x=2, y=1, z=-1$ is the solution for the given system of equations.

3.) Find the value of λ for which the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ will have infinite number of solutions and solve them with that λ value.

Sol: The matrix form for the given system of eqs is $AX=B$.

$$\Rightarrow \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

Then augumented matrix $[A|B] = \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array}\right]$

$$R_2 \to 3R_2 - R_1,\quad R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda-8 & -9 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda+5 & 0 \end{array}\right]$$

If $\lambda=-5$, $\rho(A) = \rho([A|B]) = 2 < 3$ (no. of unknowns), then the given system of eqs is consistent and it has infinite number of solutions.

If $\lambda=-5$, then the system of equations in matrix form becomes

$$\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}.$$

$$\Rightarrow 3x-y+4z=3 \quad \text{— (1)}$$
$$7y-13z=-9. \quad \text{— (2)}$$

Put $z=k$ in (2), then (2) $\Rightarrow 7y-13k=-9$

$$\Rightarrow y = \frac{13k-9}{7} = -\frac{9}{7}+\frac{13}{7}k$$

Put $z=k$ & $y=\frac{13k-9}{7}$ in (1), then (1) $\Rightarrow 3x - \frac{13k-9}{7} + 4k = 3$

$$\Rightarrow 3x = 3-4k+\frac{13k-9}{7}$$
$$= \frac{21-28k+13k-9}{7}$$
$$= \frac{-15k+12}{7}$$
$$\Rightarrow x = \frac{-5k+4}{7}$$

$\therefore x = \frac{-5k+4}{7}$, $y = \frac{-9+13k}{7}$, $z=k$ is the required solution.

4.) Prove that the following set of equations are consistent and solve them. $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$, $2x-3y-z=5$

Ans $x=2$, $y=1$, $z=-4$

5.) Solve the system of equations $x+y+z=6$, $x-y+2z=5$, $3x+y+z=-8$.

Ans $x=-7$, $y=14/3$, $z=25/3$.

6.) Test for the consistency and solve the system $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$, $2x-2y+3z=7$. Ans $x=1$, $y=2$, $z=3$

8064334910

Linear system of homogeneous equations:

Consider the 'n' homogeneous linear equations $AX=0$.

Here the matrix A and augumented matrix $[A\,O]$ is same. Therefore $\rho(A) = \rho([A\,O])$ i.e., the system of eqs is consistent always.

Let 'r' be the rank of the matrix A.

Nature of the solution: i) If $r=n$, then the system of equations have only trivial solution i.e., zero solution.

ii) If $r<n$, then the system of equations have an infinite number of non-trivial solutions.

iii) If no. of eqs < no. of unknowns then infinite no. of solutions.

Trivial solution: Zero solution is the trivial solution.

Non-trivial solution: Non-zero solution is the non-trivial solution.

Note:- for the system of eqs $AX=0$, i) A is singular $\Rightarrow X$ is nontrivial sol. if $|A|=0$

ii) A is non-singular $\Rightarrow X$ is trivial. if $|A| \neq 0$.

1.) Solve the system of equations $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$.

Sol The given system of eqs can be written in the matrix form as $AX=0$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of nonzero rows $= 2$.

$\therefore \rho(A) = 2 < 3$ (no. of unknowns).

$\therefore$ Infinitely many non-zero solutions.

Now, Matrix eqs reduced to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\Rightarrow x+3y-2z=0 \quad -(1)$$
$$-7y+8z=0 \quad -(2)$$

Put $z=k$ in (2), then (2) $\Rightarrow y = \frac{8k}{7}$

Put $z=k$, $y=\frac{8k}{7}$ in (1), then (1) $\Rightarrow x = -3\left(\frac{8k}{7}\right) + 2k = \frac{-10}{7}k$

$\therefore X = k\begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$. For different values of k, we get different 'X' matrices.

i.e., we get infinitely many solutions for given eqs for different k values

2) Determine whether the following eqs will have a non-trivial solutions if so solve them.
$4x+2y+z+3w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$

Ans: $x=k_1$, $y=-2k_1-k_2$, $z=-k_2$, $w=k_2$, where k_1, k_2 are arbitrary constants.

3) Solve the system of equations $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$.

Ans $x=\frac{-10}{7}k$, $y=8/7\,k$, $z=k$.

3) Solve the system of eqs $x+2y-z=0$, $2x+y+z=0$, $x-4y+5z=0$.

Ans $x=-k$, $y=k$, $z=k$.

4) Solve the system of eqs $x+y+w=0$, $y+z=0$, $x+y+z+w=0$, $x+y+2z=0$.

Ans: Trivial solution.

5) Determine whether the following eqs will have a non-trivial solution, if so solve them $4x+2y+z+2w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$.

Ans: $X=k\begin{bmatrix}-1/2\\1\\0\\0\end{bmatrix}$.

6)

7) Show that the eqs $x-4y+7z=14$, $3x+8y-2z=13$, $7x-8y+26z=5$ are not consistent.

8) Test for the consistency of $x+y+z=1$, $x-y+2z=1$, $x-y+2z=5$, $2x-2y+3z=1$, $3x+y+z=2$.

Ans: Not consistency.

9) Show that the system of eqs $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$ are consistent and solve them.

Ans: $x=-1$, $y=1$, $z=2$.

10) Find the values of 'a' & 'b' for which the eqs $x+y+z=3$, $x+2y+2z=6$, $x+ay+3z=b$ have i) No solution ii) a unique solution iii) infinite number of solutions.

Ans: i) $a=-1$, $b\neq 6$
ii) $a\neq -1$, $b=$ any value
iii) $a=-1$, $b=6$

Note

The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly dependent if $\exists$ the scalars $k_1, k_2, \ldots, k_n$ not all zeros such that $k_1X_1+k_2X_2+\cdots+k_nX_n=0$

In $k_1X_1+k_2X_2+\cdots+k_nX_n=0$, all the scalars $k_1=k_2=\cdots=k_n=0$, then $X_1, X_2, \ldots, X_n$ are said to be linearly independent.

If a set of vectors is linearly dependent then atleast one member of the set can be expressed as a linear combination of the rest of the members.

1) Find the values of k for which the system of eqs $(3k-8)x+3y+3z=0$, $3x+(3k-8)y+3z=0$, $3x+3y+(3k-8)z=0$ has a non-trivial solution.

Sol: $\because$ Eqs have nontrivial sol, $|A|=0$.

$$\Rightarrow \begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$C_1 \to C_1+C_2+C_3$

$$\Rightarrow \begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0$$

$$\Rightarrow (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \qquad R_2 \to R_2-R_1,\ R_3 \to R_3-R_1$$

$$\Rightarrow (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$$\Rightarrow (3k-2)(3k-11)(3k-11)=0$$

$$\Rightarrow k = 2/3,\ 11/3,\ 11/3$$

2) If the following system has non-trivial sol P.T $a+b+c=0$ or $a=b=c$; $ax+by+cz=0$, $bx+cy+az=0$, $cx+ay+bz=0$.

Sol: If given eqs has non-trivial sol, then $|A|=0$.

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$C_1 \to C_1+C_2+C_3$

$$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\left[(c-b)(b-c)-(a-b)(a-c)\right]=0$$

$\Rightarrow (a+b+c)(b-c)c-b$.

$$\Rightarrow (a+b+c)(bc-c^2-b^2+\cancel{bc}-a^2+ac+ab-\cancel{bc})=0$$

$$\Rightarrow -(a+b+c)(a^2+b^2+c^2+ab+bc+ac)=0.$$

$$\Rightarrow (a+b+c)\left(\tfrac{1}{2}\right)\left((a-b)^2+(b-c)^2+(c-a)^2\right)=0$$

$$\Rightarrow a+b+c=0;\ a=b,\ b=c,\ c=a$$

$$\Rightarrow a+b+c=0;\ a=b=c$$

① Test for consistency and solve $5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$

Ans: $\frac{7-16k}{11}$, $\frac{3+k}{11}$, k

② Investigate the values of λ and μ so that the eqs $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ have i) no solution ii) a unique sol. iii) an infinite no. of sols.

Ans i.) $\lambda=5$ & $\mu\neq 9$

ii.) $\lambda\neq 5$, $\mu=$ any value

iii.) $\lambda=5$ & $\mu=9$

③ Test for consistency the following eqs and solve them if consistent $x-2y+3t=2$, $2x+y+z+t=-4$, $4x-3y+z+7t=8$

Ans: Inconsistent

④ For what values of k the eqs $x+y+z=1$, $2x+y+4z=k$, $4x+y+10z=k^2$ have a sol. & solve them completely in each case

Ans $k=1$, $x=-3z$, $y=2z+1$

$k=2$, $x=1-3z$, $y=2z$

⑤ S.T the eqs $3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$ do not have a solution unless $a+c=2b$

⑥ Find the values of a & b for which the eqs $x+ay+z=3$, $x+2y+2z=b$, $x+5y+3z=9$ are consistent. When will these eqs have a unique sol.

Ans: $a\neq -1$, $b=$ any value.

8) Determine the eigen values and eigen vectors of $B = 2A^2 - \frac{1}{2}A + 3I$, where $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

Sol: Given $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

then $A^2 = A \cdot A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}$.

$$B = 2A^2 - \frac{1}{2}A + 3I = 2\begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}.$$

Then the characteristic equation of B is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (111-\lambda)(-6-\lambda) + 78(39) = 0$$

$$\Rightarrow \lambda^2 + 6\lambda - 111\lambda - 111(6) + 3042 = 0$$

$$\Rightarrow \lambda^2 - 105\lambda + 2376 = 0$$

$$\Rightarrow \lambda = 72, 33 \text{ are the eigen values.}$$

for $\lambda = 72$, the eigen vector of B is given by $(B - 72I)X = 0$.

$$\Rightarrow \begin{bmatrix} 111-72 & -78 \\ 39 & -6-72 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x - 78y = 0$$

$$\Rightarrow 13x - 26y = 0$$

$$\Rightarrow x - 2y = 0$$

Put $y = k$, then $x = 2k$

$\therefore X = \begin{bmatrix} 2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $k = 1$, $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 72$.

for $\lambda = 33$, the eigen vector of B is given by $(B - 33I)X = 0$

$$\Rightarrow \begin{bmatrix} 111-33 & -78 \\ 39 & -6-33 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1/78$, $R_2 \to R_2/39$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - y = 0$$

Put $y = k$, then $x = k$

$\therefore X = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If $k = 1$, $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 33$.

9) for the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigen values of $3A^3+5A^2-6A+2I$

The characteristic equation of A is given by $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda)=0$$

$\Rightarrow \lambda = 1, 3, -2$ are the eigen values.

w.k.t, if λ is the eigen value of A, then the eigen value of $a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$.

When $\lambda=1$, then the eigen value of $3A^3+5A^2-6A+2I$ is $3(1)^3+5(1)^2+6(1)+2 = 4$.

$\lambda=2$, then the eigen value of $3A^3+5A^2-6A+2I$ is $3(2)^3+5(2)^2-6(2)+2=10$.

$\lambda=3$, then the eigen value of $3A^3+5A^2-6A+2I$ is $3(3)^3+5(3)^2-6(3)+2 = 110$.

10) find the sum and product of the eigen values of $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Sum of eigen values = trace of the matrix

$= 2+4+2 = 8$

Product of the eigen values $= |A| = 2(8)-1(6-2)-1(-4) = 16-4+4 = 16$.

Note: If the given matrix A is of order n, then the cofactors of the elements of $(A-\lambda I)$ are atmost of degree $(n-1)$ and hence the elements of $B = adj(A-\lambda I)$ are also of degree $(n-1)$.

* Cayley-Hamilton Theorem:- Every square matrix satisfies its characteristic equation.

Proof:- Let A be a square matrix of order 'n', then '$A-\lambda I$' is also a square matrix of order 'n'.

$|A-\lambda I|=0$ is the characteristic equation of A.

Let $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$ is a polynomial of order 'n'.

Then $adj(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}$ is a polynomial of order 'n-1'.

where $B_0, B_1, \ldots, B_{n-1}$ are n-rowed square matrices.

Now, $(A-\lambda I)\, adj(A-\lambda I) = |A-\lambda I| \cdot I$ $\qquad \because A \cdot adj A = |A| I$

$\Rightarrow (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n) \cdot I$

$\Rightarrow -B_0\lambda^n + (AB_0-B_1)\lambda^{n-1} + (AB_1-B_2)\lambda^{n-2} + \cdots + (AB_{n-2}-B_{n-1})\lambda + AB_{n-1} = (-1)^n\lambda^n I + (-1)^n a_1\lambda^{n-1} I + (-1)^n a_2\lambda^{n-2} I + \cdots + (-1)^n a_n I$.

Comparing the co-efficients of like powers of λ, we get

$$-B_0 = (-1)^n I$$
$$AB_0 - B_1 = (-1)^n a_1 I$$
$$AB_1 - B_2 = (-1)^n a_2 I$$
$$\cdots\cdots$$
$$AB_{n-2} - B_{n-1} = (-1)^n a_{n-1} I$$
$$AB_{n-1} = (-1)^n a_n I$$

Pre multiply the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$ and then add, then we get $-\cancel{A^nB_0} + \cancel{A^nB_0} - \cancel{A^{n-1}B_1} + \cancel{A^{n-1}B_1} - \cancel{A^{n-2}B_2} + \cdots + \cancel{A^2B_{n-2}} - \cancel{AB_{n-1}} + \cancel{AB_{n-1}} = (-1)^n(A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI)$

$$\Rightarrow (-1)^n(A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI) = 0.$$

$\therefore$ Every square matrix satisfies its characteristic equation.

7 **Note:-** The important applications of Cayley-Hamilton theorem are
1.) to find the inverse of a matrix.
2.) to find the higher powers of the matrix.

1.) Using Cayley Hamilton theorem find the inverse and A^4 of the matrix $A=\begin{bmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{bmatrix}$ and also verify Cayley Hamilton theorem

Sol:- Given $A=\begin{bmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{bmatrix}$

The characteristic eq is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}7-\lambda & 2 & -2\\ -6 & -1-\lambda & 2\\ 6 & 2 & -1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (7-\lambda)\left((-1-\lambda)^2-4\right)-2\left(-6(-1-\lambda)-12\right)-2\left(-12-6(-1-\lambda)\right)=0$$

$$\Rightarrow (7-\lambda)\left(1+\lambda^2+2\lambda-4\right)-2\left(6+6\lambda-12\right)-2\left(-12+6+6\lambda\right)=0$$

$$\Rightarrow (7-\lambda)(\lambda^2+2\lambda-3)-2(-6+6\lambda)-2(-6+6\lambda)=0$$

$$\Rightarrow 7\lambda^2+14\lambda-21-\lambda^3-2\lambda^2+3\lambda+24-24\lambda=0$$

$$\Rightarrow -\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\Rightarrow \lambda^3-5\lambda^2+7\lambda-3=0.$$

By Cayley Hamilton theorem, we know that the matrix A satisfies the characteristic equation.

i.e., we have to show that $A^3-5A^2+7A-3I=0$. —①

$$A^2=\begin{bmatrix}25 & 8 & -8\\ -24 & -7 & 8\\ 24 & 8 & -7\end{bmatrix}$$

$$A^3=\begin{bmatrix}79 & 26 & -26\\ -78 & -25 & 26\\ 78 & 26 & -25\end{bmatrix}$$

$$\therefore A^3-5A^2+7A-3I=\begin{bmatrix}79 & 26 & -26\\ -78 & -25 & 26\\ 78 & 26 & -25\end{bmatrix}-5\begin{bmatrix}25 & 8 & -8\\ 24 & -7 & 8\\ 24 & 8 & -7\end{bmatrix}+7\begin{bmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{bmatrix}-3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0.$$

$\therefore$ Cayley Hamilton theorem verified.

To find A^{-1}, multiply ① with A^{-1} on bothsides, then we get $A^{-1}(A^3-5A^2+7A-3I)=0$

$$\Rightarrow A^2-5A+7I-3A^{-1}=0$$

$$\Rightarrow A^{-1}=\frac{1}{3}\left(A^2-5A+7I\right)$$

$$=\frac{1}{3}\begin{bmatrix}-3 & -2 & 2\\ 6 & 5 & -2\\ -6 & -2 & 5\end{bmatrix}$$

To find A^4, multiply ① with A on bothsides, then $A(A^3-5A^2+7A-3I)=0$

$$A^4-5A^3+7A^2-3A=0$$

$$A^4=5A^3-7A^2+3A$$

$$=\begin{bmatrix}241 & 80 & -80\\ -240 & -79 & 80\\ 240 & 80 & -79\end{bmatrix}.$$

2.) State Cayley Hamilton theorem and use it to find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$.

Ans: $A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$.

3.) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$. Verify Cayley Hamilton theorem and hence find A^{-1}.

Ans: $\lambda^3 - 13\lambda^2 + 54\lambda - 72 = 0$, $A^{-1} = \frac{1}{36}\begin{bmatrix} 12 & -3 & -3 \\ 2 & 7 & 1 \\ -2 & 2 & 8 \end{bmatrix}$.

4.) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.

Ans $A^3 - 6A^2 - A + 22I = 0$

5) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}.

Ans $A^3 - 11A^2 - 4A + I = 0$, $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

6) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, verify Cayley Hamilton theorem. Find A^4 and A^{-1} using Cayley Hamilton theorem.

Ans $A^3 - 3A^2 - 3A + 9I = 0$, $A^{-1} = \frac{1}{9}(3A + 3I - A^2) = \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$, $A^4 = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$.

7) Using Cayley Hamilton theorem, express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A, given $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0.$$

By Cayley Hamilton theorem, we know that every square matrix A satisfies the characteristic equation i.e., $A^2 - 5A + 7I = 0$

$$\Rightarrow A^2 = 5A - 7I.$$

Then $A^3 = 5A^2 - 7A = 5(5A - 7I) - 7A = 25A - 35I - 7A = 18A - 35I$

$A^4 = 5A^3 - 7A^2 = 5(18A - 35I) - 7(5A - 7I) = 55A - 126I$

$A^5 = 5A^4 - 7A^3$

Now, $2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$

$= 7A^4 - 14A^3 + A^2 - 4I$

$= 7(55A - 126I) - 14(18A - 35I) + 5A - 7I - 4I$

$= 138A - 403I$ is the required linear polynomial in A.

8) Using Cayley Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Ans $A^2 - 5I = 0$, $A^8 = 625I$

Find the matrix P which transform the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ to diagonal form & hence calculate A^4.

Unit-2 Eigen Values and Eigen Vectors

Eigen values:- If A is any square matrix of order 'n', we can form the matrix $A-\lambda I$, where I is the n^{th} order unit matrix. The equation $|A-\lambda I|=0$ is called characteristic equation. On expanding it takes the form $(-1)^n\lambda^n+k_1\lambda^{n-1}+k_2\lambda^{n-2}+\cdots+k_n=0$. The roots of this equation are known as eigen values or characteristic roots or latent roots.

Eigen vectors:- If λ is a characteristic root of the matrix A, then a non-zero vector X suchthat $AX=\lambda X$ is called the eigen vector corresponding to the eigen value λ.

Properties of Eigen values:-

1.) Any square matrix A and its transpose A' have the same eigen values.

Proof:- $|(A-\lambda I)'| = |A'-\lambda I'|$ $\quad (\because (A+B)'=A'+B')$

$\Rightarrow |A-\lambda I| = |A'-\lambda I'| = |A'-\lambda I|$ $\quad (\because |A'|=|A|)$

$\therefore |A-\lambda I|=0$ iff $|A'-\lambda I|=0$

$\Rightarrow \lambda$ is the eigen value of A iff λ is the eigen value of A'

2.) The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof:- Let $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}$ be a triangular matrix.

Then $|A-\lambda I|=0$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$

$\Rightarrow \lambda = a_{11}, a_{22}, \ldots, a_{nn}$ are the eigen values of A.

3) The eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Proof Let $A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33}\end{pmatrix}$ be a diagonal matrix of order 3.

Then $|A-\lambda I|=0$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)=0$

$\Rightarrow \lambda=a_{11}, a_{22}, a_{33}$ are the eigen values of A.

4.) The eigen values of an idempotent matrix are either '0' or unity.

Proof:- Let A be an idempotent matrix so that $A^2 = A$.

If λ is the eigen value of A then there exist a non-zero vector X suchthat $AX = \lambda X$

Now, $AX = \lambda X \Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow A^2 X = \lambda (AX)$

$\Rightarrow AX = \lambda(\lambda X) \quad \because A^2 = A \;\&\; AX = \lambda X$

$\Rightarrow \lambda X = \lambda^2 X$

$\Rightarrow \lambda X - \lambda^2 X = 0$

$\Rightarrow (\lambda - \lambda^2) X = 0$

$\Rightarrow \lambda - \lambda^2 = 0 \quad \because X \neq 0$

$\Rightarrow \lambda(1-\lambda) = 0$

$\Rightarrow \lambda = 0, 1$ are the eigen values of an Idempotent matrix.

5.) The sum of the eigen values of a matrix is its trace.

Proof Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

then $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) + \cdots \quad — (1)$

If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of A, then

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$

$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \cdots \quad — (2)$

From (1) & (2), equating λ^2 co.efficient we have

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$

$\Rightarrow$ Sum of the eigen values = Trace

6.) The product of the eigen values of a matrix A is equal to its determinant

Proof:- Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A

Then $|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$

Put $\lambda = 0$ then we get

$|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

= Product of eigen values of A

7.) If λ is an eigen value of the matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}

Proof:- we know that if λ is the eigen value of A then there exists a non-zero vector X suchthat $AX=\lambda X$

Now, $AX=\lambda X$

$$A^{-1}(AX)=A^{-1}(\lambda X)$$
$$(A^{-1}A)X=\lambda(A^{-1}X)$$
$$IX=\lambda A^{-1}X$$
$$X=\lambda A^{-1}X$$
$$\frac{1}{\lambda}X=A^{-1}X$$

$\Rightarrow \frac{1}{\lambda}$ is the eigen value of A^{-1}.

8.) If $\lambda_1, \lambda_2, ---, \lambda_n$ are the eigen values of A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ---, \lambda_n^m$, where m being a +ve integer.

Proof:- We know that if λ_i is the eigen value of A then there exists a non-zero vector X_i suchthat $AX_i=\lambda_i X_i$, $i=1,2,---,n$

Now, $AX_i=\lambda_i X_i$

$$A(AX_i)=A(\lambda_i X_i)$$
$$(AA)X_i=\lambda_i(AX_i)$$
$$A^2X_i=\lambda_i(\lambda_i X_i)$$
$$=\lambda_i^2 X_i \quad \Rightarrow \lambda_i^2 \text{ is the eigen value of } A^2$$

|||ly $A^3X_i=\lambda_i^3X_i \Rightarrow \lambda_i^3$ is the eigen value of A^3

- - - -

|||ly λ_i^m are the eigen values of A^m, where $i=1,2,---,n$

9.) If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value

Proof:- Let A be an orthogonal matrix

Then $AA'=A'A=I$

$\Rightarrow A^{-1}=A'$

we know that If λ is the eigen value of A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}

$\Rightarrow \frac{1}{\lambda}$ is the eigen value of A' $\because A^{-1}=A'$

$\Rightarrow \frac{1}{\lambda}$ is the eigen value of A $\because A$ and A' have the same eigen values

10.) If λ is the eigen value of A, then the eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$

Proof:- We know that if λ is the eigen value of A then λ^m is the eigen value of $A^m \Rightarrow A^m X = \lambda^m X$

Now, $BX = (a_0A^2 + a_1A + a_2I)X$

$= a_0A^2X + a_1AX + a_2IX$

$= a_0\lambda^2X + a_1\lambda X + a_2X \quad \because A^mX = \lambda^mX$

$= (a_0\lambda^2 + a_1\lambda + a_2)X$

$\Rightarrow a_0\lambda^2 + a_1\lambda + a_2$ is the eigen value of $B = a_0A^2 + a_1A + a_2I$

11.) If λ is the eigen value of A then $\frac{|A|}{\lambda}$ is the eigen value of Adj. A

Proof:- We know that $A^{-1} = \frac{1}{|A|} \text{Adj } A$

$\Rightarrow A^{-1}|A| = \text{Adj } A$

$\Rightarrow A(A^{-1}|A|) = A \cdot \text{Adj} A$

$\Rightarrow I\,|A| = A \cdot \text{Adj } A$

$\Rightarrow |A| = A \cdot \text{Adj } A$

We know that if λ is the eigen value of A then there exist a non-zero ~~eige~~ vector X suchthat $AX = \lambda X$

Now, $AX = \lambda X$

$\text{Adj}A\,(AX) = \text{Adj}A(\lambda X)$

$(\text{Adj}A \cdot A)X = \lambda(\text{Adj}A \cdot X)$

$|A|\,X = \lambda \text{ Adj}A \cdot X$

$\frac{|A|}{\lambda} X = \text{Adj}A \cdot X$

(4)

Theorem:- The eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

Proof:- Let x_1 and x_2 are the eigen vectors of a matrix A corresponding to distinct eigen values of λ_1 and λ_2.

Then $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$

The $x_1, x_2, \ldots x_n$ are said to be linearly independent, if $k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0$, where $k_1 = k_2 = \cdots = k_n = 0$

Let $k_1x_1 + k_2x_2 = 0 \Rightarrow k_1x_1 = -k_2x_2$ — ①

Then $A(k_1x_1 + k_2x_2) = 0$

$\Rightarrow A(k_1x_1) + A(k_2x_2) = 0$

$\Rightarrow k_1(Ax_1) + k_2(Ax_2) = 0$

$\Rightarrow k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) = 0$

$\Rightarrow \lambda_1(k_1x_1) + \lambda_2(k_2x_2) = 0$

$\Rightarrow \lambda_1(-k_2x_2) + \lambda_2(k_2x_2) = 0$

$\Rightarrow (\lambda_2 - \lambda_1)k_2x_2 = 0$

$\Rightarrow \lambda_2 - \lambda_1 = 0 \Rightarrow \lambda_1 = \lambda_2$ and $k_2x_2 = 0$ i.e., $k_2 = 0$ $(\because x_2 \neq 0)$.

If $k_2 = 0$, then from ①, $k_1 = 0$.

Hence x_1 and x_2 are lineary independent.

Algebraic multiplicity:- If λ_1 is an eigen value of order t, then t is called the algebraic multiplicity of eigen value λ_1.

Geometric multiplicity:- If 's' is the number of linearly independent eigen vectors corresponding to the eigen value λ_1, then s is called the geometric multiplicity of eigen value λ_1.

Note:- Geometric multiplicity $\leq$ algebraic multiplicity.

1) find the eigen values and the corresponding vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Sol: Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\big((7-\lambda)(3-\lambda) - 16\big) + 6\big(-6(3-\lambda) + 8\big) + 2\big(24 - 2(7-\lambda)\big) = 0$$

$$\Rightarrow (8-\lambda)(\lambda^2 + 5 - 10\lambda) + 6(-10 + 6\lambda) + 2(10 + 2\lambda) = 0$$

$$\Rightarrow -\lambda^3 - 85\lambda + 18\lambda^2 + 40 - 60 + 36\lambda + 20 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2 + 45 - 18\lambda) = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 3\lambda - 15\lambda + 45) = 0$$

$$\Rightarrow -\lambda(\lambda - 3)(\lambda - 15) = 0$$

$\Rightarrow \lambda = 0, 3, 15.$ are the eigen values.

Eigen vector corresponding to the eigen value λ ~~are~~ given by equation $(A - \lambda I)X = 0$.

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— ①}$$

If $\lambda=0$, then the eigen vector X is obtained by putting $\lambda=0$ in ①.

Then ① $\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 8x-6y+2z=0$ — ②

$-6x+7y-4z=0$ — ③

$2x-4y+3z=0$.

To get the eigen vector solve any two eqs by cross multiplication.

Suppose eqs ②&③.

$\begin{matrix} -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{matrix}$

$$\frac{x}{24-14} = \frac{y}{-12+32} = \frac{z}{56-36}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2} = k$$

$$\Rightarrow x = k,\ y=2k,\ z=2k$$

$\therefore$ The eigen vector corresponding to the eigen value $\lambda=0$ is $X = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

If $k=1$, $X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

If $\lambda=3$, then the eigen vector x is obtained by putting $\lambda=3$ in ①

Then ① $\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 5x-6y+2z=0$

$-6x+4y-4z=0$

$2x-4y=0$

Solve first two equations by cross multiplication, then we get

$\begin{matrix} -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{matrix}$

$$\frac{x}{24-8} = \frac{y}{-12+20} = \frac{z}{20-36}$$

$$\frac{x}{\cancel{16}\,2} = \frac{y}{\cancel{8}\,1} = \frac{z}{\cancel{-16}\,-2}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = k, \text{ say}$$

$$\Rightarrow x=2k,\ y=k,\ z=-2k$$

$\therefore$ The eigen vector corresponding to the eigen value $\lambda=3$ is $X = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

If $k=1$, $X = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

If $\lambda=15$, then the eigen vector X is obtained by substituting $\lambda=15$ in ①

Then ① $\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -7x-6y+2z=0$

$-6x-8y-4z=0 \Rightarrow 3x+4y+2z=0$

$2x-4y-12z=0$

By solving first two eqs by cross multiplication, we get

$\begin{matrix} -6 & 2 & -7 & -6 \\ 4 & 2 & 3 & 4 \end{matrix}$

$$\frac{x}{-12-8} = \frac{y}{6+14} = \frac{z}{-28+18}$$

5

$$\frac{x}{\frac{-28}{2}} = \frac{y}{\frac{28}{2}} = \frac{z}{-18}, = k$$

$\Rightarrow x = -2k,\ y = 2k,\ z = -k$

$$\therefore X = \begin{bmatrix} -2k \\ 2k \\ -k \end{bmatrix} = k\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$|A-\lambda I| = 0$ is Characteristic eq.

If λ is a C.R of A, then a non-zero vector X s.t. $AX=\lambda X$ is called a E. Vector of A corresponding to the eigen value λ.

If $k=1$, then $X = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 15$.

2) Find the eigen values and the eigen vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Then the characteristic equation is $|A-\lambda I| = 0$.

Note: Eigen vector must be a non-zero vector.

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left((3-\lambda)^2-1\right)+2\left(-2(3-\lambda)+2\right)+2\left(2-2(3-\lambda)\right)=0$$

$$\Rightarrow (6-\lambda)\ (\lambda^2+8-6\lambda)+2(-4+2\lambda)+2(-4+2\lambda)=0$$

$$\Rightarrow 6\lambda^2+48-36\lambda-\lambda^3-8\lambda+6\lambda^2-8+4\lambda-8+4\lambda=0$$

$$\Rightarrow -\lambda^3+12\lambda^2-36\lambda+32=0$$

$\Rightarrow \lambda = 2, 2, 8$ are the eigen values.

The eigen vectors $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the matrix A corresponding to the eigen values λ are given by the equation $(A-\lambda I)X = 0$.

If $\lambda = 2$, the eigen vector X is given by $(A-2I)X = 0$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note: For this case, all the three eqs are same i.e, $2x-y+z=0$. So, to get the corresponding eigen vector put some arbitrary constants to any two variables (unknowns).

