ANNAMACHARYA UNIVERSITY::RAJAMPET

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)

HUMANITES AND SCIENCES

DIFFERENTIAL EQUATIONS AND TRANSFORM TECHNIQUES (24AMAT21T)

I B.Tech. & II-Semester

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ANNAMACHARYA UNIVERSITY

EXCELLENCE IN EDUCATION; SERVICE TO SOCIETY

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016)

Title of the Course:	Differential Equations and Transform Techniques
Category:	BS
Semester:	II Semester
Course Code:	24AMAT21T
Branch/es:	Common to all branches

Lecture Hours	Tutorial Hours	Practice Hours	Credits
3	0	0	3

Course Objectives: The course aims to develop proficiency in solving ordinary and partial differential equations, emphasizing their application in engineering and scientific contexts. It will cover Laplace Transforms and their inverses, illustrating their role in solving engineering and real-world problems. Additionally, the course will introduce Fourier Series and Fourier Transforms, focusing on their practical applications for addressing real-life challenges

Course Outcomes:

At the end of the course, the student will be able to

1. describe the application of higher order differential equations with constant coefficients in modeling dynamic systems.
2. solve the standard partial differential equations relevant to engineering scenarios.
3. utilize the Laplace transformations to handle various types of functions in engineering context.
4. analyze ordinary differential equations by employing Laplace transformations for solutions derivation.
5. apply Fourier series and Fourier transforms in engineering context.

Unit 1 Linear differential equations of higher order with constant Coefficients 10

Basic concepts - general solution-operator D-rules for finding complimentary function-inverse operator-rules for finding particular integral for RHS term of the type $e^{ax}, \sin ax / \cos ax$, polynomials in x, $e^{ax} \sin ax / e^{ax} \cos ax / e^{ax} x^n, x \sin ax / x \cos ax$ -method of variation of parameters

Unit 2 Partial Differential Equations 8

Introduction and formation of Partial Differential Equations by elimination of arbitrary constants and arbitrary functions, solutions of first order linear equations using Lagrange's method, non-linear PDEs (Charpit's method), method of separation of variables for second order linear partial differential equations.

Unit 3 Laplace transforms 8

Laplace transforms of standard functions- first shifting theorem - change of scale property - multiplication by t^n - division by t - transforms of derivatives and integrals - Unit step function – second shifting theorem– Laplace transform of periodic functions (without proofs).

Unit 4 Inverse Laplace transforms 8

Inverse Laplace transforms (without proofs) – Convolution theorem (without proof).

Applications of Laplace transforms to ordinary differential equations of first and second order with constant coefficients.

Unit 5 Fourier series and Fourier Transforms 10

Fourier series: Dirichlet conditions - functions of any period - odd and even functions - half range series.

Fourier Transforms: Fourier integrals - Fourier cosine and sine integrals - Fourier transform - sine and cosine transform.

Prescribed Textbooks:

1. E. Kreyszig. *Advanced Engineering Mathematics*. 10^{th} Ed., John Wiley & Sons, 2011.
2. B. S. Grewal. *Higher Engineering Mathematics.* 44^{th} Ed., Khanna Publishers, 2017.

Reference Books:

1. B. V. Ramana. *Higher Engineering Mathematics.* Mc Graw Hill Education.
2. G. B. Thomas, Maurice D. Weir and Joel Hass. *Thomas Calculus*. 14^{th} Ed., Pearson Publishers, 2022.
3. D. G. Zill. *Advanced Engineering Mathematics*. 6^{th} Ed., Jones and Bartlett, 2016
4. Glyn James. *Advanced Modern Engineering Mathematics*. 5^{th} Ed., Pearson publishers, 2018.
5. R. K. Jain and S. R. K. Iyengar. *Advanced Engineering Mathematics.* 5^{th} Ed., Alpha Science International Ltd., 2021 (9th reprint).

CO-PO Mapping:

Course Outcomes	Engineering Knowledge	Problem Analysis	Design/Development of solutions	Conduct investigations of complex problems	Engineering tool usage	The Engineer and world	Ethics	Individual and Collaborative teamwork	Communication	Project management and finance	Life-long learning
24AMAT21T.1	2	2	1	1	-	-	-	-	-	-	1
24AMAT21T.2	3	2	1	2	-	-	-	-	-	-	1
24AMAT21T.3	3	2	1	2	-	-	-	-	-	-	1
24AMAT21T.4	3	3	2	2	-	-	-	-	-	-	1
24AMAT21T.5	3	2	1	2	-	-	-	-	-	-	1

Unit-I

Higher Order Linear Differential Equations with Constant Coefficients

Differential Equation:- An equation containing the derivatives of a dependent variable with respect to one (or) more Independent variables is called a Differential equation.

Differential Equations are two types.

1) Ordinary Differential Equations (ODEs)
2) Partial Differential Equations (PDEs)

D.E

↓ ODE ↓ PDE

1) Ordinary Differential Equations (ODEs):-

An equation containing the derivatives of a dependent variable with respect to one independent variable is called Ordinary Differential equations.

i.e An equation containing ordinary derivatives ($\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, --- etc) only is called a ODE.

Ex: 1) $\frac{dy}{dx} + 6y = 3$ (2) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = e^x$

where y is a dependent variable & x is an Independent variable.

2) Partial Differential Equation (PDE) :- An equation containing the derivatives of a dependent variable with respect to two (or) more independent variables is called partial Differential Equations.

Ex ① $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ 2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 5$ 3) $\frac{\partial^2 z}{\partial x \partial y} = 6$

where z is a dependent variable & x, y are independent variables..

Order & degree of a Differential Equations :-

Order of the D.E : The order of the highest derivatives of the given D.E is called order of the D.E.

Ex: ① $\frac{dy}{dx} + 2y = e^x$ → Highest order = 1

2) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \sin x$ → order = 2

Degree of the D.E :- The degree (Power) of the highest derivative of a given D.E after removing radicals and fractions is called the Degree of D.E.

Ex (1) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 7y = \cos x$

Here order = 2

degree = 1

② $\left[1+\frac{dy}{dx}\right]^{3/2} = \frac{d^2y}{dx^2}$

squaring o.b.s, we get

$\left(1+\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$

∴ order = 2, degree = 2

Solution of Differential Equation :- Any relation between the dependent and Independent Variables, which not containing their derivatives, which satisfies the given D.E is called the solution of D.E.

Complete Solution (or) General Solution :- A solution in which the number of arbitrary constants is equal to the order of the D.E is called the complete solution.

Ex: $\frac{dy}{dx} = \frac{y}{x}$

& $\frac{dy}{y} = \frac{dx}{x}$

on integrating, we get

$\int \frac{1}{y} dy = \int \frac{1}{x} dx$

$\Rightarrow \log y = \log x + \log c$

$\Rightarrow \log y = \log cx$

$\Rightarrow \boxed{y = cx}$

$\therefore y = cx$ is the Complete solution.

Linear D.E of First order :- An eqn of the form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$, where $P(x)$ & $Q(x)$ are functions in x (or) constants, is called Linear D.E of First order in y.

An eqn of the form $\frac{dx}{dy} + P(y) \cdot x = Q(y)$, where $P(y)$ & $Q(y)$ are functions in "y" (or) constants, is called Linear D.E of First order in x.

Linear Differential Equation of nth order :- An eqn of the form $\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(x)y = Q(x)$

where $P_1(x), P_2(x), \ldots, P_n(x)$ and $Q(x)$ ared functions in x (or) constants, is called a Linear D.E of n^{th} order.

Linear D.E of nth order with constant Coefficients :-

An eqn of the form $\frac{d^ny}{dx^n} + P_1 \cdot \frac{d^{n-1}y}{dx^{n-1}} + P_2 \cdot \frac{d^{n-2}y}{dx^{n-2}} + \cdots$

$+ \cdots + P_n \cdot y = Q(x) \longrightarrow (1)$, where $P_1, P_2, \ldots P_n$ are constants and $Q(x)$ is a function in x (or) constant, is called a Linear D.E of nth order with constant Coefficients.

Linear D.E with constant coefficients are classified into 2 types. They are

i) Homogeneous Linear D.E (H.L.DE)

ii) Non-Homogeneous Linear D.E (Non-Homogeneous L.DE)

i) if $Q(x) = 0$ in eqn (1), then the linear D.E is called Homogeneous Linear D.E

ii) If $Q(x) \neq 0$ in eqn (1), Then the Linear D.E is called Non-Homogeneous Linear D.E.

Operation D :- The symbol "D" represents the differentiation with respect to independent variable x. That is $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$ ----, $D^n = \frac{d^n}{dx^n}$.

$\therefore$ The eqn ① in operator D form is

$$D^n y + P_1 \cdot D^{n-1} y + P_2 \cdot D^{n-2} y + \cdots + P_n \cdot y) = Q(x)$$

$$\Rightarrow (D^n + P_1 \cdot D^{n-1} + P_2 \cdot D^{n-2} + \cdots + P_n) y = Q(x)$$

$$\Rightarrow \boxed{f(D) \cdot y = Q(x)}, \text{ where } f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n.$$

Procedure to find the solution of Homogeneous Linear D.E

Step ①

The Homogeneous Linear D.E of n^{th} order in operator D form is $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n) y = 0$

$$\Rightarrow f(D) \cdot y = 0 \longrightarrow ②, \text{ where } f(D) = D^n + P_1 D^{n-1} + \cdots + P_n.$$

Step ②:- The Auxilary eqn of eqn ② is $f(D) = 0$

replace D by m, we get $f(m) = 0$. By solving this equation, we will get n roots say $m_1, m_2, \cdots m_n$.

Step ③: Based on the nature of the roots, we can write the Complementary function (C.F (or) y_c)

$\therefore$ The Complete solution of Homogeneous Linear D.E is $y = C.F$.

Complentary functions (C.F) :-

1) The roots are real & distinct i.e $m_1, m_2, m_3 --- m_n$.

Then $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + --- + c_n e^{m_n x}$

2) If the two roots are real & equal and the remaining roots are real & different.

i.e $m_1, m_1, m_3, m_4 --- m_n$ Then

$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + --- + c_n e^{m_n x}$

3) If Three roots are real & equal and the remaining roots are real and different

i.e $m_1 = m_2 = m_3 = m, m_4, m_5, ---- m_n$ then

$y_c = (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + --- + c_n e^{m_n x}$.

4) If the two roots are conjugate complex roots and the remaining roots are real and distinct.

i.e $\alpha \pm i\beta, m_3, m_4 --- m_n$ Then

$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + --- + c_n e^{m_n x}$

5) If the complex conjugate roots are repeated twice

i.e $\alpha \pm i\beta, \alpha \pm i\beta$

$\therefore y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x) \cdot \sin\beta x\right]$

6) If the two roots are $\alpha \pm \sqrt{\beta}$ and the remaining are real and distinct i.e $\alpha \pm \sqrt{\beta}, m_3, m_4, \cdots m_n$

$$y_c = e^{\alpha x}\left[c_1 \cosh(\sqrt{\beta}x) + c_2 \sinh(\sqrt{\beta}\,x)\right] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

Problems

1) solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Sol: Given that $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0 \rightarrow (1)$

$\therefore D^2y - 8D + 15y = 0$

$\Rightarrow (D^2 - 8D + 15)y = 0 \rightarrow (2)$, $f(D) = D^2 - 8D + 15$

$\therefore$ The Auxilary eqn of eqn (2) is $f(D) = 0$

$\Rightarrow D^2 - 8D + 15 = 0 \Rightarrow m^2 - 8m + 15 = 0$

$\therefore m = 3, 5$, which are real and distinct.

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{3x} + c_2 e^{5x}$

$\therefore$ The complete solution of eqn (1) is $y = y_c$

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$.

2) solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Sol: Given that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0 \rightarrow (1)$

$\Rightarrow D^2y - 2Dy + y = 0$

$\Rightarrow (D^2 - 2D + 1)y = 0 \rightarrow (2)$

$\therefore$ The Auxilary eqn of (2) is $f(D) = 0$

$\Rightarrow D^2 - 2D + 1 = 0 \quad \therefore m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$, which are real & equal

$\therefore y_c = (c_1 + c_2 x) e^{mx} = (c_1 + c_2 x) e^{x}$.

$\therefore$ The General solution is $y = y_c$

$\Rightarrow y = (c_1 + c_2 x) e^{x}$.

3) solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

Ⓐ In operator form is $D^2 y + Dy + y = 0 \Rightarrow (D^2 + D + 1) y = 0$

$\therefore$ The Auxilary eqn is $f(D) = 0$

$\Rightarrow D^2 + D + 1 = 0$

$\therefore m^2 + m + 1 = 0 \quad \therefore m = \dfrac{-1 \pm \sqrt{1-4}}{2(1)} = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow m = \dfrac{-1 \pm \sqrt{3} \cdot \sqrt{-1}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2} \quad (\because \sqrt{-1} = i)$

$\Rightarrow m = \underbrace{-\tfrac{1}{2}}_{\alpha} \pm i \underbrace{\tfrac{\sqrt{3}}{2}}_{\beta}$, which are in the form of $\alpha \pm i\beta$

$\therefore \alpha = -\frac{1}{2}, \beta = \sqrt{3}/2$

$\therefore y_c = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

$y_c = e^{-\frac{1}{2}x} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

$\therefore$ The General solution is $y = y_c$

$\Rightarrow y = e^{-\frac{1}{2}x} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

4) solve $(D^3+6D^2+12D+8)y=0$

A: A.E is $f(m)=0 \Rightarrow m^3+6m^2+12m+8=0$

$\therefore m=-2$ is one root

$$\begin{array}{r|rrrr} & m^3 & & & \\ -2 & 1 & 6 & 12 & 8 \\ & 0 & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & 0 \\ & m^2 & m & con & \end{array}$$

$\therefore m^2+4m+4=0$

$\Rightarrow (m+2)^2=0 \Rightarrow m=-2,-2$

$\therefore$ The roots are $-2,-2,-2$

$\therefore y_c=(c_1+c_2x+c_3x^2)e^{mx}=(c_1+c_2x+c_3x^2)e^{-2x}$.

$\therefore$ General solution is $y=y_c$.

$\Rightarrow y=(c_1+c_2x+c_3x^2)e^{-2x}$

(5) solve $(D^4-7D^3+17D^2-17D+6)y=0$ A: 1,1,2,3

1) solve $\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=0$ A: $m=1,1,-2$

2) solve $(D^3-D^2-8D+12)y=0$ A: $m=2,2,-3$

3) solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$ A: $m=2\pm i$

4) solve $(D^3+D^2+4D+4)y=0$ A: $m=-1, 0\pm 2i$

Inverse operator:- The operator D^{-1} is called the inverse of the Differentiation operator "D".

$\therefore$ "D" is the Differentiation operator and D^{-1} (or) $\frac{1}{D}$ is an Integration operator.

Note:- If $Q(x)$ is any function of x then $D^{-1}Q(x)$ (or) $\frac{1}{D}Q(x)$ is called $\int Q(x)\cdot dx$.

Procedure to find the solution of Non-Homogeneous Linear D.E :-

The ~~Non~~-Homogeneous Linear D.E of nth order is

$$\frac{d^n y}{dx^n} + P_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n \cdot y = Q(x) \longrightarrow (1)$$

step (1) The operator D form of eqn (1) is

$$D^n \cdot y + P_1 D^{n-1} y + P_2 D^{n-2} y + \cdots + P_n y = Q(x)$$

$$\Rightarrow \left(D^n + P_1 \cdot D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right) y = Q(x)$$

$$\Rightarrow f(D) \cdot y = Q(x) \longrightarrow (2)$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$.

step 2 :- The General solution of eqn (1) is $y = C.F + P.I$

$$\Rightarrow y = y_c + y_p$$

step (3) : To find the particular integral P.I.

From (2), we have $P.I = y_p = \frac{1}{f(D)} \cdot Q(x)$, which is the particular Integral.

Various Methods to find particular integral

1) $P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D-\alpha} \cdot Q(x) = e^{\alpha x} \cdot \int Q \cdot e^{-\alpha x} \, dx$

where α is root

2) $P.I = \frac{1}{(D-\alpha)(D-\beta)} Q(x) = \frac{1}{D-\beta} \left[\frac{1}{D-\alpha}\right] Q(x)$

$$= \frac{1}{D-\beta}\left[e^{\alpha x} \cdot \int Q \cdot e^{-\alpha x} \, dx\right]$$

1) $\frac{1}{D}(x^2) = \int x^2\,dx = \frac{x^3}{3}$

2) $\frac{1}{D^2}e^{4x} = \frac{1}{D}\left[\frac{1}{D}e^{4x}\right] = \frac{1}{D}\left(\frac{e^{4x}}{4}\right) = \int \left(\frac{1}{4}e^{4x}\right)dx$

$= \frac{1}{4}\int e^{4x}\,dx = \frac{1}{4}\left(\frac{e^{4x}}{4}\right) = \frac{1}{16}e^{4x}$

3) Find The particular value of $\frac{1}{D-2}e^{3x}$

A $y_p = \frac{1}{D-2}e^{3x} = e^{2x}\int e^{3x}\cdot e^{-2x}\,dx$

$= e^{2x}\cdot\int e^{x}\,dx = e^{2x}\cdot e^{x} = e^{3x}$

4) $\frac{1}{(D-2)(D-3)}e^{2x} = \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right] = \frac{1}{D-2}\left[e^{3x}\int e^{-3x}\cdot e^{2x}\,dx\right]$

$= \frac{1}{D-2}\left[e^{3x}\cdot\int e^{-x}\,dx\right] = \frac{1}{D-2}\left[e^{3x}(-e^{-x})\right]$

$= \frac{1}{D-2}(-e^{2x}) = e^{2x}\left[\int(-e^{2x})\cdot e^{-2x}\,dx\right]$

$= e^{2x}\int(-1)\,dx = -e^{2x}\int dx = -e^{2x}x = -xe^{2x}$

Method ①:- $f(D)y = e^{ax}$ (or) e^{ax+b} (or) Constant

$\therefore P.I = y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{f(D)}\cdot e^{ax}$ put $D=a$ Then

$y_p = \frac{1}{f(a)}\cdot e^{ax}$ if $f(a)\neq 0$

Suppose if $f(a)=0$, Then $y_p = x\cdot\frac{1}{f'(D)}\cdot e^{ax}$ & put $D=a$

Then $P.I = y_p = x\cdot\frac{1}{f'(a)}e^{ax}$ if $f'(a)\neq 0$

Suppose $f'(a)=0$ then $P.I = x^2 \cdot \frac{1}{f''(D)} e^{ax}$ & put $x=a$

Then $y_p = x^2 \cdot \frac{1}{f''(a)} \cdot e^{ax}$ if $f''(a) \neq 0$

Continuing this process until $f^n(a) \neq 0$

1) solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

A: Given that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x} \longrightarrow (1)$

In operator form is $D^2y + 4Dy + 3y = e^{2x}$

$\Rightarrow (D^2+4D+3)y = e^{2x} \longrightarrow (2)$

$\therefore f(D) = D^2+4D+3$, $\phi(x) = e^{2x}$

$\therefore$ The A.E of (2) is $f(D)=0 \Rightarrow f(m)=0$

$\Rightarrow m^2+4m+3=0 \quad \therefore m = -1, -3$

which are real & distinct

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{-x} + c_2 e^{-3x}$

$P.I = y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2+4D+3} e^{2x}$

Put $D=a \Rightarrow D=2$

$\therefore y_p = \frac{1}{2^2+4(2)+3} e^{2x} = \frac{1}{15} e^{2x}$

$\therefore$ The General solution is $y = y_c + y_p$

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-3x} + \frac{1}{15} e^{2x}$.

2) solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1+e^x)^2$

A: operator D form is $D^2y + Dy + y = (1+e^x)^2$

$\Rightarrow (D^2+D+1)y = (1+e^x)^2$

$\therefore f(D) = D^2+D+1, \quad \phi(x) = (1+e^x)^2$

$\therefore$ A.E is $f(D) = 0 \Rightarrow D^2+m+1=0 \Rightarrow m^2+m+1=0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\Rightarrow m = \underbrace{-\frac{1}{2}}_{\alpha} \pm i \underbrace{\frac{\sqrt{3}}{2}}_{\beta}$ which are complex roots.

$\therefore y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] = \cancel{e^{-\frac{1}{2}x}[c_1\cos\frac{\sqrt{3}}{2}x + c_2 \sin}$

$\Rightarrow y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

$\therefore$ P.I $= y_p = \frac{1}{f(D)} \cdot \phi(x) = \frac{1}{D^2+D+1}(1+e^x)^2$

$\Rightarrow y_p = \frac{1}{D^2+D+1}(1+e^{2x}+2e^x)$

$= \frac{1}{D^2+D+1}(1) + \frac{1}{D^2+D+1}(e^{2x}) + \frac{1}{D^2+D+1}(2e^{x})$

$= \frac{1}{D^2+D+1}(e^{0x}) + \frac{1}{D^2+D+1}e^{2x} + 2\cdot\frac{1}{D^2+D+1}(e^x)$

Put $D = a$ $\Rightarrow$ Put $D = 0$ Put $D = a$ $\Rightarrow$ Put $D = 2$ Put $D = a$ $\Rightarrow$ Put $D = 1$

$$= \frac{1}{0^2+0+1}(e^{0x}) + \frac{1}{2^2+2+1}(e^{2x}) + 2\cdot\frac{1}{1^2+1+1}(e^{x})$$

$$y_p = \frac{1}{1}e^{0x} + \frac{1}{7}e^{2x} + \frac{2}{3}e^{x} = 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^{x}$$

$\therefore$ The General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^{x}$$

BS

3) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

A: operator D form is $D^2y + 4Dy + 5y = -2\cosh x$

$\Rightarrow (D^2+4D+5)y = -2\cosh x$

$\therefore f(D) = D^2+4D+5, \quad Q(x) = -2\cosh x$

A.E is $f(D)=0 \Rightarrow D^2+4D+5=0 \Rightarrow m^2+4m+5=0$

$$\Rightarrow m = \frac{-4 \pm \sqrt{16-4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm i2}{2}$$

$\Rightarrow m = -2 \pm i1$ which are complex roots.

$\therefore \alpha = -2, \beta = 1$

$$\therefore y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x] = e^{-2x}\left[c_1\cos(1x) + c_2\sin(1\cdot x)\right]$$

$$\therefore P.I = y_p = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{D^2+4D+5}(-2\cosh x)$$

$$y_p = \frac{1}{D^2+4D+5}\left\{-\cancel{2}\left(\frac{e^{x}+e^{-x}}{\cancel{2}}\right)\right\}$$

$$y_p = \frac{1}{D^2+4D+5}(-e^x) + \frac{1}{D^2+4D+5}(-e^{-x})$$

Put $D=a$ $\Rightarrow$ Put $D=1$ Put $D=a$ $\Rightarrow$ Put $D=-1$

$$\therefore y_p = \frac{1}{1^2+4(1)+5}(-e^x) + \frac{1}{(-1)^2+4(-1)+5}(-e^{-x}) = \frac{-e^x}{10} - \frac{e^{-x}}{2}$$

$$\therefore y_p = \frac{-e^x}{10} - \frac{e^{-x}}{2}$$

$\therefore$ General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{-2x}\left[c_1 \cos x + c_2 \sin x\right] - \frac{e^x}{10} - \frac{e^{-x}}{2}$$

4) Solve $(D^2-4D+4)y = e^{2x}$

Sol: $(D^2-4D+4)y = e^{2x} \rightarrow (1)$

$\therefore f(D) = D^2-4D+4$, $Q(x) = e^{2x}$

A.E is $f(D)=0 \Rightarrow D^2-4D+4=0$, Replace D by m

$\Rightarrow m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow (m-2)(m-2)=0$

$\Rightarrow m = 2, 2$ which are real & equal.

$\therefore y_c = (c_1+c_2x)e^{mx} = (c_1+c_2x)e^{2x}$.

P.I: $y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2-4D+4}(e^{2x})$

Put $D=2$ Then $f(2)=0$ Then

$$P.I = x \cdot \frac{1}{f'(D)} \cdot e^{2x} = x \cdot \frac{1}{2D-4}(e^{2x})$$

Put $D = a \Rightarrow$ Put $D = 2$ then $f'(2) = 0$

Then $P.I = x^2 \cdot \frac{1}{f''(D)}(e^{2x}) = x^2 \cdot \frac{1}{2}(e^{2x})$

$\Rightarrow y_p = \frac{1}{2}(x^2 e^{2x})$

$\therefore$ General solution is $y = y_c + y_p$

$\Rightarrow y = (c_1 + c_2 x)e^{2x} + \frac{1}{2}(x^2 e^{2x})$

5) solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$

A: operator D form is $D^2y - 5Dy + 6y = e^{3x}$

$\Rightarrow (D^2 - 5D + 6)y = e^{3x} \longrightarrow (2)$

$f(D)\,y = Q(x)$ form, $f(D) = D^2 - 5D + 6$, $Q(x) = e^{3x}$

$\therefore$ A.E is $f(D) = 0 \Rightarrow D^2 - 5D + 6 = 0 \Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$ are real & distinct

$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{2x} + c_2 e^{3x}$.

$\therefore P.I = y_p = \frac{1}{f(D)} e^{3x} = \frac{1}{D^2 - 5D + 6}(e^{3x})$, Put $D = a$

$\Rightarrow$ Put $D = 3$

$\therefore P.I = \frac{1}{3^2 - 5(3) + 6}(e^{3x}) = \frac{1}{0}(e^{3x})$

$\therefore f(3) = 0$ Then $P.I = x \cdot \frac{1}{f'(D)}(e^{3x}) = x \cdot \frac{1}{2D - 5}(e^{3x})$

Again put $D = a \Rightarrow$ put $D = 3$

$\therefore y_p = x \cdot \frac{1}{2(3)-5}(e^{3x}) = x\frac{1}{1}e^{3x} = xe^{3x}$

$\therefore$ General solution is $y = y_c + y_p$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{3x} + xe^{3x}$

13§ 8) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Ans $y_p = \frac{1}{9}xe^{-2x} + \frac{1}{6}x^2e^{x} + \frac{1}{4}e^{-x}$

5) solve $(D^2 + D + 13)y = 2e^{-x}$

6) solve $(4D^2 - 4D + 1)y = 100$

7) solve $(D^2 - 4D + 3)y = e^{x} + e^{3x}$ → BS

(8) solve $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$.