We can also find the eigen vectors by reducing the above eq into echelon form by using Elementary row transformations.

$$R_2 \to 2R_2 + R_1,\ R_3 \to 2R_3 - R_1$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x - 2y + 2z = 0$$

$$\Rightarrow 2x - y + z = 0 \ \text{--}\ (1)$$

Put $y = k_1,\ z = k_2$

Then (1) $\Rightarrow 2x - k_1 + k_2 = 0 \Rightarrow x = \frac{k_1 - k_2}{2}$

$\therefore X = \begin{bmatrix} \frac{k_1-k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ are the eigen vectors of A corresponding to the eigen value $\lambda = 2$, where k_1 and k_2 are arbitrary constants.

If $\lambda=8$, then the eigen vector X is given by $(A-8I)X=0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1/-2$, $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2/-3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x+y-z=0 \Rightarrow x=z-y$$
$$y+z=0 \Rightarrow y=-z$$

Put $z=k$, then $y=-k$ and
$x=k+k=2k$

$\therefore X=\begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}=k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, if $k=1$ is the eigen vector of A corresponding to the eigen value $\lambda=8$.

3) Find the eigen values and corresponding eigen vectors of $A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Ans: $\lambda=-3,-3,5$, $X_1=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2=\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $X_3=\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

4) Find the eigen values and the eigen vectors of $A=\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Ans: $\lambda=3,6,9$, $X_1=\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, $X_2=\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $X_3=\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

5) Verify that the sum of the eigen ~~vectors~~ values is equal to the trace of A for the matrix $A=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vectors.

Ans: $\lambda=2,3,6$, $X_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_2=$

6) Find the eigen values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans: $\lambda=1,3,6$, $X_1=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $X_3=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

7) Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$.

$\lambda=1,1,0$, $X_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2=\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$, $X_3=\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Diagonalization of a matrix:-

A matrix A is diagonalizable if there exists an invertible matrix P suchthat $P^{-1}AP = D$, where 'D' is a diagonal matrix or canonical diagonal form.

Note: 1.) the matrix P which diagonalizes A is called the modal matrix of A and the resulting diagonal D is called the spectral matrix of A.

2) The transformation of a matrix A to $P^{-1}AP$ is known as a similarity transformation.

Power of a matrix:-

Diagonalization is useful in finding the powers of a matrix.

Let $D = P^{-1}AP$.

Then
$$D^2 = (P^{-1}AP)(P^{-1}AP)$$
$$= P^{-1}A(PP^{-1})AP$$
$$= P^{-1}AIAP$$
$$= P^{-1}A^2P$$

Hly $D^n = P^{-1}A^nP$
$$\Rightarrow A^n = PD^nP^{-1}.$$

Hence the power of a matrix can be obtained by $A^n = PD^nP^{-1}$.

1) Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence find A^4.

Sol: The characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left((2-\lambda)(3-\lambda)+2\right) - 1\left(2 - 2(2-\lambda)\right) = 0$$

$$\Rightarrow (1-\lambda)(8+\lambda^2-5\lambda) - (-2+2\lambda) = 0$$

$$\Rightarrow 8+\lambda^2-5\lambda-8\lambda-\lambda^3+5\lambda^2+2-2\lambda = 0$$

$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6 = 0.$$

$\Rightarrow \lambda = 1, 2, 3$ are the eigen values of A.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigen vector.

If $\lambda = 1$, then the corresponding eigen vector of the matrix A is given by $(A-I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z = 0 \Rightarrow z = 0$

$x+y+z = 0 \Rightarrow x+y = 0$

Put $y = -k$, then $x = +k$.

$\therefore X = \begin{bmatrix} +k \\ -k \\ 0 \end{bmatrix} = k\begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}$. If $k=1$, $X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 1$.

If $\lambda=2$, then the corresponding eigen vector X is given by $(A-2I)X=0$

$$\Rightarrow \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2+R_1$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3+2R_1$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x-z=0 \Rightarrow x+z=0$$
$$2y-z=0$$

Put $z=k$, then $y=k/2$ and $x=-k$.

$$\therefore X = \begin{bmatrix} -k \\ k/2 \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

If $k/2=1$, $X=\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda=2$.

If $\lambda=3$, then the corresponding eigen vector X is given by $(A-3I)X=0$

$$\Rightarrow \begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x-z=0$ — ① } solving x.
$x-y+z=0$ — ②
$2x+2y=0$

solve ① & ②,

$$\begin{matrix} 0 & -1 & -2 & 0 \\ -1 & 1 & 1 & -1 \end{matrix}$$

$$\frac{x}{-1} = \frac{y}{-1+2} = \frac{z}{2-0}$$

$$\Rightarrow \frac{x}{-1} = \frac{y}{1} = \frac{z}{2} = k$$

$$\Rightarrow x=-k,\ y=k,\ z=2k$$

$$\therefore X = \begin{bmatrix} -k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

If $k=1$, $X=\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda=3$.

Writing the three eigen vectors of a matrix A as three columns, the required transformation matrix is $P=\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ which is the modal matrix.

Then $P^{-1} = \frac{-1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$

Now, $P^{-1}AP = \begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$, say.

Hence $A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1/2 \\ -1 & -1 & 0 \\ 1 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$

9

2) Diagonalize $A=\begin{bmatrix}1&1&1\\0&2&1\\-4&4&3\end{bmatrix}$ and hence find a) A^8 b) A^4.

Ans $P=\begin{bmatrix}1&1&1\\2&1&1\\-2&0&1\end{bmatrix}$, $D=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$

3) Diagonalize the matrix $A=\begin{bmatrix}8&-8&-2\\4&-3&-2\\3&-4&1\end{bmatrix}$

Ans $P=\begin{bmatrix}2&4&3\\1&3&2\\1&2&1\end{bmatrix}$, $D=\begin{bmatrix}3&0&0\\0&1&0\\0&0&2\end{bmatrix}$

4) Diagonalize the matrix $A=\begin{bmatrix}-1&2&-2\\1&2&1\\-1&-1&0\end{bmatrix}$

5) Diagonalize the matrix $A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$ Ans $D=\begin{bmatrix}0&0&0\\0&3&0\\0&0&15\end{bmatrix}$, $P=\begin{bmatrix}1&2&2\\2&1&-2\\2&-2&1\end{bmatrix}$

6) Diagonalize the matrix $A=\begin{bmatrix}5&-4&4\\12&-11&12\\4&-4&5\end{bmatrix}$ Ans $\lambda=1,1,-3$, $X_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $X_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $X_3=\begin{bmatrix}1\\3\\1\end{bmatrix}$,

$D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\end{bmatrix}$, $P=\begin{bmatrix}0&1&1\\1&0&3\\1&-1&1\end{bmatrix}$

7) Find a matrix P which transform the matrix $A=\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$ to diagonal form. Hence calculate A^4.

Ans: $\lambda=-2,3,6$, $X_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $X_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $X_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$, $D=\begin{bmatrix}-2&0&0\\0&3&0\\0&0&6\end{bmatrix}$, $P=\begin{bmatrix}-1&1&1\\0&-1&2\\1&1&1\end{bmatrix}$,

$$A^4=PD^4P^{-1}=\begin{bmatrix}251&405&235\\405&891&405\\235&405&251\end{bmatrix}.$$

Quadratic form :- A homogeneous expression of the second degree in any number of variables is called a quadratic form.

Eg:- If $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $X' = (x \;\; y \;\; z)$

then $X'AX = ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz$ is a quadratic form.

Note:- Symmetric matrix A corresponding to the quadratic form $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz$ is

$$A = \begin{bmatrix} \text{Co.eff. of } x^2 & \frac{1}{2}\text{ co.eff of } xy & \frac{1}{2}\text{ co.eff of } xz \\ \frac{1}{2}\text{ co.eff of } xy & \text{co.eff. of } y^2 & \frac{1}{2}\text{co.eff. of } yz \\ \frac{1}{2}\text{ Coeff. of } zx & \frac{1}{2}\text{ Coeff of } zy & \text{Coeff of } z^2 \end{bmatrix}$$

1.) find the symmetric matrix corresponding to the quadratic form $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx$

Sol. $A = \text{Co.eff. of} \begin{bmatrix} x^2 & \frac{1}{2}xy & \frac{1}{2}xz \\ \frac{1}{2}xy & y^2 & \frac{1}{2}yz \\ \frac{1}{2}xz & \frac{1}{2}yz & z^2 \end{bmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & \frac{5}{2} \\ 3 & \frac{5}{2} & 3 \end{pmatrix}$

2.) find the quadratic form for the real symmetric matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Sol Quadratic form $= X'AX$

$$= (x \;\; y \;\; z)\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (x \;\; y \;\; z)\begin{pmatrix} 3x - y + z \\ -x + 5y - z \\ x - y + 3z \end{pmatrix}$$

$$= 3x^2 - xy + xz - xy + 5y^2 - yz + xz - yz + 3z^2$$

$$= 3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$$

Reduction of quadratic form to canonical form by orthogonal reduction :-

1.) Write the matrix A of the given quadratic form (say A is of order 3)

2.) find the eigen values and eigen vectors for A i.e., $\lambda_1, \lambda_2, \lambda_3$; X_1, X_2, X_3

3.) Check whether X_1, X_2, X_3 are mutually orthogonal or not.
If not, find the orthogonal vectors

4.) find the normalized eigen vectors e_1, e_2, e_3

5.) Write the modal matrix $P = [e_1 \; e_2 \; e_3]$

6.) find $P^{-1}AP = P'AP = D$, say

7.) find the canonical form say $Y'DY$

8.) Orthogonal transformation is $X = PY$

Rank of quadratic form (r) :- Rank of the quadratic form is equal to the number of terms in canonical form.

Index of quadratic form (s) :- It is equal to the no. of positive terms in canonical form

Signature of the quadratic form :- It is the difference of no. of positive terms and no. of negative terms in canonical form.

Note :- Let n be the order of the matrix.

Nature of the quadratic form :-

i.) Positive definite, if $r = n$ & $s = n$ (or) if all the eigen values are positive

ii.) Negative definite, if $r = n$ & $s = 0$ (or) if all the eigen values are negative

iii.) Positive semidefinite, if $r < n$ & $s = r$ (or) if all the eigen values are positive & atleast one eigen value is zero.

iv.) Negative semidefinite, if $r < n$ & $s = 0$ (or) if all the eigen values are negative & atleast one eigen value is zero

v.) Indefinite, if eigen values are positive and negative.

1.) Reduce the quadratic form $2xy+2yz+2zx$ by orthogonal transformation and obtain the corresponding transformation. find the index, signature and nature of the quadratic form.

Sol Given quadratic form $2xy+2yz+2zx$.

then matrix of the quadratic form, $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3+3\lambda+2=0$$

$\Rightarrow \lambda = -1, -1, 2$ are the eigen values

for $\lambda=-1$, the eigen vector is given by $(A-(-1)I)X=0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x+y+z=0$$

Let $y=k_1$, $z=k_2$ then $x=-k_1-k_2$

Then $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k_1-k_2 \\ k_1 \\ k_2 \end{pmatrix} = k_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\therefore \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ are the eigen vectors at $\lambda=-1$

But these vectors are not orthogonal

Let $a\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -a-b \\ b \\ a \end{pmatrix}$ is orthogonal to $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow a+b+a=0 \Rightarrow b=-2a$

$\therefore a\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - 2a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are orthogonal vectors at $\lambda=-1$

Let $X_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ then $e_1 = \frac{X_1}{\|X_1\|} = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ then $e_2 = \frac{X_2}{\|X_2\|} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

for $\lambda = 2$, the eigen vector is given by $(A-2I)X = 0$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 + R_2$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2x + y + z = 0$$
$$-3y + 3z = 0$$

Let $z = k$ then $y = k$ &
$x = k$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the eigen vector at $\lambda = 2$

Let $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then $e_3 = \frac{X_3}{\|X_3\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

Now, $P = [e_1 \ e_2 \ e_3] = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$

Then $P^{-1} = P'$ $\because P$ is orthogonal

Now, $P^{-1}AP = P'AP = \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D, \text{ say}$$

Canonical form $= Y'DY = (y_1 \ y_2 \ y_3)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -y_1^2 - y_2^2 + 2y_3^2$

Orthogonal transformation, $X = PY = \begin{pmatrix} \frac{1}{\sqrt{6}}y_1 - \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ -\frac{2}{\sqrt{6}}y_1 + \frac{1}{\sqrt{3}}y_3 \\ \frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{pmatrix}$

Index = no. of positive terms in canonical form = 1

Signature = difference between no. of +ve terms & no. of −ve terms
$= 1 - 2 = -1$

Nature = Indefinite.

2.) Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal reduction.

Sol:- Given quadratic form is $3x^2+5y^2+3z^2-2yz+2zx-2xy$

Then matrix $A=\begin{pmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{pmatrix}$

The characteristic eq is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}3-\lambda&-1&1\\-1&5-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0$$

$$\Rightarrow -\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\Rightarrow \lambda=2,3,6 \text{ are eigen values}$$

At $\lambda=2$, the eigen vector is given by $(A-2I)X=0$

$$\Rightarrow \begin{pmatrix}1&-1&1\\-1&3&-1\\1&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow x-y+z=0$$
$$-x+3y-z=0$$
$$x-y+z=0$$

solving first two equations, $\begin{matrix}-1&1&1&-1\\3&-1&-1&3\end{matrix}$

$$\frac{x}{-2}=\frac{y}{0}=\frac{z}{2}=k \Rightarrow x=-k,\ y=0,\ z=k$$

$\therefore X_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ is the eigen vector at $\lambda=2$ & $e_1=\begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

At $\lambda=3$, the eigen vector is given by $(A-3I)X=0$

$$\begin{pmatrix}0&-1&1\\-1&2&-1\\1&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow -y+z=0 \quad -(1)$$
$$-x+2y-z=0 \quad -(2)$$
$$x-y=0 \quad -(3)$$

From (1) & (2) $\begin{matrix}-1&1&0&-1\\2&-1&-1&2\end{matrix}$

$$\frac{x}{-1}=\frac{y}{-1}=\frac{z}{-1}=k \Rightarrow x=k,\ y=k,\ z=k$$

$\therefore X_2=\begin{pmatrix}1\\1\\1\end{pmatrix}$ is the eigen vector at $\lambda=3$ and $e_2=\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$

At $\lambda=6$, the eigen vector is given by $(A-6I)X=0$

$$\begin{pmatrix}-3&-1&1\\-1&-1&-1\\1&-1&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-3x-y+z=0 \quad -(1)$$
$$-x-y-z=0 \quad -(2)$$
$$x-y-3z=0 \quad -(3)$$

from (1) & (2) $\begin{matrix}-1&1&-3&-1\\-1&-1&-1&-1\end{matrix}$

$$\frac{x}{2}=\frac{y}{-4}=\frac{z}{2}=k \Rightarrow x=k,\ y=-2k,\ z=k$$

$\therefore X_3=\begin{pmatrix}1\\-2\\1\end{pmatrix}$ is the eigen vector at $\lambda=6$ and $e_3=\begin{pmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$

Here X_1, X_2, X_3 are pairwise orthogonal.

Then $P = [e_1 \ e_2 \ e_3] = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$

Now, $P^{-1}AP = P'AP = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D$

Canonical form $= Y'DY$

$= (y_1 \ y_2 \ y_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$= 2y_1^2 + 3y_2^2 + 6y_3^2$

Unit-3 Mean Value Theorems & Multivariable Calculus

Continuous function:- A function $f(x)$ is said to be continuous at a point $x=a$, if right hand limit at $x=a$ is equal to the left hand limit at $x=a$ and equal to the actual value of the function at $x=a$

i.e., $\underset{x\to a^+}{Lt} f(x) = \underset{x\to a^-}{Lt} f(x) = f(a)$

Continuity in an interval:- A function $f(x)$ is said to be continuous on an interval I, if f is continuous at each point of I.

Discontinuity at a point:- A function $f(x)$ is said to be discontinuous at $x=a$, if i.) $f(x)$ is not defined at $x=a$

ii.) $\underset{x\to a}{Lt} f(x)$ does not exist

iii.) $f(x)$ is defined at $x=a$ and $\underset{x\to a}{Lt} f(x) \neq f(a)$

Derivable function:- A function $f(x)$ is said to be derivable at the point $x=a$, if $\underset{x\to a^+}{Lt} \frac{f(x)-f(a)}{x-a} = \underset{x\to a^-}{Lt} \frac{f(x)-f(a)}{x-a}$

Taylor's theorem:- If i.) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a+h]$ and

ii.) $f^n(x)$ exists for every value of x in $(a, a+h)$

then there is atleast one number θ say $(0<\theta<1)$ such that

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^n(a+\theta h)$ —— ①

is called Taylor's theorem with Lagrange's form remainder

If we put $a=0$ and $h=x$ in ①, then we get

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(\theta x)$ is known as Maclaurin's theorem with Lagrange's form of remainder

If we put $a+h=x$ in ①, we get

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$ is known as Taylor's series

If we put $a=0$ in the above we get

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$ is known as Maclaurin's series

1.) Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e 1.1$ correct to 4 decimal places

Sol:- Let $f(x) = \log_e x$ $\qquad f(1) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{IV}(x) = \frac{-6}{x^4}$ $\qquad f^{IV}(1) = -6$

----

w.k.t by Taylor's series

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Put $a=1$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$$\Rightarrow \log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Put $x = 1.1$, then we get

$$\log_e 1.1 = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \cdots = 0.0953$$

2.) Calculate approximate value of $\sqrt{10}$ correct to 4 decimal places using Taylor's theorem

Sol:- Let $f(x) = \sqrt{x}$ then $f(a+h) = \sqrt{a+h}$

Assume $a = 9$ & $h = 1$

Now, $f(x) = \sqrt{x}$ $\qquad f(9) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'(9) = \frac{1}{6}$

$f''(x) = \frac{-1}{4x^{3/2}}$ $\qquad f''(9) = \frac{-1}{108}$

$f'''(x) = \frac{3}{8x^{5/4}}$ $\qquad f'''(9) = \frac{1}{648}$

w.k.t by Taylor's theorem, $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$

$$\Rightarrow \sqrt{10} = f(9) + 1 \cdot f'(9) + \frac{1^2}{2!} f''(9) + \frac{1^3}{3!} f'''(9) + \cdots$$

$$= 3 + \frac{1}{6} + \frac{1}{2}\left(\frac{-1}{108}\right) + \frac{1}{3!}\left(\frac{1}{648}\right) + \cdots = 3.1624$$

3.) Obtain the Taylor's series expansion of $\sin x$ in powers of $x-\frac{\pi}{4}$

Sol:- Let $f(x)=\sin x$, $a=\frac{\pi}{4}$ then $f(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f'(x) = \cos x \qquad f'(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \qquad f''(\pi/4) = -1/\sqrt{2}$

$f'''(x) = -\cos x \qquad f'''(\pi/4) = -1/\sqrt{2}$

$f^{IV}(x) = \sin x \qquad f^{IV}(\pi/4) = 1/\sqrt{2}$

- - - - - - - -

w.k.t Taylor's series is given by

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

put $a=\pi/4$

$$= f(\pi/4) + \frac{x-\pi/4}{1!} f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \cdots$$

$$\Rightarrow \sin x = \frac{1}{\sqrt{2}} + (x-\pi/4)\frac{1}{\sqrt{2}} + \frac{(x-\pi/4)^2}{2!}(-1/\sqrt{2}) + \frac{(x-\pi/4)^3}{3!}(-1/\sqrt{2}) + \cdots$$

4.) show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

Sol:- Let $f(x) = \log(1+e^x)$ & $a=0$ then $f(0) = \log 2$

Now, $f'(x) = \frac{e^x}{1+e^x} \qquad f'(0) = \frac{1}{2}$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \qquad f''(0) = \frac{1}{4}$$

$$= \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3} \qquad f'''(0) = 0$$

- - - - - -

By Maclaurin's series we know that

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots$$

$$\Rightarrow \log(1+e^x) = \log 2 + \frac{x}{1!}\cdot\frac{1}{2} + \frac{x^2}{2!}\frac{1}{4} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \cdots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

D.w.r.t 'x', we get

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

5) Using Maclaurin's series, expand $\tan x$ upto the term containing x^5

Sol:- $f(x) = \tan x$ $\quad f(0) = 0$

$f'(x) = \sec^2 x = 1+\tan^2 x$ $\quad f'(0) = 1$

$f''(x) = 2\tan x \sec^2 x$ $\quad f''(0) = 0$

$= 2\tan x(1+\tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$ $\quad f'''(0) = 2$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 8\tan^2 x + 6\tan^4 x$

lly $f^{IV}(x) = 16\tan x + 40\tan^3 x + 24\tan^5 x$ $\quad f^{IV}(0) = 0$

$f^{V}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x$ $\quad f^{V}(0) = 16$

$- - - - -$

w.k.t by Maclaurin's series

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$\Rightarrow \tan x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

6) Expand $\sin x$ in powers of $(x-\pi/2)$. Hence find the value of $\sin 91^\circ$ correct to 4 decimal places

Sol:- $f(x) = \sin x$ $\quad f(\pi/2) = 1$

$f'(x) = \cos x$ $\quad f'(\pi/2) = 0$

$f''(x) = -\sin x$ $\quad f''(\pi/2) = (-1)$

$f'''(x) = -\cos x$ $\quad f'''(\pi/2) = 0$

$f^{IV}(x) = \sin x$ $\quad f^{IV}(\pi/2) = 1$

w.k.t by Taylor's series at $x = \pi/2$

$$f(x) = f(\pi/2) + \frac{x-\pi/2}{1!}f'(\pi/2) + \frac{(x-\pi/2)^2}{2!}f''(\pi/2) + \frac{(x-\pi/2)^3}{3!}f'''(\pi/2) + \cdots$$

$$\sin x = 1 + (x-\pi/2)(0) + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}(0) + \frac{(x-\pi/2)^4}{4!}(1) + \cdots$$

Put $x = 91^\circ$

$$\Rightarrow \sin 91^\circ = 1 + 0 + \frac{(1^\circ)^2}{2!}(-1) + 0 + \frac{(1^\circ)^4}{4!} + \cdots$$

$$= 1 - \frac{(0.0174)^2}{2} + \frac{(0.0174)^4}{24} + \cdots$$

$$= 0.9998$$

Partial derivatives:- Let $z=f(x,y)$ be a function of two variables 'x' and 'y'. The derivative of z w.r.t 'x', treating y as constant is called partial derivative of z w.r.t 'x' and it is denoted by $\frac{\partial z}{\partial x}$ or z_x.
Illy the derivative of z w.r.t 'y', treating x as constant is called partial derivative of z w.r.t 'y' and it is denoted by $\frac{\partial z}{\partial y}$ or z_y.

1.) Find the first and second order partial derivatives of $z=x^3+y^3-3axy$

Sol:- $z=x^3+y^3-3axy$

I order partial derivatives

$$\frac{\partial z}{\partial x}=3x^2+0-3ay(1)=3x^2-3ay$$

$$\frac{\partial z}{\partial y}=0+3y^2-3ax(1)=3y^2-3ax$$

II order partial derivatives

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(3x^2-3ay\right)=6x$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(3y^2-3ax\right)=6y$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(3y^2-3ax\right)=-3a$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(3x^2-3ay\right)=-3a$$

2.) If $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ then show that $\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}$ and $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

Sol:- $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial y}=x^2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)-2y\tan^{-1}\left(\frac{x}{y}\right)-y^2\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)$$

$$=\frac{x^3}{x^2+y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)+\frac{xy^2}{x^2+y^2}$$

$$=\frac{x(x^2+y^2)}{x^2+y^2}-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$=x-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= 1 - 2y\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

|||ly $\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2x\tan^{-1}\left(\frac{y}{x}\right) - y\right) = \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

3) If $z = f(x+ct) + \phi(x-ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

Sol:- $z = f(x+ct) + \phi(x-ct)$

$$\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$$

$$\frac{\partial z}{\partial t} = f'(x+ct)\cdot c + \phi'(x-ct)\,(-c) = c\left(f'(x+ct) - \phi'(x-ct)\right)$$

$$\frac{\partial^2 z}{\partial t^2} = c\left(f''(x+ct)\cdot c - \phi''(x-ct)\,(-c)\right) = c^2\left(f''(x+ct) + \phi''(x-ct)\right)$$

$$= c^2\frac{\partial^2 z}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$$

4) If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$?

Sol:- $\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\cdot\left(\frac{-2r}{4t}\right) = -\frac{r}{2}t^{n-1}e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial t} = nt^{n-1}e^{-r^2/4t} + t^n e^{-r^2/4t}\cdot\frac{r^2}{4t^2} = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$$

Now, $r^2\frac{\partial\theta}{\partial r} = r^2\left(-\frac{r}{2}t^{n-1}e^{-r^2/4t}\right) = -\frac{r^3}{2}t^{n-1}e^{-r^2/4t}$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3}{2}t^{n-1}e^{-r^2/4t}\right) = -\frac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \frac{r^3}{2}t^{n-1}e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

Now, $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$

$$\Rightarrow\left(-\frac{3}{2}t^{n-1}+\frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}=\left(nt^{n-1}+\frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$$

$$\Rightarrow n=-3/2$$

5) If $u=\log(x^3+y^3+z^3-3xyz)$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\frac{-9}{(x+y+z)^2}$

Sol:- $u=\log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x}=\frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

|||ly $$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$=\frac{3}{x+y+z}$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$

Total derivative :- If $u=f(x,y)$, where $x=\phi(t)$ and $y=\psi(t)$, then we can express 'u' as a function of 't' alone by substituting values of 'x' and 'y' in $f(x,y)$. Thus, we can find the ordinary derivative $\frac{du}{dt}$ which is known as total derivative.

To find $\frac{du}{dt}$ without substituting the value of x and y, we can establish chainrule as $\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$.

1.) Given $u=\sin\left(\frac{x}{y}\right)$, $x=e^t$ and $y=t^2$, find $\frac{du}{dt}$ as a function of t. Verify your answer by direct substitution.

Sol:- w.k.t if $u=f(x,y)$, $\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$

Given $u=\sin\left(\frac{x}{y}\right)$ $\qquad x=e^t \qquad y=t^2$

$\frac{\partial u}{\partial x}=\cos\left(\frac{x}{y}\right)\cdot\frac{1}{y} \qquad \frac{dx}{dt}=e^t \qquad \frac{dy}{dt}=2t$

$\frac{\partial u}{\partial y}=\cos\left(\frac{x}{y}\right)\cdot -\frac{x}{y^2}$

Now, $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$

$$=\frac{1}{y}\cos\left(\frac{x}{y}\right)\cdot e^t+\cos\left(\frac{x}{y}\right)\cdot\left(\frac{-x}{y^2}\right)\cdot 2t$$

$$=\frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)\cdot e^t-\frac{e^t\cdot 2t}{(t^2)^2}\cos\left(\frac{e^t}{t^2}\right)$$

$$=\left(\frac{t-2}{t^3}\right)e^t\cos\left(\frac{e^t}{t^2}\right)$$

Direct substitution method :- $u=\sin\left(\frac{x}{y}\right)=\sin\left(\frac{e^t}{t^2}\right)$

$$\frac{du}{dt}=\cos\left(\frac{e^t}{t^2}\right)\cdot\left(\frac{t^2\cdot e^t-e^t\cdot 2t}{(t^2)^2}\right)$$

$$=\frac{te^t(t-2)}{t^4}\cos\left(\frac{e^t}{t^2}\right)$$

$$=e^t\left(\frac{t-2}{t^3}\right)\cos\left(\frac{e^t}{t^2}\right)$$

2.) If $z = u^2+v^2$ and $u = at^2$, $v = 2at$, find $\frac{dz}{dt}$

Sol:- Given $z = u^2+v^2$

$$\frac{\partial z}{\partial u} = 2u$$

$$\frac{\partial z}{\partial v} = 2v$$

$$u = at^2 \Rightarrow \frac{du}{dt} = 2at$$

$$v = 2at \Rightarrow \frac{dv}{dt} = 2a$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

$$= 2u \cdot 2at + 2v \cdot 2a$$

$$= 2(at^2)\, 2at + 2(2at) \cdot 2a$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4a^2t\,(t^2+2)$$

Note:- If $f(x,y) = c$ is an implicit function, then $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

3.) If $u = x \log xy$ where $x^3+y^3+3xy = 1$, find $\frac{du}{dx}$

Sol:- Given $u = x \log(xy)$

Let $f = x^3+y^3+3xy-1$

Now, $\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = \frac{-(3x^2+3y)}{3y^2+3x} = \frac{-(x^2+y)}{y^2+x}$

Now, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$$= \log(xy) + \cancel{x} \cdot \frac{1}{\cancel{x}y} \cdot y + \cancel{x} \cdot \frac{1}{\cancel{x}y} \cdot x \cdot \frac{-(x^2+y)}{y^2+x}$$

$$= 1 + \log(xy) - \frac{x(x^2+y)}{y(y^2+x)}$$

Change of variables:-

If $u=f(x,y)$ where $x=\phi(s,t)$ and $y=\psi(s,t)$ then sometimes it is necessary to change the variables.

If 't' is treated as constant,

then $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$

If 's' is treated as constant, then $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$

Problems

1.) If $u=f(x-y, y-z, z-x)$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Sol:- Given $u=f(x-y, y-z, z-x)$

Put $x-y=r$, $y-z=s$, $z-x=t$

Then $\frac{\partial r}{\partial x}=1$ $\quad\frac{\partial s}{\partial x}=0$ $\quad\frac{\partial t}{\partial x}=-1$

$\frac{\partial r}{\partial y}=-1$ $\quad\frac{\partial s}{\partial y}=1$ $\quad\frac{\partial t}{\partial y}=0$

$\frac{\partial r}{\partial z}=0$ $\quad\frac{\partial s}{\partial z}=-1$ $\quad\frac{\partial t}{\partial z}=1$

Now, $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$

$=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)$

$=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$

$=\frac{\partial u}{\partial r}\cdot(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}\cdot(0)$

$=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$

$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$

$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}\cdot(1)$

$=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$

Now, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$

$=0$

Jacobians:- If u, v are functions of 2 independent variables 'x' and 'y' then $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v with respect to x, y.