Model (2) :- $f(D)y = Q(x)$; where $Q(x) = \sin ax$ (or) $\sin(ax+b)$ (or) $\cos ax$ (or) $\cos(ax+b)$

$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)}(\sin ax / \cos ax)$

put $D^2 = -a^2$ then $y_p = \frac{1}{f(-a^2)}[\sin ax / \cos ax]$ if $f(-a^2) \neq 0$

if $f(-a^2) = 0$ then $y_p = x \cdot \frac{1}{f'(D)}[\sin ax \text{ (or) } \cos ax]$

put $D^2 = -a^2$, $y_p = x \cdot \frac{1}{f'(-a^2)}(\sin ax \text{ (or) } \cos ax)$ if $f'(-a^2) \neq 0$

if $f'(-a^2) = 0$ then $P.I = x^2 \cdot \frac{1}{f''(D)}(\sin ax \text{ (or) } \cos ax)$

Continuing this process untill $f^n(-a^2) \neq 0$

Note :- 1) $\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$

2) $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$

1) Solve $(D^2+3D+2)y = \sin 3x$

A: Given that $(D^2+3D+2)y = \sin 3x \longrightarrow (1)$

$f(D)\, y = \phi(x)$ form

$\therefore f(D) = D^2+3D+2, \quad \phi(x) = \sin 3x.$

$\therefore$ A-E is $f(D)=0 \Rightarrow D^2+3D+2=0$

replace D by m, $f(m)=0 \Rightarrow m^2+3m+2=0 \Rightarrow m=-1,-2$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{-x} + c_2 e^{-2x}$

$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2+3D+2}\sin 3x$, put $D^2=-a^2$

$\Rightarrow D^2 = -(3)^2 = -9$

$\therefore P.I = \frac{1}{-9+3D+2}\sin 3x = \frac{1}{3D-7}\sin 3x$

$y_p = \frac{1}{3D-7}\times\frac{3D+7}{3D+7}(\sin 3x) = \frac{3D+7}{9D^2-49}(\sin 3x)$

put $D^2=-a^2 \Rightarrow D^2 = -(3)^2 = -9 \Rightarrow D^2=-9$

$\therefore y_p = \frac{3D+7}{9(-9)-49}(\sin 3x) = \frac{3D+7}{-130}(\sin 3x)$

$= \frac{-1}{130}\left[3D(\sin 3x) + 7\sin 3x\right] = \frac{-1}{130}\left[3\frac{d}{dx}(\sin 3x) + 7\sin 3x\right]$

$= \frac{-1}{130}\left[3(3\cos 3x) + 7\sin 3x\right]$

$y_p = -\frac{1}{130}[9\cos 3x + 7\sin 3x]$

$\therefore$ General solution is $y = y_c + y_p$

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{130}[9\cos 3x + 7\sin 3x]$

2) Solve $(D^2+9)y = \cos 3x$

S: $f(D)y = Q(x)$ form, $f(D) = D^2+9$, $Q(x) = \cos 3x$

$\therefore$ A.E is $f(D) = 0 \Rightarrow D^2+9=0$, Replace D by m,

We get $m^2+9=0 \Rightarrow m^2 = -9, \Rightarrow m = \pm\sqrt{-9}$

$\Rightarrow m = \pm\sqrt{(-1)9} = \pm i3 \quad \therefore m = \pm 3i = 0 \pm 3i$

$\therefore m = \underset{\alpha}{0} \pm \underset{\beta}{3} i$ which are Complex roots

$y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x] = e^{0x}[c_1\cos 3x + c_2\sin 3x]$

$\Rightarrow y_c = 1\cdot[c_1\cos 3x + c_2\sin 3x] = c_1\cos 3x + c_2\sin 3x$

$\therefore P.I = y_p = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+9}\cos 3x = \frac{1}{D^2+3^2}(\cos 3x)$

$\Rightarrow y_p = \frac{x}{2(3)}\sin 3x = \frac{x}{6}\sin 3x$

$\therefore$ General solution is $y = y_c + y_p$

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x$

3) Solve $(D^3+1)y = \cos(2x+1)$

Sol: Given that $(D^3+1)y = \cos(2x+1)$

$\therefore f(D)\cdot y = \phi(x)$ form.

$\therefore f(D) = D^3+1, \quad \phi(x) = \cos(2x+1)$

A-E is $f(D)=0 \Rightarrow D^3+1=0$ replace D by m

$\therefore f(m)=0 \Rightarrow m^3+1=0$

$\therefore m=-1$ is one root

$$\therefore m=-1 \;\Big|\; \begin{array}{cccc} m^3 & & & \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & 0 \\ m^2 & m & c & \end{array}$$

$\therefore m^2-m+1=0$

$\therefore m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore$ The roots are $-1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, in which one root is real & other two are complex roots.

$\therefore y_c = c_1 e^{mx} + e^{\alpha x}\left[c_2\cos\beta x + c_3\sin\beta x\right]$.

$$y_c = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$P.I = y_p = \frac{1}{f(D)}\cdot\phi(x) = \frac{1}{D^3+1}\cos(2x+1) \quad \langle \cos(\underline{a}x+b)$

Put $D^2=-a^2 = D^2=-(2)^2 \Rightarrow D^2=-4$

$\therefore y_p = \frac{1}{DD^2+1}\cos(2x+1) = \frac{1}{D(-4)+1}\cos(2x+1)$

$= \frac{1}{1-4D}\cos(2x+1) = \frac{1}{1-4D}\times\frac{1+4D}{1+4D}\cos(2x+1)$

$= \frac{1+4D}{1-16D^2}\cdot\cos(2x+1)$, put $D^2=-a^2$

$\Rightarrow D^2=-4$

$$y_p = \frac{1+4D}{1-16(-4)}\cos(2x+1) = \frac{1+4D}{65}\cos(2x+1)$$

$$= \frac{1}{65}\left[\cos(2x+1) + 4D\left[\cos(2x+1)\right]\right]$$

$$= \frac{1}{65}\left[\cos(2x+1) + 4\cdot\frac{d}{dx}\left(\cos(2x+1)\right)\right]$$

$$= \frac{1}{65}\left[\cos(2x+1) + 4\left[-\sin(2x+1)\cdot 2\right]\right]$$

$$y_p = \frac{1}{65}\left[\cos(2x+1) - 8.\sin(2x+1)\right]$$

$\therefore$ The General solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + \frac{1}{65}\left[\cos(2x+1) - 8\cdot\sin(2x+1)\right]$$

BS

4) Solve $\dfrac{d^2y}{dx^2} + 3\cdot\dfrac{dy}{dx} + 2y = 4\cos^2 x$

Σ operator form is $(D^2+3D+2)y = 4\cos^2 x \rightarrow (1)$

$\therefore f(D) = D^2+3D+2)$, $\theta(x) = 4\cos^2 x$

$\therefore$ A.E is $f(D)=0 \Rightarrow D^2+3D+2=0 \Rightarrow m^2+3m+1=0$

$\therefore m = -1, -2$, which are real & distinct.

$$\therefore y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+3D+2} (4\cos^2 x) = \frac{1}{D^2+3D+2} \cancel{4}^{2} \left(\frac{1+\cos 2x}{\cancel{2}_1}\right)$$

$$= \frac{1}{D^2+3D+2} (2+2\cos 2x) = \frac{1}{D^2+3D+2} (2 \cdot e^{0x}) + \frac{1}{D^2+3D+2} (2\cos 2x)$$

put $D = a$ $\Rightarrow D = 0$ put $D^2 = -a^2$ $\Rightarrow D^2 = -2^2 = -4$

$$\therefore P.I = \frac{1}{0+0+2} (2e^{0x}) + \frac{1}{-4+3D+2} (2\cos 2x)$$

$$= \frac{1}{\cancel{2}} \cancel{2}(e^{0x}) + \frac{1}{3D-2} (2\cos 2x) = 1 + 2 \cdot \frac{1}{3D-2} \times \frac{3D+2}{3D+2} (\cos 2x)$$

$$= 1 + 2 \frac{(3D+2)}{9D^2-4} (\cos 2x) \quad \text{put } D^2 = -a^2 = -4$$

$$\therefore P.I = 1 + \frac{6D+4}{9(-4)-4} (\cos 2x) = 1 - \frac{1}{40} [6D(\cos 2x) + 4\cos 2x]$$

$$= 1 - \frac{1}{40} \left[6 \frac{d}{dx}(\cos 2x) + 4\cos 2x\right]$$

$$= 1 - \frac{1}{40} [6(-2\sin 2x) + 4\cos 2x] = 1 - \frac{1}{40} (-12\sin 2x + 4\cos 2x]$$

$$\therefore P.I = 1 + \frac{1}{20} [6\sin 2x - 2\cos 2x] = 1 + \frac{1}{10} [3\sin 2x - \cos 2x]$$

$\therefore$ General solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{1}{10} [3\sin 2x - \cos 2x]$$

5) $$\frac{d^3 y}{dx^3} + 2 \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$$

A: A.E is $m^3 + 2m^2 + m = 0 \Rightarrow m(m^2+2m+1) = 0$

$\therefore m = 0, -1, -1$

$$y_c = c_1 e^{0x} + (c_2 + c_3 x)\, e^{-1\cdot x} = c_1 + (c_2 + c_3 x) e^{-x}.$$

$$P.I = y_p = \frac{1}{f(D)}\,\theta(x) = \frac{1}{D^3+2D^2+D}\left(e^{-x} + \sin 2x\right)$$

$$= \frac{1}{D^3+2D^2+D}\left(e^{-x}\right) + \frac{1}{D D^2+2D^2+D}(\sin 2x)$$

put $D = a$ $\Rightarrow D = -1$ put $D^2 = -a^2$ $\Rightarrow D^2 = -(2)^2 = -4$

$$\therefore y_p = \frac{1}{-1+2(-1)^2-1}\left(e^{-x}\right) + \frac{1}{D(-4)+2(-4)+D}(\sin 2x)$$

$$= x \cdot \frac{1}{f'(D)}\left(e^{-x}\right) + \frac{1}{-3D-8}(\sin 2x)$$

$$= x \cdot \frac{1}{3D^2+4D+1}\left(e^{-x}\right) - \frac{1}{3D+8} \times \frac{3D-8}{3D-8}(\sin 2x)$$

Put $D = -1$ ∴ $f(-1) = 0$ Then $\quad - \dfrac{3D-8}{9D^2-64}(\sin 2x)$

put $D^2 = -4$

$$\therefore y_p = x^2 \cdot \frac{1}{f''(D)}\, e^{-x} - \frac{(3D-8)}{9(-4)-64}(\sin 2x)$$

$$= x^2 \cdot \frac{1}{6D+4}\left(e^{-x}\right) - \frac{(3D-8)}{-100}(\sin 2x)$$

put $D = -1$

$$\therefore y_p = x^2 \cdot \frac{1}{6(-1)+4}\, e^{-x} + \frac{1}{100}\left[3 \cdot D(\sin 2x) - 8 \sin 2x\right]$$

$$y_p = \frac{-x^2 \cdot e^{-x}}{2} + \frac{1}{100}\left[3 \cdot (2\cos 2x) - 8\sin 2x\right]$$

$$y_p = \frac{-x^2 e^{-x}}{2} + \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

$\therefore$ The General solution is $y = y_c + y_p$

$$\therefore y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{x^2 \cdot e^{-x}}{2} + \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

$= \quad =$

6) Solve $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

Sol: $m = 1, 3$, $y_c = c_1 e^{x} + c_2 e^{3x}$

$$y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2 - 4D + 3}(\sin 3x \cdot \cos 2x)$$

$$= \frac{1}{D^2 - 4D + 3}\left(\frac{1}{2} \times 2\sin 3x \cdot \cos 2x\right) = \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}(\sin 5x + \sin x)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3}(\sin 5x) + \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3}(\sin x)$$

put $D^2 = -a^2$ $\qquad$ put $D^2 = -a^2$

$D^2 = -5^2 = -25$ $\qquad$ $\Rightarrow D^2 = -1^2 = -1$

$$\therefore y_p = \frac{1}{2} \cdot \frac{1}{-25 - 4D + 3}(\sin 5x) + \frac{1}{2} \cdot \frac{1}{-1 - 4D + 3}(\sin x)$$

$$= \frac{1}{2} \cdot \frac{1}{-4D - 22}(\sin 5x) + \frac{1}{2} \cdot \frac{1}{-4D + 2}(\sin x)$$

$$= \frac{-1}{2} \cdot \frac{1}{22 + 4D}(\sin 5x) + \frac{1}{2} \cdot \frac{1}{2 - 4D}(\sin x)$$

$$= -\frac{1}{2} \cdot \frac{1}{22+4D} \times \frac{22+4D}{22+4D} (\sin 5x) + \frac{1}{2} \cdot \frac{1}{2-4D} \times \frac{2+4D}{2+4D} (\sin x)$$

$$= -\frac{1}{2} \cdot \frac{22+4D}{484-16D^2} (\sin 5x) + \frac{1}{2} \cdot \frac{2+4D}{4-16D^2} (\sin x)$$

Put $D^2 = -25$ Put $D^2 = -1$

$$\therefore y_p = -\frac{11+2D}{484-16(-25)} (\sin 5x) + \frac{1+2D}{4-16(-1)} (\sin x)$$

$$= \frac{-11+2D}{884} (\sin 5x) + \frac{1+2D}{20} (\sin x)$$

$$= \frac{1}{884} \left[11 \sin 5x + 2 D (\sin 5x) \right] + \frac{1}{20} \left[\sin x + 2 D (\sin x) \right]$$

$$= \frac{1}{884} \left[11 \cdot \sin 5x + 2 (5 \cos 5x) \right] + \frac{1}{20} \left[\sin x + 2 \cos x \right]$$

$$y_p = \frac{-1}{884} \left[11 \sin 5x + 10 \cos 5x \right] + \frac{1}{20} \left[\sin x + 2 \cos x \right]$$

$\therefore$ General solution is $y = y_c + y_p$

7) Solve $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = e^{2x} - \cos^2 x$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

8) Solve $(D^2+1) y = \sin x \cdot \sin 2x$

9) $(D^2 - 3D + 2) y = \cos 3x \cdot \cos 2x$

*10) $y'' + 4y' + 4y = 4 \cos x + 3 \sin x$, $y(0) = 1$ & $y'(0) = 0$

11) 13G $\frac{d^3y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x$

& $y_p = -\frac{1}{8} x \sin 2x$

25-A

10) solve $y'' + 4y' + 4y = 4\cos x + 3\sin x$,

$y(0) = 1, y'(0) = 0$

Sol Given that $(D^2 + 4D + 4)y = 4\cos x + 3\sin x \longrightarrow (1)$

$f(D) = D^2 + 4D + 4 = (D+2)^2$, $Q(x) = 4\cos x + 3\sin x$

A-E is $f(D) = 0 \Rightarrow (D+2)^2 = 0$, Replace D by m,

we get $(m+2)^2 = 0 \Rightarrow (m+2)(m+2) = 0 \Rightarrow m = -2, -2$

$\therefore y_c = (c_1 + c_2 x) e^{mx} = (c_1 + c_2 x) e^{-2x}$

$\therefore P.I = y_p = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{(D+2)^2} (4\cos x + 3\sin x)$