It is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Illy If u, v, w are functions of x, y, z then $J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties

1.) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

2.) If u, v are functions of r, s and r, s are functions of x, y then

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$. It is known as chain rule

Note:- 1.) The functions u and v in x and y are functionally dependent if $J\left(\frac{u,v}{x,y}\right) = 0$

2.) The functions u, v, w in x, y, z are functionally dependent if

$J\left(\frac{u,v,w}{x,y,z}\right) = 0$

Problems

1.) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

Sol:- $x = r\cos\theta$ $\qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

Now, $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta$

$= r\left((\cos^2\theta) + \sin^2\theta\right) = r(1) = r$

2.) In spherical polar coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Sol:- $x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial y}{\partial r} = \sin\theta\sin\phi \qquad \frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi \qquad \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi\,(-r^2\sin^2\theta\sin\phi)$$
$$+ \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\,(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin^3\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin\theta\cos^2\theta\,(1)$$

$$= r^2\sin^3\theta\,(1) + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta\,(1)$$

$$= r^2\sin\theta$$

3.) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

Sol:- $u = x + 3y^2 - z^3$ $\quad v = 4x^2yz$ $\quad w = 2z^2 - xy$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 8xyz \qquad \frac{\partial w}{\partial x} = -y$$

$$\frac{\partial u}{\partial y} = 6y \qquad \frac{\partial v}{\partial y} = 4x^2z \qquad \frac{\partial w}{\partial y} = -x$$

$$\frac{\partial u}{\partial z} = -3z^2 \qquad \frac{\partial v}{\partial z} = 4x^2y \qquad \frac{\partial w}{\partial z} = 4z$$

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

At $(1,-1,0)$, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 1(0-4) - 0 + 1(24-0)$$
$$= 20$$

4.) If $u=x^2-y^2$, $v=2xy$ & $x=r\cos\theta$, $y=r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Sol:- $u=x^2-y^2$ $\quad v=2xy$ $\quad x=r\cos\theta$ $\quad y=r\sin\theta$

$\frac{\partial u}{\partial x}=2x$ $\quad \frac{\partial v}{\partial x}=2y$ $\quad \frac{\partial x}{\partial r}=\cos\theta$ $\quad \frac{\partial y}{\partial r}=\sin\theta$

$\frac{\partial u}{\partial y}=-2y$ $\quad \frac{\partial v}{\partial y}=2x$ $\quad \frac{\partial x}{\partial \theta}=-r\sin\theta$ $\quad \frac{\partial y}{\partial \theta}=r\cos\theta$

Now, $$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (4x^2+4y^2)(r\cos^2\theta+r\sin^2\theta)$$

$$= 4(x^2+y^2)\, r(\cos^2\theta+\sin^2\theta)$$

$$= 4(r^2\cos^2\theta+r^2\sin^2\theta)\, r(1) \quad (\because x=r\cos\theta,\ y=r\sin\theta)$$

$$= 4r^2(\cos^2\theta+\sin^2\theta)\cdot r$$

$$= 4r^3(1)$$

$$= 4r^3$$

5) If $u=x\sqrt{1-y^2}+y\sqrt{1-x^2}$, $v=\sin^{-1}x+\sin^{-1}y$ then show that u,v are functionally related and find the relationship.

Sol:- $u=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x}=\sqrt{1-y^2}+y\frac{1}{2\sqrt{1-x^2}}\cdot -2x = \sqrt{1-y^2}-\frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial u}{\partial y}=x\frac{1}{2\sqrt{1-y^2}}\cdot -2y+1\cdot\sqrt{1-x^2} = \frac{-xy}{\sqrt{1-y^2}}+\sqrt{1-x^2}$$

$v=\sin^{-1}x+\sin^{-1}y$

$$\frac{\partial v}{\partial x}=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial v}{\partial y}=\frac{1}{\sqrt{1-y^2}}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \left(\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right)\frac{1}{\sqrt{1-y^2}} - \left(\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right)\frac{1}{\sqrt{1-x^2}}$$

$$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} - 1$$

$$= 0$$

$\therefore$ u,v are functionally related

To find relationship, Assume $\sin^{-1}x = A$ & $\sin^{-1}y = B$

$\Rightarrow x = \sin A$ & $y = \sin B$

Now, $v = \sin^{-1}x + \sin^{-1}y = A+B$

$u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$= \sin A\sqrt{1-\sin^2 B} + \sin B\sqrt{1-\sin^2 A}$

$= \sin A \cos B + \sin B \cos A$

$= \sin(A+B)$

$\Rightarrow u = \sin v$ or $v = \sin^{-1}u$

6) If $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Are u,v functionally dependent. If so, find the relation between them

Sol $u = \frac{x+y}{x-y}$

$$\frac{\partial u}{\partial x} = \frac{(x-y)1 - (x+y)(1)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \frac{x-y+x+y}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$v = \frac{xy}{(x-y)^2}$

$$\frac{\partial v}{\partial x} = \frac{(x-y)^2 \cdot y - xy \cdot 2(x-y)\cdot 1}{((x-y)^2)^2} = \frac{(x-y)(xy-y^2-2xy)}{(x-y)^4} = \frac{-y^2-xy}{(x-y)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)^2 \cdot x - xy\, 2(x-y)(-1)}{((x-y)^2)^2} = \frac{(x-y)\left(x^2 - xy + 2xy\right)}{(x-y)^4}$$

$$= \frac{x^2+xy}{(x-y)^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{-y^2-xy}{(x-y)^3} & \frac{x^2+xy}{(x-y)^3} \end{vmatrix}$$

$$= \frac{-2y(x^2+xy)}{(x-y)^5} - \frac{2x(-y^2-xy)}{(x-y)^5}$$

$$= \frac{-2x^2y - 2xy^2 + 2xy^2 + 2x^2y}{(x-y)^5}$$

$$= 0$$

$\therefore$ u, v are functionally dependent

To find relationship $\quad u^2 = \left(\frac{x+y}{x-y}\right)^2$

$$= \frac{x^2+y^2+2xy}{(x-y)^2}$$

$$= \frac{x^2+y^2}{(x-y)^2} + \frac{2\,xy}{(x-y)^2}$$

$$= \frac{x^2+y^2+2xy-2xy}{(x-y)^2} + 2 \cdot v$$

$$= \frac{(x-y)^2+2xy}{(x-y)^2} + 2v$$

$$= 1 + \frac{2xy}{(x-y)^2} + 2v$$

$$= 1+2v+2v$$

$$= 1+4v$$

Jacobian of implicit functions:- If u_1, u_2, u_3 instead of being explicitly in terms of x_1, x_2, x_3 be connected with the equations such as $f_1(u_1,u_2,u_3,x_1,x_2,x_3)=0$, $f_2(u_1,u_2,u_3,x_1,x_2,x_3)=0$ & $f_3(u_1,u_2,u_3,x_1,x_2,x_3)=0$ then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}$$

Problem

If $u=xyz$, $v=x^2+y^2+z^2$, $w=x+y+z$, then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

solution:- Let $f_1 = u - xyz$

$f_2 = v - x^2 - y^2 - z^2$

$f_3 = w - x - y - z$

$\frac{\partial f_1}{\partial u}=1$ $\quad \frac{\partial f_2}{\partial u}=0$ $\quad \frac{\partial f_3}{\partial u}=0$

$\frac{\partial f_1}{\partial v}=0$ $\quad \frac{\partial f_2}{\partial v}=1$ $\quad \frac{\partial f_3}{\partial v}=0$

$\frac{\partial f_1}{\partial w}=0$ $\quad \frac{\partial f_2}{\partial w}=0$ $\quad \frac{\partial f_3}{\partial w}=1$

$\frac{\partial f_1}{\partial x}=-yz$ $\quad \frac{\partial f_2}{\partial x}=-2x$ $\quad \frac{\partial f_3}{\partial x}=-1$

$\frac{\partial f_1}{\partial y}=-xz$ $\quad \frac{\partial f_2}{\partial y}=-2y$ $\quad \frac{\partial f_3}{\partial y}=-1$

$\frac{\partial f_1}{\partial z}=-xy$ $\quad \frac{\partial f_2}{\partial z}=-2z$ $\quad \frac{\partial f_3}{\partial z}=-1$

Now, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$= -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \div \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= -1 \Big/ -2\begin{vmatrix} yz & xz & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2(x-y)(y-z)(z-x)}$$

Maxima and Minima of functions of two variables:-

Definition:- A function $f(x,y)$ is said to have a maximum & minimum at $x=a, y=b$ according as $f(a+h, b+k) < \text{ or } > f(a,b)$, $\forall$ small values of 'h' and 'k'.

Working rule:- Let $f=f(x,y)$ be a function

1.) find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

2.) Solve $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ to get the values of x, y say a, b and these obtained values are the stationary values.

3.) Find $r=\frac{\partial^2 f}{\partial x^2}$, $s=\frac{\partial^2 f}{\partial x \partial y}$, $t=\frac{\partial^2 f}{\partial y^2}$ at each stationary point.

4.) If i.) $rt-s^2>0$, $r<0$ at (a,b) then f has maximum value at (a,b)

ii.) $rt-s^2>0$, $r>0$ at (a,b) then f has minimum value at (a,b)

iii.) $rt-s^2<0$ at (a,b) then f has no maximum & no minimum and the stationary point (a,b) is known as saddle point.

iv.) $rt-s^2=0$ at (a,b), further investigation is required

1.) find the maximum and minimum values of $f(x,y)=x^3+3xy^2-3x^2-3y^2+4$

Sol:- Given $f(x,y)=x^3+3xy^2-3x^2-3y^2+4$

Then $\frac{\partial f}{\partial x}=3x^2+3y^2-6x$

$\frac{\partial f}{\partial y}=6xy-6y$

Now, $\frac{\partial f}{\partial x}=0 \Rightarrow 3x^2+3y^2-6x=0$

$\Rightarrow x^2+y^2-2x=0$ — (1)

$\frac{\partial f}{\partial y}=0 \Rightarrow 6xy-6y=0$

$\Rightarrow 6y(x-1)=0$

$\Rightarrow y=0, x=1$

Put $y=0$ in (1), (1) $\Rightarrow x^2-2x=0$

$\Rightarrow x(x-2)=0$

$\Rightarrow x=0,2$

$\Rightarrow (0,0), (2,0)$

Put $x=1$ in ①, ① $\Rightarrow 1+y^2-2=0$

$\Rightarrow y^2-1=0$

$\Rightarrow y^2=1$

$\Rightarrow y=\pm 1$

$\Rightarrow (1,-1),(1,1)$

$\therefore (1,-1),(1,1),(0,0),(2,0)$ are the stationary points.

Now, $r=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}(3x^2+3y^2-6x)=6x-6$

$s=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(6xy-6y)=6y$

$t=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}(6xy-6y)=6x-6$

$rt-s^2=(6x-6)(6x-6)^2-(6y)^2=(6x-6)^2-(6y)^2$

At $(1,-1)$, $rt-s^2=(6(1)-6)^2-(6(-1))^2=-36<0$

$\therefore f$ has no extreme value & $(1,-1)$ is a saddle point

At $(1,1)$, $rt-s^2=(6(1)-6)^2-(6(1))^2=-36<0$

$\therefore f$ has no extreme value & $(1,1)$ is a saddle point

At $(0,0)$, $rt-s^2=(6(0)-6)^2-(6(0))^2=36>0$, $r=-6<0$

$\therefore f$ has maximum value and $f_{max}=0^3+3(0)(0)^2-3(0)^2-3(0)^2+4=4$

At $(2,0)$, $rt-s^2=(6(2)-6)^2-(6(0))^2=36>0$, $r=6>0$

$\therefore f$ has minimum value and $f_{min}=2^3+3(2)(0)^2-3(2)^2-3(0)^2+4=0$

2.) Examine the following function for extreme values $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

Sol:- Let $f=x^4+y^4-2x^2+4xy-2y^2$

Then $\frac{\partial f}{\partial x}=4x^3-4x+4y$

$\frac{\partial f}{\partial y}=4y^3+4x-4y$

Now, $\frac{\partial f}{\partial x}=0 \Rightarrow 4x^3-4x+4y=0$

$\Rightarrow x^3-x+y=0$ —①

$\frac{\partial f}{\partial y}=0 \Rightarrow 4y^3+4x-4y=0$

$\Rightarrow y^3+x-y=0$ —②

by solving ① & ②, we get $y=-x$

Put $y=-x$ in ①, we get $x^3-2x=0$

$x(x^2-2)=0$

$x=0,\sqrt{2},-\sqrt{2}$

$\therefore (0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$ are the stationary points.

Now, $r = \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 4$

$t = \dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

$rt - s^2 = (12x^2-4)(12y^2-4) - 4^2$

At $(0,0)$, $rt - s^2 = 0$

$\therefore$ further investigation is needed

At $(\sqrt{2},\sqrt{2})$, $rt - s^2 = 20^2 - 4^2 > 0$, $r = 20 > 0$

$\therefore f$ has minimum value and $f_{min} = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$

$= -8$

At $(-\sqrt{2},\sqrt{2})$, $rt - s^2 = (20)(20) - 4^2 > 0$, $r = 20 > 0$

$\therefore f$ has minimum value and $f_{min} = -8$

3.) Examine for maxima and minima of $\sin x + \sin y + \sin(x+y)$

Sol:- Let $f = \sin x + \sin y + \sin(x+y)$

Then $\dfrac{\partial f}{\partial x} = \cos x + \cos(x+y)$

$\dfrac{\partial f}{\partial y} = \cos y + \cos(x+y)$

Now, $\dfrac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0$

$\Rightarrow \cos x = -\cos(x+y)$

$\Rightarrow \cos(x+y) = -\cos x$ —①

$\dfrac{\partial f}{\partial y} = 0 \Rightarrow \cos y + \cos(x+y) = 0$

$\Rightarrow \cos(x+y) = -\cos y$ —②

from ① & ② $\Rightarrow -\cos x = -\cos y$

$\Rightarrow x = y$

Now, Put $y = x$ in $\cos x + \cos(x+y) = 0$

$\Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow 2\cos\left(\frac{x+2x}{2}\right)\cos\left(\frac{x-2x}{2}\right) = 0$

$\Rightarrow 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \cos\left(\frac{3x}{2}\right) = 0 = \cos \pm\frac{\pi}{2}$; $\cos\frac{x}{2} = 0 = \cos\pm\frac{\pi}{2}$

$\Rightarrow \frac{3x}{2} = \pm\frac{\pi}{2}$; $\Rightarrow \frac{x}{2} = \pm\frac{\pi}{2}$

$\Rightarrow x = \pm\frac{\pi}{3}$; $\Rightarrow x = \pm\pi$

$\therefore (\pi,\pi), (-\pi,-\pi), (\pi/3, \pi/3), (-\pi/3, -\pi/3)$ are the stationary points.

$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

$rt - s^2 = (\sin x + \sin(x+y))(\sin y + \sin(x+y)) - \sin^2(x+y)$

At $(\pm\pi, \pm\pi)$, $rt - s^2 = 0 \Rightarrow$ Further investigation is required

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $rt - s^2 = \frac{9}{4} > 0$ and $r = -\sqrt{3} < 0$

$\therefore$ f has maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $f_{max} = \frac{3\sqrt{3}}{2}$

At $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$, $rt - s^2 > 0$, $r > 0$

$\therefore$ f has minimum value at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$ and $f_{min} = -\frac{3\sqrt{3}}{2}$

4) Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$

Sol:- Given $f = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

Then $\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

Now, $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(3 - 4x - 3y) = 0$

$\Rightarrow x^2 = 0, y^2 = 0, 3 - 4x - 3y = 0$

$\Rightarrow x = 0, y = 0, 4x + 3y = 3$ — ①

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(2 - 2x - 3y) = 0$

$\Rightarrow x = 0, y = 0, 2x + 3y = 2$ — ②

Solving ① & ②, we get $x = \frac{1}{2}$ & $y = \frac{1}{3}$

$\therefore (0,0), (1/2, 1/3)$ are the stationary points

$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$

$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

At $(0,0)$, $rt - s^2 = 0 \Rightarrow$ Further investigation is needed

At $\left(\frac{1}{2}, \frac{1}{3}\right)$, $rt - s^2 = \frac{1}{14} > 0$ & $r = -\frac{1}{9} < 0$

$\therefore$ f has maximum value and $f_{max} = \frac{1}{432}$

Lagrange's method of undetermined multipliers:-

This method is used to find the extremum of the function $f(x,y,z)$ subjected to the condition $\phi(x,y,z)=0$.

Working rule

① Form a Lagrangian function, $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$, where '$\lambda$' is called Lagrange multiplier

② Obtain $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$

③ Solve the above equations to get the stationary point (x,y,z)

④ Find the extreme value at the stationary point.

Problems:-

1.) Find the minimum value of $x^2+y^2+z^2$ given $x+y+z=3a$

Sol:- Let $f=x^2+y^2+z^2$, $\phi=x+y+z-3a$

Then Lagrangian function, $F=f+\lambda\phi$

$$=x^2+y^2+z^2+\lambda(x+y+z-3a)$$

Now, $\frac{\partial F}{\partial x}=0 \Rightarrow 2x+\lambda=0 \Rightarrow -\lambda=2x$ — ①

$\frac{\partial F}{\partial y}=0 \Rightarrow 2y+\lambda=0 \Rightarrow -\lambda=2y$ — ②

$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+\lambda=0 \Rightarrow -\lambda=2z$ — ③

① = ② = ③ $\Rightarrow 2x=2y=2z$

$\Rightarrow x=y=z$

Put $y=x, z=x$ in $x+y+z=3a \Rightarrow x+x+x=3a$

$\Rightarrow 3x=3a$

$\Rightarrow x=a$

$\therefore y=z=a$

$\therefore$ Minimum value of $f = a^2+a^2+a^2 = 3a^2$.

2.) A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Sol:- Let x,y,z be the edges of the box and S be its surface and V be its volume

Then $S=xy+2yz+2zx$ & $V=xyz=32$

Let $f=xy+2yz+2zx$ & $\phi=xyz-32=0$

Now, $F = f + \lambda\phi$

$= xy + 2yz + 2zx + \lambda(xyz - 32)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0$

$\Rightarrow -\lambda yz = y + 2z$

$\Rightarrow -\lambda xyz = xy + 2xz$ — ①

$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2yz + \lambda xz = 0$

$\Rightarrow -\lambda xz = x + 2yz$

$\Rightarrow -\lambda xyz = xy + 2y^2z$ — ②

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2y + 2x + \lambda xy = 0$

$\Rightarrow -\lambda xy = 2y + 2x$

$\Rightarrow -\lambda xyz = 2yz + 2xz$ — ③

① = ② $\Rightarrow \cancel{xy} + 2xz = \cancel{xy} + 2yz$

$\Rightarrow 2xz = 2yz$

$\Rightarrow x = y$

② = ③ $\Rightarrow xy + \cancel{2yz} = \cancel{2yz} + 2xz$

$\Rightarrow xy = 2xz$

$\Rightarrow y = 2z$

$\therefore x = y = 2z$

so, put $x = y = 2z$ in $xyz = 32$

$\therefore V = xyz = 32 \Rightarrow (2z)(2z)z = 32$

$\Rightarrow 4z^3 = 32$

$\Rightarrow z^3 = 8$

$\Rightarrow z = 2$

$\therefore x = y = 2(2) = 4$

3) If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by $x = \frac{\Sigma a}{a}$, $y = \frac{\Sigma a}{b}$, $z = \frac{\Sigma a}{c}$

Sol:- Let $u = a^3x^2 + b^3y^2 + c^3z^2$

and $\phi = x^{-1} + y^{-1} + z^{-1} - 1 = 0$

Then Lagrangian function, $F = f + \lambda\phi = u + \lambda\phi$

$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2a^3x - \frac{\lambda}{x^2} = 0 \Rightarrow \lambda = 2a^3x^3$ — ①

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2b^3y - \frac{\lambda}{y^2} = 0 \Rightarrow \lambda = 2b^3y^3$ — ②

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2c^3z - \frac{\lambda}{z^2} = 0 \Rightarrow \lambda = 2c^3z^3$ — ③

① = ② = ③ $\Rightarrow 2a^3x^3 = 2b^3y^3 = 2c^3z^3$

$\Rightarrow ax = by = cz$

$\Rightarrow \frac{a}{\frac{1}{x}} = \frac{b}{\frac{1}{y}} = \frac{c}{\frac{1}{z}} = \frac{a+b+c}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$\Rightarrow ax = by = cz = \frac{\Sigma a}{1}$

$\Rightarrow x = \frac{\Sigma a}{a}, \; y = \frac{\Sigma a}{b}, \; z = \frac{\Sigma a}{c}$

4) Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol:- Let $2x, 2y, 2z$ be the length, breadth, height of a rectangular solid.

Then volume $= 8xyz = f$, say

Equation of sphere is $x^2+y^2+z^2 = r^2 \Rightarrow x^2+y^2+z^2-r^2 = 0$

$\Rightarrow \phi = x^2+y^2+z^2-r^2$, say

Now, $F = f + \lambda\phi$

$= 8xyz + \lambda(x^2+y^2+z^2-r^2)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + 2\lambda x = 0 \Rightarrow 1/\lambda = \frac{-x}{4yz} \Rightarrow \lambda = \frac{-4yz}{x}$ — ①

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8xz + 2\lambda y = 0 \Rightarrow 1/\lambda = \frac{-y}{4xz} \Rightarrow \lambda = \frac{-4xz}{y}$ — ②

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xy + 2\lambda z = 0 \Rightarrow \lambda = -\frac{4xy}{z}$ — ③

① = ② = ③ $\Rightarrow \frac{-\cancel{4}yz}{x} = \frac{-\cancel{4}xz}{y} = \frac{-\cancel{4}xy}{z}$

$\Rightarrow \frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$

$\Rightarrow \frac{xyz}{x^2} = \frac{xyz}{y^2} = \frac{xyz}{z^2}$

$\Rightarrow \frac{1}{x^2} = \frac{1}{y^2} = \frac{1}{z^2}$

$\Rightarrow x^2 = y^2 = z^2$

$\Rightarrow x = y = z$

$\therefore$ Rectangular solid is a cube.

5) Given $x+y+z = a$, find the maximum value of $x^m y^n z^p$

Sol:- Let $f = x^m y^n z^p$ and $\phi = x+y+z-a$

Now, $F = f + \lambda\phi$

$= x^m y^n z^p + \lambda(x+y+z-a)$

$$\frac{\partial F}{\partial x}=0 \Rightarrow mx^{m-1}y^n z^p+\lambda=0 \Rightarrow -\lambda = mx^{m-1}y^n z^p \quad \text{--- (1)}$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow nx^{m}y^{n-1} z^p+\lambda=0 \Rightarrow -\lambda = nx^{m}y^{n-1} z^p \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow px^{m}y^{n} z^{p-1}+\lambda=0 \Rightarrow -\lambda = px^{m}y^{n} z^{p-1} \quad \text{--- (3)}$$

$$(1)=(2)=(3) \Rightarrow mx^{m-1}y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\Rightarrow mx^{-1} = ny^{-1} = pz^{-1}$$

$$\Rightarrow \frac{m}{x}=\frac{n}{y}=\frac{p}{z} = \frac{m+n+p}{x+y+z}$$

$$= \frac{m+n+p}{a}$$

$$\Rightarrow x = \frac{am}{m+n+p}\ ;\ y=\frac{an}{m+n+p}\ ;\ z=\frac{ap}{m+n+p}$$

$$\therefore f_{max} = \frac{a^{m+n+p}\ m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

Unit-4 Multiple Integrals

<u>Double Integral</u> :- An expression of the form $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx$ or $\int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)\, dx\, dy$ is called an iterated integral or double integrals.

Suppose R is in the form $a \le x \le b$, $y_1(x) \le y_2(x)$ then

$$\iint_R f(x,y)\, dy\, dx = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \right) dx$$

Similarly, if R is of the form $a \le y \le b$, $x_1(y) \le x \le x_2(y)$, then

$$\iint_R f(x,y)\, dx\, dy = \int_a^b \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \right) dy$$

<u>Problems</u>

1.) Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\, dy\, dx$

<u>Sol</u>:-
$$\int_0^3 \int_1^2 xy(1+x+y)\, dy\, dx = \int_{x=0}^3 \int_{y=1}^2 xy + x^2y + xy^2\, dy\, dx$$

$$= \int_{x=0}^3 \left[\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=1}^2 dx$$

$$= \int_{x=0}^3 \left(2x + 2x^2 + \frac{8}{3}x - \frac{x}{2} - \frac{x^2}{2} - \frac{x}{3} \right) dx$$

$$= \left[x^2 + \frac{2x^3}{3} + \frac{8}{3}\frac{x^2}{2} - \frac{x^2}{4} - \frac{x^3}{6} - \frac{x^2}{6} \right]_0^3$$

$$= \frac{123}{4}$$

2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$

<u>Sol</u>:-
$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} x^2+y^2\, dy\, dx$$

$$= \int_{x=0}^1 \left[x^2y + \frac{y^3}{3} \right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^1 x^2\sqrt{x} + \frac{1}{3}x^{3/2} - x^3 - \frac{x^3}{3}\, dx$$

$$= \int_{x=0}^1 x^{5/2} + \frac{1}{3}x^{3/2} - \frac{4}{3}x^3\, dx$$

$$=\left[\frac{x^{7/2}}{7/2}-\frac{4}{3}\frac{x^4}{4}+\frac{1}{3}\frac{x^{5/2}}{5/2}\right]_0^1$$

$$=\frac{2}{7}-\frac{1}{3}+\frac{2}{15}=\frac{3}{35}$$

3.) Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol:- $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_{y=0}^1\int_{x=0}^1 \frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{\sqrt{1-y^2}}\,dx\,dy$

$$=\int_{y=0}^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}x\right]_{x=0}^1 dy$$

$$=\int_{y=0}^1 \frac{1}{\sqrt{1-y^2}}\left(\sin^{-1}1-\sin^{-1}0\right)dy$$

$$=\int_{y=0}^1 \frac{1}{\sqrt{1-y^2}}\left(\frac{\pi}{2}-0\right)dy$$

$$=\frac{\pi}{2}\left[\sin^{-1}y\right]_{y=0}^1$$

$$=\frac{\pi}{2}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$

$$=\frac{\pi}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{4}$$

4.) Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol:- $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_{x=0}^1\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\,dy\,dx$

$$=\int_{x=0}^1\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_{y=0}^{\sqrt{1+x^2}}dx$$

$$=\int_{x=0}^1\left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}}\right)-\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{0}{\sqrt{1+x^2}}\right)\right]dx$$

$$=\int_{x=0}^1\left(\frac{1}{\sqrt{1+x^2}}\frac{\pi}{4}-0\right)dx$$

$$=\frac{\pi}{4}\int_{x=0}^1\frac{1}{\sqrt{1+x^2}}\,dx$$

$$=\frac{\pi}{4}\left[\sinh^{-1}x\right]_{x=0}^1$$

$$=\frac{\pi}{4}\sinh^{-1}(1)$$

5) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy dx$

Sol:- $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, dy\, dx$

$$= \int_{x=0}^{a} \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2} \sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) \right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \left(0 + \frac{a^2-x^2}{2} \cdot \frac{\pi}{2} - 0 - 0\right) dx$$

$$= \frac{\pi}{4}\left[a^2x - \frac{x^3}{3}\right]_{x=0}^{a}$$

$$= \frac{\pi}{4}\left(a^2 \cdot a - \frac{a^3}{3} - 0 + 0\right)$$

$$= \frac{\pi a^3}{6}$$

6) Evaluate $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\, d\theta dr$

Sol:- $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\, d\theta\, dr = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r\, d\theta\, dr$

$$= \int_{r=0}^{\infty} e^{-r^2} \cdot r \left(\int_{\theta=0}^{\pi/2} d\theta\right) dr$$

$$= \int_{r=0}^{\infty} e^{-r^2} \cdot r\, [\theta]_0^{\pi/2}\, dr$$

$$= \frac{\pi}{2} \int_{r=0}^{\infty} e^{-r^2} r\, dr$$

$$= \frac{\pi}{2}\left(\frac{-1}{2}\right) \int_{r=0}^{\infty} e^{-r^2} \cdot -2r\, dr$$

$$= -\frac{\pi}{4} \int_{r=0}^{\infty} e^{-r^2} d(-r^2)$$

$$= -\frac{\pi}{4}\left[e^{-r^2}\right]_0^\infty$$

$$= -\frac{\pi}{4}\left(e^{-\infty} - e^0\right)$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4}$$

7.) Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$

Sol:- $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$

$$= \frac{-1}{2}\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} \frac{-2r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= \frac{-1}{2}\int_{\theta=0}^{\pi/4}\left[2\sqrt{a^2-r^2}\right]_{r=0}^{a\sin\theta} d\theta$$

$$= \frac{-1}{2}\int_{\theta=0}^{\pi/4}\left(2\sqrt{a^2-a^2\sin^2\theta} - 2\sqrt{a^2-0}\right)d\theta$$

$$= -\frac{1}{2}\cdot 2\int_{\theta=0}^{\pi/4} a\cos\theta - a\,d\theta$$

$$= -a\int_{\theta=0}^{\pi/4} \cos\theta - 1\,d\theta$$

$$= -a\left[\sin\theta - \theta\right]_0^{\pi/4}$$

$$= -a\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right)$$

$$= a\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

8.) Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

Sol:- Given $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2} = \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{2r}{(r^2+a^2)^2}\,dr\,d\theta$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{-1}{r^2+a^2}\right]_{r=0}^{\infty} d\theta$$

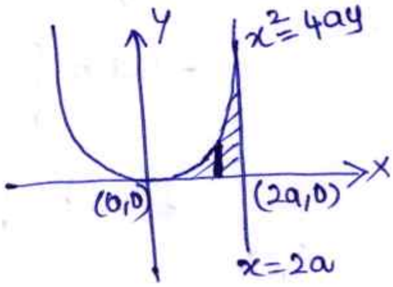
$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left(0+\frac{1}{a^2}\right)d\theta$$

$$= \frac{1}{2a^2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2a^2}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi}{4a^2}$$

9) Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$



Sol:- $\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dx\,dy$

$$= \int_{x=0}^{2a} x\left(\int_{y=0}^{x^2/4a} y\,dy\right)dx$$

$$= \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_{y=0}^{x^2/4a} dx$$

$$= \int_{x=0}^{2a} x\left(\frac{1}{2}\left(x^2/4a\right)^2 - 0\right)dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_{x=0}^{2a}$$

$$= \frac{1}{32a^2}\left(\frac{1}{6}(2a)^6 - 0\right)$$

$$= \frac{64a^6}{32a^2(6)} = \frac{a^4}{3}$$

10.) Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol:- Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

Region of integration is $-a \le x \le a$ and $-\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore \iint (x+y)^2\,dx\,dy = \int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\,dx$$

$$= 2\int_{x=0}^{a} 2\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$$ $\because xy$ is odd function & x^2, y^2 are even functions

$$= 4\int_{x=0}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\int_{x=0}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(a^2-x^2\right)^{3/2}\right]dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

As $x\to 0,\ \theta\to 0$ &

$x\to a,\ \theta\to \pi/2$

$$= 4\int_0^{\pi/2}\left(\frac{b}{a}a^2\sin^2\theta\, a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right) a\cos\theta\, d\theta$$

$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= 4a^3b\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} + \frac{4ab^3}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2}$$

$$\because \int_0^{\pi/2}\sin^m\theta\cos^n\theta\, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots}\cdot\frac{\pi}{2},$$

since both m, n are even

$$\int_0^{\pi/2}\cos^n\theta\, d\theta = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots}\cdot\frac{\pi}{2}, \text{ if } n \text{ is even}$$

$$= \cancel{4}a^3b\cdot\frac{\pi}{\cancel{16}\,4} + \frac{\cancel{4}ab^3}{\cancel{3}}\cdot\frac{\cancel{3}\pi}{\cancel{16}\,4}$$

$$= \frac{\pi ab}{4}(a^2+b^2)$$

Change of variables: Sometimes double or triple integral is easy to evaluate if we change the variables suitably.

Suppose, if we change x, y to u, v then we have to change the vars. accordingly i.e., $dxdy$ is to be replaced by $|J|\,dudv$, where $|J|$ is the Jacobian of x, y w.r.t u, v.

To change cartesian co-ordinates to polar co-ordinates, put $x = r\cos\theta$, $y = r\sin\theta$

then $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$$\therefore \iint f(x,y)\,dx\,dy = \iint f(r,\theta)\, r\,dr\,d\theta.$$

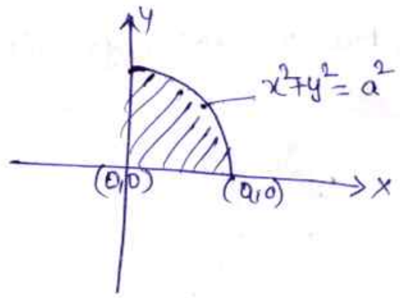
1) Evaluate the integral by transforming into polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$.

Given x — 0 to a

y — 0 to $\sqrt{a^2-x^2}$

$\Rightarrow y = \sqrt{a^2-x^2}$

$\Rightarrow x^2+y^2 = a^2$.



Put $x = r\cos\theta$, $y = r\sin\theta$. and $dxdy = r\,dr\,d\theta$.
Then $x^2+y^2 = r^2$

r — 0 to a

θ — 0 to $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r\sin\theta \; r .\, r\,dr\,d\theta$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r^3 \sin\theta \, dr\,d\theta$$

$$= \int_{r=0}^{a} r^3 \left[-\cos\theta\right]_0^{\pi/2} dr$$

$$= \int_0^a r^3 (+1)\,dr$$

$$= \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}.$$

2) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$

x — 0 to $\sqrt{a^2-y^2}$

$\Rightarrow x = \sqrt{a^2-y^2}$

$\Rightarrow x^2+y^2 = a^2$

y — 0 to a

Put $x = r\cos\theta$, $y = r\sin\theta$, then $x^2+y^2 = r^2$ & $dxdy = r\,dr\,d\theta$

r — 0 to a

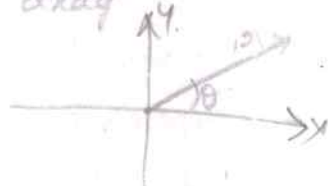
θ — 0 to $\pi/2$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^2 \, r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4} d\theta = \frac{a^4}{4} \cdot [\theta]_0^{\pi/2} = \frac{a^4 \pi}{8}$$

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)} dx dy$$



$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$
θ — 0 to $\pi/2$
r — 0 to ∞

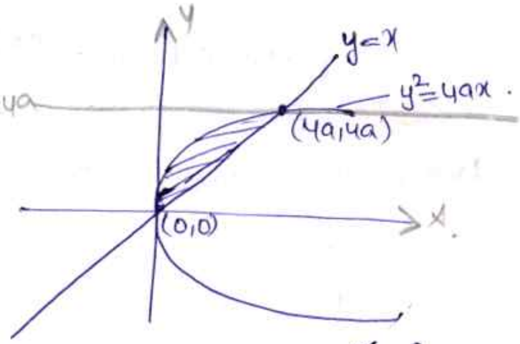
3) S.T $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx dy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$

Given x — $\frac{y^2}{4a}$ to y

$x = \frac{y^2}{4a}$ $\quad x = y$

$\Rightarrow y^2 = 4ax$

y — 0 to $4a$



Put $x = r\cos\theta$, $y = r\sin\theta$ then $x^2+y^2 = r^2$ and $dxdy = r dr d\theta$.

$r^2\sin^2\theta = 4a\cancel{r}\cos\theta$

θ — $\pi/4$ to $\pi/2$

r — 0 to $\frac{4a\cos\theta}{\sin^2\theta}$.

$x^2-y^2 = r^2(\cos^2\theta - \sin^2\theta)$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx dy = \int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta - \sin^2\theta)}{\cancel{r^2}} r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{\overset{8}{\cancel{16}}a^2\cos^2\theta}{\cancel{2}\sin^4\theta} - 0\right) d\theta$$

$\frac{1}{3} - 2$
$-\frac{5}{3}$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta) \, d\theta$$

$\int \cot^n\theta \, d\theta = \frac{-\cot^{n-1}\theta}{n-1} - \int \cot^{n-2}\theta \, d\theta$

$$= 8a^2\left(\left[-\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} - 2\int_{\pi/4}^{\pi/2} \cot^2\theta \, d\theta\right)$$

$1 - \frac{\pi}{4} + \frac{1}{3}$

$\frac{4}{3} + \frac{\pi}{4} \times 2$

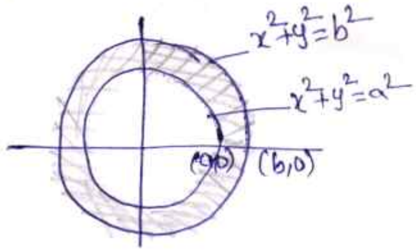
$$= 8a^2\left(0 + \frac{1}{3} - 2\left(-\cot\theta - \theta\right)_{\pi/4}^{\pi/2}\right)$$

$$= 8a^2\left(\frac{1}{3} - 2\left(0 - \pi/2 + 1 + \pi/4\right)\right)$$

$$= 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right).$$

⑥

4.) By changing into polar co-ordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region b/n the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$



Put $x=r\cos\theta$, $y=r\sin\theta$ then $x^2+y^2=r^2$ &

$dx\,dy=r\,dr\,d\theta$

r — a to b

θ — 0 to 2π

$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy=\int_{\theta=0}^{2\pi}\int_{r=a}^{b}\frac{\cancel{r^4}\,r^2\cos^2\theta\sin^2\theta}{\cancel{r^2}}\,r\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\frac{1}{4}\sin^2 2\theta\left[\frac{r^4}{4}\right]_a^b d\theta$$

$$=\frac{1}{16}(b^4-a^4)\cdot\int_{\theta=0}^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$=\frac{1}{32}(b^4-a^4)\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$=\frac{b^4-a^4}{32}(2\pi-0)$$

$$=(b^4-a^4)\frac{\pi}{16}$$

<u>change of order of integration:</u>

Working rule for $\int_{x=a}^{b}\int_{y=f(x_1)}^{f(x_2)} f(x,y)\,dy\,dx$.

Draw the region of integration by drawing the curves $y=f_1(x)$ & $y=f_2(x)$ & $x=a$ & $x=b$.

If these curves & lines intersect then draw straight lines ||el to x-axis to get various subregions.

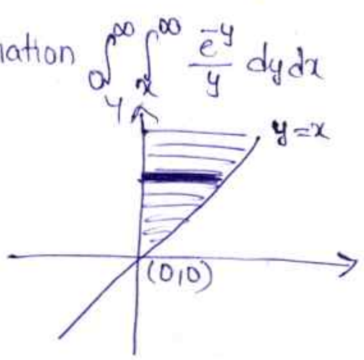
In each of these sub regions draw elementary strips ||el to x-axis & obtain the limits for 'x' interms of 'y' & then lts for 'y' as 'constants'.

||ly, the order of integration can be changed for $\int_{y=a}^{b}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$

5) Evaluate the integral by changing the order of integration $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

Given x — 0 to ∞

y — x to ∞



y — 0 to ∞

x — 0 to y

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \left[x\right]_0^y dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}} \cancel{y}\, dy$$

$$= \left[-e^{-y}\right]_0^\infty = 1$$

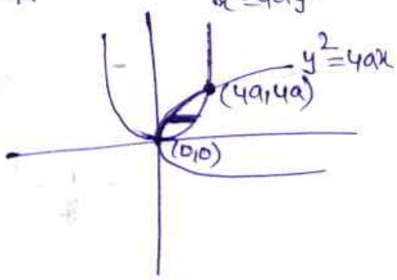
✓ 2) change the order of integration and evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

Given x — 0 to $4a$

y — $\frac{x^2}{4a}$ to $2\sqrt{ax}$

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$



$2\sqrt{ay}$

$x = \frac{y^2}{4a}$

y — 0 to $4a$

x — $\frac{y^2}{4a}$ to $2\sqrt{ay}$

$$\therefore \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left(8\, a\sqrt{a}\right) - \frac{\cancel{64}^{16} a^{\cancel{3}\,2}}{\cancel{12a}_{3}}$$

$4^{3/2} = 4\sqrt{4} = 4 \cdot 2$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

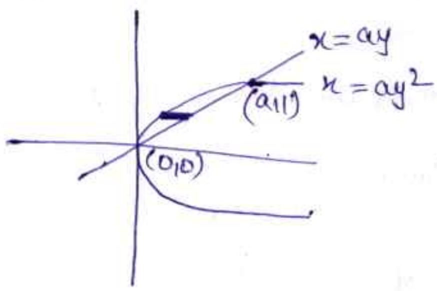
3) change the order of integration and evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dx\, dy$

x — 0 to a

y — x/a to $\sqrt{x/a}$

$\Rightarrow y = x/a \quad y = \sqrt{x/a}$

$\Rightarrow x = ay \quad x = ay^2$



y — 0 to 1

x — ay^2 to ay

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\frac{x^3}{3} + y^2 x\right]_{ay^2}^{ay} dy$$

$$= \int_{y=0}^{1} \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \left[\frac{a^3}{3}\frac{y^4}{4} + a\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - a\frac{y^5}{5}\right]_0^1$$

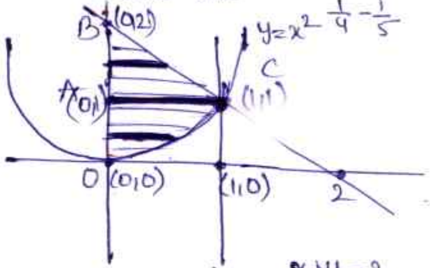
$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5}$$

$$= \frac{a^3}{28} + \frac{a}{20}.$$

$\frac{1}{12} - \frac{1}{21}$

$21 - 12 = \frac{9}{12 \cdot 21} = \frac{9}{12 \cdot 21} = \frac{1}{28}.$

* 4) Change the order of integration in $\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integral.



$\frac{1}{4} - \frac{1}{5}$

Given x — 0 to 1

y — x^2 to $2-x$

In OAC $\quad y$ — 0 to 1

x — 0 to $\sqrt{y}$

In ABC $\quad y \to 1$ to 2

$x \to 0$ to $2-y$

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy + \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 (2-y)^2 y\,dy$$

$(4+y^2-4y)y$

$y^3 - 4y^2 + 4y$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{\cancel{4}^{2}y^2}{\cancel{2}}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(4 - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2\right)$$

$10 - \frac{1}{4} = \frac{39}{4}$

$-\frac{28}{3}$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{39}{4} - \frac{28}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{117-112}{12}\right)$$

$$= \frac{1}{6} + \frac{1}{2}\cdot\frac{5}{12} = \frac{1}{6}\left(1 + \frac{5}{4}\right) = \frac{9^3}{4 \times \cancel{6}_2} = \frac{3}{8}$$

5.) change the order of integration, evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$.

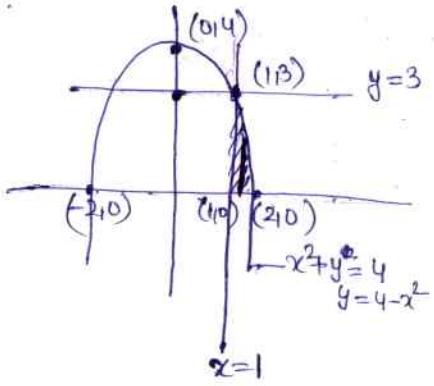
x — 1 to $\sqrt{4-y}$

y — 0 to 3

$x = \sqrt{4-y}$

$x^2 = 4-y$

$x^2+y=4$



$\cancel{y}x$ — 1 to 2

y — 0 to $4-x^2$

$$\therefore \int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy = \int_1^2 \int_{y=0}^{4-x^2} (x+y)\, dx\, dy$$

$$= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} dx$$

$$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2} \right) dx$$

$$= \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2 \right) dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3} \right]_1^2$$

$$= \cancel{8} - \frac{16}{4} + 16 + \frac{32}{10} - \frac{32}{3} - 2 + \frac{1}{4} - \cancel{8} - \frac{1}{10} + \frac{4}{3}$$

$$= 14 + \frac{31}{10} - \frac{15}{4} - \frac{28}{3}$$

$$= \frac{1680 + 372 - 450 - 1120}{120} = \frac{\cancel{482}^{241}}{\cancel{120}_{60}} = \frac{241}{60}.$$

$\frac{\overset{4}{14} \times 12}{16\ 8}$

$\frac{31 \times 12}{2}$

6.) change the order of integration and evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$.

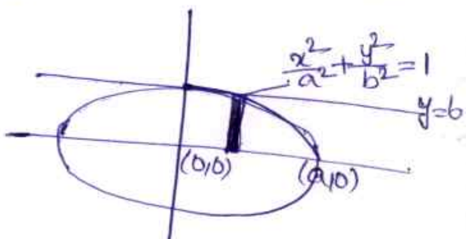
x — 0 to $\frac{a}{b}\sqrt{b^2-y^2}$

y — 0 to b

$x = \frac{a}{b}\sqrt{b^2-y^2}$

$\frac{b^2x^2}{a^2} = b^2-y^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



x — 0 to a.

y — 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$y^2 = b^2\sqrt{1-\frac{x^2}{a^2}}$

$y = \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dx\, dy$$

$$= \int_0^a x \left[\frac{y^2}{2} \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x \cdot \frac{b^2}{a^2}(a^2-x^2)\, dx$$

$$= \frac{b^2}{2a^2} \left[\cancel{b}a^2\frac{x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4} - 0 \right) = \frac{b^2}{2a^2} \cdot \frac{a^4}{4} = \frac{a^2b^2}{8}$$

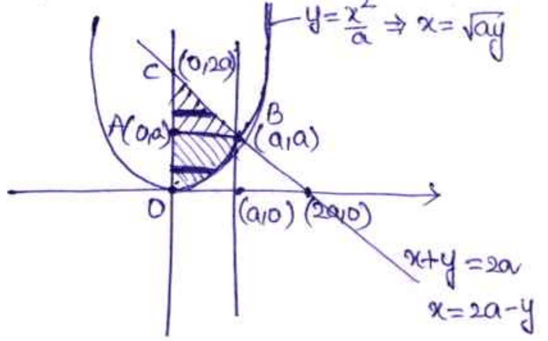
7) Change the order of integration and solve $\int_0^a \int_{x^2/a}^{2a-x} xy^2\,dy\,dx$.

Given x — 0 to a

y — x^2/a to $2a-x$.

$y = x^2/a \Rightarrow x^2 = ay$

$y = 2a - x \Rightarrow x + y = 2a$



In OAB y — 0 to a

x — 0 to $\sqrt{ay}$

In ABC, y — a to $2a$

x — 0 to $2a-y$

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy^2\,dy\,dx = \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy^2\,dx\,dy + \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy^2\,dx\,dy$$

$$= \int_{y=0}^{a} y^2 \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_{y=a}^{2a} y^2 \left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{a} ay^3\,dy + \frac{1}{2}\int_{y=a}^{2a} (2a-y)^2 y^2\,dy$$

$(2a-y)^2y^2 = (4a^2+y^2-4ay)y^2$
$= 4a^2y^2 + y^4 - 4ay^3$

$$= \frac{a}{2}\left[\frac{y^4}{4}\right]_0^a + \frac{1}{2}\left[\frac{4a^2y^3}{3} + \frac{y^5}{5} - \frac{\cancel{4}ay^4}{\cancel{4}}\right]_a^{2a}$$

$$= \frac{a^5}{8} + \frac{1}{2}\left(\frac{4a^2}{3}\cdot 8a^3 + \frac{32a^5}{5} - 16a^5 - \frac{4a^4}{3} - \frac{a^5}{5} + a^5\right)$$

$$= \frac{a^5}{8} + \frac{1}{2}\left(\frac{28}{3}a^5 + \frac{31}{5}a^5 - 15a^5\right)$$

$$= \frac{a^5}{8} + \frac{a^5}{2}\left(\frac{140+93-225}{15}\right)$$

$$= \frac{a^5}{8} + \frac{a^5}{2}\left(\frac{+8}{15}\right)$$

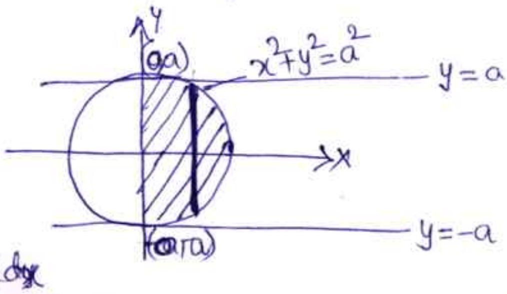
$$= a^5\left(\frac{15+32}{120}\right) = \frac{47a^5}{120}.$$

$\cancel{225}$
-233
225
8

8, 2, 15

0

8) change the order the integration and evaluate $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$.



x — 0 to $\sqrt{a^2-y^2}$

y — $-a$ to a

x — 0 to a

y — $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x = \sqrt{a^2-y^2} \Rightarrow x^2+y^2 = a^2$

$$\therefore \int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy = \int_{x=0}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \ldots \, dy \, dx$$

$$\ldots \, dy \, dx$$

As $x \to 0$, $\theta \to 0$; $x \to a$, $\theta \to \pi/2$

$$\int_0^{\pi/2} \left(\ldots - \frac{a^6}{5} \cos^5\theta \right) a\cos\theta \, d\theta$$

$$- \frac{a^7}{5} \int_0^{\pi/2} \cos^5\theta \sin\theta \, d\theta$$

$$\frac{a^7}{3} \int_0^{\pi/2} (\cos^4\theta - \cos^6\theta) \sin\theta \, d\theta$$

$$- \frac{a^7}{5} \int_0^{\pi/2} \cos^6\theta \sin\theta \, d\theta$$

$$\ldots - \frac{a^7}{5} \int_0^1 t^6 \, dt \Big)$$

$$= \frac{1}{2} \ldots - \frac{a^7}{5} \left[\frac{t^7}{7} \right]_0^1 \Big)$$

$$= \frac{1}{2} \left(\frac{1}{15} a^7 \ldots \frac{1}{35} \right) a^7$$

$$\frac{35 \times 3}{105}$$

$$\frac{-2}{105} a^7$$

$$\frac{1}{2} \left(\frac{7 - 2 - 3}{105} \right) a^7$$

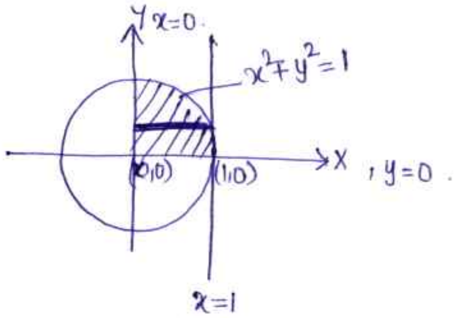
$$= \frac{1}{2} \frac{2}{105} a^7 = \frac{1}{105} a^7.$$

$$\int_1^e \left(\log y + \ldots - y + \log y \ldots \right) dy$$

9.) By changing the order of integration, evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$.

y — 0 to $\sqrt{1-x^2}$ i.e., $y=0$, $y=\sqrt{1-x^2}$
$\Rightarrow x^2+y^2=1$

x — 0 to 1



$\therefore$ y — 0 to 1

x — 0 to $\sqrt{1-y^2}$

$$\therefore \int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy$$

$$= \int_{y=0}^{1} y^2\left[x\right]_0^{\sqrt{1-y^2}}\,dy$$

$$= \int_{y=0}^{1} y^2\sqrt{1-y^2}\,dy$$

Put $y=\sin\theta$
$dy=\cos\theta\,d\theta$

As $y\to 0$, $\theta\to 0$
$y\to 1$, $\theta\to \pi/2$

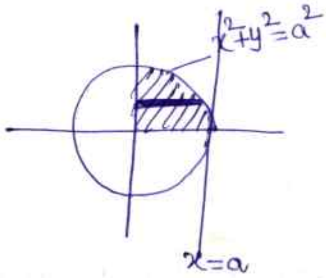
$$= \int_0^{\pi/2} \sin^2\theta\,\cos\theta\cdot\cos\theta\,d\theta.$$

$$= \frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{16}.$$

10.) By the order of integration, evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$

Given x — 0 to a i.e., $x=0$, $x=a$

y — 0 to $\sqrt{a^2-x^2}$ i.e., $y=0$, $y=\sqrt{a^2-x^2}$
$\Rightarrow x^2+y^2=a^2$



$\therefore$ y — 0 to a

x — 0 to $\sqrt{a^2-y^2}$

$$\therefore \int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy.$$

$$= \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx\,dy.$$

$\left(\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)$

$$= \int_{y=0}^{a}\left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2}+\frac{a^2-y^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$$

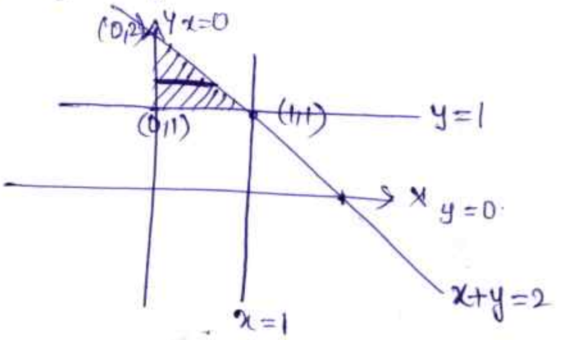
$$= \int_{y=0}^{a} \left(0 + \frac{a^2-y^2}{2} \sin^{-1}(1) - 0\right) dy$$

$$= \int_{y=0}^{a} \frac{a^2-y^2}{2} \cdot \frac{\pi}{2} \cdot dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left(a^3 - \frac{a^3}{3}\right) = \frac{2a^3\pi}{12} = \frac{a^3\pi}{6}.$$

11.) By changing the order of integration, evaluate $\int_0^1 \int_1^{2-x} xy \, dx \, dy$.



Given x — 0 to 1 i.e., $x=0$, $x=1$

y — 1 to $2-x$ i.e., $y=1$, $y=2-x$ $\Rightarrow x+y=2$

$\therefore y$ — 1 to 2

x — 0 to $2-y$.

$$\therefore \int_0^1 \int_1^{2-x} xy \, dx \, dy = \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=1}^{2} y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_{y=1}^{2} y \frac{(2-y)^2}{2} dy$$

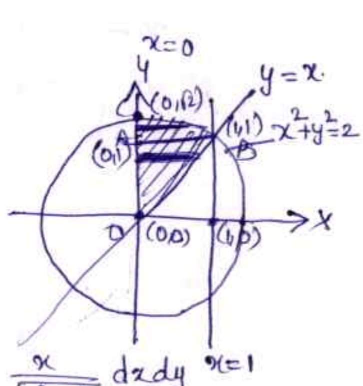
$$= \frac{1}{2} \int_{y=1}^{2} \left(4y + y^3 - 4y^2\right) dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2 = \frac{1}{2}\left(8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right)$$

$$= \frac{1}{2}\left(10 - \frac{1}{4} - \frac{28}{3}\right)$$

$$= \frac{1}{2}\left(\frac{120-3-112}{12}\right) = \frac{1}{2} \cdot \frac{5}{12} = \frac{5}{24}$$

✓ * 12) Evaluate by changing the order of integration $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x \, dy \, dx}{\sqrt{x^2+y^2}}$



Given x — 0 to 1

y — x to $\sqrt{2-x^2}$ i.e., $y=x$, $y=\sqrt{2-x^2} \Rightarrow x^2+y^2=2$

In ABC

$\therefore y$ — 1 to $\sqrt{2}$

x — 0 to $\sqrt{2-y^2}$

In OAB, y — 0 to 1

x — 0 to y.

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy \, dx = \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dx \cdot dy + \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}} dx \, dy$$

$$= \int_{y=1}^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_{x=0}^{\sqrt{2-y^2}} dy + \int_{y=0}^{1} \left[\sqrt{x^2+y^2}\right]_0^y$$

$$= \int_{y=1}^{\sqrt{2}} (\sqrt{2}-y)\,dy + \int_{y=0}^{1} (\sqrt{2}y - y)\,dy$$

$$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} + \left[\sqrt{2}\frac{y^2}{2} - \frac{y^2}{2}\right]_0^1$$

$$= 2 - \frac{2}{2} - \sqrt{2} + \cancel{\frac{1}{2}} + \frac{1}{\sqrt{2}} - \cancel{\frac{1}{2}}$$

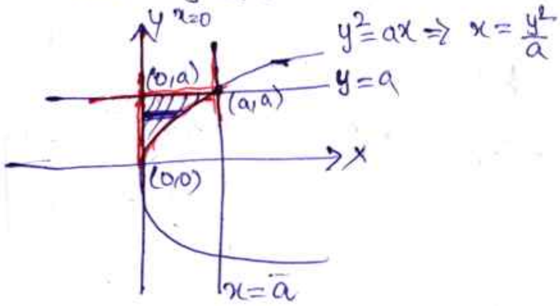
$$= 1 - \frac{1}{\sqrt{2}}.$$

13.) Evaluate by changing the order of integration $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$

x — 0 to a i.e., $x=0$, $x=a$

y — $\sqrt{ax}$ to a i.e., $y=\sqrt{ax}$, $y=a$

$\Rightarrow y^2 = ax$



$\therefore y$ — 0 to a

x — 0 to $\frac{y^2}{a}$.

$$\therefore \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}} = \int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2}{\sqrt{y^4-a^2x^2}}\,dx\,dy$$

$$= \int_{y=0}^{a} y^2 \cdot \frac{1}{a} \cdot \left[\sin^{-1}\left(\frac{x}{\left(\frac{y^2}{a}\right)}\right)\right]_0^{y^2/a} dy \qquad \left(\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2 \left(\frac{\pi}{2}\right) dy$$

$$= \frac{\pi}{2a} \cdot \left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{2a} \cdot \frac{a^3}{3} = \frac{\pi a^2}{6}$$

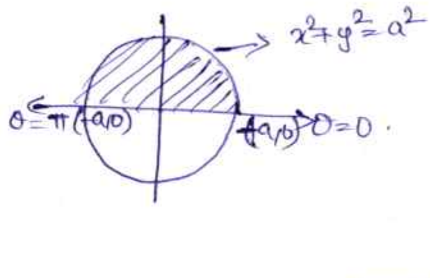
14.) Transform the following to cartesian form & hence evaluate $\int_0^{\pi}\int_0^a r^3 \sin\theta\cos\theta\,dr\,d\theta$.

Given θ — 0 to π

r — 0 to a.

w.k.t $x = r\cos\theta$ $dx\,dy = r\,dr\,d\theta$.

$y = r\sin\theta$



$x^2+y^2 = r^2 = a^2$ $(\because r=a)$

x — '$-a$' to 'a'

y — 0 to $\sqrt{a^2-x^2}$

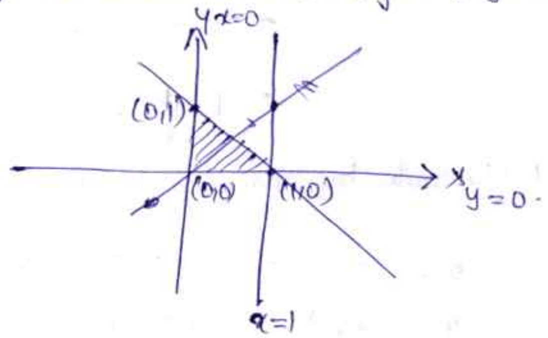
$$\therefore \int_0^{\pi}\int_0^a r^3\sin\theta\cos\theta\,dr\,d\theta = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,.\,dy\,dx = \int_{x=-a}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=-a}^{a} x\left(a^2-x^2\right) dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{-a}^{a}$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4} - \frac{a^4}{2} + \frac{a^4}{4}\right) = 0.$$

~~Evaluate~~ S.T $\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{1}{2}(e-1)$ by using the transformation $x+y=u$ & $y=uv$.



Given x — 0 to 1

y — 0 to $1-x$. i.e., $y=0$, $y=1-x$ $\Rightarrow x+y=1$

Given $x+y=u \Rightarrow x = u-y = u-uv$

$y = uv \Rightarrow v = \frac{y}{u} = \frac{y}{x+y}$.

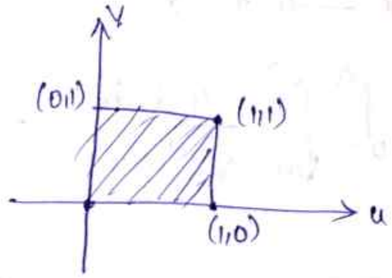
$dx\,dy = |J|\,du\,dv$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v)+uv = u$$

$x = u(1-v)$

$y = uv$

$\therefore dx\,dy = u\,du\,dv$.



$\therefore$ At $(1,0)$, $u=1$, $v=0$

At $(0,1)$, $u=1$, $v=1$

At $(0,0)$, $u=0$, $v=0$

When $x=0$, $u = y = 0 \Rightarrow u=1$ $\Rightarrow u(1-v)=0 \Rightarrow u=0$. $v=1$.

$x=1$, $u = y+1 \Rightarrow$. $x=1$, $u(1-v)=1$

$y=0$, $u=x$, $v=0$.