$= \frac{1}{D^2 + 4D + 4} (4\cos x) + \frac{1}{D^2 + 4D + 4} (3\sin x)$

put $D^2 = -a^2$ $\Rightarrow D^2 = -1$ put $D^2 = -a^2$ $D^2 = -1$

$\therefore y_p = 4 \cdot \frac{1}{-1 + 4D + 4} (\cos x) + 3 \cdot \frac{1}{-1 + 4D + 4} (\sin x)$

$= 4 \cdot \frac{1}{4D+3} \cos x + 3 \cdot \frac{1}{4D+3} \sin x$

$= 4 \times \frac{1}{4D+3} \times \frac{4D-3}{4D-3} (\cos x) + 3 \cdot \frac{1}{4D+3} \times \frac{4D-3}{4D-3} (\sin x)$

$= 4 \cdot \frac{1}{16D^2 - 9} (4D-3) \cos x + 3 \cdot \frac{4D-3}{16D^2 - 9} (\sin x)$

put $D^2 = -1$ put $D^2 = -1$

$\therefore y_p = 4 \cdot \frac{1}{16(-1) - 9} (4D-3)(\cos x) + 3 \cdot \frac{1}{16(-1) - 9} (4D-3)(\sin x)$

$$y_p = 4 \times \frac{1}{-25}\left[4D(\cos x) - 3\cos x\right] + 3 \times \frac{1}{-25}\left[4 \cdot D(\sin x) - 3\sin x\right]$$

25-B

$$= \frac{-4}{25}\left[4(-\sin x) - 3\cos x\right] - \frac{3}{25}\left[4\cos x - 3\sin x\right]$$

$$y_p = \frac{1}{25}\left[16\sin x + \cancel{12\cos x} - \cancel{12}\cos x + 9\sin x\right]$$

$$y_p = \frac{1}{25}\left[25\sin x\right] = \sin x.$$

$$\therefore y_p = \sin x$$

$\therefore$ General solution is $y = y_c + y_p$

$$\therefore y(x) = (c_1 + c_2 x)\, e^{-2x} + \sin x \longrightarrow (2)$$

$$y'(x) = \frac{dy}{dx} = (c_1 + c_2 x)(-2e^{-2x}) + e^{-2x}(0 + c_2) + \cos x$$

$$y'(x) = e^{-2x}\left[(c_1 + c_2 x)(-2) + c_2\right] + \cos x$$

$$y'(x) = e^{-2x}\left[-2c_1 - 2c_2 x + c_2\right] + \cos x \longrightarrow (3)$$

Given that $y(0) = 1$ i.e at $x = 0$, $y = 1$

substitute $y(0) = 1$ in eqn (2), we get

$$(2) \Rightarrow y(0) = \left[c_1 + c_2(0)\right] e^{0} + \sin 0$$

$$\Rightarrow 1 = c_1(1) + 0 \Rightarrow \boxed{c_1 = 1}$$

substitute $y'(0) = 0$ i.e at $x = 0$, $y' = 0$ in (3), we get

$$(3) \Rightarrow y'(0) = e^{0}\left[-2c_1 - 2c_2(0) + c_2\right] + \cos 0$$

$\Rightarrow 0 = 1[-2(1) - 0 + c_2] + 1 \quad \langle \because c_1 = 1 \rangle$ 25-c

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$\Rightarrow -2 + c_2 + 1 = 0 \Rightarrow c_2 - 1 = 0 \Rightarrow \boxed{c_2 = 1}$

substitute $c_1 = 1$ & $c_2 = 1$ in eqn (2), we get

$\therefore (2) \Rightarrow y(x) = [1 + (1)x]\, e^{-2x} + \sin x$

$\Rightarrow y(x) = (1 + x)\, e^{-2x} + \sin x.$

which is the required solution.

Note :- (1) The Binomial expansion is

$$(x+y)^n = nC_0\, x^n \cdot y^0 + nC_1\, x^{n-1} y^1 + nC_2\, x^{n-2} \cdot y^2 + \cdots + nC_n\, x^0 \cdot y^n$$

$$\Rightarrow (x+y)^n = nC_0\, x^n + nC_1\, x^{n-1} y^1 + nC_2\, x^{n-2} \cdot y^2 + \cdots + nC_n \cdot y^n$$

2) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2} \cdot x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \cdots + x^n$

3) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

4) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

5) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

6) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

7) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$

8) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$

Model (3) :- If $f(D)y = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$, which is a polynomial in "x".

$$\therefore y_p = \frac{1}{f(D)} (a_0 x^n + a_1 x^{n-1} + \cdots + a_n) = \frac{1}{[1 \pm \phi(D)]} (a_0 x^n + a_1 x^{n-1} + \cdots + a_n)$$

since $f(D)$ canbe convert into $[1 \pm \phi(D)]$ by taking the Lowest ~~term~~ degree term of Common from $f(D)$.

$$\therefore y_p = [1 \pm \phi(D)]^{-1} (a_0 x^n + a_1 x^{n-1} + \cdots + a_n)$$

1) Solve $(D^2+D+1)y = x^3$

Sol: $m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, $y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

$$y_p = \frac{1}{D^2+D+1}(x^3) = \frac{1}{1\left[1+D+D^2\right]}x^3 = \left[1+(D^2+D)\right]^{-1}(x^3)$$

$$Q(x) = x^3 \rightarrow D \rightarrow 3x^2$$

$$D^2 = 6x$$

$$D^3 = 6, \quad D^4 = D^5 = \cdots = 0$$

$$\therefore y_p = \left[1-(D^2+D)+(D^2+D)^2 - (D^2+D)^3 + \cdots\right]x^3$$

$$= \left[1 - D^2 - D + D^4 + D^2 + 2D^3 - (D^6 + 3D^5 + 3D^4 + D^3)\right]x^3$$

$$= \left[1 - \cancel{D^2} - D + \cancel{D^2} + 2D^3 - D^3\right]x^3 = \left[1 - D + D^3\right]x^3$$

$$= x^3 - D(x^3) + D^3(x^3) = x^3 - 3x^2 + 6$$

$$\therefore y_p = x^3 - 3x^2 + 6$$

$\therefore$ The General solution is $y = y_c + y_p$

(4)

$\boxed{\text{given}}$

2) Solve $(D^2+D)y = x^2+2x+4$

Sol: $m = 0, -1$, $y_c = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$

$$D(x^2+2x+4) = 2x+2, \quad D^2(x^2+2x+4) = 2, \quad D^3 \Rightarrow 0$$

$$\& \therefore D^4 = D^5 = \cdots = 0$$

$$y_p = \frac{1}{D^2+D}(x^2+2x+4) = \frac{1}{D(D+1)}(x^2+2x+4)$$

$$= \frac{1}{D}(1+D)^{-1}(x^2+2x+4) = \frac{1}{D}(1-D+D^2-D^3\ldots)(x^2+2x+4)$$

$$= \frac{1}{D}\left[x^2+2x+4-D(x^2+2x+4)+D^2(x^2+2x+4)\right]$$

$$= \frac{1}{D}\left(x^2+\cancel{2x}+4-\cancel{2x}-\cancel{2}+2\right] = \frac{1}{D}(x^2+4)$$

$$y_p = \int(x^2+4)\,dx = \underline{\frac{x^3}{3}+4x} \quad (\text{or}) \quad P.I = \underline{\frac{x^3}{3}+4x-4}$$

$\therefore$ The General Solution is $y = y_c + y_p$

*** 3) Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = e^{2x} + \sin x + x$

A: $m = 3 \pm 4i$, $y_c = e^{3x}\left[c_1\cos 4x + c_2 \sin 4x\right]$

$$y_p = \frac{1}{D^2-6D+25}\left(e^{2x} + \sin x + x\right)$$

$$= \frac{1}{D^2-6D+25}(e^{2x}) + \frac{1}{D^2-6D+25}(\sin x) + \frac{1}{D^2-6D+25}(x)$$

put $D = a$ $\Rightarrow D = 2$ put $D^2 = -a^2$, $D^2 = -1$ $D \to 1$, $D^2 = D^3 = \ldots = 0$

$$\therefore y_p = \frac{1}{4-12+25}e^{2x} + \frac{1}{-1-6D+25}(\sin x) + \frac{1}{25\left[\frac{D^2-6D}{25}+1\right]}(x)$$

$$= \frac{1}{17} e^{2x} + \frac{1 (\sin x)}{-6D+24} + \frac{1}{25} \cdot \left[1 + \left(\frac{D^2 - 6D}{25} \right) \right]^{-1} (x)$$

$$= \frac{1}{17} e^{2x} + \frac{1}{6} \cdot \frac{1}{4-D} \sin x + \frac{1}{25} \left[1 - \left(\frac{D^2 - 6D}{25} \right) \right] x$$

$$= \frac{1}{17} e^{2x} + \frac{1}{6} \cdot \frac{1}{4-D} \times \frac{4+D}{4+D} (\sin x) + \frac{1}{25} \left[1 - \left(\frac{0 - 6D}{25} \right) \right] x$$

$$= \frac{1}{17} e^{2x} + \frac{1}{6} \times \frac{4+D}{16 - D^2} (\sin x) + \frac{1}{25} \left[1 + \frac{6D}{25} \right] x$$

$$\text{Put } D^2 = -1$$

$$= \frac{1}{17} e^{2x} + \frac{1}{6} \cdot \frac{4+D}{16-(-1)} \sin x + \frac{1}{25} \left[x + \frac{6}{25} D(x) \right]$$

$$= \frac{1}{17} e^{2x} + \frac{1}{102} \left[4 \sin x + D (\sin x) \right] + \frac{1}{25} \left[x + \frac{6}{25} (1) \right]$$

$$y_p = \frac{1}{17} e^{2x} + \frac{1}{102} \left[4 \sin x + \cos x \right] + \frac{1}{25} \left(x + \frac{6}{25} \right)$$

$$= \quad =$$

$\therefore$ General Solution is $y = y_c + y_p$

4) Solve $(D^2 - 3D + 2) y = 2x^2$

*5) Solve $(D-2)^2 y = 8 (e^{2x} + \sin 2x + x^2)$

6) Solve $(D^3 - D) y = 2x + 1 + 4 \cos x + 2 e^x$

7) Solve $(D^2 + 1)^2 y = x^4 + 2 \sin x \cdot \cos 3x$

Model(iv) :- If $f(D)y = e^{ax} \cdot \sin bx$ (or) $e^{ax} \cos bx$ (or) $e^{ax} \cdot x^m$

$$P.I = y_p = \frac{1}{f(D)} e^{ax} \sin bx \text{ (or) } e^{ax} \cos bx \text{ (or) } e^{ax} x^m$$

Put $D = D + a$ then.

$$y_p = e^{ax} \cdot \frac{1}{f(D+a)} \sin bx \text{ (or) } \cos bx \text{ (or) } x^m$$

Continue the remaining Process of Model ② or Model ③

Q

1) solve $(D^2 - 2D + 4)y = e^x \cos x$

A $m = 1 \pm i\sqrt{3}$, $y_c = e^{1 \cdot x}[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x]$

$\alpha = 1$, $\beta = \sqrt{3}$

$$P.I = \frac{1}{D^2 - 2D + 4} e^x \cos x \quad ; \text{ put } D = D + a$$
$$D = D + 1$$

Then $P.I = e^x \cdot \dfrac{1}{(D+1)^2 - 2(D+1) + 4} \cos x = e^x \cdot \dfrac{1}{D^2 + 2D + 1 - 2D - 2 + 4}(\cos x)$

$$= e^x \cdot \frac{1}{D^2 + 3} \cos x \quad , \text{ Put } D^2 = -a^2$$
$$\Rightarrow D^2 = -1^2 = -1$$

$$\therefore y_p = e^x \cdot \frac{1}{-1+3}(\cos x) = e^x \cdot \frac{1}{2} \cos x = \frac{1}{2} e^x \cdot \cos x$$

General Solution is $y = y_c + y_p$

2) solve $\dfrac{d^2y}{dx^2} - 3 \cdot \dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

$m = 1, 2, \quad y_c = c_1 e^{x} + c_2 e^{2x}$

$$y_p = \frac{1}{D^2-3D+2}\left(xe^{3x} + \sin 2x\right) = \frac{1}{D^2-3D+2}\left(e^{3x}\cdot x\right) + \frac{1}{D^2-3D+2}(\sin 2x)$$

put $D = D+a$ $\Rightarrow D = D+3$ put $D^2 = -a^2$ $\Rightarrow D^2 = -2^2 = -4$

$$\therefore y_p = e^{3x}\cdot\frac{1}{(D+3)^2-3(D+3)+2}(x) + \frac{1}{-4-3D+2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{D^2+3D+2}(x) - \frac{1}{3D+2}(\sin 2x) \quad \left\langle \begin{matrix} D \to 1 \\ D^2 = D^3 = \dots = 0 \end{matrix} \right\rangle$$

$$= e^{3x}\cdot\frac{1}{2\left[\frac{D^2+3D}{2}+1\right]}(x) - \frac{1}{3D+2}\times\frac{3D-2}{3D-2}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\cdot\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x - \frac{(3D-2)}{9D^2-4}(\sin 2x)$$

put $D^2 = -4$

$$\therefore y_p = \frac{e^{3x}}{2}\left[1-\left(\frac{D^2+3D}{2}\right)\right] x - \frac{(3D-2)}{9(-4)-4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1-\left(\frac{3D}{2}\right)\right] x + \frac{(3D-2)(\sin 2x)}{40}$$

$$= \frac{e^{3x}}{2}\left(\frac{2-3D}{2}\right) x + \frac{1}{40}\left[3D(\sin 2x) - 2\sin 2x\right]$$

$$= \frac{e^{3x}}{4}\left(2x-3(1)\right) + \frac{1}{40}\left[3(2\cos 2x) - 2\sin 2x\right]$$

$$y_p = \frac{e^{3x}}{4}(2x-3) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

$\therefore$ Complete solution is $y = y_c + y_p$

3) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2^x + e^{-2x}\sin 2x$.