$y = 1-x$, $x+y=1$, $u=1$

$$\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \int_{u=0}^{1}\int_{v=0}^{1} e^{uv/u}\,u\,du\,dv$$

$$= \int_{u=0}^{1}\int_{v=0}^{1} e^{v}\,u\,du\,dv$$

$$= \int_{u=0}^{1} u.\left[e^v\right]_0^1\,du$$

$$= \int_{u=0}^{1} u\,(e^1-1)\,du$$

$$= (e-1)\left[\frac{u^2}{2}\right]_0^1 = \frac{e-1}{2}$$

$$\int \cot^n\theta\,d\theta = -\frac{\cot^{n-1}\theta}{n-1} - \int \cot^{n-2}\theta\,d\theta$$

Tripple Integral:-

1) $\int_a^b \int_c^d \int_e^f f(x,y,z)\, dx\, dy\, dz = \int_{z=a}^{b} \int_{y=c}^{d} \int_{x=e}^{f} f(x,y,z)\, dx\, dy\, dz$

2) $\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dx\,dy\,dz = \int_{x=x_1}^{x_2} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx$

1.) Evaluate $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} x+y+z \; dx\, dy\, dz$

Sol: $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} x+y+z \; dx\, dy\, dz = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-z}^{x+z} x+y+z \; dy\, dx\, dz$

$= \int_{z=-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + yz \right]_{y=x-z}^{x+z} dx\, dz$

$= \int_{z=-1}^{1} \int_{x=0}^{z} x(x+z) + \frac{(x+z)^2}{2} + (x+z)z - x(x-z)$
$- \frac{(x-z)^2}{2} - (x-z)z \;\; dx\, dz$

$= \int_{z=-1}^{1} \int_{x=0}^{z} \cancel{x^2} + zx + \cancel{\frac{x^2}{2}} + \cancel{\frac{z^2}{2}} + zx + zx + z^2 - \cancel{x^2} + zx$
$- \cancel{\frac{x^2}{2}} - \cancel{\frac{z^2}{2}} + z\cancel{x} - x\cancel{z} + z^2 \;\; dx\, dz$

$= \int_{z=-1}^{1} \int_{x=0}^{z} 2z^2 + 4zx \;\; dx\, dz$

$= \int_{z=-1}^{1} \left[2xz^2 + \frac{\cancel{4}^{2} z x^2}{\cancel{2}} \right]_{x=0}^{z} dz$

$= \int_{z=-1}^{1} 2z^3 + 2z^3 - 0 \;\; dz$

$= \left[\frac{\cancel{4} z^4}{\cancel{4}} \right]_{z=-1}^{1}$

$= 1 - 1 = 0$

2) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} x^2+y^2+z^2 \; dx\, dy\, dz$

$\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} x^2+y^2+z^2 \; dx\, dy\, dz = \int_{z=-c}^{c} \int_{y=-b}^{b} \int_{x=-a}^{a} x^2+y^2+z^2 \, dx\, dy\, dz$

$= 2\int_{z=0}^{c} 2 \int_{y=0}^{b} 2 \int_{x=0}^{a} x^2+y^2+z^2 \; dx\; dy\, dz$

$$= 8 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^3}{3} + xy^2 + xz^2 \right]_{x=0}^{a} dy\, dz$$

$$= 8 \int_{z=0}^{c} \int_{y=0}^{b} \frac{a^3}{3} + ay^2 + az^2 - 0 \; dy\, dz$$

$$= 8 \int_{z=0}^{c} \left[\frac{a^3 y}{3} + \frac{ay^3}{3} + ayz^2 \right]_{y=0}^{b} dz$$

$$= 8 \int_{z=0}^{c} \frac{a^3 b}{3} + \frac{ab^3}{3} + abz^2 \; dz$$

$$= 8 \left[\frac{a^3 bz}{3} + \frac{ab^3 z}{3} + \frac{abz^3}{3} \right]_{z=0}^{c}$$

$$= 8 \left(\frac{a^3 bc}{3} + \frac{ab^3 c}{3} + \frac{abc^3}{3} - 0 \right)$$

$$= \frac{8abc}{3} \left(a^2 + b^2 + c^2 \right)$$

3) Evaluate $\int_0^a \int_0^b \int_0^c x^2 + y^2 + z^2 \; dx\, dy\, dz$ Ans:- $\frac{abc}{3}(a^2 + b^2 + c^2)$

4) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dx\, dy\, dz$

Sol:- $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dx\,dy\,dz = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_{z=0}^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left(\frac{1-x^2-y^2}{2} - 0 \right) dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[x\frac{y^2}{2} - x^3 \frac{y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} - 0 \; dx$$

$$= \frac{1}{8} \int_{x=0}^{1} 2x - 2x^3 - \cancel{2}x^3 + 2x^5 - x - x^5 + \cancel{2}x^3 \; dx$$

$$= \frac{1}{8} \int_{x=0}^{1} 2x^5 + x^4 - 2x^3 + 2x^2 + 2x - 4 \; dx$$

$$\cancel{= \frac{1}{8} \left[\frac{2x^6}{6_3} - \frac{x^5}{5} - \frac{4x^4}{4} + 2\frac{x^3}{3} + 2\frac{x^2}{2} - x \right]_{x=0}^{1}}$$

$$\cancel{= \frac{1}{8} \left(\frac{1}{3} - \frac{1}{5} - 1 + \frac{2}{3} + 1 - 1 - 0 \right)}$$

$$= \frac{1}{8} \left[\frac{x^6}{6} - \frac{\cancel{2}x^4}{\cancel{4}_2} + \frac{x^2}{2} \right]_{x=0}^{1}$$

$$= \frac{1}{8} \left(\frac{1}{6} - \frac{\cancel{1}}{\cancel{2}} + \frac{\cancel{1}}{\cancel{2}} - 0 \right)$$

$$= \frac{1}{48}$$

5) Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dx\,dy\,dz$

Sol:- $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dx\,dy\,dz = \int_{x=0}^{\log 2}\int_{y=0}^{x}\int_{z=0}^{x+\log y} e^{x}.e^{y}.e^{z}\, dz\,dy\,dx$

$= \int_{x=0}^{\log 2} e^{x}\int_{y=0}^{x} e^{y}\left[e^{z}\right]_{z=0}^{x+\log y} dy\, dx$

$= \int_{x=0}^{\log 2} e^{x}\int_{y=0}^{x} e^{y}\left(e^{x}.y-1\right) dy\, dx$

$= \int_{x=0}^{\log 2} e^{x}\int_{y=0}^{x} e^{x}.\, ye^{y}-e^{y}\, dy\, dx$

$= \int_{x=0}^{\log 2} e^{x}.\left[e^{x}.(y-1)e^{y}-e^{y}\right]_{y=0}^{x} dx$

$= \int_{x=0}^{\log 2} e^{x}\left(e^{x}(x-1)e^{x}-e^{x}-e^{x}(0-1)+e^{0}\right) dx$

$= \int_{x=0}^{\log 2} e^{x}\left((x-1)e^{2x}-\cancel{e^{x}}+\cancel{e^{x}}+1\right) dx$

$= \int_{x=0}^{\log 2} (x-1)e^{3x}+e^{x}\, dx$

$= \left[x\cdot\frac{e^{3x}}{3}-1\cdot\frac{e^{3x}}{9}-\frac{e^{3x}}{3}+e^{x}\right]_{x=0}^{\log 2}$

$= \log 2\cdot\left(\frac{8}{3}\right)-\frac{8}{9}-\frac{8}{3}+2-0+\frac{1}{9}+\frac{1}{3}-1$

$= \frac{8}{3}\log 2-\frac{19}{9}$

6) Evaluate $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dz\,dy\,dx$

Sol:- $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dz\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x}.e^{y}.e^{z}\, dz\,dy\,dx$

$= \int_{x=0}^{a}\int_{y=0}^{x} e^{x}.e^{y}\left[e^{z}\right]_{z=0}^{x+y} dy\, dx$

$= \int_{x=0}^{a}\int_{y=0}^{x} e^{x}.e^{y}\left(e^{x+y}-1\right) dy\, dx$

$= \int_{x=0}^{a}\int_{y=0}^{x}\left(e^{2x}.e^{2y}-e^{x}e^{y}\right) dy\, dx$

$= \int_{x=0}^{a}\left[e^{2x}.\frac{e^{2y}}{2}-e^{x}.e^{y}\right]_{y=0}^{x} dx$

$$= \int_{x=0}^{a} \frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \; dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3}{2} \cdot \frac{e^{2x}}{2} + e^{x} \right]_{x=0}^{a}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{1}{8} + \frac{3}{4} - 1$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$

7) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \; dz dx dy$

Sol:- $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \; dz\, dx dy = \int_{y=1}^{e} \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z \; dz dx dy$

$\left(\because \int \log x \, dx = x \log x - x \right)$

$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left[z \log z - z \right]_{z=1}^{e^x} dx \, dy$$

$$= \int_{y=1}^{e} \int_{x=1}^{\log y} \left(e^{x} \cdot x - e^{x} - 0 + 1 \right) dx \, dy$$

$$= \int_{y=1}^{e} \left[(x-1) e^{x} - e^{x} + x \right]_{x=1}^{\log y} dy$$

$$= \int_{y=1}^{e} (\log y - 1) \cdot y - y + \log y - 0 + e - 1 \; dy$$

$$= \int_{y=1}^{e} \left(y \log y - 2y + \log y + e - 1 \right) dy$$

$\left(\because \int x \log x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4} \right)$

$$= \left[\frac{y^2}{2} \log y - \frac{y^2}{4} - \cancel{2} \frac{y^2}{\cancel{2}} + y \log y - y + e y - y \right]_{y=1}^{e}$$

$$= \frac{e^2}{2} - \frac{e^2}{4} \cancel{- e^2} + \cancel{e} - \cancel{e} \cancel{+ e^2} - e - 0 + \frac{1}{4} + 1 - 0 + 1 - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

$$= \frac{1}{4} \left(e^2 - 8e + 13 \right)$$

~~8) Evaluate~~ $\cancel{\int_0^a \int_0^b \int_0^c x^2 + y^2 +}$

8) Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \; dz \, dx dy$

Sol: $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \; dz \, dx dy = \int_{y=0}^{1} \int_{x=y^2}^{1} \int_{z=0}^{1-x} x \; dz \, dx dy$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x \left[z \right]_{z=0}^{1-x} dx \; dy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x(1-x-0)\, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{1} x - x^2\, dx\, dy$$

$$= \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=y^2}^{1} dy$$

$$= \int_{y=0}^{1} \frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3}\, dy$$

$$= \cancel{\int_{y=0}^{1}} \left[\frac{1}{6}y - \frac{1}{2}\frac{y^5}{5} + \frac{y^7}{3\times 7}\right]_{y=0}^{1}$$

$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} - 0$$

$$= \frac{4}{35}$$

9) Evaluate $\iiint xy^2z\, dxdydz$ taken through the +ve octant of the sphere $x^2+y^2+z^2=a^2$

Sol: Given $x^2+y^2+z^2=a^2$ in +ve octant

$$\Rightarrow z^2 = a^2-x^2-y^2$$

$$\Rightarrow z = \pm\sqrt{a^2-x^2-y^2}$$

$\therefore$ z varies from 0 to $\sqrt{a^2-x^2-y^2}$

Put $z=0$ in $x^2+y^2+z^2=a^2 \Rightarrow x^2+y^2=a^2$

$$\Rightarrow y = \pm\sqrt{a^2-x^2}$$

$\therefore$ y varies from 0 to $\sqrt{a^2-x^2}$

Put $y=z=0$ in $x^2+y^2+z^2=a^2 \Rightarrow x^2=a^2$

$$\Rightarrow x = \pm a$$

$\therefore$ x varies from 0 to a

$$\therefore \iiint xy^2z\, dxdydz = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy^2z\, dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy^2\left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy^2(a^2-x^2-y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{a} \left[x(a^2-x^2)\frac{y^3}{3} - x\frac{y^5}{5}\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} \frac{x(a^2-x^2)\left(\sqrt{a^2-x^2}\right)^3}{3} - \frac{x}{5}\left(\sqrt{a^2-x^2}\right)^5\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \frac{x(a^2-x^2)^{5/2}}{3} - \frac{x}{5}(a^2-x^2)^{5/2} \, dx$$

$$= \frac{1}{15} \int_{x=0}^{a} x(a^2-x^2)^{5/2} \, dx$$

$$= \frac{1}{15} \cdot \frac{1}{-2} \int_{x=0}^{a} (a^2-x^2)^{5/2} \, -2x \, dx$$

$$= \frac{-1}{30} \int_{x=0}^{a} (a^2-x^2)^{5/2} \, d(a^2-x^2)$$

$$= \frac{-1}{30} \left[\frac{(a^2-x^2)^{5/2+1}}{5/2+1} \right]_{x=0}^{a}$$

$$= \frac{-1}{30} \left(0 - \frac{(a^2)^{7/2}}{7/2} \right)$$

$$= \frac{-1}{30} \left(-\frac{2}{7} a^7 \right) = \frac{a^7}{105}$$

10) Evaluate $\iiint z^2 \, dx\,dy\,dz$ taken over the volume bounded by the surfaces $x^2+y^2=a^2$, $x^2+y^2=z$ & $z=0$.

Sol: Given $x^2+y^2=a^2$, $x^2+y^2=z$ & $z=0$

$\therefore$ z varies from 0 to x^2+y^2

y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

x varies from $-a$ to a

$$\therefore \iiint z^2 \, dx\,dy\,dz = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{x^2+y^2} z^2 \, dz \, dy \, dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3} \right]_0^{x^2+y^2} dy \, dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{(x^2+y^2)^3}{3} \, dy \, dx$$

$$= \frac{1}{3} \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^6 + 3x^4y^2 + 3x^2y^4 + y^6 \, dy \, dx$$

$$= \frac{1}{3} \int_{x=-a}^{a} 2 \left[x^6 y + \cancel{3} x^4 \frac{y^3}{\cancel{3}} + 3x^2 \frac{y^5}{5} + \frac{y^7}{7} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{2}{3} \int_{x=-a}^{a} x^6 \sqrt{a^2-x^2} + x^4 (a^2-x^2)^{3/2} + \frac{3}{5} x^2 (a^2-x^2)^{5/2} + \frac{1}{7}(a^2-x^2)^{7/2} \, dx$$

Unit-5 Vector Calculus

Scalar:- A scalar is a quantity that has magnitude (numerical value) but no direction.
Eg:- Mass, temperature

Vector:- A vector is a quantity that has both magnitude and direction.
Eg:- Displacement, Momentum.

Scalar Point function:- A scalar point function is a function that assigns a real number (i.e., a scalar) to each point of some region of space i.e., to each point (x,y,z) of a region R in space there is assigned a real number $u=\phi(x,y,z)$ then ϕ is called a scalar point function. Eg:- $xy^3z^2=4$

Vector point function:- In the region of space, if each point $P(x,y,z)$ is associated with a unit vectors $\bar{i},\bar{j},\bar{k}$ as $\bar{f}(x,y,z) = f_1(x,y,z)\bar{i} + f_2(x,y,z)\bar{j} + f_3(x,y,z)\bar{k}$ then $\bar{f}(x,y,z)$ is known as vector point function. Here $f_1(x,y,z)$, $f_2(x,y,z)$, $f_3(x,y,z)$ are scalar functions.

Eg:- 1) If a bird is flying in the sky then speed is a scalar point function and, velocity & acceleration are vector point functions.

2) $\bar{f} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Vector differential operator:- Vector differential operator is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

Gradient of scalar Point function :- Let $\phi(x,y,z)$ be the scalar Point function then gradient of ϕ is denoted by grad ϕ (or) $\nabla\phi$ and is defined as grad ϕ (or) $\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Directional derivative :- the rate of change of a function w.r.to the distance in a specified direction is called "directional ~~derivati~~ derivative."

the directional derivative of a scalar function ϕ at the point (x, y, z) in the direction of a vector $\bar{a}$ is $\nabla\phi = \frac{\bar{a}}{|\bar{a}|}$

Note :- 1. the maximum value of directional derivative of ϕ is $|\nabla\phi|$

2. the Minimum value of directional derivative of ϕ is $-|\nabla\phi|$

Normal vector :- the Normal vector of the surface (i.e a scalar point function) ϕ is given by $\nabla\phi$.

Unit normal vector :- the unit normal vector to the surface ϕ is given by $\frac{\nabla\phi}{|\nabla\phi|}$

Angle between the two surfaces (i.e scalar Point functions) :-

Let f and g be two surfaces at the point (x,y,z) then the angle between f and g is defined as $\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$

Note :- the angle between the normals to the surface ϕ at the Point (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

where $\bar{n}_1 = \nabla\phi$ at (x_1, y_1, z_1)

$\bar{n}_2 = \nabla\phi$ at (x_2, y_2, z_2)

1) find grad f, where $f = x^3+y^3+3xyz$

Sol:- grad f $= \nabla f$

$$= \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(x^3+y^3+3xyz) + \bar{j}\frac{\partial}{\partial y}(x^3+y^3+3xyz) + \bar{k}\frac{\partial}{\partial z}(x^3+y^3+3xyz)$$

$$= \bar{i}(3x^2+3yz) + \bar{j}(3y^2+3xz) + \bar{k}(3xy)$$

2) Find $\nabla\phi$, if $\phi = \log(x^2+y^2+z^2)$

Sol:- $$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}(\log(x^2+y^2+z^2)) + \bar{j}\frac{\partial}{\partial y}(\log(x^2+y^2+z^2)) + \bar{k}(\log(x^2+y^2+z^2))$$

$$= \bar{i}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x + \bar{j}\frac{1}{x^2+y^2+z^2}\cdot 2y + \bar{k}\frac{1}{x^2+y^2+z^2}\cdot 2z$$

$$= \frac{2}{x^2+y^2+z^2}(x\bar{i}+y\bar{j}+z\bar{k})$$

$$\nabla\phi = \frac{2\bar{r}}{r^2}$$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and

$$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$$

3) If $\phi = 2xz^4 - x^2y$ find $|\nabla\phi|$ at the point $(2,-2,1)$

Sol:- Given $\phi = 2xz^4 - x^2y$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2z^4 - 2xy) + \bar{j}(-x^2) + \bar{k}(8xz^3)$$

At $(2,-2,1)$, $\nabla\phi = \bar{i}(2(1)^4 - 2(2)(-2)) + \bar{j}(-(2^2)) + \bar{k}(8(2)(1)^3)$

$$= 10\bar{i} - 4\bar{j} + 16\bar{k}$$

Now $|\nabla\phi| = \sqrt{(10)^2+(-4)^2+16^2}$

$$= \sqrt{372}$$

$$|\nabla\phi| = 2\sqrt{83}$$

4) find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,1)$ in the direction of the vector $2\bar{i}-\bar{j}-2\bar{k}$

Sol:- Given $\phi = x^2yz + 4xz^2$

then $$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2xyz+4z^2) + \bar{j}(x^2z) + \bar{k}(x^2y+8xz)$$

at $(1,-2,1)$; $\nabla\phi = \bar{i}\left(2(1)(-2)(1)+4(1)^2\right) + \bar{j}\left((1)^2(1)\right) + \bar{k}\left((1)^2(-2)+8(1)(1)\right)$
$x\ y\ z$

$$= (0)\bar{i} + \bar{j} + 6\bar{k}$$

$$= \bar{j} + 6\bar{k}$$

Let the given vector be $\bar{a} = 2\bar{i}-\bar{j}-2\bar{k}$

then $a = |\bar{a}| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{9} = 3.$

$\therefore$ Required directional derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (\bar{j}+6\bar{k}) \cdot \frac{(2\bar{i}-\bar{j}-2\bar{k})}{3}$$

$$= \frac{(2)(0) + 1(-1) + 6(-2)}{3}$$

$$= \frac{-1-12}{3} = \frac{-13}{3}.$$

5) Find the directional derivative of $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$

Sol:- the position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i}+2\bar{j}+3\bar{k}$ and

$$\overline{OQ} = 5\bar{i}+0\cdot\bar{j}+4\bar{k} = 5\bar{i}+4\bar{k}$$

then $\overline{PQ} = \overline{OQ}-\overline{OP} = 4\bar{i}-2\bar{j}+\bar{k} = \bar{a}$ (say) given $f = x^2 - y^2 + 2z^2$

then $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= \bar{i}(2x) + \bar{j}(-2y) + \bar{k}(4z)$

$\therefore$ At $(1,2,3)$, $\boxed{\nabla f = 2\bar{i} - 4\bar{j} + 12\bar{k}}$

Now $\bar{a} = 4\bar{i} - 2\bar{j} + \bar{k}$

$\boxed{|\bar{a}| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{16+4+1} = \sqrt{21}}$

$\therefore$ Directional derivative $= \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$

$= (2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot \frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}}$

$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$

6) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

Sol:- Let the given surfaces be $f = x^2+y^2+z^2-9$

$g = x^2+y^2-z-3$

then $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

At $(2,-1,2)$ $= 2(2)\bar{i} + 2(-1)\bar{j} + 2(2)\bar{k}$

$\boxed{\nabla f = 4\bar{i} - 2\bar{j} + 4\bar{k}}$

$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1)$

At $(2,-1,2)$ $= \bar{i}(2(2)) + \bar{j}(2(-1)) + \bar{k}(-1)$

$\boxed{\nabla g = 4\bar{i} - 2\bar{j} - \bar{k}}$

Now $|\nabla f| = \sqrt{4^2+(-2)^2+4^2} = 6$

$|\nabla g| = \sqrt{4^2+(-2)^2+(-1)^2} = \sqrt{21}$

$\therefore$ Required angle can be obtained by using

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$$

$$\Rightarrow \cos\theta = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2\bar{j}-\bar{k})}{6\times\sqrt{21}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{\cancel{16}^{8}}{\cancel{6}_{3}\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

7) find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$

Sol:- Let the given surface be $f = x^2 - yz$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to the surface f at the points $(1,1,1)$ and $(2,4,1)$ i.e $\bar{n}_1 = \nabla f$ at $(1,1,1)$

$\bar{n}_2 = \nabla f$ at $(2,4,1)$

Now $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= 2x\bar{i} - z\bar{j} - y\bar{k}$

$\therefore \bar{n}_1 = \nabla f$ at $(1,1,1) = 2(1)\bar{i} - (1)\bar{j} - (1)\bar{k} = 2\bar{i} - \bar{j} - \bar{k}$

$\bar{n}_2 = \nabla f$ at $(2,4,1) = 2(2)\bar{i} - (1)\bar{j} - 4\bar{k} = 4\bar{i} - \bar{j} - 4\bar{k}$

$|\bar{n}_1| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$

$|\bar{n}_2| = \sqrt{4^2 + (-1)^2 + (-4)^2} = \sqrt{33}$

Let θ be the angle between the two normal then

$$\cos\theta = \frac{\bar{n}_1 \bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}$$

$$= \frac{(2\bar{i}-\bar{j}-\bar{k})(4\bar{i}-\bar{j}-4\bar{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{198}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)}$$

Problems for practice :-

1) find the directional derivative of $f = xy^3 + yz^3$ at the point $(2,-1,1)$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$

Ans :- $\frac{7}{3}$

2) find the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at $(-1,2,1)$.

Ans :- $\frac{15}{\sqrt{17}}$

3) find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$.

Ans :-

4) Calculate the angle between the normals to the surface $xy = z^2$ at $(4,1,2)$ and $(3,3,-3)$.

Ans :-

5) P.T $\nabla r^n = n r^{n-2} \bar{r}$

6) If $\nabla u = 2r^4 \bar{r}$, find u Ans $u = \frac{r^6}{3} + c$ Hint $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

7) S.T $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$

8) P.T $\nabla(\log r) = \frac{\bar{r}}{r^2}$

9) If $\bar{a}$ is a constant vector then prove that $\text{grad}(\bar{a}\cdot\bar{r}) = \bar{a}$

10) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, then prove that grad u, grad v and grad w are coplanar.

Hint :- w.k.t Three vectors are coplanar, if their scalar tripple product is zero i.e., [grad u, grad v, grad w] = 0.

$$[\text{grad}\,u\ \ \text{grad}\,v\ \ \text{grad}\,w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$R_3 \to R_3 + R_2$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 2(x+y+z)$$

$= 0$

Divergence of a vector:- Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be a continuously differentiable vector point function.

Then the divergence of $\bar{f}$ is defined as

$$\text{div}\,\bar{f} = \nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note:- 1.) $\text{div}\,\bar{f}$ is always a scalar

2.) $\text{div}(\bar{f} \pm \bar{g}) = \text{div}\,\bar{f} \pm \text{div}\,\bar{g}$

3.) If $\bar{f}$ is a constant vector then $\text{div}\,\bar{f} = 0$.

Solenoidal vector:- A vector point function $\bar{f}$ is said to be a solenoidal vector if $\text{div}\,\bar{f} = 0$.

1.) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find $\text{div}\,\bar{f}$ at $(1,-1,1)$.

sol:- Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

i.e., $f_1 = xy^2$, $f_2 = 2x^2yz$, $f_3 = -3yz^2$

w.k.t $\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= y^2 + 2x^2z - 6yz$

$\left(\because \frac{\partial f_1}{\partial x} = \frac{\partial(xy^2)}{\partial x} = y^2 \right.$

$\frac{\partial f_2}{\partial y} = \frac{\partial}{\partial y}(2x^2yz) = 2x^2z$

$\left. \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial z}(-3yz^2) = -6yz \right)$

$\therefore$ At $(1,-1,1)$, $\text{div}\,\bar{f} = (-1)^2 + 2(1)^2(1) - 6(-1)(1) = 9$

2.) If $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$, find $\text{div}\,\bar{f}$.

sol:- Let $\phi = x^3+y^3+z^3-3xyz$

Now, $\text{grad}\,\phi = \nabla\phi$

$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$= \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$

Given that $\bar{f} = \text{grad}\,\phi$

$\Rightarrow \bar{f} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$

$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, say then $\operatorname{div}\bar{f} = \nabla\cdot\bar{f}$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

$$\left(\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial}{\partial x}(3x^2 - 3yz) = 6x \\ \frac{\partial f_2}{\partial y} &= \frac{\partial}{\partial y}(3y^2 - 3xz) = 6y \\ \frac{\partial f_3}{\partial z} &= \frac{\partial}{\partial z}(3z^2 - 3xy) = 6z \end{aligned}\right)$$

Curl of a vector:- Let $\bar{f}$ be any continuously differentiable vector point function. then vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\,\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and it is denoted by curl $\bar{f}$ (or) $\nabla\times\bar{f}$

i.e. $\operatorname{curl}\bar{f} = \nabla\times\bar{f}$

$$= \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$$

$$= \Sigma\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$$

If $\bar{f}$ is a differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then $\operatorname{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla\times\bar{f}$

$$\Rightarrow \operatorname{Curl}\bar{f} = \nabla\times\bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Note:- 1. if $\bar{f}$ is a constant vector then $\operatorname{curl}\bar{f} = \bar{0}$

2. $\operatorname{curl}(\bar{a}\pm\bar{b}) = \operatorname{curl}\bar{a} \pm \operatorname{curl}\bar{b}$

Irrotational vector:- A vector point function $\bar{f}$ is said to be irrotational vector if $\operatorname{curl}\bar{f} = \bar{0}$

if $\bar{f}$ is irrotational there will always exists a scalar function $\phi(x,y,z)$ such that $\bar{f} = \operatorname{grad}\phi$ then ϕ is called a scalar potential function of $\bar{f}$.

Note:- $\operatorname{curl}(\operatorname{grad}\phi) = \bar{0}$.

1) If $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$ find $\text{curl}\,\bar{f}$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz$$

$$\frac{\partial \phi}{\partial y} = 3y^2 - 3xz$$

$$\frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

Given $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$= \nabla\phi$$

$$= \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$= (3x^2-3yz)\bar{i} + (3y^2-3xz)\bar{j} + (3z^2-3xy)\bar{k}$$

$$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k} \quad (\text{say})$$

Now $\text{curl}\,\bar{f} = \nabla \times \bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \bar{i}\left(\partial/\partial y\,(3z^2-3xy) - \partial/\partial z\left(3y^2-3xz\right)\right) - \bar{j}\left(\partial/\partial x\,(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right) + \bar{k}\left(\partial/\partial x\,(3y^2-3xz) - \partial/\partial y\,(3x^2-3yz)\right)$$

$$= (-3x+3x)\bar{i} - (-3y+3y)\bar{j} + (-3z+3z)\bar{k}$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$\boxed{\text{curl}\,\bar{f} = \bar{0}}$$

2) Find curl $\bar{f}$ for $\bar{f} = 2xz^2\bar{i} - yz\bar{j} + 3xz^3\bar{k}$

Sol:- Given $\bar{f} = 2xz^2\bar{i} - yz\bar{j} + 3xz^3\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ say

We know that curl $f = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \bar{i}\left(\partial/\partial y(3xz^3) - \partial/\partial z(-yz)\right) - \bar{j}\left(\partial/\partial x(3xz^3) - \partial/\partial z(2xz^2)\right)$$
$$+ \bar{k}\left(\partial/\partial x(-yz) - \partial/\partial y(2xz^2)\right).$$

$$= \bar{i}(0+y) - \bar{j}(3z^3 - 4xz) + \bar{k}(0-0)$$

$$\boxed{\text{Curl } \bar{f} = y\bar{i} + (4xz - 3z^3)\bar{j}}$$

3) If $\bar{f}$ $(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ then show that $\bar{f}\cdot\text{curl }\bar{f} = 0$

Sol:- Given $\bar{f} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$\therefore$ Curl $\bar{f} = \nabla \times \bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \bar{i}(-1-0) - \bar{j}(-1-0) + \bar{k}(0-1)$$
$$= -\bar{i} + \bar{j} - \bar{k}$$

Now $\bar{f}\cdot\text{curl }\bar{f} = \left((x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}\right)\left(-\bar{i} + \bar{j} - \bar{k}\right)$

$$= -(x+y+1) + 1 + x + y$$
$$= -x - y - 1 + 1 + x + y$$

$$\boxed{\bar{f}\cdot\text{curl }\bar{f} = 0}$$

Del applied twice to scalar point function (or) secound order equation or Laplacian operator:

the operator $\nabla^2 = \nabla.\nabla$

$= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z}$ is called Laplacian operator

If ϕ is any scalar point function, then $\nabla^2\phi = \nabla.\nabla.\phi$

$= div(grad\,\phi)$

Note:- If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation

1). Evaluate $\nabla^2(\log r)$

sol Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

then $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ and $r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}.\cancel{2}x$

$= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}.\cancel{2}y$

$= \frac{y}{\sqrt{x^2+y^2+z^2}}$

$= \frac{y}{r}$

Ily $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\nabla^2(\log r) = \nabla.\nabla(\log r)$

$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\log r$

$= \frac{\partial^2}{\partial x^2}\log r+\frac{\partial^2}{\partial y^2}\log r+\frac{\partial^2}{\partial z^2}\log r$

$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\log r\right)+\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\log r\right)+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\log r\right)$

$= \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$

$= \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{1}{r}.\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{z}{r}\right)$

$= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$

$= \frac{1}{r^2}+x\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial x}+\frac{1}{r^2}+y\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial y}+\frac{1}{r^2}+z\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial z}$

$= \frac{3}{r^2}+x\left(\frac{-2}{r^3}\right)\frac{x}{r}+y\left(\frac{-2}{r^3}\right)\frac{y}{r}+z\left(\frac{-2}{r^3}\right)\frac{z}{r}$

$= \frac{3}{r^2}-\frac{2x^2}{r^4}-\frac{2y^2}{r^4}-\frac{2z^2}{r^4}$

$= \frac{3}{r^2} - \frac{2}{r^4}(x^2+y^2+z^2)$

$= \frac{3}{r^2} - \frac{2}{r^{4/2}}(r^2)$

$= \frac{1}{r^2}$

2). If $\bar{r}$ is the position vector of the point (x, y, z), prove that $\text{div}(\text{grad } r^n) = \nabla^2(r^n) = n(n+1) r^{n-2}$

Sol

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ and $r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

Now, $\text{grad } r^n = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}(r^n)$

$= \bar{i} n r^{n-1}\frac{\partial r}{\partial x} + \bar{j} n r^{n-1}\frac{\partial r}{\partial y} + \bar{k} n r^{n-1}\frac{\partial r}{\partial z}$

$= \bar{i} n r^{n-1}\frac{x}{r} + \bar{j} n r^{n-1}\frac{y}{r} + \bar{k} n r^{n-1}\frac{z}{r}$

$= \frac{n r^{n-1}}{r}(x\bar{i} + y\bar{j} + z\bar{k})$

$= n r^{n-2} x\bar{i} + n r^{n-2} y\bar{j} + n r^{n-2} z\bar{k}$

$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ say

$\text{div}(\text{grad } r^n) = \nabla\cdot(\nabla r^n)$

$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= \frac{\partial}{\partial x}(n x^{n-2} x) + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z)$

$= n r^{n-2} + x\, n(n-2) r^{n-3}\frac{\partial r}{\partial x} + n r^{n-2} + y\, n(n-2) r^{n-3}\frac{\partial r}{\partial y} + z\, n(n-2) r^{n-3}\frac{\partial r}{\partial z}$

$= 3 n r^{n-2} + n(n-2) r^{n-3}\frac{x^2}{r} + n(n-2) r^{n-3}\frac{y^2}{r} + n(n-2) r^{n-3}\frac{z^2}{r}$

$= 3 n r^{n-2} + n(n-2) r^{n-4}(x^2+y^2+z^2)$

$= 3 n r^{n-2} + n(n-2) r^{n-4} r^2$

$= 3 n r^{n-2} + n(n-2) r^{n-2}$

Line integral:-

Any integral which is to be evaluated along a curve is called Line integral

Let $\bar{f}$ be a continuous vector point function defined at each point of a curve C, then the line integral of $\bar{f}$ along C is defined as $\int_C \bar{f}\cdot d\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

If curve 'c' is closed then the integral can be written as $\oint_C$

Note:- If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then

$$\int_C \bar{f}\cdot d\bar{r} = \int_C f_1 dx + f_2\, dy + f_3\, dz$$

1) If $\bar{f} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_C \bar{f}\cdot d\bar{r}$ while C is the curve in the xy-plane from $(0,0)$ to $(1,2)$

Sol:- Given $\bar{f} = 3xy\bar{i} - y^2\bar{j}$

Given equation of the curve $y = 2x^2$

$$\Rightarrow dy = 4x\,dx$$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

Now $\bar{f}\cdot d\bar{r} = (3xy\bar{i} - y^2\bar{j})(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$$= 3xy\,dx - y^2 dy$$

$$= 3x(2x^2)\,dx - (2x^2)^2 4x\,dx \qquad (\because y = 2x^2,\ dy = 4x\,dx)$$

$$= 6x^3 dx - 16x^5 dx$$

$$\int_C \bar{f}\cdot d\bar{r} = \int_C 6x^3 dx - 16x^5 dx$$

$$= \int_{x=0}^{1} (6x^3 - 16x^5)\,dx \qquad (\because \text{along } C,\ x \text{ varies } 0 \text{ to } 1)$$

$$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_{x=0}^{1}$$

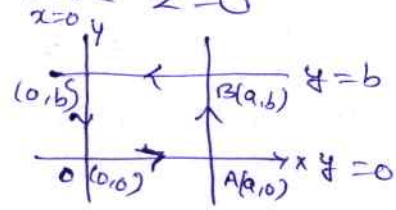
$$\int_C \bar{f}\cdot d\bar{r} = \left(\frac{3}{2} - \frac{8}{3}\right) - (0) = \frac{9-16}{6} = -7/6$$

2) If $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ then evaluate $\oint_C \bar{f}\cdot d\bar{r}$ where curve 'C' is the rectangle in xy-plane bounded by $y=0, y=b, x=0, x=a$

Sol:- Given curve C is in xy-plane that means here $z=0$

$$\therefore \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$= x\bar{i} + y\bar{j} \qquad (\because z=0)$$



then $d\bar{r} = dx\bar{i} + dy\bar{j}$ & $\bar{f}\cdot d\bar{r} = (x^2+y^2)\,dx - 2xy\,dy$

$$\therefore \oint_C \bar{f}\cdot d\bar{r} = \int_{OA} \bar{f}\cdot d\bar{r} + \int_{AB} \bar{f}\cdot d\bar{r} + \int_{BC} \bar{f}\cdot d\bar{r} + \int_{CD} \bar{f}\cdot d\bar{r} \rightarrow (1)$$

Along OA:- $y=0 \Rightarrow dy=0$

x varies from 0 to a

$$\therefore \int_{OA} \bar{f}\cdot d\bar{r} = \int_{x=0}^{a} (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{x=0}^{a} x^2\,dx \qquad (\because y=0)$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB:- $x=a \Rightarrow dx=0$ and

y varies from 0 to b

$$\bar{f}\cdot d\bar{r} = (a^2+y^2)(0) - 2a\cdot y\cdot dy = -2ay\,dy$$

$$\therefore \int_{AB} \bar{f}\cdot d\bar{r} = \int_{y=0}^{b} -2ay\,dy$$

$$= -\left[-\frac{\cancel{2}ay^2}{\cancel{2}}\right]_{y=0}^{b} = -ab^2$$

Along BC:- $y=b \Rightarrow dy=0$

x varies from a to 0

$$\bar{f}\cdot d\bar{r} = (x^2+b^2)\,dx - 2xb(0) = (x^2+b^2)\,dx$$

$$\therefore \int_{BC} \bar{f}\cdot d\bar{r} = \int_{x=a}^{0} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_{x=a}^{0}$$

$$= \left[0+0-\frac{a^3}{3} - ab^2\right]$$

$$= -\frac{a^3}{3} - ab^2$$

Along CO :- $x=0 \Rightarrow dx=0$

y varies from b to 0

$$\bar{F}\cdot d\bar{r} = (0^2+y^2)0 - 2(0)y\,dy = 0$$

$$\therefore \int_{CO} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 0 = 0$$

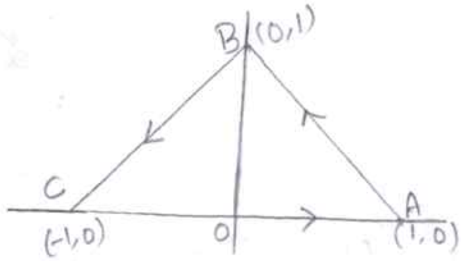
$$\therefore ① \Rightarrow \oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 - 0$$

$$= -2ab^2$$

③ Compute the line integral $\int_C y^2dx - x^2dy$ round the triangle whose vertices are $(1,0)$, $(0,1)$, $(-1,0)$ in the xy-plane.

Sol: Let $A=(1,0)$, $B=(0,1)$, $C(-1,0)$

Here curve $C = AB + BC + CA$



$A=(1,0)$, $B=(0,1)$
$x_1\ y_1$ $\quad x_2\ y_2$

Along AB :- Equation of AB is

$$y-0 = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow y = -(x-1)$$

$$= -x+1$$

i.e, $y=-x+1$

then $dy = -dx$

and x varies from 1 to 0

$$\therefore \int_{AB} y^2dx - x^2dy = \int_{x=1}^{0} (-x+1)^2dx - x^2(-dx)$$

$$= \int_{x=1}^{0} x^2+1-2x+x^2\,dx$$

$$= \int_{x=1}^{0} 2x^2-2x+1\,dx$$

$$= \left[\frac{2x^3}{3} - 2\frac{x^2}{2} + x\right]_1^0$$

$$= 0 - 0 + 0 - \frac{2}{3} + 1 - 1$$

$$= -2/3$$

Along BC :- Eq of BC is $y-1=\frac{0-1}{-1-0}\cdot(x-0)$

$$y-1=1(x)$$
$$y=x+1$$

i.e, Here $y=x+1$

then $dy=dx$ & x varies from 0 to -1

$$\therefore \int_{BC} y^2dx-x^2dy=\int_{x=0}^{-1}(x+1)^2dx-x^2dx$$

$$=\int_{x=0}^{-1}(\cancel{x^2}+2x+1-\cancel{x^2})dx$$

$$=\int_{x=0}^{-1} 2x+1\,dx$$

$$=\Big[\cancel{2}\frac{x^2}{\cancel{2}}+x\Big]_0^{1}$$

$$=1-1-0-0=0.$$

Along CA :- $y=0 \Rightarrow dy=0$

x varies from -1 to 1

$$\therefore \int_{CA} y^2dx-x^2dy=\int_{x=-1}^{1}0\,dx-x^2.0=0$$

$$\therefore \int_C y^2dx-x^2dy=\int_{AB}y^2dx-x^2dy+\int_{BC}y^2dx-x^2dy+\int_{CA}y^2dx-x^2dy$$

$$=-2/3+0+0$$

$$=-\frac{2}{3}.$$

(4) If $\bar{f}=xy\bar{i}-z\bar{j}+x^2\bar{k}$ and c is the curve $x=t^2, y=2t$, $z=t^3$ from $t=0$ to $t=1$. Evaluate $\int_c \bar{f}.\,d\bar{r}$

sol: Given $\bar{f}=xy\bar{i}-z\bar{j}+x^2\bar{k}$

and $x=t^2 \Rightarrow dx=2t\,dt$

$y=2t \Rightarrow dy=2\,dt$

$z=t^3 \Rightarrow dz=3t^2dt$

Workdone:- If $\bar{F}$ is a vector point function, moving along an arc AB then the workdone d a small displacement. $\delta\bar{r}$ is $\bar{F}\cdot\delta\bar{r}$.

Hence the total workdone by $\bar{F}$ during displacement from A to B is given by

$$\boxed{\text{workdone} = \int_A^B \bar{F}\cdot d\bar{r}}$$

If the force $\bar{F}$ is conservative i.e. $\bar{F} = \nabla\phi$, then the workdone is independent of the path and vice. In this case $\text{curl}\,\bar{F} = \text{curl}\,(\text{grad}\,\phi) = \bar{0}$ where ϕ is called scalar potential.

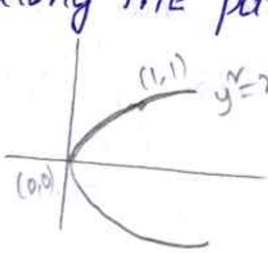
Note:- If $\bar{F}$ is conservative force field then $\nabla\times\bar{F} = \bar{0}$ (irrotational)

① Find the workdone by a force $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ which moves a particle in xy-plane from (0,0) to (1,1) along the parabola $y^2 = x$.

Sol: Given $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$

c is $y^2 = x \Rightarrow x = y^2$

$dx = 2y\,dy$



let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$= x\bar{i} + y\bar{j} + 0\bar{k}$ ($\because$ particle is moving on xy-plane i.e. on xy-plane, $z = 0$)

$= x\bar{i} + y\bar{j}$

Then $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$d\bar{r} = 2y\,dy\,\bar{i} + dy\,\bar{j}$

Now, $\bar{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j} = ((y^2)^2 - \cancel{y^2} + \cancel{y^2})\bar{i} - (2(y^2)y + y)\bar{j}$

$= y^4\bar{i} - (2y^3 + y)\bar{j}$

$\therefore \bar{F}\cdot d\bar{r} = (y^4\bar{i} - (2y^3 + y)\bar{j})\cdot(2y\,dy\,\bar{i} + dy\,\bar{j})$

$= 2y^5\,dy - (2y^3 + y)\,dy$

$= (2y^5 - 2y^3 - y)\,dy$

Given (x, y) varies from $(0,0)$ to $(1,1)$

$\Rightarrow y$ varies from 0 to 1

$\therefore \text{workdone} = \int_{(0,0)}^{(1,1)} \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} (2y^5 - 2y^3 - y)\,dy$

$= \left[\frac{2y^6}{\cancel{6}_3} - \frac{2y^4}{\cancel{4}_2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = \frac{-2}{3}.$

Then $\bar{f} = xy\,\bar{i} - z\,\bar{j} + x^2\,\bar{k}$

$= t^2(2t)\,\bar{i} - t^3\,\bar{j} + (t^2)^2\,\bar{k}$

$= 2t^3\,\bar{i} - t^3\,\bar{j} + t^4\,\bar{k}$

and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$= 2t\,dt\,\bar{i} + 2\,dt\,\bar{j} + 3t^2\,dt\,\bar{k}$

$\therefore\ \bar{f}\cdot d\bar{r} = (2t^3\,\bar{i} - t^3\,\bar{j} + t^4\,\bar{k})\cdot(2t\,dt\,\bar{i} + 2dt\,\bar{j} + 3t^2dt\,\bar{k})$

$= (2t^3)(2t\,dt) - t^3(2dt) + t^4(3t^2dt)$

$= 4t^4dt - 2t^3dt + 3t^6dt$

$= (4t^4 - 2t^3 + 3t^6)\,dt$

Given that t varies from 0 to 1

$\therefore \int_c \bar{f}\cdot d\bar{r} = \int_{t=0}^{1} (4t^4 - 2t^3 + 2t^6)\,dt$

$= \left[\frac{4t^5}{5} - \frac{2t^4}{4} + \frac{3t^7}{7}\right]_{t=0}^{1}$

$= \frac{4}{5} - \frac{2}{4} + \frac{3}{7} - 0 + 0 - 0$

$= \frac{51}{70}$

For practice

1) Evaluate the line integral $\int_c (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by the lines $x = \pm 1$ and $y = \pm 1$.

② find the workdone by the force $\bar{f}=(3x^2-6yz)\bar{i}+(2y+3xz)\bar{j}+(1-4xyz^2)\bar{k}$ in moving particle from the point $(0,0,0)$ to the points $(1,1,1)$ along the curve $c: x=t,\ y=t^2,\ z=t^3$. (106)

Sol: Given $\bar{f}=(3x^2-6yz)\bar{i}+(2y+3xz)\bar{j}+(1-4xyz^2)\bar{k}$

Given curve c is $x=t \Rightarrow dx=dt$

$y=t^2 \Rightarrow dy=2t\,dt$

$z=t^3 \Rightarrow dz=3t^2dt$

Given that (x,y,z) varies from $(0,0,0)$ to $(1,1,1)$

$\Rightarrow t$ varies from 0 to 1

Now, put $x=t,\ y=t^2,\ z=t^3$ in $\bar{f}$ and $d\bar{r}$

Then $\bar{f}=(3t^2-6t^5)\bar{i}+(2t^2+3(t)(t^3))\bar{j}+(1-4(t)(t^2)(t^3)^2)\bar{k}$

$=(3t^2-6t^5)\bar{i}+(2t^2+3t^4)\bar{j}+(1-4t^9)\bar{k}$

$d\bar{r}=dt\,\bar{i}+2t\,dt\,\bar{j}+3t^2\,dt\,\bar{k}$ $\quad (\because d\bar{r}=dx\bar{i}+dy\bar{j}+dz\bar{k})$

& $\bar{f}\cdot d\bar{r}=(3t^2-6t^5)dt+(2t^2+3t^4)(2t\,dt)+(1-4t^9)(3t^2dt)$

$=(3t^2-\cancel{6t^5}+4t^3+\cancel{6t^5}+3t^2-12t)\,dt$

$=(6t^2+4t^3-12t)\,dt$

$\therefore$ workdone $=\int_{(0,0,0)}^{(1,1,1)}\bar{f}\cdot d\bar{r}$

$=\int_{t=0}^{1} 6t^2+4t^3-12t\ dt$

$=\left[\frac{6t^3}{3}+\frac{4t^4}{4}-\frac{12t^2}{2}\right]_0^1$

$=[2+1-1-0-0+0]$

$=2$.

③. find the workdone in moving a particle in the force field $\bar{f} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along a straight line from $(0,0,0)$ to $(2,1,3)$. (107)

Sol: Given $\bar{f} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

W.K.T $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

Then $\bar{f}\cdot d\bar{r} = 3x^2 dx + (2xz - y)\,dy + z\,dz$

The eq'n of the straight line from $(0,0,0)$ to $(2,1,3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ say}$$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$

$\Rightarrow dx = 2dt,\ dy = dt,\ dz = 3dt$

$$\bar{f}\cdot d\bar{r} = 3(2t)^2\, 2dt + (2(2t)(3t) - t)dt + 3t\,(3dt)$$

$$= 24t^2\,dt + (12t^2 - t)\,dt + 9t\,dt$$

$$= (36t^2 + 8t)\,dt$$

$\therefore$ (x, y, z) varies from $(0,0,0)$ to $(2,1,3)$, t varies from 0 to 1

$$\therefore \text{workdone} = \int_{(0,0,0)}^{(2,1,3)} \bar{f}\cdot d\bar{r}$$

$$= \int_{t=0}^{1} (36t^2 + 8t)\,dt$$

$$= \left[\cancel{36}^{12} \frac{t^3}{\cancel{3}} + 8\frac{t^2}{2} \right]_{t=0}^{1}$$

$$= 12(1) + 4(1) - 0 - 0$$

$$= 12 + 4 = 16.$$

4.) find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ from $(0,0,0)$ to $(2,1,3)$

Sol:- Given $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

w.k.t $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

Then $\bar{F}\cdot d\bar{r} = 3x^2 dx + (2xz-y)\,dy + z\,dz$

Given that (x,y,z) varies from $(0,0,0)$ to $(2,1,3)$.

$$\therefore \text{workdone} = \int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r}$$

$$= \int_{(0,0,0)}^{(2,1,3)} 3x^2\,dx + (2xz-y)\,dy + z\,dz$$

$$= \left[\frac{\cancel{3}x^3}{\cancel{3}} + 2xyz - \frac{y^2}{2} + \frac{z^2}{2} \right]_{(0,0,0)}^{(2,1,3)}$$

$$= (2)^3 + 2(2)(1)(3) - \frac{(1)^2}{2} + \frac{3^2}{2} - 0 - 0 + 0 - 0$$

$$= 8 + 12 - \frac{1}{2} + \frac{9}{2}$$

$$= 8 + 12 + 4$$

$$= 24.$$

5.) If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\,\bar{k}$ then prove that $\int_C \bar{F}\cdot d\bar{r}$ i.e, workdone is independent of the curve joining two points.

Sol:- w.k.t If the force is conservative i.e, $\bar{F} = \nabla\phi$ then the workdone is independent of the path.

So, we have to prove that $\bar{F} = \nabla\phi \Rightarrow \text{Curl}\,\bar{F} = \bar{0}$

Now,
$$\text{Curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2) \right) - \bar{j}\left(\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy - 3x^2z^2) \right)$$
$$+ \bar{k}\left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - 3x^2z^2) \right)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} \qquad (\because -6x^2z + 6x^2z = 0)$$

$$= \bar{0}$$

$\therefore$ workdone is independent of the path

6) Find the workdone by $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$ along a curve C in the xy-plane given by $x^2+y^2=9$, $z=0$.

Sol:- Given, $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$.

We know that, $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$.

Then $\bar{F}\cdot d\bar{r} = (2x-y-z)\,dx + (x+y-z)\,dy + (3x-2y-5z)\,dz$.

$= (2x-y)dx + (x+y)\,dy$ [$\because$ Given that $z=0 \Rightarrow dz=0$]

Given that C is the circle $x^2+y^2=9$.

So take, $x = r\cos\theta = 3\cos\theta$. , Here $r=3$.

$y = r\sin\theta = 3\sin\theta$

$\Rightarrow dx = -3\sin\theta\,d\theta$.

$dy = 3\cos\theta\,d\theta$.

and θ varies from 0 to 2π.

$\therefore$ Workdone $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_C (2x-y)dx + (x+y)dy$

$= \int_{\theta=0}^{2\pi} (2(3\cos\theta) - 3\sin\theta)(-3\sin\theta)d\theta + (3\cos\theta + 3\sin\theta)\,3\cos\theta d\theta$.

$= \int_{\theta=0}^{2\pi} (-18\cos\theta\sin\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta)d\theta$

$= \int_{\theta=0}^{2\pi} (-9\sin\theta\cos\theta + 9)\,d\theta$.

$= 9\int_{\theta=0}^{2\pi} 1 - \sin\theta\cos\theta\,d\theta$.

$= 9\int_{\theta=0}^{2\pi} 1 - \frac{\sin2\theta}{2}\,d\theta$.

$= 9\left[\theta - \left(\frac{-\cos2\theta}{4}\right)\right]_{\theta=0}^{2\pi} = 9\left[\theta + \frac{\cos2\theta}{4}\right]_0^{2\pi}$

$= 9\left(2\pi + \frac{1}{4}(0)\right]$. $= 9\left[2\pi + \frac{1}{4}(0)\right]$

$= 18\pi$.

For practice

1.) Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the curve defined by $x^2 = 4y$, $3x^2 = 8z$ from $x=0$ to $x=2$

Ans:- 16

Hint:- $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$; $3x^2 = 8z \Rightarrow z = \frac{3x^2}{8}$

Substitute y & z values in $\bar{F}$ & $d\bar{r}$.

2.) Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Ans:- $\frac{27}{2}$

3.) Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x-3y)\bar{i} + (y-2x)\bar{j}$ and C is the closed curve in xy plane $x = 2\cos t$, $y = 2\sin t$ from $t=0$ to $t=2\pi$

Ans:- 6π

4.) Find the circulation of $\bar{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} + (2x-2y-5z)\bar{k}$ along the circle $x^2+y^2=4$ in the xy-plane.

Ans:- 8π

Hint:- Circulation $= \int_C \bar{F}\cdot d\bar{r}$

5.) If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, then evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x=t$, $y=t^2$, $z=t^3$.

Ans:- 5

6.) If $\phi = 2xy^2z + x^2y$. Evaluate $\int_C \phi\, d\bar{r}$, where C consists of the straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

Ans: $\frac{1}{2}\bar{j} + 2\bar{k}$

Circulation :- If C is a simple closed curve, then the tangent line integral of $\bar{F}$ around 'c' is called the circulation of $\bar{F}$ about c and is denoted by $\oint_c \bar{F}\cdot d\bar{r} = \oint_c F_1 dx + F_2 dy + F_3 dz$

If $\oint_c \bar{F}\cdot d\bar{r} = 0$, then $\bar{F}$ is called irrotational.

Surface integral: Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be any continuous differentiable vector function, then the surface integral is $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{n}$ is the outward drawn unit normal vector.

1.) If S is the surface projected on xy plane, then $\iint \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$.

2.) If S is the surface projected on yz plane, then $\iint \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$.

3.) If S is the surface projected on zx plane, then $\iint \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$.

1.) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

Solution: Let $\phi = 2x+3y+6z-12 = 0 \Rightarrow 6z = 12-2x-3y$

Then $\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$, $|\nabla\phi| = \sqrt{4+9+36} = 7$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$$

Let S be the surface projected on xy-plane, then $\int_S \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,\dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

On xy plane $z=0$, then $2x+3y=12 \Rightarrow y = \dfrac{12-2x}{3}$

$\therefore$ y varies from '0' to '$\frac{12-2x}{3}$'

If $y=0$, then $2x=12 \Rightarrow x=6$

$\therefore$ x varies from '0' to '6'

$\bar{F}\cdot\bar{n} = (18z\bar{i} - 12\bar{j} + 3y\bar{k})\cdot\left(\dfrac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\right) = \dfrac{36z-36+18y}{7}$

$\bar{n}\cdot\bar{k} = \left(\dfrac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\right)\cdot\bar{k} = \dfrac{6}{7}$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \frac{\left(\frac{36z-36+18y}{7}\right)}{\frac{6}{7}}\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6z-6+3y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12-2x-3y-6+3y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dx\,dy$$

$$= \int_{x=0}^{6} (6-2x)\,[y]_0^{\frac{12-2x}{3}}\,dx$$

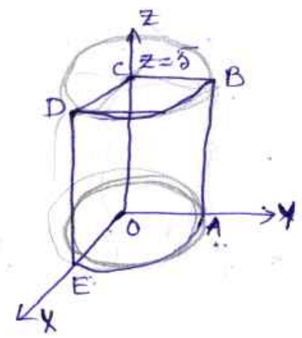
$$= \int_{x=0}^{6} (6-2x)\left(\frac{12-2x}{3}\right)dx$$

$$= \frac{1}{3}\left[72x - \cancel{36}^{18}\frac{x^2}{\cancel{2}} + 4\frac{x^3}{3}\right]_0^6$$

$$= \frac{1}{3}(432-648+288) = 24.$$

-24
-12

2) Evaluate $\int \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.



Solution: Let the surface S be ABDEA.

Let $\phi = x^2+y^2-16=0$

Then $\nabla\phi = 2x\bar{i} + 2y\bar{j}$

$$|\nabla\phi| = 2\sqrt{x^2+y^2} = 2\sqrt{16} = 2\times4 = 8$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}}{2\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{4}$$

$$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}-3y^2z\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right)$$

$$= \frac{xz}{4} + \frac{xy}{4} = \frac{x}{4}(y+z)$$

Let R be the projection of S on yz plane, then $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$$|\bar{n}\cdot\bar{i}| = \left|\frac{x\bar{i}+y\bar{j}}{4}\cdot\bar{i}\right| = \frac{x}{4}$$

Here z varies from 0 to 5

y varies from 0 to 4 ($\because$ by putting $x=0$)

For the surface $x^2+y^2=16$

$x^2 = 16-y^2$

$x = \sqrt{16-y^2}$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{\cancel{x}}{\cancel{4}}(y+z)\cdot\frac{dy\,dz}{\cancel{x}/\cancel{4}}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\,dy\,dz$$

$$= \int_{y=0}^{4} \left[yz+\frac{z^2}{2}\right]_0^5 dy$$

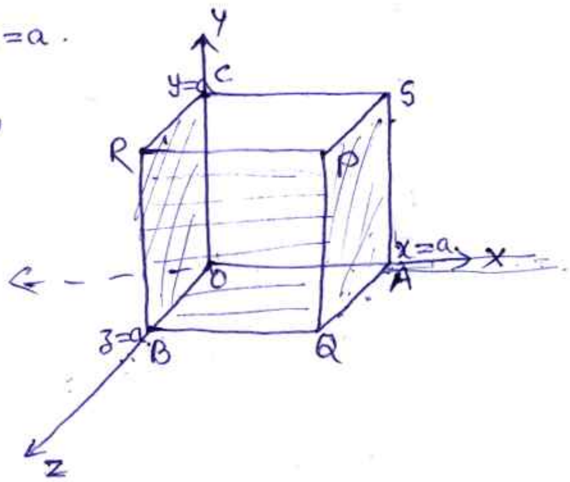
$$= \int_{y=0}^{4} \left(5y+\frac{25}{2}\right)dy$$

$$= \left[5\frac{y^2}{2} + \frac{25}{2}y\right]_0^4$$

$$= 40 + 50$$

$$= 90$$

3.) If $\bar{F} = 4xz\,\bar{i} - y^2\bar{j} + yz\,\bar{k}$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where S is the surface of the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$.



Solution: Consider the volume within the cube PQASCRBO bounded by $x=0, x=a, y=0, y=a, z=0, z=a$

Given $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$.

Now, calculate $\int_S \bar{F}\cdot\bar{n}\,ds$ for each face.

R_1 = Along PQAS : $x=a$, $ds = dy\,dz$

Outward drawn unit normal vector, $\bar{n} = \bar{i}$

$0 \le y \le a$ & $0 \le z \le a$

$\bar{F}\cdot\bar{n} = 4xz = 4az$ and $ds = dy\,dz$

$$\therefore \iint_{R_1} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} 4az\,dy\,dz$$

$$= 4a\int_{y=0}^{a}\left[\frac{z^2}{2}\right]_0^a dy$$

$$= 4a\int_0^a \frac{a^2}{2}\,dy = \frac{\cancel{4}^{2}a^3}{\cancel{2}}[y]_0^a = 2a^4$$

R_2 = OCRB: $x=0$, $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$ Here $\bar{n} = -\bar{i}$

Outward drawn unit normal vector $\bar{n} = -\bar{i}$ and y — 0 to a & z — 0 to a

$\therefore \bar{F}\cdot\bar{n} = -4xz = 0$.

$$\therefore \iint_{R_2} \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} 0\,dy\,dz = 0$$

R_3 = RBQP : $z=a$, $ds = dx\,dy$

Outward drawn unit normal vector $\bar{n} = \bar{k}$ & x — 0 to a & y — 0 to a

$\therefore \bar{F}\cdot\bar{n} = yz = ay$

$$\therefore \iint_{R_3} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} ay\,dx\,dy = a\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^a dx = \frac{a^3}{2}[x]_0^a = \frac{a^4}{2}$$

R_4 = OASC $z=0$, $ds = dx\,dy$

Outward drawn unit normal vector $\bar{n} = -\bar{k}$ & x — 0 to a & y — 0 to a

$\therefore \bar{F}\cdot\bar{n} = -yz = 0$

$$\therefore \iint_{R_4} \bar{F}\cdot\bar{n}\,ds = 0.$$

$R_5 = RPSC$: $y=a$, $ds = dx\,dz$

Outward drawn unit normal vector $\bar{n} = \bar{j}$, $x - 0$ to a & $z - 0$ to a

$\bar{F}\cdot\bar{n} = -y^2 = -a^2$

$\therefore \iint_{R_5} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a -a^2\,dx\,dz = -a^2\int_0^a [x]_0^a\,dz = -a^3\int_0^a dz = -a^4$

$R_6 = BQAO$: $y=0$, $ds = dx\,dz$, $\bar{n} = -\bar{j}$, $x - 0$ to a & $z - 0$ to a

$\bar{F}\cdot\bar{n} = y^2 = 0$

$\therefore \iint_{R_6} \bar{F}\cdot\bar{n}\,ds = 0.$

$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1}\bar{F}\cdot\bar{n}\,ds + \iint_{R_2}\bar{F}\cdot\bar{n}\,ds + \iint_{R_3}\bar{F}\cdot\bar{n}\,ds + \iint_{R_4}\bar{F}\cdot\bar{n}\,ds + \iint_{R_5}\bar{F}\cdot\bar{n}\,ds + \iint_{R_6}\bar{F}\cdot\bar{n}\,ds$

$= 2a^4 + 0 + \frac{a^4}{2} + 0 + (-a^4) + 0$

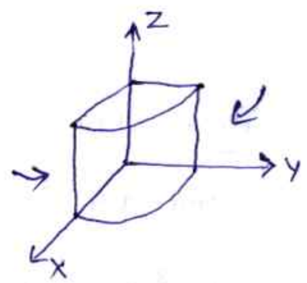
$= 2a^4 + \frac{a^4}{2} - a^4 = \frac{3a^4}{2}$

4) Evaluate $\iint_S \bar{F}\cdot d\bar{s}$ if $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the plane $z=0$ & $z=2$.

Given $\phi = x^2+y^2-9=0$

Then $\nabla\phi = 2x\bar{i}+2y\bar{j}$ and $|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{9} = 6$

$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}}{6} = \frac{x\bar{i}+y\bar{j}}{3}$



Now, $\bar{F}$. $\bar{n}\cdot\bar{i} = \frac{x}{3}$

Let R be the projection of S on YZ plane

$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S \bar{F}\cdot\bar{n}\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \int_{y=0}^{3}\int_{z=0}^{2} \frac{xyz+2y^3}{3}\cdot\frac{dy\,dz}{\left|\frac{x}{3}\right|}$

$= \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{x}\right)dy\,dz$

$= \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{\sqrt{9-y^2}}\right)dy\,dz$

$= 9 + 4\int_{y=0}^{3}\frac{y^3}{\sqrt{9-y^2}}\,dy$

$= 9 + 4\times 18$

$= 81.$

$x^2+y^2=9$

$y^2=9$

$y=\pm 3$

$x = \sqrt{9-y^2}$

Put $y = 3\sin\theta$

$dy = 3\cos\theta\,d\theta$

$\int_0^3 \frac{y^3}{\sqrt{9-y^2}}dy = \int_0^{\pi/2}\frac{27\sin^3\theta}{\cancel{3\cos\theta}}\cdot\cancel{3\cos\theta}\,d\theta$

$= \cancel{27}^{9}\cdot\frac{2}{\cancel{3}} = 18$

$\left[\frac{y^2}{2}\right]_0^3\left[\frac{z^2}{2}\right]_0^2$

$\frac{9}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{2}}$

Volume Integrals: Let $\bar{f}(r) = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$, where f_1, f_2, f_3 are functions of x, y, z

we know that $dv = dx\,dy\,dz$, then the volume integral is given by,

$$\int_V \bar{f}\,dv = \iiint \left(f_1\bar{i}+f_2\bar{j}+f_3\bar{k}\right)dx\,dy\,dz$$

$$= \bar{i}\iiint f_1\,dx\,dy\,dz + \bar{j}\iiint f_2\,dx\,dy\,dz + \bar{k}\iiint f_3\,dx\,dy\,dz$$

1.) If $\bar{f} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$, then evaluate $\int_V \bar{f}\,dv$ where V is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$.