$m = -2, -3$, $y_c = c_1 e^{-2x} + c_2 e^{-3x}$

$P.I = y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+5D+6}\left(2^x + e^{-2x}\sin 2x\right)$

$\Rightarrow y_p = \frac{1}{D^2+5D+6}(2^x) + \frac{1}{D^2+5D+6}\left(e^{-2x}\sin 2x\right)$

$= \frac{1}{D^2+5D+6}\left(e^{\log 2^x}\right) + \frac{1}{D^2+5D+6}\left(e^{-2x}\cdot\sin 2x\right)$

$= \frac{1}{D^2+5D+6}\left(e^{x\cdot\log 2}\right) + \frac{1}{D^2+5D+6}\left(e^{-2x}\cdot\sin 2x\right)$

$= \frac{1}{D^2+5D+6} e^{(\log_e 2)x} + \frac{1}{D^2+5D+6}\left(e^{-2x}\cdot\sin 2x\right)$

put $D = a$ $\Rightarrow D = \log 2$

put $D = D+a$ $\Rightarrow D = D-2$

$\therefore y_p = \frac{1}{(\log 2)^2 + 5(\log 2) + 6}\left(e^{(\log 2)x}\right) + e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}(\sin 2x)$

$= \frac{1}{(\log 2)^2 + 5\log 2 + 6} e^{(\log 2)x} + e^{-2x}\cdot\frac{1}{D^2+D}(\sin 2x)$

put $D^2 = -a^2 = -4$

$= \frac{1}{(\log 2)^2 + 5\log 2 + 6} e^{(\log 2)x} + e^{-2x}\cdot\frac{1}{-4+D}\sin 2x$

$$= \frac{1}{(\log 2)^2 + 5(\log 2 + 6} \cdot e^{(\log 2)x} + e^{-2x} \cdot \frac{1}{D-4} \times \frac{D+4}{D+4}(\sin 2x)$$

$$= \frac{1}{(\log 2)^2 + 5\log 2 + 6} e^{(\log 2)x} + e^{-2x} \frac{(D+4)}{D^2 - 16} (\sin 2x)$$

put $D^2 = -4$

$$= \frac{1}{(\log 2)^2 + 5 \cdot \log 2 + 6} e^{(x\log 2)} + e^{-2x} \frac{D+4}{-4-16} (\sin 2x)$$

$$= \frac{1}{(\log 2)^2 + 5(\log 2) + 6} e^{(\log 2^x)} + e^{-2x} \cdot \frac{1}{-20} [D(\sin 2x) + 4\sin 2x]$$

$$y_p = \frac{1}{(\log 2)^2 + 5(\log 2) + 6} (2^x) - \frac{e^{-2x}}{20} (2\cos 2x + 4\sin 2x)$$

$$y_p = \frac{1}{(\log 2)^2 + 5(\log 2) + 6} (2^x) - \frac{e^{-2x}}{10} (\cos 2x + 2\sin 2x)$$

$\therefore$ The Complete solution is $y = y_c + y_p$

4) Solve $\frac{d^2y}{dx^2} - 4y = x \cdot \sinh x$

$\Rightarrow (D^2 - 4)y = x \cdot \sinh x$, $m = -2, 2$, $y_p = c_1 e^{-2x} + c_2 e^{2x}$

$$\therefore y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{D^2 - 4} (x \cdot \sinh x) = \frac{1}{D^2 - 4} \left(x \cdot \left(\frac{e^x - e^{-x}}{2} \right) \right)$$

$$= \frac{1}{D^2-4}\left(\frac{1}{2}e^{x}x\right) - \frac{1}{D^2-4}\left(\frac{1}{2}e^{-x}x\right)$$

Put $D = D+1$ $\Rightarrow D = D+1$ Put $D = D-1$ $\Rightarrow D = D-1$

$$= \frac{1}{2}e^{x}\cdot\frac{1}{(D+1)^2-4}(x) - \frac{1}{2}e^{-x}\cdot\frac{1}{(D-1)^2-4}(x)$$

$$= \frac{1}{2}e^{x}\cdot\frac{1}{D^2+2D-3}(x) - \frac{1}{2}e^{-x}\cdot\frac{1}{D^2-2D-3}(x)$$

$$= \frac{1}{2}e^{x}\cdot\frac{1}{-3\left[\frac{D^2+2D}{-3}+1\right]}(x) - \frac{1}{2}e^{-x}\cdot\frac{1}{-3\left[\frac{D^2-2D}{-3}+1\right]}(x)$$

$$= -\frac{e^{x}}{6}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1}(x) + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1}(x)$$

$$= -\frac{e^{x}}{6}\left[1+\left(\frac{D^2+2D}{3}\right)\right](x) + \frac{e^{-x}}{6}\left[1+\left(\frac{D^2-2D}{3}\right)\right](x) \quad \left\langle \begin{matrix} D \xrightarrow{x} 1 \\ D^2 = 0 \end{matrix} \right\rangle$$

$$= -\frac{e^{x}}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$$

$$= -\frac{e^{x}}{6}\left(x+\frac{2}{3}D(x)\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}D(x)\right)$$

$$y_p = -\frac{e^{x}}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$$

$$= \frac{1}{6}\left\{-e^{x}\cdot x - \frac{2}{3}e^{x} + xe^{-x} - \frac{2}{3}e^{-x}\right\} = \frac{1}{3}\left(\frac{-e^{x}+e^{-x}}{2}\right)x + \frac{1}{6}\times\frac{2}{3}\left(-e^{x}-e^{-x}\right)$$

$$= -\frac{1}{3}\left(\frac{e^{x}-e^{-x}}{2}\right)x + \frac{1}{6}\left(-\frac{2}{3}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)2$$

$\therefore y_p = \frac{-1}{3} x \cdot \sinh x - \frac{2}{9} \cosh x$

$\therefore$ General Solution is $y = y_c + y_p$

(5) Solve $(D^2+4D+3)y = e^{-x}\sin x + x e^{3x}$

$\underline{A}$ $m = -1, -3$, $y_c = c_1 e^{-x} + c_2 e^{-3x}$

$$y_p = \frac{1}{D^2+4D+3}\left(e^{-x}\sin x + x \cdot e^{3x}\right)$$

$$= \frac{1}{D^2+4D+3}\left(e^{-x} \cdot \sin x\right) + \frac{1}{D^2+4D+3}\left(e^{3x} \cdot x\right)$$

Put $D = D+a$ $\Rightarrow D = D-1$ Put $D = D+a \Rightarrow D = D+3$

$$\therefore y_p = e^{-x} \cdot \frac{1}{(D-1)^2+4(D-1)+3}(\sin x) + e^{3x} \cdot \frac{1}{(D+3)^2+4(D+3)+3}(x)$$

$$= e^{-x} \cdot \frac{1}{D^2-2D+1+4D-4+3}(\sin x) + e^{3x} \cdot \frac{1}{D^2+6D+9+4D+12+3}(x)$$

$$= e^{-x} \cdot \frac{1}{D^2+2D}(\sin x) + e^{3x} \cdot \frac{1}{D^2+10D+24}(x)$$

Put $D^2 = -a^2$ $\Rightarrow D^2 = -1$

$$\therefore y_p = e^{-x} \cdot \frac{1}{-1+2D}\sin x + e^{3x} \cdot \frac{1}{24\left[\frac{D^2+10D}{24}+1\right]}(x)$$

$$= e^{-x} \cdot \frac{1}{2D-1}(\sin x) + \frac{e^{3x}}{24}\left[1+\left(\frac{D^2+10D}{24}\right)\right]^{-1}(x)$$

$$= e^{-x} \cdot \frac{1}{2D-1} \times \frac{2D+1}{2D+1} (\sin x) + \frac{e^{3x}}{24}\left[1 - \left(\frac{D^2+10D}{24}\right)\right](x)$$

$\because x,\ D \to 1$

$D^2 = D^3 = \cdots = 0$

$$y_p = e^{-x} \frac{2D+1}{4D^2-1} (\sin x) + \frac{e^{3x}}{24}\left[1 - \frac{10D}{24}\right](x) \quad \left(\because D^2(x) = 0\right)$$

Put $D^2 = -1$

$$= e^{-x} \frac{2D+1}{4(-1)-1} (\sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12} D(x)\right)$$

$$= \frac{e^{-x}}{-5}\left[2D(\sin x) + \sin x\right] + \frac{e^{3x}}{24}\left[x - \frac{5}{12}(1)\right]$$

$$y_p = -\frac{e^{-x}}{5}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$$

$\therefore$ The General solution of (1) is $y = y_c + y_p$

6) solve $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$

& $m = 0, -1, -1$, $y_c = c_1 + (c_2 + c_3 x)e^{-x}$

$$y_p = \frac{1}{D^3+2D^2+D}\left(x^2 e^{2x} + \sin^2 x\right) = \frac{1}{D^3+2D^2+D}\left(e^{2x} x^2\right) + \frac{1}{D^3+2D^2+D}(\sin^2 x)$$

put $D = D + a$

$D = D + 2$

$$\therefore y_p = e^{2x} \frac{1}{(D+2)^3 + 2(D+2)^2 + (D+2)}(x^2) + \frac{1}{D^3+2D^2+D}\left(\frac{1-\cos 2x}{2}\right)$$

$$= e^{2x} \cdot \frac{1}{D^3 + 8D^2 + 21D + 18}(x^2) + \frac{1}{D^3+2D^2+D}\left(\frac{1}{2}e^{0x}\right) + \frac{1}{D D^2 + 2D^2 + D}\left(-\frac{1}{2}\cos 2x\right)$$

$D = a$

$D = 0$

$D^2 = -a^2$

$D^2 = -4$

(37)

$$= e^{2x} \cdot \frac{1}{18\left[\frac{D^3+8D^2+21D}{18}+1\right]}(x^2) + \frac{1}{2}x \cdot \frac{1}{f'(D)}(e^{0x}) + \frac{1}{D(-4)+2(-4)+D}\left(-\frac{1}{2}\cos 2x\right)$$

$$= \frac{e^{2x}}{18}\left[1+\left(\frac{D^3+8D^2+21D}{18}\right)\right]^{-1}(x^2) + \frac{1}{2}x \cdot \frac{1}{3D^2+4D+1}(e^{0x})$$

$$-\frac{1}{2} \cdot \frac{1}{-3D-8}(\cos 2x)$$

x^2, $D \to 2x$

$D^2 = 2$

$D^3 = D^4 = \dots = 0$

$$\therefore y_p = \frac{e^{2x}}{18}\left[1-\left(\frac{D^3+8D^2+21D}{18}\right)\right](x^2) + \frac{1}{2} \cdot x \cdot \frac{1}{0+0+1}(e^{0x}) - \frac{1}{2} \cdot \frac{1}{-(3D+8)}(\cos 2x)$$

$$= \frac{e^{2x}}{18}\left[1 - \frac{8D^2}{18} - \frac{21D}{18}\right](x^2) + \frac{1}{2}x \cdot \frac{1}{1} + \frac{1}{2}\frac{1}{3D+8} \times \frac{3D-8}{3D-8}(\cos 2x)$$

$$= \frac{e^{2x}}{18}\left(x^2 - \frac{4}{9}(2) - \frac{7}{6}(2x)\right) + \frac{x}{2} + \frac{1}{2}\frac{3D-8}{9D^2-64}(\cos 2x)$$

put $D^2 = -a^2 = -4$

$$\therefore y_p = \frac{e^{2x}}{18}\left(x^2 - \frac{8}{9} - \frac{7}{3}x\right) + \frac{x}{2} + \frac{1}{2}\frac{3D-8}{9(-4)-64}(\cos 2x)$$

$$= \frac{e^{2x}}{18}\left(x^2 - \frac{8}{9} - \frac{7x}{3}\right) + \frac{x}{2} + \frac{1}{2} \times \frac{1}{-100}\left[3D(\cos 2x) - 8\cos 2x\right]$$

$$= \frac{e^{2x}}{18}\left(x^2 - \frac{8}{9} - \frac{7x}{3}\right) + \frac{x}{2} + \frac{1}{-100} \times \frac{1}{2}\left[3(-2\sin 2x) - 8\cos 2x\right]$$

$$y_p = \frac{e^{2x}}{18}\left(x^2 - \frac{8}{9} - \frac{7x}{3}\right) + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)$$

$\therefore$ General solution is $y = y_c + y_p$

Model (v) :- If $Q(x) = x \cdot \sin ax$ (or) $x \cdot \cos ax$, then

P.I $= y_p = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} x \cdot \sin ax / \cos ax$

$\therefore y_p = \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} \sin ax$, if $f(-a^2) \neq 0$.