Solution: $$\int_V \bar{f}\,dv = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}\left(2xz\bar{i} - x\bar{j} + y^2\bar{k}\right)dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left(\cancel{2}x\frac{z^2}{\cancel{2}}\bar{i} - xz\bar{j} + y^2z\bar{k}\right)_{x^2}^{4}dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left(16x\bar{i} - 4x\bar{j} + 4y^2\bar{k} - x^5\bar{i} + x^3\bar{j} - x^2y^2\bar{k}\right)dx\,dy$$

$$= \int_{y=0}^{6}\left(\frac{\overset{8}{\cancel{16}}x^2}{\cancel{2}}\bar{i} - \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}}\bar{j} + 4xy^2\bar{k} - \frac{x^6}{6}\bar{i} + \frac{x^4}{4}\bar{j} - \frac{x^3y^2}{3}\bar{k}\right)_0^2 dy$$

$$= \int_{y=0}^{6}\left(32\bar{i} - 8\bar{j} + 8y^2\bar{k} - \frac{\overset{32}{\cancel{64}}}{\cancel{6}_3}\bar{i} + \frac{\overset{4}{\cancel{16}}}{\cancel{4}}\bar{j} - \frac{8y^2}{3}\bar{k}\right)dy$$

$$= \int_{y=0}^{6}\left(\frac{64}{3}\bar{i} - 4\bar{j} + \left(8y^2 - \frac{8y^2}{3}\right)\bar{k}\right)dy$$

$$= \left(\frac{64}{3}y\bar{i} - 4y\bar{j} + \left(\frac{8y^3}{3} - \frac{8y^3}{3\times 3}\right)\bar{k}\right)_0^6$$

$$= \frac{64}{3}(6)\bar{i} - 4(6)\bar{j} + \left(\frac{8(6^3)}{3} - \frac{8(6^3)}{9}\right)\bar{k}$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}.$$

2.) If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate i.) $\int_V \nabla\cdot\bar{F}\,dv$ and ii.) $\int_V \nabla\times\bar{F}\,dv$ where V is the closed region bounded by $x=0, y=0, z=0, 2x+2y+z=4$.

Solution: Given $\bar{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$.

Given that $x=0, y=0, z=0, 2x+2y+z=4$

$z = 4-2x-2y$

'z' varies from '0' to '$4-2x-2y$'

Put $z=0$ then $y = 2-x$

$\therefore$ 'y' varies from '0' to '$2-x$'

Put $y=z=0$, then $x=2$

$\therefore$ x varies from '0' to '2'

$6^3 \cdot \frac{6^2}{3}$ $\quad 4\cdot 6^3$

$8\cdot 6^2\cdot\frac{6^2}{3} - 2\cdot 6^3\cdot\frac{6^2}{3}$

$16(36) - 36(24)$

$-\frac{36\times 8}{288}$

$576 - 864$

$\left(\frac{8\cdot 6^3}{3}\right)\left(\frac{2}{3}\right)$

$\frac{8\cdot 6\cdot 6\cdot 6\times 2}{3\cdot 3} = 64\times 6 = 384$

5.) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Solution: Let $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$

$$\phi = x+y+z-1 = 0$$

$$\nabla\phi = x\bar{i} + \bar{j} + \bar{k}$$

$$|\nabla\phi| = \sqrt{3}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$$

Let S be the projection on xy-plane, then $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$\therefore\ z = (1-x-y)$

$$\bar{F}\cdot\bar{n} = \left(12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}\right)\cdot\left(\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(12x^2y - 3yz + 2z\right)$$

$$= \frac{1}{\sqrt{3}}\left(12x^2y - 3y(1-x-y) + 2(1-x-y)\right)$$

$$= \frac{1}{\sqrt{3}}\left(12x^2y - 3y + 3xy + 3y^2 + 2 - 2x - 2y\right)$$

$$= \frac{1}{\sqrt{3}}\left(12x^2y + 3xy + 3y^2 - 2x - 5y + 2\right)$$

$|\bar{n}\cdot\bar{k}| = \left|\dfrac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\cdot\bar{k}\right| = \dfrac{1}{\sqrt{3}}$ and 'y' varies from '0' to '1-x'. ($\because z=0$ on xy-plane)
'x' varies from '0' to '1'.

$$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1-x} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \cancel{\frac{1}{\sqrt{3}}}\left(12x^2y + 3xy + 3y^2 - 2x - 5y + 2\right)\frac{dx\,dy}{\cancel{\frac{1}{\sqrt{3}}}}$$ $-13+\frac{3}{2}$ $-26+3=-23$

$$= \int_{x=0}^{1}\left[\cancel{12}^{\,6}x^2\frac{y^2}{\cancel{2}} + 3x\frac{y^2}{2} + \cancel{3}\frac{y^3}{\cancel{3}} - 2xy - 5\frac{y^2}{2} + 2y\right]_0^{1-x} dx$$ $8-\frac{5}{2}$ $16-5$

$$= \int_{x=0}^{1}\left(6x^2(1-x)^2 + \frac{3}{2}x(1-x)^2 + (1-x)^3 - 2x(1-x) - \frac{5}{2}(1-x)^2 + 2(1-x)\right) dx$$ $\frac{3}{2}-\cancel{4}+\cancel{8}-2$ $3-\frac{5}{2}$

$$= \int_{x=0}^{1}\Big(6x^2 + 6x^4 - 12x^3 + \frac{3}{2}x + \frac{3}{2}x^3 - \cancel{3x^2} + 1 - \cancel{3x} + \cancel{3x^2} - x^3 - \cancel{2x} + 2x^2 - \frac{5}{2} - \frac{5}{2}x^2 + \cancel{5x}$$
$$+ 2 - 2x\Big) dx$$

$$= \int_{x=0}^{1}\left(6x^4 - \frac{23}{2}x^3 + \frac{11}{2}x^2 - \frac{1}{2}x + \frac{1}{2}\right) dx$$

$$= \left[\frac{6x^5}{5} - \frac{23}{2}\frac{x^4}{4} + \frac{11}{2}\frac{x^3}{3} - \frac{1}{2}\frac{x^2}{2} + \frac{x}{2}\right]_0^1 = \frac{6}{5} - \frac{23}{8} + \frac{11}{6} - \frac{1}{4} + \frac{1}{2}$$

$$= \frac{144 - 345 + 220 - 30 + 60}{120} = \frac{49}{120}$$ $\frac{-55}{24}$

i) $\int_V \nabla\cdot\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}\left(\frac{\partial}{\partial x}(2x^2-3z)+\frac{\partial}{\partial y}(-2xy)+\frac{\partial}{\partial z}(-4x)\right)dx\,dy\,dz$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}(4x-2x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}2x\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}x\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}x\,(4-2x-2y)\,dy\,dx$$

$$= 2\int_{x=0}^{2}x\left[\left(4y-2xy-\frac{\cancel{2}y^2}{\cancel{2}}\right)\right]_{y=0}^{2-x}dx$$

$$= 2\int_{x=0}^{2}x\left(4(2-x)-2x(2-x)-(2-x)^2\right)dx$$

$$= 2\int_{x=0}^{2}\left(8x-4x^2-4x^2+2x^3-4x-x^3+4x^2\right)dx$$

$$= 2\int_{x=0}^{2}\left(2x^3-9x^2+12x-4\right)dx$$

$$= 2\left[\frac{2x^4}{4}-\frac{9x^3}{3}+\frac{12x^2}{2}-4x\right]_0^2$$

$$= 2\left(\frac{16}{2}-3(8)+6(4)-4\right)$$

$$= 2(-16+20) = 8$$

$$= 2\int_{x=0}^{2}\left(x^3-4x^2+4x\right)dx$$

$$= 2\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{4x^2}{2}\right]_0^2$$

$$= 2\left(\frac{16}{4}-\frac{4(8)}{3}+8\right)$$

$$= 2\left(12-\frac{32}{3}\right)$$

$$= 2\left(\frac{4}{3}\right) = \frac{8}{3}.$$

ii) $\nabla\times\bar{F} = \begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x^2-3z & -2xy & -4x\end{vmatrix} = \bar{i}(0-0)-\bar{j}(-4+3)+\bar{k}(-2y-0)$

$$= \bar{j}-2y\bar{k}$$

$\therefore \int_V \nabla\times\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}(\bar{j}-2y\bar{k})\,dz\,dy\,dx$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left[z\bar{j}-2yz\bar{k}\right]_{z=0}^{4-2x-2y}dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left((4-2x-2y)\bar{j}-2y(4-2x-2y)\bar{k}\right)dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\left((4-2x-2y)\bar{j}-(8y-4xy-4y^2)\bar{k}\right)dy\,dx$$

$$= \int_{x=0}^{2} \left[\left(4y - 2xy - \cancel{2}\frac{y^2}{\cancel{2}}\right)\bar{j} - \left(\cancel{8}^{4}\frac{y^2}{\cancel{2}} - \cancel{4}^{2}\frac{xy^2}{\cancel{2}} - 4\frac{y^3}{3}\right)\bar{k}\right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \left(\left(4(2-x) - 2x(2-x) - (2-x)^2\right)\bar{j} - \left(4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\right)\bar{k} \right) dx$$

$$= \int_{x=0}^{2} \left(8 - \cancel{4x} - 4x + 2x^2 - 4 - x^2 + \cancel{4x}\right)\bar{j} - \left(16 + 4x^2 - 16x - 8x + 2x^3 + \cancel{8}x^2 - \frac{4}{3}(8 - x^3 + 6x^2 - 12x)\right)\bar{k}\, dx$$

$$= \int_{x=0}^{2} \left((x^2 - 4x + 4)\bar{j} - \left(-\frac{2x^3}{3} + 4x^2 - 8x + \frac{16}{3}\right)\bar{k} \right) dx$$

$$= \left[\left(\frac{x^3}{3} - \cancel{4}^{2}\frac{x^2}{\cancel{2}} + 4x\right)\bar{j} - \left(-\frac{\cancel{2}x^4}{\cancel{12}_{6}} + 4\frac{x^3}{3} - \cancel{8}^{4}\frac{x^2}{\cancel{2}} + \frac{16}{3}x\right)\bar{k} \right]_0^2$$

$$= \left(\frac{8}{3} - \cancel{8} + \cancel{8}\right)\bar{j} - \left(\frac{-16^{8}}{6_{3}} + \frac{32}{3} - 16 + \frac{32}{3}\right)\bar{k}$$

$$= \frac{8}{3}\bar{j} - \frac{8}{3}\bar{k}$$

$$= \frac{8}{3}(\bar{j} - \bar{k}).$$

1) find $(\bar{A}\times\nabla)\phi$, if $A = yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k}$ and $\phi = xyz$

Solution: $(\bar{A}\times\nabla)\phi = (\bar{A}\times\bar{i})\left(\frac{\partial\phi}{\partial x}\right) + (\bar{A}\times\bar{j})\left(\frac{\partial\phi}{\partial y}\right) + (\bar{A}\times\bar{k})\frac{\partial\phi}{\partial z}$

$$\bar{A}\times\bar{i} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ 1 & 0 & 0 \end{vmatrix} \qquad \bar{A}\times\bar{j} = \qquad \bar{A}\times\bar{k} =$$

$$= \bar{i}(0) - \bar{j}(-2xyz) + \bar{k}(3xz^2) \qquad = \qquad =$$

$$= 2xyz\bar{j} + 3xz^2\bar{k} \qquad = yz^2\bar{k} - 2xyz\bar{i} \qquad = -yz^2\bar{j} - 3xz^2\bar{i}$$

$$\therefore (\bar{A}\times\nabla)\phi = (2xyz\bar{j} + 3xz^2\bar{k})yz + (yz^2\bar{k} - 2xyz\bar{i})xz + (-yz^2\bar{j} - 3xz^2\bar{i})xy$$

$$= -x^2yz^2\bar{i} + xy^2z^2\bar{j} + 4xyz^3\bar{k}$$

2) find $(A\cdot\nabla)\phi$ at $(1,-1,1)$ if $A = 3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz k$ and $\phi = 3x^2 - yz$

$$(A\cdot\nabla)\phi = (\bar{A}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{A}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{A}\cdot\bar{k})\frac{\partial\phi}{\partial z}$$

$$= (3xyz^2)\,6x + 2xy^3\cdot(-z) + (-x^2yz)(-y)$$

$$= 18x^2yz^2 - 2xy^3z + x^2y^2z.$$

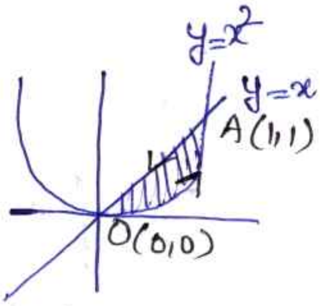
$$(\bar{A}\cdot\nabla)\phi\Big|_{(1,-1,1)} = -15.$$

3) P.T $\nabla\left(\nabla\cdot\frac{\bar{r}}{r}\right) = \frac{-2}{r^3}\bar{r}$

Green's theorem:- If $\phi(x,y)$, $\psi(x,y)$, ϕ_y and ψ_x be continuous in a region R of xy-plane bounded by a closed curve c, then

$$\oint_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy.$$

1) Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$, where c is bounded by $y=x$ and $y=x^2$.



Sol: Let $\phi = xy+y^2$ and $\psi = x^2$

Then $\frac{\partial \phi}{\partial y} = x+2y$ and $\frac{\partial \psi}{\partial x} = 2x$

Given $y=x$ and $y=x^2$

By Green's theorem, $\int_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy.$

To find L.H.S:-

$C =$ OA along $y=x^2$ + AO along $y=x$. $\Rightarrow \int_C = \int_{OA} + \int_{AO}$

OA along $y=x^2$:- $y=x^2 \Rightarrow dy = 2x\,dx$

x varies from 0 to 1

$$\therefore \int_{OA} \phi\, dx + \psi\, dy = \int_{x=0}^{1} \left(x\cdot x^2 + (x^2)^2\right) dx + x^2\cdot 2x\,dx$$

$$= \int_{x=0}^{1} \left(x^3 + x^4 + 2x^3\right) dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}$$

AO along $y=x$:- $y=x \Rightarrow dy = dx$

x varies from 1 to 0

$$\therefore \int_{AO} \phi\, dx + \psi\, dy = \int_{x=1}^{0} \left(x\cdot x + x^2\right) dx + x^2\, dx$$

$$= \int_{x=1}^{0} 3x^2\, dx$$

$$= \left[\cancel{3}\frac{x^3}{\cancel{3}}\right]_1^0 = -1$$

$$\therefore \int_C \phi\, dx + \psi\, dy = \int_{OA} \phi\, dx + \psi\, dy + \int_{AO} \phi\, dx + \psi\, dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$\therefore$ L.H.S $= \frac{-1}{20}$

To find R.H.S :- x varies from 0 to 1
y varies from x^2 to x

$\therefore$ R.H.S $= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$

$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\, dy\, dx$

$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\, dy\, dx$

$= \int_{x=0}^{1} \left[xy - \frac{\cancel{2}y^2}{\cancel{2}} \right]_{y=x^2}^{x} dx$

$= \int_{x=0}^{1} \left(\cancel{x^2} - \cancel{x^2} - x^3 + x^4\right) dx$

$= \left[-\frac{x^4}{4} + \frac{x^5}{5} \right]_{x=0}^{1}$

$= \frac{-1}{20}$

$\therefore$ L.H.S = R.H.S

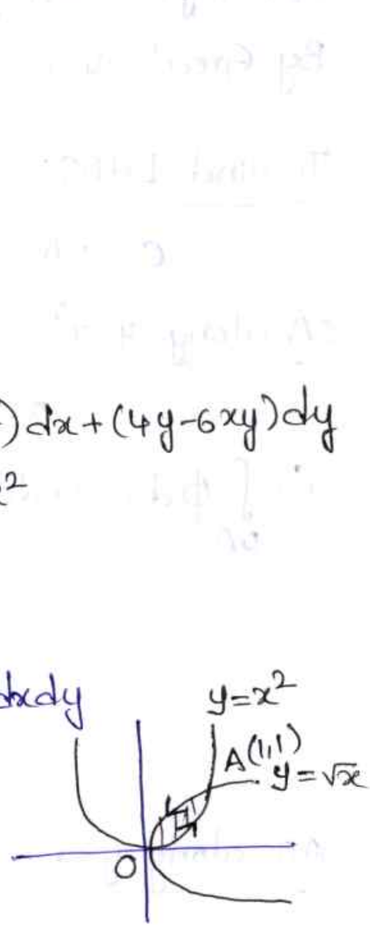
Hence Green's theorem is verified.

2) Verify Green's theorem in the plane for $\oint (3x^2-8y^2)dx + (4y-6xy)dy$ where 'c' is the region defined by $y=\sqrt{x}$ and $y=x^2$

Sol:- Let $\phi = 3x^2 - 8y^2$ and $\psi = 4y - 6xy$

$\frac{\partial \phi}{\partial y} = -16y$ and $\frac{\partial \psi}{\partial x} = -6y$

w.k.t by Green's theorem, $\int_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$



Given c is $y=\sqrt{x}$ and $y=x^2$

$\Rightarrow$ C = OA along $y=x^2$ + AO along $y=\sqrt{x}$

L.H.S

$\int_C \phi\, dx + \psi\, dy = \int_{\substack{OA \text{ along} \\ y=x^2}} \phi\, dx + \psi\, dy + \int_{\substack{AO \text{ along} \\ y=\sqrt{x}}} \phi\, dx + \psi\, dy$

OA along $y=x^2$:- $y = x^2 \Rightarrow dy = 2x\, dx$
x varies from 0 to 1

$\therefore \int_{\substack{OA \text{ along} \\ y=x^2}} \phi\, dx + \psi\, dy = \int_{x=0}^{1} (3x^2 - 8(x^2)^2)\, dx + (4x^2 - 6x(x^2))\, 2x\, dx$

$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\, dx$

$$= \int_{x=0}^{1} \left(3x^2 + 8x^3 - 20x^4\right) dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1$$

$$= 1 + 2 - 4 - 0 = -1$$

Ao along $y=\sqrt{x}$:- $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}} dx$ and x varies from 1 to 0

$$\therefore \int_{\substack{AO \text{ along} \\ y=\sqrt{x}}} \phi\, dx + \psi\, dy = \int_{x=1}^{0} (3x^2 - 8x) dx + (4\sqrt{x} - 6x\sqrt{x}) \frac{1}{2\sqrt{x}} dx$$

$$= \int_{x=1}^{0} 3x^2 - 8x + 2 - 3x \; dx$$

$$= \int_{x=1}^{0} 2 - 11x + 3x^2 \; dx$$

$$= \left[2x - \frac{11x^2}{2} + \frac{3x^3}{3}\right]_{x=1}^{0}$$

$$= 0 - 2 + \frac{11}{2} - 1$$

$$= -3 + \frac{11}{2} = \frac{5}{2}$$

$$\therefore \text{L.H.S} = \int_C \phi\, dx + \psi\, dy = -1 + \frac{5}{2} = \frac{3}{2}$$

R.H.S :- $= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\, dy\, dx$

x varies from 0 to 1

y varies from x^2 to $\sqrt{x}$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y \; dy\, dx$$

$$= \int_{x=0}^{1} 5\left[y^2\right]_{y=x^2}^{\sqrt{x}} dx$$

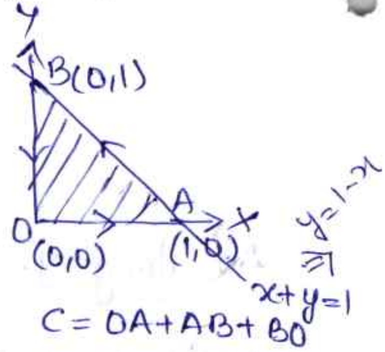
$$= 5 \int_{x=0}^{1} x - x^4 \; dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5 \times \frac{3}{10} = \frac{3}{2}$$

$\therefore$ L.H.S = R.H.S

3. Verify Green's theorem from $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the region bounded by $x=0$, $y=0$ and $x+y=1$



Sol:- w.k.t by Green's theorem,

$$\int_C \phi\,dx+\psi\,dy = \iint_R \left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dx\,dy$$

Let $\phi = 3x^2-8y^2$ and $\psi = 4y-6xy$

Then $\frac{\partial\phi}{\partial y} = -16y$ and $\frac{\partial\psi}{\partial x} = -6y$

Given $C: x=0,\ y=0$ and $x+y=1$

$\Rightarrow C = OA+AB+BO$

<u>L.H.S</u> $= \int_C \phi\,dx+\psi\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

<u>Along OA</u>:- x varies from 0 to 1

$y=0. \Rightarrow dy=0$

$$\therefore \int_{OA} \phi\,dx+\psi\,dy = \int_{x=0}^{1} (3x^2-0)\,dx + 0$$

$$= \left[\cancel{3}\frac{x^3}{\cancel{3}}\right]_0^1 = 1$$

<u>Along AB</u>:- $x+y=1 \Rightarrow y=1-x$

$\Rightarrow dy=-dx$

x varies from 1 to 0

$$\therefore \int_{AB} \phi\,dx+\psi\,dy = \int_{x=1}^{0} (3x^2-8(1-x)^2)\,dx + (4(1-x)-6x(1-x))(-dx)$$

$$= \int_{x=1}^{0} (3x^2-8-8x^2+16x-4+4x+6x-6x^2)\,dx$$

$$= \int_{x=1}^{0} (-12+26x-11x^2)\,dx$$

$$= \left[-12x+\cancel{26}^{13}\frac{x^2}{\cancel{2}}-11\frac{x^3}{3}\right]_{x=1}^{0} = 0+12-13+\frac{11}{3} = \frac{8}{3}$$

<u>Along BO</u>:- $x=0 \Rightarrow dx=0$

y varies from 1 to 0

$$\therefore \int_{BO} \phi\,dx+\psi\,dy = \int_{y=1}^{0} (0-8y^2)0+(4y-6(0))\,dy$$

$$= \left[\cancel{4}^{2}\frac{y^2}{\cancel{2}}\right]_{y=1}^{0} = 0-2 = -2$$

$$\therefore L.H.S = 1+\frac{8}{3}-2 = \frac{5}{3}$$

$$R.H.S = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

x varies from 0 to 1

y varies from 0 to $1-x$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (-6y+16y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[\cancel{10}^{5} \frac{y^2}{\cancel{2}}\right]_0^{1-x} dx$$

$$= 5\int_{x=0}^{1} (1-x)^2\, dx$$

$$= 5\int_{x=0}^{1} 1+x^2-2x\, dx$$

$$= 5\left[x + \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^1$$

$$= 5\left(1+\frac{1}{3}-1\right) = \frac{5}{3} = L.H.S$$

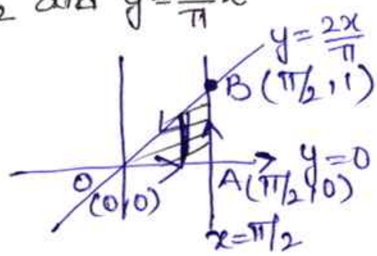
$\therefore L.H.S = R.H.S$

Hence Green's theorem is verified.

4) Verify Green's theorem in the plane for $\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$, where 'c' is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.

Ans :- 8

*5) Using Green's theorem, evaluate $\int_C (y-\sin x)\,dx + \cos x\,dy$, where C is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=\frac{2}{\pi}x$



Sol :- Let $\phi = y-\sin x$ and $\psi = \cos x$

Then $\frac{\partial \phi}{\partial y} = 1$ and $\frac{\partial \psi}{\partial x} = -\sin x$

x varies from 0 to $\pi/2$

y varies from 0 to $\frac{2x}{\pi}$

$\therefore$ By Green's theorem, $\int_C \phi\,dx + \psi\,dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$\Rightarrow \int_C (y-\sin x)\,dx + \cos x\,dy = \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2} \left[-(\sin x+1)\,y\right]_{y=0}^{\frac{2x}{\pi}} dx$$

$$= \int_{x=0}^{\pi/2} -(\sin x+1)\frac{2x}{\pi} - 0\,dx$$

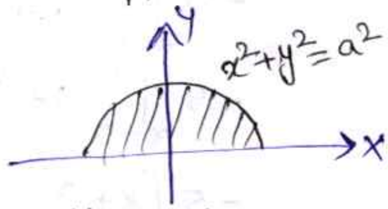
$$= -\frac{2}{\pi}\int_{x=0}^{\pi/2} x\sin x + x \ dx$$

$$= -\frac{2}{\pi}\left[x.(-\cos x) - 1(-\sin x) + \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[0+1+\frac{\pi^2}{8}-0\right]$$

$$= -\left(\frac{2}{\pi}+\frac{\pi}{4}\right)$$

6) Apply Green's theorem to evaluate $\int_C (2x^2-y^2)dx + (x^2+y^2)dy$ where 'c' is the boundary of the area enclosed by the x-axis and the upperhalf of the circle $x^2+y^2=a^2$



Sol:- Let $\phi = 2x^2-y^2$ and $\psi = x^2+y^2$

Then $\frac{\partial\phi}{\partial y} = -2y$ and $\frac{\partial\psi}{\partial x} = 2x$

w.k.t by Green's theorem, $\int_C \phi dx + \psi dy = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$

$$\therefore \int_C (2x^2-y^2)dx + (x^2+y^2)dy = \iint_R (2x+2y)\,dx\,dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

r varies from 0 to a

θ varies from 0 to π

$$= 2\int_{r=0}^{a}\int_{\theta=0}^{\pi}(r\cos\theta + r\sin\theta)\, r\, dr\,d\theta$$

$$= 2\int_{r=0}^{a} r^2\left[\sin\theta - \cos\theta\right]_0^{\pi} dr$$

$$= 2\int_{r=0}^{a} r^2(0+1-0+1)\,dr$$

$$= 4\left[\frac{r^3}{3}\right]_0^a = \frac{4a^3}{3}$$

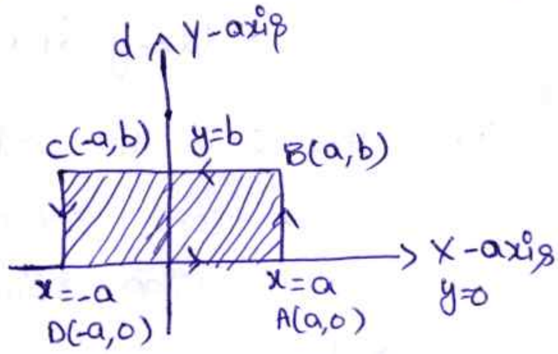
7) If C is a simple closed curve in XY plane not enclosing the origin, show that $\int_C f\cdot dR = 0$, where $f = \frac{yI - xJ}{x^2+y^2}$

Hint By Green's theorem, $\int_C \phi dx + \psi dy = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$

$$\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{-(x^2+y^2)+2x^2}{(x^2+y^2)^2}$$

Stoke's theorem :- If 's' be an open surface bounded by a closed curve 'C' and $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be any continuously differentiable vector point function then, $\int_C \bar{f}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{f}\cdot\bar{n}\, ds$. {where $\bar{n}$ is the outward drawn unit normal vector}.

1) Verify Stoke's theorem for $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken around the rectangle bounded by the line $x = \pm a$, $y=0$, $y=b$.



Sol :- Given, $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

Given that 'c' is the rectangle bounded by $x = \pm a$, $y=0$, $y=b$.

We know that, By Stoke's theorem

$$\int_C \bar{f}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{f}\cdot\bar{n}\, ds.$$

Now, $L.H.S = \int_C \bar{f}\cdot d\bar{r} = \int_{AB} \bar{f}\cdot d\bar{r} + \int_{BC} \bar{f}\cdot d\bar{r} + \int_{CD} \bar{f}\cdot d\bar{r} + \int_{DA} \bar{f}\cdot d\bar{r} \longrightarrow (1)$

Now, $\bar{f}\cdot d\bar{r} = \left((x^2+y^2)\bar{i} - 2xy\bar{j}\right)\cdot\left(dx\bar{i} + dy\bar{j}\right)$

$$= (x^2+y^2)dx - 2xy\,dy.$$

Along AB :- $x=a \Rightarrow dx=0$.

y varies from 0 to b.

$\therefore \bar{f}\cdot d\bar{r} = (a^2+y^2)0 - 2ay\,dy = -2ay\,dy$

$\therefore \int_{AB} \bar{f}\cdot d\bar{r} = \int_{y=0}^{b} -2ay\,dy = -2a\int_{y=0}^{b} y\,dy.$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2 \longrightarrow (2)$$

Along BC :- $y=b \Rightarrow dy=0$

x varies from a to -a.

$\therefore \bar{f}\cdot d\bar{r} = (x^2+b^2)dx - 2xb\cdot 0 = (x^2+b^2)dx.$

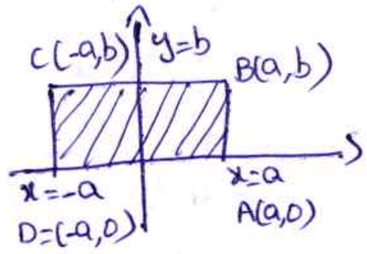
$\therefore \int_{BC} \bar{f}\cdot d\bar{r} = \int_{x=a}^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_{-a}^{a} \Rightarrow -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$

$$\Rightarrow \frac{2a^3}{3} - 2ab^2 \longrightarrow (3).$$

Along CD :- $x = -a \Rightarrow dx = 0$

y varies from b to 0

$\therefore \bar{F} \cdot d\bar{r} = (a^2+y^2)\,0 - 2(-a)y\,dy = 2ay\,dy$



$$\therefore \int_{CD} \bar{F} \cdot d\bar{r} = \int_{y=b}^{0} 2ay\,dy$$

$$= 2a\left[\frac{y^2}{2}\right]_{y=b}^{0}$$

$$= 2a\left(0 - \frac{b^2}{2}\right) = -ab^2 \rightarrow (4)$$

Along DA :- $y=0 \Rightarrow dy=0$.

x varies from -a to a.

$\therefore \bar{F} \cdot d\bar{r} = (x^2+0^2)\,dx - 2x(0)0 = x^2dx$

$$\therefore \int_{DA} \bar{F} \cdot d\bar{r} = \int_{x=-a}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{2a^3}{3} \rightarrow (5)$$

By substituting (2), (3), (4), (5) in (1) we get.

$$\int_C \bar{F} \cdot d\bar{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab + \frac{2a^3}{3} = -4ab^2.$$

Now, $R.H.S = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\,ds$.