1) Solve $(D^2 + 2D + 1)y = x \cos x$

A. $m = -1, -1$, $y_c = (c_1 + c_2 x) e^{-x}$.

$y_p = \frac{1}{D^2+2D+1} (x \cos x) = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} \cos x$

$y_p = \left[x - \frac{2D+2}{D^2+2D+1}\right] \cdot \frac{1}{D^2+2D+1} \cos x$ ⟨ By Model 2 ⟩

Put $D^2 = -a^2 = -1$

$\therefore y_p = \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{1}{\cancel{-1}+2D+\cancel{1}} \cos x = \left[x - \frac{2D+2}{D^2+2D+1}\right] \cdot \frac{1}{2D} \cos x$

$= \left[x - \frac{2(D+1)}{(D+1)^2}\right] \cdot \frac{1}{2} (\sin x)$ $\left\langle \because \frac{1}{D} \cos x = \int \cos x\,dx = \sin x \right\rangle$

$= \left[x - \frac{2}{D+1}\right] \cdot \frac{1}{2} \sin x = x \cdot \left(\frac{1}{2} \sin x\right) - \cancel{2} \cdot \frac{1}{D+1} \left(\frac{1}{\cancel{2}} \sin x\right)$

$= \frac{x}{2} \sin x - \frac{1}{D+1} \times \frac{D-1}{D-1} \sin x = \frac{x}{2} \sin x - \frac{(D-1)}{D^2-1} \sin x$

Put $D^2 = -1$

$\therefore y_p = \frac{x}{2} \sin x - \frac{(D-1)}{-1-1} \sin x = \frac{x}{2} \sin x + \frac{1}{2}\left[D(\sin x) - \sin x\right]$

$y_p = \frac{x}{2} \sin x + \frac{1}{2}\left[\cos x - \sin x\right]$

$\therefore$ General solution is $y = y_c + y_p$

2) $\frac{d^2y}{dx^2} + 16y = x \sin 3x$

A: $m = 0 \pm 4i$ $\quad y_c = [c_1 \cos 4x + c_2 \sin 4x]$

$\downarrow \alpha \quad \downarrow \beta$

$$y_p = \frac{1}{D^2+16}(x \sin 3x) = \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{D^2+16}(\sin 3x)$$

$$= \left[x - \frac{2D}{D^2+16}\right] \cdot \frac{1}{D^2+16}(\sin 3x) \quad , \text{put } D^2 = -a^2 = -3^2 = -9$$

$$\therefore y_p = \left[x - \frac{2D}{D^2+16}\right] \cdot \frac{1}{-9+16}(\sin 3x) = \left[x - \frac{2D}{D^2+16}\right] \cdot \frac{1}{7}(\sin 3x)$$

$$= x \cdot \frac{1}{7} \cdot \sin 3x - \frac{2D}{D^2+16}\left(\frac{1}{7}\sin 3x\right) \quad ; \text{put } D^2 = -a^2 = -9$$

$$\therefore y_p = \frac{x}{7}\sin 3x - \frac{2D}{-9+16}\left(\frac{1}{7}\sin 3x\right) = \frac{x}{7}\sin 3x - \frac{2D}{7}\left(\frac{1}{7}\sin 3x\right)$$

$$= \frac{x}{7}\sin 3x - \frac{2}{49} \cdot D(\sin 3x) = \frac{x}{7}\sin 3x - \frac{2}{49}(3\cos 3x)$$

$$\therefore y_p = \frac{x}{7}\sin 3x - \frac{6}{49}\cos 3x$$

$\therefore$ The Complete solution is $y = y_c + y_p$.

3) Solve $(D^2 - 1)y = x \sin x + (1+x^2)e^x$

A: ~~$m = 0 \pm i$, $y_c = c_1 \cos x + c_2 \sin x$~~

$m = -1, 1$, $\quad y_c = c_1 e^{-x} + c_2 e^{x}$.

$$y_p = \frac{1}{D^2-1}\left(x \sin x + (1+x^2)e^x\right)$$

$$y_p = \frac{1}{D^2-1}(x\sin x) + \frac{1}{D^2-1}e^x(1+x^2)$$

put $D = D+a = D+1$

$$= \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}\sin x + e^x \cdot \frac{1}{(D+1)^2-1}(1+x^2)$$

$$= \left[x - \frac{2D}{D^2-1}\right]\frac{1}{D^2-1}(\sin x) + e^x \cdot \frac{1}{D^2+2D+\cancel{1}-\cancel{1}}(1+x^2)$$

put $D^2 = -1$

$\emptyset = 1+x^2 \Rightarrow D \to 2x$

$D^2 \to 2$

$D^3 = D^4 = \dots = 0$

$$= \left[x - \frac{2D}{D^2-1}\right] \cdot \frac{1}{-1-1}(\sin x) + e^x \cdot \frac{1}{D^2+2D}(x^2+1)$$

$$= x\left(-\frac{1}{2}\sin x\right) - \frac{\cancel{2}D}{D^2-1}\left(-\frac{1}{\cancel{2}}\sin x\right) + e^x \cdot \frac{1}{2D\left[\frac{D^2}{2D}+1\right]}(x^2+1)$$

put $D^2 = -1$

$$= -\frac{x}{2}\sin x + \frac{D}{-1-1}(\sin x) + \frac{e^x}{2D} \cdot \frac{1}{\left[1+\frac{D}{2}\right]}(x^2+1)$$

$$= -\frac{x}{2}\sin x + \frac{D}{-2}(\sin x) + \frac{e^x}{2} \cdot \frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(x^2+1)$$

$$= \frac{-x\sin x}{2} - \frac{1}{2}(\cos x) + \frac{e^x}{2}\frac{1}{D}\left[1 - \frac{D}{2} + \left(\frac{D}{2}\right)^2 - \dots\right](x^2+1)$$

$$= \frac{-x\sin x}{2} - \frac{1}{2}\cos x + \frac{e^x}{2} \cdot \frac{1}{D}\left(1 - \frac{D}{2} + \frac{D^2}{4}\right)(x^2+1)$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{2} \cdot \frac{1}{D}\left[x^2+1 - \frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{2}}{\cancel{4}_2}\right]$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{2} \cdot \frac{1}{D}\left(x^2+1-x+\frac{1}{2}\right)$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{2}\cdot\frac{1}{D}\left(x^2 - x + \frac{3}{2}\right)$$

$-3/4$

$$y_p = -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right]$$

$-\frac{3}{4}$

$\therefore$ General Solution is $y = y_c + y_p$

Model VI :-

1) Solve $(D^2 - 2D + 1)y = x e^x \sin x$

A.E. $m = 1, 1$, $y_c = (c_1 + c_2 x)e^x$

$$y_p = \frac{1}{D^2 - 2D + 1}(x e^x \sin x) = \frac{1}{D^2 - 2D + 1} e^x (x \sin x)$$

Put $D = D + a = D + 1$

$$\therefore y_p = e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 1}(x \sin x) = e^x \cdot \frac{1}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - \cancel{2} + \cancel{1}}(x \sin x)$$

$$= e^x \cdot \frac{1}{D^2}(x \sin x) = e^x\left[x - \frac{f'(D)}{f(D)}\right]\cdot\frac{1}{f(D)}(\sin x)$$

$$= e^x\left[x - \frac{2D}{D^2}\right]\cdot\frac{1}{D^2}\sin x, \quad \text{put } D^2 = -a^2 = -1$$

$$\therefore y_p = e^x\left[x - \frac{2}{D}\right]\cdot\frac{1}{-1}\sin x = -e^x\left[x\sin x - 2\cdot\frac{1}{D}(\sin x)\right]$$

$$y_p = -e^x\left[x\sin x - 2\cdot(-\cos x)\right] = -e^x\left[x\sin x + 2\cos x\right]$$

$\therefore$ The Complete solution is $y = y_c + y_p$

(2) Solve $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$

Sol: $m = 2, 2$, $y_c = (c_1 + c_2 x)e^{2x}$

$$\therefore y_p = \frac{1}{(D-2)^2}(8x^2 e^{2x} \sin 2x) = \frac{1}{(D-2)^2} 8 e^{2x}(x^2 \sin x)$$

$D = D + a = D + 2$

$$= 8 e^{2x} \cdot \frac{1}{(D+2-2)^2}(x^2 \sin 2x) = 8 e^{2x} \cdot \frac{1}{D^2}(x^2 \sin 2x)$$

$$= 8 e^{2x} \cdot \frac{1}{D}\left[\frac{1}{D}(x^2 \sin 2x)\right] = 8 e^{2x} \cdot \frac{1}{D}\left[\int \underset{u}{x^2} \underset{v}{\sin 2x}\, dx\right]$$

$$= 8 e^{2x} \cdot \frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - (2x)\left(-\frac{\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right) - 0\right]$$

x^2 +	$\sin 2x$
$2x$ −	$-\frac{\cos 2x}{2}$
2 +	$-\frac{\sin 2x}{4}$
0	$\frac{\cos 2x}{8}$

$$= 8 e^{2x} \cdot \frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]$$

$$= 8 e^{2x}\left[-\frac{1}{2}\int x^2 \cos 2x\, dx + \frac{1}{2}\int x \sin 2x\, dx + \frac{1}{4}\int \cos 2x\, dx\right]$$

x^2 +	$\cos 2x$
$2x$ −	$\frac{\sin 2x}{2}$
2 +	$-\frac{\cos 2x}{4}$
0	$-\frac{\sin 2x}{8}$

$$= 8 e^{2x}\left[-\frac{1}{2}\left[x^2 \frac{\sin 2x}{2} - (2x)\left(-\frac{\cos 2x}{4}\right) + 2\left(-\frac{\sin 2x}{8}\right)\right] + \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right) + 0\right] + \frac{1}{4}\left(\frac{\sin 2x}{2}\right)\right]$$

$$= 8 e^{2x}\left\{-\frac{1}{2}\left[\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right] + \frac{1}{2}\left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{4}\right] + \frac{1}{8}\sin 2x\right\}$$

$\therefore y_p = 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x - \cancel{\frac{1}{8}\sin 2x}\right.$

$\left. - \frac{x\cos 2x}{4} + \cancel{\frac{\sin 2x}{8}} + \frac{1}{8}\sin 2x\right]$

$= 8e^{2x}\left[-\frac{x^2}{4}\sin 2x + x\cos 2x\left(-\frac{1}{4}-\frac{1}{4}\right) + \frac{\sin 2x}{8}\right]$

$y_p = 8e^{2x}\left[-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{1}{8}\sin 2x\right]$

$\therefore$ The complete solution is $y = y_c + y_p$

3) solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$

& $m = -1, -2$

$y_p = \frac{e^x}{10}\left[\left(x - \frac{1}{10}\right)\sin x + \left(x - \frac{7}{10}\right)\cos x\right]$

Prob 1

4) solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Model VI :- Method of variation of parameters

This Method is used to solve the second order Linear differential Equation with constant coefficients.

Procedure

Step ①: The Second order Linear D.E is

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 \cdot y = \alpha(x) \longrightarrow ①$$

where P_1, P_2 are constants & $\alpha(x)$ is function in 'x' (or) constant.

$$\Rightarrow (D^2 + P_1 D + P_2) y = \alpha(x) \longrightarrow ②$$

Step ②: Let the C.F be is $y_c = C_1 \cdot u(x) + C_2 \cdot v(x)$.

Step ③: P.I is $y_p = -u \int \frac{v \cdot \alpha}{W} dx + v \cdot \int \frac{u \cdot \alpha}{W} dx$

where $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Step ④: The Complete solution is $y = y_c + y_p$

* ① Solve $(D^2 + a^2) y = \tan ax$, by using Method of variation of parameter

S: $(D^2 + a^2) y = \tan ax \longrightarrow ①$

$\therefore f(D) = D^2 + a^2, \quad \alpha = \tan ax.$

$\therefore$ A.E is $f(m) = 0 \Rightarrow m^2 + a^2 = 0$, Replace D by m,

we get $m^2 + a^2 = 0 \Rightarrow m^2 = -a^2 \Rightarrow m = \pm ai$

$\therefore m = 0 \pm ai$, $\quad$ C.F is $y_c = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$ 45

$\underset{\alpha}{\downarrow}$ $\quad \underset{\beta}{\downarrow}$

$\Rightarrow y_c = e^{0x}\left[c_1 \cos ax + c_2 \sin ax\right] = \underline{\underline{c_1 \cos ax + c_2 \sin ax}}$

compare with $y_c = c_1 u(x) + c_2 \cdot v(x)$.

$\therefore u(x) = \cos ax$, $v(x) = \sin ax$.

$$\therefore W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cdot\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax$$

$\Rightarrow W = a\left(\cos^2 ax + \sin^2 ax\right) = a(1) = \underline{\underline{a}}$.

$\longrightarrow = y_p = -u\int \frac{v\cdot\theta}{W}dx + v\int\frac{u\cdot\theta}{W}dx$

$$\therefore P.I = y_p = -\cos ax \cdot \int \frac{(\sin ax)\,\text{Tan}\,ax}{a}dx + \sin ax\int\frac{\cos ax\cdot \text{Tan}\,ax}{a}dx$$

$$y_p = -\frac{1}{a}\cos ax\cdot\int \sin ax\cdot\frac{\sin ax}{\cos ax}dx + \frac{\sin ax}{a}\int \cos ax\cdot\frac{\sin ax}{\cos ax}dx$$

$$= -\frac{1}{a}\cos ax\int\frac{\sin^2 ax}{\cos ax}dx + \frac{1}{a}\sin ax\int \sin ax\, dx$$

$$= -\frac{1}{a}\cos ax\int\frac{1-\cos^2 ax}{\cos ax}dx + \frac{1}{a}\sin ax\cdot\int \sin ax\, dx$$

$$= -\frac{1}{a}\cos ax\int(\sec ax - \cos ax)dx + \frac{1}{a}\sin ax\cdot\left(-\frac{\cos ax}{a}\right)$$

$$= -\frac{1}{a}\cos ax\left[\frac{\log(\sec ax + \text{Tan}\,ax)}{a} - \frac{\sin ax}{a}\right] + \frac{1}{a^2}\sin ax(-\cos ax)$$

$$= -\frac{1}{a^2}\cos ax\cdot\log(\sec ax + \text{Tan}\,ax) + \frac{1}{a^2}\cos ax\cdot\sin ax$$

$$-\frac{1}{a^2}\sin ax\cdot\cos ax$$

$\Rightarrow y_p = -\frac{1}{a^2}\cos ax \cdot \log(\sec ax + \tan ax)$

$\therefore$ The General solution is $y = y_c + y_p$

(2) Solve $(D^2+a^2)y = \sec ax$, by the Method of variation of parameters

Ans $m = 0 \pm ai$, $y_c = c_1 \cos ax + c_2 \sin ax$, $X = \sec ax$

$\therefore y_c = c_1 u(x) + c_2 \cdot v(x)$.

$\therefore u = \cos ax$, $v = \sin ax$.

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = \underline{\underline{a}}$$

$$\therefore P.I = -u \cdot \int \frac{v \cdot X}{W}\,dx + v \cdot \int \frac{u \cdot X}{W}\,dx$$

$$= -\cos ax \int \frac{\sin ax \cdot \sec ax}{a}\,dx + \sin ax \cdot \int \frac{\cos ax \cdot \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\cos ax \int \sin ax \cdot \frac{1}{\cos ax}\,dx + \frac{1}{a}\sin ax \int \cos ax \cdot \frac{1}{\cos ax}\,dx$$

$$= -\frac{1}{a}\cos ax \cdot \int \tan ax\,dx + \frac{1}{a}\sin ax \int dx$$

$$= -\frac{1}{a}\cos ax\left[-\frac{\log|(\cos ax)|}{a}\right] + \frac{1}{a}\sin ax\,(x)$$

$$y_p = \frac{1}{a^2}\cos ax \cdot \log|\cos ax| + \frac{1}{a}x \cdot \sin ax$$

$\therefore$ complete solution is $y = y_c + y_p$.

3) Using Method of variation of parameters,
Solve $(D^2+3D+2)y = e^{e^x}$

Ans: $m=-1,-2$, $y_c = c_1 e^{-x} + c_2 e^{-2x}$, $\phi(x) = e^{e^x}$

$\therefore y_c = c_1 \cdot u(x) + c_2 \cdot v(x)$

$u = e^{-x}$, $v = e^{-2x}$

$\therefore W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2\cdot e^{-2x} \end{vmatrix} = -2e^{-2x}(e^{-x}) - e^{-2x}(-e^{-x})$

$= -2e^{-3x} + e^{-3x} = \underline{\underline{-e^{-3x}}}$

P.I is $y_p = -u\int \frac{v\cdot\phi}{W}dx + v\int \frac{u\cdot\phi}{W}dx$

$\Rightarrow y_p = -e^{-x}\int \frac{e^{-2x}\cdot e^{e^x}}{-e^{-3x}}dx + e^{-2x}\int \frac{e^{-x}\cdot e^{e^x}}{-e^{-3x}}dx$

$= -e^{-x}\int e^{-2x}\cdot(-e^{3x})\cdot e^{e^x}dx + e^{-2x}\int e^{-x}(-e^{3x})\cdot e^{e^x}dx$

$= e^{-x}\int e^{x}\cdot e^{e^x}dx - e^{-2x}\int e^{2x}\cdot e^{e^x}dx$

$= e^{-x}\int e^{x}\cdot e^{e^x}dx - e^{-2x}\int e^{x}\cdot e^{x}\cdot e^{e^x}dx$

Put $e^x = t$ | Put $e^x = t$
$e^x dx = dt$ | $e^x dx = dt$

$\therefore y_p = e^{-x}\int e^t dt - e^{-2x}\int t\cdot e^t dt$

$= e^{-x}e^t - e^{-2x}(t-1)e^t = e^{-x}\cdot e^{e^x} - e^{-2x}(e^x-1)e^{e^x}$

$$y_p = e^{e^x}\left[e^{-x} - e^{-2x}\cdot e^{x} + e^{-2x}\right] = e^{e^x}\left[\cancel{e^{-x}} - \cancel{e^{-x}} + e^{-2x}\right]$$

$$\therefore y_p = e^{-2x}\cdot e^{e^x}$$

$\therefore$ The complete solution is $y = y_c + y_p$

4) Use Method of variation of parameter, solve

$$(D^2 - 2D)y = e^x \cdot \sin x$$

A: $m = 0, 2$, $y_C = c_1 + c_2 e^{2x}$, $\phi(x) = e^x \cdot \sin x$

$$y_c = c_1 \cdot u(x) + c_2 \cdot v(x)$$

$$\therefore u = 1, \quad v = e^{2x}$$

$$\therefore W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} - 0 = 2e^{2x}.$$

$$\therefore y_p = -u\cdot\int \frac{v\cdot\phi}{W}\,dx + v\int \frac{u\cdot\phi}{W}\,dx$$

$$= -1\cdot\int \frac{\cancel{e^{2x}}\cdot e^x\cdot\sin x}{2\cancel{e^{2x}}}\,dx + e^{2x}\int \frac{(1\cdot)\,e^x\cdot\sin x}{2e^{2x}}\,dx$$

$$= -\frac{1}{2}\int e^x \sin x\,dx + \frac{1}{2}\cdot e^{2x}\cdot\int e^{-x}\sin x\,dx$$

$$= -\frac{1}{2}\left[\frac{e^x}{1^2+1^2}(1\cdot\sin x - 1\cdot\cos x)\right] + \frac{1}{2}e^{2x}\left[\frac{e^{-x}}{(-1)^2+1^2}(-\sin x - \cos x)\right]$$

$$\left\langle \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right\rangle$$

$$\therefore y_p = -\frac{1}{2}\,\frac{e^x}{2}(\sin x - \cos x) + \frac{1}{2}\cdot e^{2x}\cdot\frac{e^{-x}}{2}\left[-(\sin x + \cos x)\right]$$

$$y_p = -\frac{1}{4}e^x(\sin x - \cos x) - \frac{1}{4}e^x(\sin x + \cos x)$$

$$= -\frac{1}{4}e^x\left[\sin x - \cancel{\cos x} + \sin x + \cancel{\cos x}\right] = -\frac{1}{4}e^x(2\sin x)$$

$$y_p = -\frac{1}{2}e^x \cdot \sin x$$

∴ The Complete solution is $y = y_c + y_p$

5) Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by Method of variation of parameter.

Sol: $m = -1, 1$, $y_c = c_1 e^{-x} + c_2 e^x$

$$\therefore u = e^{-x},\quad v = e^x,\quad X = \frac{2}{1+e^x}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = e^{-x}\cdot e^x + e^{-x}\cdot e^x = e^0 + e^0 = 1+1 = 2$$

$$y_p = -u\cdot\int \frac{v\cdot X}{W}\,dx + v\cdot\int \frac{u\cdot X}{W}\,dx$$

$$= -e^{-x}\int \frac{e^x \cdot \frac{\cancel{2}}{1+e^x}}{\cancel{2}}\,dx + e^x\int \frac{e^{-x}\cdot\frac{\cancel{2}}{1+e^x}}{\cancel{2}}\,dx$$

$$= -e^{-x}\int \frac{e^x}{1+e^x}\,dx + e^x\int \frac{e^{-x}}{1+e^x}\,dx$$

$$= -e^{-x}\cdot\log|1+e^x| + e^x\int \frac{1}{e^x(1+e^x)}\,dx$$

$$= -e^{-x}\cdot\log|1+e^x| + e^x\int\left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx$$

$$= -e^{-x}\cdot\log|1+e^x| + e^x\left[\int e^{-x}\,dx - \int\frac{1}{1+e^x}\,dx\right]$$

$$= -e^{-x}\cdot\log|1+e^x| + e^x\left[\frac{e^{-x}}{-1} - \int\frac{1}{e^x\left(\frac{1}{e^x}+1\right)}\,dx\right]$$

$$= -e^{-x} \cdot \log|1+e^x| + e^x\left[-e^{-x} + \int \frac{-e^{-x}}{1+e^{-x}}\,dx\right]$$

$$= -e^{-x} \cdot \log|1+e^x| + e^x\left[-e^{-x} + \log(1+e^{-x})\right]$$

$$\therefore y_p = -e^{-x} \cdot \log|1+e^x| - 1 + e^x \cdot \log|1+e^{-x}|$$

$\therefore$ The Complete Solution is $y = y_c + y_p$.

6) Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ by the Method of variation of parameters

A: $y_c = (c_1 + c_2 x)e^{3x}$, $y_p = -e^{3x}(\log x + 1)$

7) Solve $y'' - 2y' + y = e^x \cdot \log x$, by Method of variation of parameters

A: $y_c = (c_1 + c_2 x)e^x$, $y_p = \frac{1}{4} x^2 \cdot e^x (2\log x - 3)$

8) Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$ by the Method of variation of parameters

9) $\frac{d^2y}{dx^2} + a^2 y = \operatorname{cosec} ax$

10) $\frac{d^2y}{dx} + y = x \cdot \sin x$

11) $y'' - 2y' + 2y = e^x \cdot \tan x$.

12) $\frac{d^2y}{dx^2} + y = \sec x$.

(2)

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Homogeneous L.D.E

1) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = 0$ A $-3, -3$

2) $(D^3 + D^2 + 4D + 4)y = 0$ A $-1, \pm 2i$

3) $(D^4 - 4D^2 + 4)y = 0 \Rightarrow (D^2 - 2)^2 = 0$ A $\pm\sqrt{2}, \pm\sqrt{2}$

4) $\frac{d^4x}{dt^4} + 4x = 0$ A $-1 \pm i$ & $1 \pm i$

5) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0,\ x(0) = \frac{d(x(0))}{dt} = 2$

6) $(D^4 + 18D^2 + 81)y = 0$

7) $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$

Non - Homogeneous

M1 1) $(D^2 + 5D + 6)y = e^x$ A - $y_c =$ $\quad y_p = \frac{e^x}{12}$

2) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$ $\quad y_p = \frac{x}{9}e^{-2x} + \frac{x^2}{6} + \frac{1}{4}e^{-x}$

M2: 1) $(D^3+1)y = \cos(2x-1)$ A $y_p = \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$

2) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$ A $y_p = -\frac{x}{8}\sin 2x$

M3

1) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ A $y_p = \frac{x^3}{3} + 4x$

M4 1) $(D^2 - 2D + 4)y = e^x\cos x$ A $y_p = \frac{1}{2}e^x\cos x$

M5 1) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = (1-e^x)^2$ A $y_c = \frac{1}{2}(-1+\sqrt{3}i),\ y_p = 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$

M2 ① $y'' + 4y' + 4y = 3\sin x + 4\cos x$, $y(0) = 1$ & $y'(0) = 0$

A: roots $-2, -2$, $y_p = \sin x$, $c_1 = 1$, $c_2 = 1$

A: $(1+x)e^{-2x} + \sin x$

M5 ② $\dfrac{d^2y}{dx^2} - 4y = x\sinh x$, $y_p = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

M5 3) $(D^2 - 1)y = x\sin 3x + \cos x$ A. $m = \pm 1$,

A: $y_p = \frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{\cos x}{2}$

M5 ④ $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^{x}\sin x$ A: $m = 1, 1$

A: $y_p = -e^{x}(x\sin x + 2\cos x)$

Assignment questions

1) Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$
2) Solve $(D^4 - 1)y = \sin 2x$
3) Solve $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = x^2$
4) $(D+1)^3 y = x \cdot e^{-x}$
5) Solve $(D^2 - 1)y = xe^{x} + \cos^2 x$
6) $(D^4 - D^2)y = 2$
7) Solve $(D^2 + 9)y = x\sin x$
8) Solve $(D^3 + D)y = \sin(2x + 3)$
9) $\dfrac{d^4y}{dx^4} - y = \cos x \cdot \cosh x$
10) Solve $(D^2 + 9)y = \operatorname{cosec} 3x$ by the Method of Variation of Parameters.

Unit-II: Partial Differential Equations (1)

Def:- An equation containing the derivatives of one dependent variable with respect to two (or) more independent variables is called a PDE.

Ex (1):- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$, (2) $\frac{\partial^2 u}{\partial x^2} = c^2 \cdot \frac{\partial^2 u}{\partial y^2}$

Order of the PDE & degree of the PDE:-

(i) **Order:-** The order of the highest partial derivative involved in the given equation is called an order of PDE.

(ii) **Degree:-** The degree (The power) of the highest partial derivatives occurs in a given PDE is called the degree of the PDE.

Ex:

	Order	degree
1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$	1	1
2) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial y^2} = 0$	2	1
3) $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2u$	1	1

Formation of PDE :- The formation of PDE can be obtained by the following ways

i) By elimination of arbitrary constants

2) By elimination of arbitrary functions.

1) Formation of PDE by elimination of arbitrary constants :-

1) Consider $z(x, y)$ be a function of two independent variables x and y, which is defined by $z = f(x, y, a, b)$ where a, b are arbitrary constants

2) If the No: of arbitrary constants to be eliminated is equal to the Number of independent variables involved in the given relation. Then we get First order PDE.

3) If the No: of arbitrary constants to be eliminated is greater than the No: of independent variables, then we get a second (or) higher order PDE.

Notations :- (1) If $z = f(x, y)$ Then

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial^2 z}{\partial x^2} = r \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = s.$$

$$\frac{\partial z}{\partial y} = q \quad , \quad \frac{\partial^2 z}{\partial y^2} = t$$

Note :- The formation of PDE by eliminating the arbitrary constants from the given relation is ~~...~~ may not be unique.