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \bar{i}(0) - \bar{j}(0) + \bar{k}(-2y-2y)$$

$$= -4y\bar{k}$$

Let the projection be on xy plane $\Rightarrow ds = \dfrac{dxdy}{|\bar{n} \cdot \bar{k}|}$

Here, $\bar{n} = \bar{k}$ then $ds = \dfrac{dxdy}{|\bar{k} \cdot \bar{k}|} = dxdy$.

Clearly x varies from -a to a

y varies from 0 to b.

$$\therefore R.H.S = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\bar{k} \cdot \bar{k}\,dxdy.$$

$$= \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dy \cdot dx.$$

$$= \int_{x=-a}^{a}\left[-\frac{4y^2}{2}\right]_{y=0}^{b} dx.$$

$$= \int_{x=-a}^{a} (-2b^2 - 0)\, dx$$

$$= -2b^2 \int_{x=-a}^{a} dx$$

$$= -2b^2 [x]_{-a}^{a}$$

$$= -2b^2 (a - (-a))$$

$$= -2b^2 (2a)$$

$$= -4ab^2$$

$\therefore$ L.H.S = R.H.S

$\Rightarrow \int_C \bar{f} \cdot d\bar{r} = \int_S \text{curl} \bar{f} \cdot \bar{n}\, ds.$

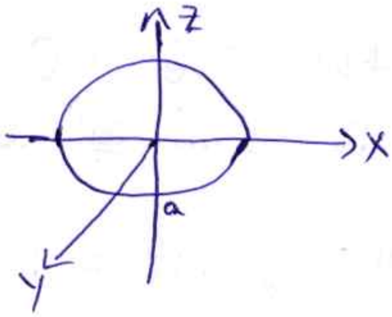
Hence, Stoke's theorem is verified.

2) Verify Stoke's theorem for the vector field $\bar{f} = (2x - y)\bar{i} - yz^2\bar{j} - y^2 z\bar{k}$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy-plane.

Sol:- Given, $\bar{f} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$

and region is upper half surface of

$x^2+y^2+z^2 = 1$ on xy-plane.



W.K.T By stoke's theorem $\int_C \bar{f} \cdot d\bar{r} = \int_S \text{curl} \bar{f} \cdot \bar{n}\, ds.$

L.H.S:- $\int_C \bar{f} \cdot d\bar{r}$

Now, $\bar{f} \cdot d\bar{r} = \left((2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k} \right) \cdot \left(dx\bar{i} + dy\bar{j} + dz\bar{k} \right)$

$= (2x-y)\,dx - yz^2\,dy - y^2z\,dz.$

Given that, projection is on xy-plane i.e; $z=0$.

$\Rightarrow x^2+y^2+0^2 = 1 \quad \because z=0.$

$\Rightarrow$ ~~x^2+~~ $\Rightarrow x^2+y^2 = 1$ is a circle.

$\Rightarrow x = \cos\theta,\ y = \sin\theta$ are the parametric curves varies from 0 to 2π.

$\therefore \int \bar{f} \cdot d\bar{r} = (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta. \qquad \because x = \cos\theta$

$\Rightarrow dx = -\sin\theta\, d\theta.$

$$\therefore \int_C \bar{f}\cdot d\bar{r} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} -2\sin\theta\cos\theta + \sin^2\theta\,d\theta$$

$$= \int_{\theta=0}^{2\pi} -\sin 2\theta + \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{\theta - \frac{\sin 2\theta}{2}}{2}\right]_{\theta=0}^{2\pi}$$

$$= \frac{1}{2} + \pi - 0 - \frac{1}{2} - 0 + 0$$

$$= \pi = L.H.S$$

Now, $R.H.S = \int_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds$.

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2x-y & 1-yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\bar{i} - (0)\bar{j} + (0+1)\bar{k}$$

$\because$ projection is on xy-plane, $\bar{n} = \bar{k}$

then, $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \dfrac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = dx\,dy$

$\because x^2+y^2=1 \Rightarrow y^2 = 1-x^2$

$\Rightarrow y = \pm\sqrt{1-x^2}$

$\Rightarrow y$ varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

Put, $y=0$ then $x^2=1 \Rightarrow x = \pm 1$

$\Rightarrow x$ varies from -1 to 1

$$\therefore R.H.S = \int_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \bar{k}\cdot\bar{k}\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx$$

$$= \int_{x=0}^{1} 2\cdot\int_{y=0}^{\sqrt{1-x^2}} dy\,dx$$

$$= 4\int_{x=0}^{1} [y]_{y=0}^{\sqrt{1-x^2}}$$

$$= 4 \int_{x=0}^{1} (\sqrt{1-x^2} - 0)\, dx$$

$$= 4 \int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

$$= 4 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x \right]_{x=0}$$

$$= 4 \left(0 + \frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(0) \right)$$

$$= 4 \left(\frac{1}{2} \cdot \frac{\pi}{2} - 0 \right)$$

$$= \pi$$

$\therefore$ L.H.S = R.H.S $\Rightarrow \int_C \bar{F} \cdot d\bar{r} = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$.

Hence Stoke's theorem is verified.

3) Use stoke's theorem, evaluate $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$ where 'c' is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.

Sol: Given, $\bar{F} \cdot d\bar{r} = (x+y)dx + (2x-z)dy + (y+z)dz$.

Then, $\bar{F} = (x+y)\bar{i} + (2x-z)\bar{j} + (y+z)\bar{k}$.

Given 'c' is a triangle with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$.

W.K.T By stoke's theorem, $\int_C \bar{F} \cdot d\bar{r} = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$.

To evaluate, $\int_C \bar{F} \cdot d\bar{r}$,

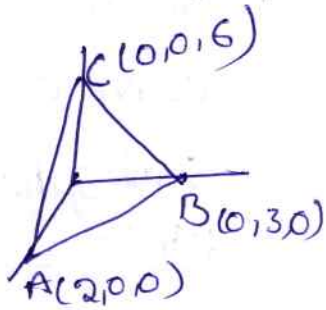
we have to find it by using $\int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$.

Now, $\bar{F} = (x+y)\bar{i} + (2x-z)\bar{j} + (y+z)\bar{k}$.

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= (1+1)\bar{i} - (0+0)\bar{j} + (2-1)\bar{k}$$

$$= 2\bar{i} + \bar{k}.$$



Equation of the plane passing through ABC we get $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

$\Rightarrow 3x+2y+z=6$

$\Rightarrow 3x+2y+z-6=0$

$\phi = 3x+2y+z$

$\nabla\phi = 3i+2j+\bar{k} \quad \therefore \left[\nabla = i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\partial/_{\partial z}\right]$

$|\nabla\phi| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$

unitvector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3i+2j+k}{\sqrt{14}}$

$\text{curl}\,\bar{f}\cdot\bar{n} = (\nabla\times\bar{F})\cdot\bar{n} = (2i+\bar{k})\cdot\frac{(3\bar{i}+2\bar{j}+\bar{k})}{\sqrt{14}}$

$= \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$

since $\oint_C \bar{f}\cdot d\bar{r} = \int_S \text{curl}\,\bar{f}\cdot\bar{n}\,ds$

$= \int_S \frac{7}{\sqrt{14}}\,ds$

$= \frac{7}{\sqrt{14}}\int_S ds$

$= \frac{7}{\sqrt{14}}$ area of the Δ^{le} ABC

$= \frac{7}{\sqrt{14}}\times\frac{1}{2}\,||\overrightarrow{AB}\times\overrightarrow{AC}||$

$= \frac{7}{\sqrt{14}}\times\frac{1}{2}\left\|\begin{matrix} i & j & \bar{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{matrix}\right|$

$= \frac{7}{\sqrt{14}}\times\frac{1}{2}\left|\left(18\bar{i}+12\bar{j}+6\bar{k}\right)\right|$

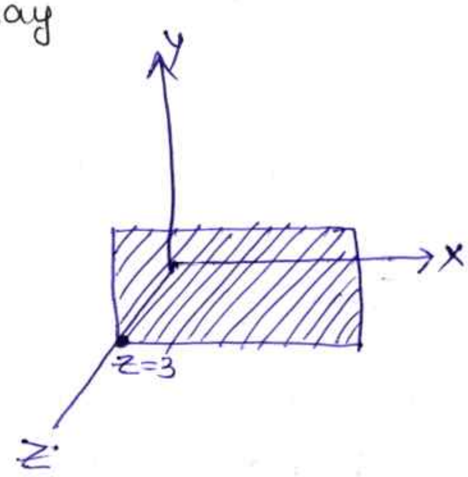
$= \frac{7}{\sqrt{14}}\times\frac{1}{2}\sqrt{(18)^2+(12)^2+6^2}$

$= \frac{7}{\cancel{\sqrt{14}}}\times\frac{1}{\cancel{2}}\,\overset{3}{\cancel{6}}\cancel{\sqrt{14}}$

$= 21$

4.) Apply stoke's theorem to evaluate $\int_C \sin z\,dx - \cos x\,dy + \sin y\,dz$, where 'C' is the boundary $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$

<u>Sol</u>:- Given $\int_C \sin z\,dx - \cos x\,dy + \sin y\,dz = \int_C \bar{F}\cdot d\bar{r}$, say



$\Rightarrow \bar{F} = \sin z\,\bar{i} - \cos x\,\bar{j} + \sin y\,\bar{k}$

Then $\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$

$= \bar{i}\cos y - \bar{j}(-\cos z) + \bar{k}(-(-\sin x))$

$= \cos y\,\bar{i} + \cos z\,\bar{j} + \sin x\,\bar{k}$

In xy-plane, $\bar{n} = \bar{k}$

$\therefore \operatorname{curl}\bar{F}\cdot\bar{n} = (\cos y\,\bar{i} + \cos z\,\bar{j} + \sin x\,\bar{k})\cdot\bar{k}$

$= \sin x$

and $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = dx\,dy$

and x varies from 0 to π

y varies from 0 to 1

$\therefore$ By stoke's theorem, $\oint_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$

$\Rightarrow \int_C \sin z\,dx - \cos x\,dy + \sin y\,dz = \int_{x=0}^{\pi}\int_{y=0}^{1} \sin x\,dx\,dy$

$= \int_{x=0}^{\pi} \sin x\,[y]_{y=0}^{1}\,dx$

$= \int_{x=0}^{\pi} \sin x\,dx$

$= [-\cos x]_0^{\pi}$

$= -\cos\pi + \cos 0$

$= -(-1) + 1 = 2$

Assignment Questions

(1) verify stokes theorem for $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ where the curve c is the unit circle in xy-plane bounded by the semi sphere $z = \sqrt{1-x^2-y^2}$ (Ans = π)

Hint: $z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2=1$ and do this same as problem

(2) verify stokes theorem for $\bar{f} = -y^3\bar{i} + x^3\bar{j}$ where s is the circular disc $x^2+y^2 \leq 1$, $z=0$

Ans: $\frac{3\pi}{2}$

(3) verify stokes theorem for $\bar{f} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0$ $x=a$, $y=0$ $y=b$

Ans: $2ab^2$

(4) find by stokes theorem $\oint_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$, and c is the curve $x^2+y^2=1$, $z=y^2$

Ans: 0

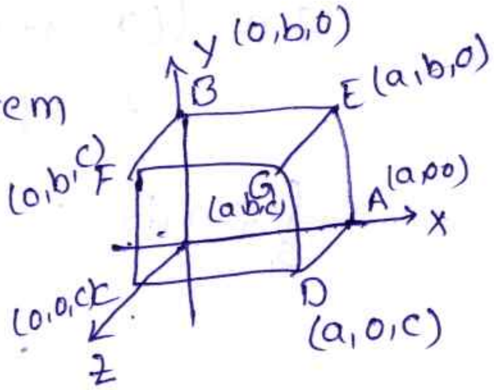
Gauss Divergence theorem:- Let s be a closed surface enclosing a volume v. Let $\bar{f}$ is a continuously differential vector point function. Let $\bar{n}$ is the outward drawn unit normal to the surfaces s then

$$\int_S \bar{f}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{f}\,dv$$

(1) verify Divergence theorem for $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ taken over the rectangular parallopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$

sol: Given $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$



We know that by Gauss divergence theorem

$$\int_S \bar{f}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{f}\,dV$$

to find RHS:- $\int_V \operatorname{div}\bar{f}\,dv$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 2x+2y+2z = 2(x+y+z)$$

x varies from 0 to a
y varies from 0 to b
z varies from 0 to c

$$\text{LHS} = \int_V \operatorname{div}\bar{f}\,dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xz+yz+\frac{z^2}{2}\right]_0^c dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left(cx+cy+\frac{c^2}{2}-0\right)dy\,dx$$

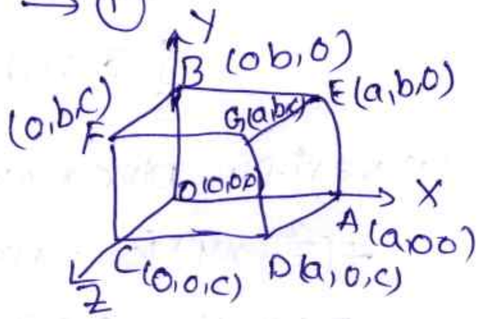
$$= 2\int_{x=0}^{a}\left(cbx+\frac{b^2}{2}c+\frac{bc^2}{2}-0\right)dx$$

$$= 2\left(\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}-0\right) = abc(a+b+c)$$

To find LHS: ie $\int_S \bar{f}\cdot\bar{n}\,ds$

$S = S_1+S_2+S_3+S_4+S_5+S_6$

$$\Rightarrow \int_S \bar{f}\cdot\bar{n}\,ds = \int_{S_1}\bar{f}\cdot\bar{n}\,ds + \int_{S_2}\bar{f}\cdot\bar{n}\,ds + \int_{S_3}\bar{f}\cdot\bar{n}\,ds + \int_{S_4}\bar{f}\cdot\bar{n}\,ds + \int_{S_5}\bar{f}\cdot\bar{n}\,ds + \int_{S_6}\bar{f}\cdot\bar{n}\,ds \rightarrow (1)$$



Along S_1 = FCDG: $\bar{n}=\bar{k}$ $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = dx\,dy$

x varies from 0 to a

y varies from 0 to b

S_1 = FCDG
S_2 = OAEB
S_3 = AEGB
S_4 = OBFC
S_6 = OADC

$$\bar{f}\cdot\bar{n} = ((x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k})\cdot\bar{k}$$
$$= z^2-xy = c^2-xy \quad (\because z=c)$$

$$\int_{S_1}\bar{f}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b}(c^2-xy)\,dx\,dy$$
$$= \int_{x=0}^{a}\left(c^2y - \frac{xy^2}{2}\right]_{y=0}^{b} dx$$
$$= \int_{x=0}^{a}\left[bc^2 - \frac{b^2}{2}x\right]dx$$
$$= \left(bc^2x - \frac{b^2}{2}\frac{x^2}{2}\right]_0^a = abc^2 - \frac{a^2b^2}{4} \rightarrow (1)$$

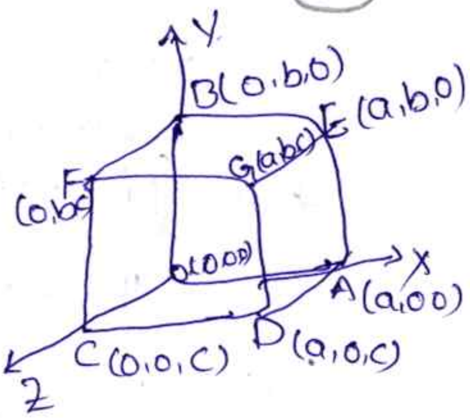
Along S_2 = OAEB $\Rightarrow z=0$ $\bar{n}=-\bar{k}$; $ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = dx\,dy$

x varies from 0 to a

y varies from 0 to b

$$\bar{f}\cdot\bar{n} = (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}\cdot(-\bar{k}) = -(z^2-xy) = xy$$

$$\int_{S_2}\bar{f}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b} xy\,dx\,dy$$
$$= \int_{x=0}^{a}\left(\frac{xy^2}{2}\right]_0^b dx = \int_{x=0}^{a}\left(x\frac{b^2}{2}\right)dx = \frac{b^2}{2}\left(\frac{x^2}{2}\right] = \frac{a^2b^2}{4} \rightarrow (2)$$



Along S_3 = AEGD :- $x=a$ $\bar{n}=\bar{i}$ $ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$$ds = dy\,dz$$

y varias from 0 to b

z varias from 0 to c

$$\bar{f}\cdot\bar{n} = \left((x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}\right)\cdot\bar{i}$$

$$= x^2 - yz$$

$$= a^2 - yz \quad (\because x=a)$$

$$\int_{S_3} \bar{f}\cdot\bar{n}\,ds = \int_{y=0}^{b}\int_{z=0}^{c} (a^2 - yz)\,dy\,dz$$

$$= \int_{y=0}^{b} \left[a^2 z - y\frac{z^2}{2}\right]_0^c dy$$

$$= \left[a^2 cy - \frac{y^2c^2}{4}\right]_{y=0}^{b}$$

$$= a^2bc - \frac{b^2c^2}{4} \rightarrow (3)$$

Along S_4 = OBFC :- $x=0$ $\bar{n}=-\bar{i}$ $d\theta = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = dy\,dz$

y varies from 0 to b

z varies from 0 to c

$$\int_{S_4} \bar{f}\cdot\bar{n}\cdot ds = \int_{y=0}^{b}\int_{z=0}^{c} yz\,dy\,dz$$

$$= \int_{y=0}^{b} \left[y\frac{z^2}{2}\right]_{z=0}^{c} dy$$

$$= \int_{y=0}^{b} \frac{c^2y}{2}\,dy$$

$$= \left[\frac{c^2y^2}{4}\right]_0^b$$

$$= \frac{b^2c^2}{4} \rightarrow (4)$$

Along $S_5 \Rightarrow BEGF$:- $y = b$ $\bar{n} = \bar{j}$ $ds = \frac{dx\,dz}{|\bar{n}\cdot\bar{j}|} = dx\,dz$

x varies from 0 to a

z varies from 0 to c

$\bar{f}\cdot\bar{n} = \left[(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}\right]\cdot j$

$= y^2 - zx = b^2 - zx \quad [\because y = b]$

$$\int_S \bar{f}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{c}(b^2-zx)\,dx\,dz$$

$$= \int_{x=0}^{a}\left[b^2 z - \frac{xz^2}{2}\right]_{z=0}^{c} dx$$

$$= \int_{x=0}^{a}\left[b^2c - \frac{c^2x}{2}\right] dx$$

$$= \int_{x=0}^{a}\left[b^2c - \frac{c^2x}{2}\right] dx$$

$$= \left[b^2cx - \frac{c^2x^2}{4}\right]_{x=0}^{a}$$

$$= ab^2c - \frac{a^2c^2}{4}$$

Along S_6 - OADC :- $y = 0$ $\bar{n} = -\bar{j}$ $ds = \frac{dx\,dz}{|\bar{n}\cdot j|} = dx\,dz$

x varies from 0 to a

z varies from 0 to c

$\bar{f}\cdot\bar{n} = ((x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k})\cdot(-\bar{j})$

$= -y^2 + zx = zx \quad (\because y = 0)$

$$\int_{S_6} \bar{f}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{c} zx\,dx\,dz$$

$$= \int_{x=0}^{a}\left[\frac{xz^2}{2}\right]_{z=0}^{c} dx$$

$$= \int_{x=0}^{a}\frac{c^2x}{2}dx = \left[\frac{c^2x^2}{4}\right]_0^a = \frac{a^2c^2}{4} \rightarrow (6)$$

put (1) (2) (3) (4) (5) (6) in Eq (1)

LHS $= abc^2 - \cancel{\frac{c^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}} + a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}}$

$= abc(a+b+c) = $ RHS $\quad \therefore \int_V \mathrm{div}\,\bar{f}\,dV = \int_S \bar{f}\cdot\bar{n}\,ds$

Hence Divergence theorem is verified.

8) Evaluate $\int_S f \cdot ds$ where $F = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$ using divergence theorem

Sol:- By Divergence theorem

w.k.t $\int_S \bar{f}\cdot\bar{n}\, ds = \int_V \mathrm{div}\bar{f}\, dv$

Given $\bar{f} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ say

then $\mathrm{div}\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 4 - 4y + 2z$

Given $x^2+y^2=4 \Rightarrow y^2 = 4-x^2 \Rightarrow y = \pm\sqrt{4-x^2}$

$\Rightarrow y$ — $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$ $\Rightarrow x$ — -2 to 2

put $y=0$ then $x^2=4 \Rightarrow x = \pm 2$

Given that z — 0 to 3

$\therefore \int_S \bar{f}\cdot ds = \int_V \mathrm{div}\bar{f}\, dv$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z\, dz)\, dy\, dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + 2\cdot\frac{z^2}{2}\right]_{z=0}^{3}$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\, dy\, dx$$

$$= \int_{x=-2}^{2} \left[21y - \frac{\overset{6}{\cancel{12}}\, y^2}{\cancel{2}}\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} (21\sqrt{4-x^2} - 6(\cancel{4-x^2}) + 21\sqrt{4-x^2} + 6(\cancel{4-x^2}))dx$$

$$= 42\int_{x=-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 42\times 2\int_{x=0}^{2} \sqrt{4-x^2}\, dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{x=0}^{2}$$

$$= 84(0 + 2\sin^{-1}(1)) = 84\pi$$

3) verify divergence theorem for $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over the region of the first octant of the cylinder $y^2+z^2=9$ and $x=0, x=2$

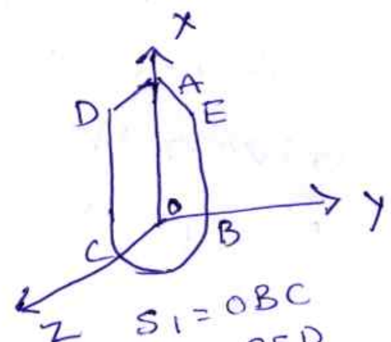
Sol:- w.k.t by divergence theorem $\int_S \bar{f}\cdot\bar{n}\,ds = \int \operatorname{div}\bar{f}\,dv$

Given that $\bar{f} = 2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$

and I octant of cylinder $y^2+z^2=9$, $x=0$, $x=2$

To find L.H.S

$$L.H.S = \int_S \bar{f}\cdot\bar{n}\,ds = \int_{S_1} \bar{f}\cdot\bar{n}\,ds + \int_{S_2} \bar{f}\cdot\bar{n}\,ds + \int_{S_3} \bar{f}\cdot\bar{n}\,ds + \int_{S_4} \bar{f}\cdot\bar{n}\,ds + \int_{S_5} \bar{f}\cdot\bar{n}\,ds \longrightarrow (1)$$



$S_1 = OBC$
$S_2 = AED$
$S_3 = AEBD$
$S_4 = AOCD$
$S_5 = BCDE$

Along $S_1 = OBC$ $x=0$, $\bar{n} = -\bar{i}$

$$\bar{f}\cdot\bar{n} = -2x^2y = 0$$

$$\therefore \int_{S_1} \bar{f}\cdot\bar{n}\,ds = \int_{S_1} 0\,ds = 0 \longrightarrow (2)$$

Along $S_2 = AED$ $x=2$, $\bar{n} = \bar{i}$

$$\bar{f}\cdot\bar{n} = 2x^2y = 2(2)^2y = 8y$$

$$ds = \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|} = dy\,dz$$

w.k.t $y^2+z^2=9 \Rightarrow z^2 = 9-y^2$

$\Rightarrow z = \pm\sqrt{9-y^2}$ $\because$ In I octant x, y, z are all +ve

$\therefore z$ — 0 to $\sqrt{9-y^2}$

put $z=0$ then $y^2=9 \Rightarrow y=\pm 3$

$\therefore y$ — 0 to 3

$$\therefore \int_{S_2} \bar{f}\cdot\bar{n}\,ds = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} 8y\,dz\,dy$$

$$= \int_{y=0}^{3} 8y\,[z]_0^{\sqrt{9-y^2}}\,dy$$

$$= 8\int_{y=0}^{3} y\sqrt{9-y^2}\,dy$$

put $9-y^2=t^2 \Rightarrow -\cancel{2}y\,dy = \cancel{2}t\,dt$

$$y\,dy = -t\,dt$$

when $y=0$ then $t=3$

when $y=3$ then $t=0$

$$= 8 \int_{t=3}^{0} -t \cdot t\, dt$$

$$= 8 \int_{t=0}^{3} t^2\, dt$$

$$= 8 \left[\frac{t^3}{3} \right]_{t=0}^{3}$$

$$= 8 \left(\frac{3^3}{3} \right) = 72 \longrightarrow (3)$$

Along $S_3 = AEBD$ $z = 0$, $\bar{n} = -\bar{k}$

$$\bar{f} \cdot \bar{n} = -4xz^2 = -4x(0)^2 = 0$$

$$\therefore \int_{S_3} \bar{f} \cdot \bar{n}\, ds = 0 \longrightarrow (4)$$

Along $S_4 = ADCD$ $y = 0$, $\bar{n} = -\bar{j}$,

$$\bar{f} \cdot \bar{n} = y^2 = 0^2 =$$

$$\therefore \int_{S_4} \bar{f} \cdot \bar{n}\, ds = 0 \longrightarrow (5)$$

Along $S_5 = BCDE$ Let $\phi = y^2 + z^2 - 9$

$$\nabla\phi = 2y\bar{j} + 2z\bar{k}$$

$$|\nabla\phi| = \sqrt{4y^2 + 4z^2}$$

$$= \sqrt{4(9)} = 6$$

$$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2yj + 2zk}{6} = \frac{yj + zk}{3}$$

$$= -\frac{y^3}{3} + \frac{4xz^3}{3}$$

Let the projection be on xy plane

then $ds = \dfrac{dx\,dy}{|\bar{n} \cdot \bar{k}|} = \dfrac{dx\,dy}{z/3}$

x — 0 to 2

y — 0 to 3

$$\therefore \int_{S_5} \bar{f} \cdot \bar{n}\, ds = \int_{x=0}^{2} \int_{y=0}^{3} \left(-\frac{y^3}{3} + \frac{4xz^3}{3} \right) \frac{dx\,dy}{z/3}$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left(-\frac{y^3}{z} + 4xz^2 \right) dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \frac{-y^3}{\sqrt{9-y^2}} + 4x(9-y^2)\, dy\, dx$$

$$= \int_{x=0}^{2}\left(\int_{y=0}^{3} \frac{-y^3}{\sqrt{9-y^2}}\, dy + 4x \int_{y=0}^{3} (9-y^2)\, dy \right) dx$$

put $9-y^2 = t^2$

$y^2 = 9-t^2$

$\Rightarrow \cancel{2}y\,dy = -\cancel{2}t\,dt \Rightarrow y\,dy = -t\,dt$

when $y=0,\ t=3$

when $y=3,\ t=0$

$$= \int_{x=0}^{2}\left(\int_{t=3}^{0} \frac{(9-t^2)t\,dt}{t} + 4x\left[9y - \frac{y^3}{3}\right]_{y=0}^{3} dx\right.$$

$$= \int_{x=0}^{2} \left[9t - \frac{t^3}{3}\right]_{t=3}^{0} + 4x\left(9(3) - \frac{3^3}{3} - 0\right) dx$$

$$= \int_{x=0}^{2} 0 - 27 + 9 + 4x(27-9)\, dx$$

$$= \int_{x=0}^{2} -18 + 72x\, dx$$

$$= \left[-18x + \cancel{72}^{\,36} \cdot \frac{x^2}{\cancel{2}}\right]_{x=0}^{2}$$

$$= \left[-18 \times 2 + 36(2)^2\right]$$

$$= -36 + 36(4)$$

$$= 108 \longrightarrow (6)$$

put (2), (3), (4), (5), (6) in (1)

$\therefore (1) \Rightarrow \int_S \bar{f}\cdot\bar{n}\, ds = 0 + 72 + 0 + 0 + 108$

$= 180 = L.H.S$

TO find R.H.S :-

$$R.H.S = \int_V \text{div}\,\bar{f}\cdot dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dz\, dy\, dx$$

$$\text{div}\bar{f} = \nabla\cdot\bar{f}$$

$$= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8zx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left[4xyz - 2yz + \cancel{8}^{4}\frac{xz^2}{\cancel{2}}\right]_{z=0}^{\sqrt{9-y^2}}$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left(4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2)\right) dy\, dx$$

put $9-y^2 = t^2 \Rightarrow -2y\,dy = 2t\,dt$

$-y\,dy = t\,dt$

when $y=0$ then $t=3$

when $y=3$ then $t=0$

$$= \int_{x=0}^{2}\left(\int_{t=3}^{0} 4x\,t(-t\,dt) + 2t^2dt + \int_{y=0}^{3} 4x(9-y^2)\,dy\right)dx$$

$$= \int_{x=0}^{2}\left(\left[-4x\cdot\frac{t^3}{3} + \frac{2t^3}{3}\right]_{t=3}^{0} + 4x[9-y^2)\,dy\right)dx$$

$$= \int_{x=0}^{2} 0 + 4x(9) - 2(9) + 4x(27-9)\, dx$$

$$= \int_{x=0}^{2} (36x - 18 + 72x)\, dx$$

$$= \int_{x=0}^{2} (108x - 18)\, dx = \left[\cancel{108}^{54}\cdot\frac{x^2}{\cancel{2}} - 18x\right]_0^2$$

$$= 54(2)^2 - 18(2)$$

$$= 180$$

L.H.S = R.H.S

$$\Rightarrow \int_S \bar{f}\cdot\bar{n}\, ds = \int_V \text{div}\bar{f}\, dv$$

Hence divergence theorem is verified

Apply divergence theorem to evaluate

$\iint_S (x+z)\,dy\,dz + (y+z)\,dx\,dz + (x+y)\,dx\,dy$ where 'z' is the surface of the sphere $x^2+y^2+z^2=4$ 243

<u>sol</u>:- Let $f_1 = x+z$, $f_2 = y+z$, $f_3 = x+y$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1+1+0 = 2$$

Given $x^2+y^2+z^2=4$

By divergence theorem $\int_S \bar{f}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{f}\,dv$

$$\Rightarrow \iint_S (x+z)\,dy\,dz + (y+z)\,dx\,dz + (x+y)\,dx\,dy = \int_V 2\,dV$$

$$= 2\int_V dV$$

$= 2$ volume of the sphere

$$= 2\cdot\frac{4}{3}\pi(2)^3 = \frac{64\pi}{3}$$

<u>Assignment</u> :-

i) Evaluate $\int_S \bar{f}\cdot\bar{n}\,ds$, if $\bar{f} = xy\,\bar{i} + z^2\bar{j} + 2yz\,\bar{k}$ over the tetrahedran bounded by $x=0$, $y=0$, $z=0$ & the plane $x+y+z=1$

2) Verify Gauss divergence theorem for $\bar{f} = 4xy\,\bar{i} - y^2\bar{j} + yz\,\bar{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Ans : $3/2$ [