1) Form a PDE by eliminating the arbitrary constants from the following relations.

i) $z = ax + by + a^2 + b^2$; a, b are arbitrary constants

ii) $z = ax + by + \frac{a}{b} - b$; a, b are " "

iii) $z = (x^2+a)(y^2+b)$; a, b are " "

H.W iv) $z = ax + a^2 y^2 + b$; a, b are " "

v) $(x-a)^2 + (y-b)^2 = r^2 - z^2$; a, b are " "

S (i) $z = ax + by + a^2 + b^2 \longrightarrow (1)$

$\therefore \frac{\partial z}{\partial x} = a \Rightarrow \boxed{p = a}$ & $\frac{\partial z}{\partial y} = b \Rightarrow \boxed{q = b}$

Substitute a & b values in (1), we get

$\therefore z = px + qy + p^2 + q^2$

(ii) $z = ax + by + \frac{a}{b} - b \longrightarrow (1)$

$\frac{\partial z}{\partial x} = a \Rightarrow \boxed{p = a}$ & $\frac{\partial z}{\partial y} = b \Rightarrow \boxed{q = b}$

$(1) \Rightarrow z = px + qy + \frac{p}{q} - b$

iii) $z = (x^2+a)(y^2+b) \longrightarrow (1)$

$\therefore \frac{\partial z}{\partial x} = (2x)(y^2+b) \Rightarrow p = (2x)(y^2+b)$

$\Rightarrow \frac{p}{2x} = y^2 + b \longrightarrow (2)$

& $\frac{\partial z}{\partial y} = (x^2+a)(2y) \Rightarrow q = (x^2+a)(2y)$

$\Rightarrow \frac{q}{2y} = x^2 + a \rightarrow (3)$

Substitute (2) & (3) in (1), we get

$(1) \Rightarrow z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right) \Rightarrow \underline{4xyz = pq}$

v) $(x-a)^2 + (y-b)^2 = r^2 - z^2 \rightarrow (1)$

$\Rightarrow z^2 = r^2 - (x-a)^2 - (y-b)^2 \rightarrow (1)$

Diff w.r.t x o.b.s of eqn (1), we get

$\therefore 2z \cdot \frac{\partial z}{\partial x} = 0 - 2(x-a)(1) - 0 \Rightarrow \cancel{2}zp = -\cancel{2}(x-a)$

$\Rightarrow x - a = -zp \rightarrow (2)$

$\therefore$ Diff partially the eqn (1) w.r.t 'y' o.b.s, we get

$2z \cdot \frac{\partial z}{\partial y} = 0 - 0 - 2(y-b)(1) \Rightarrow \cancel{2}zq = -\cancel{2}(y-b)$

$\Rightarrow y - b = -zq \rightarrow (3)$

Substitute (2) & (3) in eqn (1), we get

$(1) \Rightarrow (-pz)^2 + (-zq)^2 = r^2 - z^2$

$\Rightarrow p^2z^2 + q^2z^2 = r^2 - z^2 \Rightarrow p^2z^2 + z^2q^2 + z^2 = r^2$

$\Rightarrow z^2(p^2+q^2+1) = r^2$. which is the required PDE

6) $\log(az-1) = x + ay - b \rightarrow (1)$, a, b are arbitrary constants.

Diff w.r.t x partially o.b.s of (1), we get

$\frac{1}{az-1} \cdot \frac{\partial}{\partial x}(az-1) = 1 \Rightarrow \frac{1}{az-1}\left(a \cdot \frac{\partial z}{\partial x}\right) = 1$

$\Rightarrow \frac{1}{az-1} ap = 1$

$\Rightarrow ap = az - 1 \Rightarrow a(p-z) = -1 \Rightarrow a = \frac{-1}{p-z}$

$\Rightarrow a = \frac{1}{z-p} \rightarrow (2)$

Diff eqn (1) partially w.r.t "y" o.b.s, we get

$\frac{1}{az-1}\left(a \cdot \frac{\partial z}{\partial y}\right) = a \Rightarrow \frac{1}{az-1} \cancel{a} \cdot q = \cancel{a} \Rightarrow \frac{q}{az-1} = 1$

$\Rightarrow q = az - 1 \rightarrow (3)$.

Substitute (2) in (3), we get, $q = \frac{1}{z-p} \cdot z - 1$

$\Rightarrow q = \frac{\cancel{z} - \cancel{z} + p}{z-p} \Rightarrow q = \frac{p}{z-p} \Rightarrow \boxed{p = q(z-p)}$

7) $2z = (x+a)^{1/2} + (y-a)^{1/2} + b \rightarrow (1)$, where a, b are arbitrary constants.

A: Diff the eqn (1) partially w.r.t "x"

$\therefore 2 \cdot \frac{\partial z}{\partial x} = \frac{1}{2}(x+a)^{\frac{1}{2}-1} + 0 \Rightarrow 2 \cdot p = \frac{1}{2}(x+a)^{-1/2}$

$\Rightarrow 4p = \frac{1}{(x+a)^{1/2}} \Rightarrow (x+a)^{1/2} = \frac{1}{4p}$.

$\Rightarrow (x+a) = \frac{1}{16p^2} \Rightarrow a = \frac{1}{16p^2} - x \rightarrow (2)$

Also Diff the eqn (1) w.r.t y" partially, we get

$2 \cdot \frac{\partial z}{\partial y} = \frac{1}{2}(y-a)^{1/2 - 1} \Rightarrow 2q = \frac{1}{2}(y-a)^{-1/2}$

$\Rightarrow 4q = \frac{1}{(y-a)^{\frac{1}{2}}} \Rightarrow (y-a)^{1/2} = \frac{1}{4q}$

S.O.B.S, we get $(y-a) = \frac{1}{16q^2}$

$\Rightarrow a = y - \frac{1}{16q^2} \rightarrow (3)$

from (2) & (3), we get, $y - \frac{1}{16q^2} = \frac{1}{16p^2} - x$

$\Rightarrow x+y = \frac{1}{16p^2} + \frac{1}{16q^2} \Rightarrow x+y = \frac{1}{16}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)$

8) Form a PDE by eliminating arbitrary constants a, b from $z = a \cdot \log\left[\frac{b(y-1)}{1-x}\right]$

Sol: Given that $z = a \cdot \log\left[\frac{b(y-1)}{1-x}\right]$

$\Rightarrow z = a \cdot \left[\log b(y-1) - \log(1-x)\right] \longrightarrow (1)$

Diff eqn (1) partially w.r.t x o.b.s, we get

$\therefore \frac{\partial z}{\partial x} = a\left[0 - \frac{1}{1-x}(-1)\right] \Rightarrow p = \frac{a}{1-x}$

$\Rightarrow a = p(1-x) \longrightarrow (2)$

Diff eqn (1) w.r.t "y" partially, we get

$\therefore \frac{\partial z}{\partial y} = a\left[\frac{1}{b(y-1)} \cdot b(1) - 0\right] \Rightarrow q = \frac{a\cancel{b}}{\cancel{b}(y-1)} = \frac{a}{y-1}$

$\Rightarrow a = q(y-1) \longrightarrow (3)$

from (2) & (3), we get, $p(1-x) = q(y-1)$

$\Rightarrow p - px = -q + qy \Rightarrow -px - qy = -q - p$

$\Rightarrow px + qy = p + q$

9) Form a PDE by eliminating the arbitrary constants a, b from $(x-a)^2 + (y-b)^2 = z^2 \cdot \cot^2\alpha$, where α is a parameter.

A Given that $(x-a)^2+(y-b)^2 = z^2 \cdot \cot^2\alpha \rightarrow (1)$

Diff partially w.r.t "x" of eqn (1), o b.s, we get

$2(x-a)^1 \cdot (1) + 0 = 2z \cdot \cot^2\alpha \cdot \frac{\partial z}{\partial x} \Rightarrow \cancel{2}(x-a) = \cancel{2} z p \cot^2\alpha$

$\Rightarrow pz \cdot \cot^2\alpha = (x-a) \qquad \Rightarrow (x-a) = pz\cot^2\alpha \rightarrow (2)$

Diff eqn (1) partially w.r.t "y," we get

$0 + 2\cdot(y-b)(1) = 2z \cdot \frac{\partial z}{\partial y} \cdot \cot^2\alpha \Rightarrow \cancel{2}(y-b) = \cancel{2} z q \cdot \cot^2\alpha$

$\Rightarrow (y-b) = qz \cot^2\alpha \rightarrow (3)$

Substitute (2) & (3) in (1), we get

$(1) \Rightarrow (pz\cot^2\alpha)^2 + (qz \cdot \cot^2\alpha)^2 = z^2 \cdot \cot^2\alpha$

$\Rightarrow p^2z^2\cot^4\alpha + q^2z^2 \cdot \cot^4\alpha = z^2\cot^2\alpha$

$\Rightarrow \cancel{z^2} \cdot \cot^{\cancel{4}\,2}\alpha\left[p^2+q^2\right] = \cancel{z^2}\cot^{\cancel{2}}\alpha$

$\Rightarrow (p^2+q^2)\cot^2\alpha = 1 \Rightarrow p^2+q^2 = \frac{1}{\cot^2\alpha}$

$\Rightarrow p^2+q^2 = \tan^2\alpha.$

10) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a, b, c are arbitrary constants.

A Number arbitrary constants = 3

Number of Independent variable = 2,

$\therefore$ The No: of arbitrary constants to be eliminated is greater than the No: of Independent variables.

So we get second (or) higher order PDE.

$\dot{O}$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow z^2=c^2\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right] \rightarrow (1)$

Diff eqn (1) Partially w.r.t "x", we get

$$2 \cdot z\frac{\partial z}{\partial x}=c^2\left[0-\frac{1}{a^2}(2x)-0\right] \Rightarrow \cancel{2}zp=-\frac{c^2}{a^2}(\cancel{2}x)$$

$$\Rightarrow z \cdot p=-\frac{c^2}{a^2} \cdot x \longrightarrow (2)$$

Diff eqn (1) Partially w.r.t "y" o.b.s, we get

$$2z\frac{\partial z}{\partial y}=c^2\left[0-0-\frac{1}{b^2}(2y)\right] \Rightarrow \cancel{2}zq=-\frac{c^2}{b^2}(\cancel{2}y)$$

$$\Rightarrow z \cdot q=-\frac{c^2}{b^2} \cdot y \longrightarrow (3)$$

Diff eqn (2) partially w.r.t y o.b.s, we get

$$\frac{\partial}{\partial y}(zp)=-\frac{c^2}{a^2}\frac{\partial}{\partial y}(x) \Rightarrow z \cdot \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)+p \cdot \frac{\partial z}{\partial y}=0$$

$$\Rightarrow z \cdot \frac{\partial^2 z}{\partial y \partial x}+pq=0 \Rightarrow \boxed{z \cdot s+pq=0}$$

(OR)

Diff eqn (3) partially w.r.t "x" o.b.s, we get

$$\frac{\partial}{\partial x}(zq)=-\frac{c^2}{b^2}\frac{\partial}{\partial x}(y) \Rightarrow z \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)+q \cdot \frac{\partial z}{\partial x}=0$$

$$\Rightarrow z \cdot \frac{\partial^2 z}{\partial x \partial y}+qp=0 \Rightarrow \boxed{z \cdot s+pq=0}$$

H-W

ii) $(x-a)^2+(y-b)^2+z^2=c^2$, a, b, c are arbitrary constants

<Repeated>

2) Formation of PDE by eliminating arbitrary functions

If the order of the PDE is the number of arbitrary functions involved in the given relation.

1) Form a PDE by eliminating the arbitrary functions from the following relations.

(i) $z = f(x^2+y^2)$ (ii) $z = y\cdot f(x) + x\cdot g(y)$

(iii) $z = f(x) + e^{y}\cdot g(x)$ iv) $z = f(x+at) + g(x-at)$

Sol (i) $z = f(x^2+y^2) \rightarrow (1)$.

$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot(2x) \Rightarrow p = f'(x^2+y^2)\cdot 2x \rightarrow (2)$

& $\frac{\partial z}{\partial y} = f'(x^2+y^2)(2y) \Rightarrow q = f'(x^2+y^2)(2y) \rightarrow (3)$

$\therefore \frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{\cancel{f'(x^2+y^2)}\,\cancel{2}x}{\cancel{f'(x^2+y^2)}\,(\cancel{2}y)} \Rightarrow \frac{p}{q} = \frac{x}{y}$

$\Rightarrow \boxed{py = qx}$

(ii) $z = y\cdot f(x) + x\cdot g(y) \rightarrow (1)$

$\therefore \frac{\partial z}{\partial x} = y\cdot f'(x) + g(y) \Rightarrow p = y\cdot f'(x) + g(y) \rightarrow (2)$

$\frac{\partial z}{\partial y} = f(x) + x\cdot g'(y) \Rightarrow q = f(x) + x\cdot g'(y) \rightarrow (3)$.

$\frac{\partial^2 z}{\partial x^2} = y\cdot f''(x) + 0 \Rightarrow r = y\cdot f''(x) \rightarrow (4)$

$\frac{\partial^2 z}{\partial y^2} = \cancel{0} + x\cdot g''(y) \Rightarrow t = x\cdot g''(y) \rightarrow (5)$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 1 \cdot f'(x) + g'(y) \Rightarrow s = f'(x) + g'(y) \rightarrow (6)$$

$$\therefore px + qy = xy\, f'(x) + x \cdot g(y) + y \cdot f(x) + xy \cdot g'(y)$$

$$\Rightarrow px + qy = xy\left[f'(x) + g'(y)\right] + \left[x\, g(y) + y \cdot f(x)\right]$$

$$\Rightarrow \boxed{px + qy = xy \cdot s + z} \quad \langle \text{from } (1) \text{ \& } 6 \rangle$$

iii) $z = f(x) + e^{y} \cdot g(x) \rightarrow (1)$

$$\frac{\partial z}{\partial x} = f'(x) + e^{y} \cdot g'(x) \Rightarrow p = f'(x) + e^{y} \cdot g'(x) \rightarrow (2)$$

$$\text{\&} \quad \frac{\partial z}{\partial y} = 0 + e^{y} \cdot g(x) \Rightarrow q = e^{y} \cdot g(x) \rightarrow (3)$$

$$\text{\&} \quad \frac{\partial^2 z}{\partial y^2} = e^{y} \cdot g(x) \Rightarrow \boxed{t = q} \quad \langle \text{by } (3) \rangle$$

iv) $z = f(x+at) + g(x-at) \rightarrow (1)$

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at) \Rightarrow p = f'(x+at) + g'(x-at) \rightarrow (2)$$

$$\frac{\partial z}{\partial t} = f'(x+at) \cdot a + g'(x-at)(-a) \Rightarrow q = a\left[f'(x+at) - g'(x-at)\right] \rightarrow (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \rightarrow (4)$$

$$t = \frac{\partial^2 z}{\partial t^2} = a\left[f''(x+at) \cdot a - g''(x-at) \cdot (-a)\right]$$

$$t = a^2 \cdot \left[f''(x+at) + g''(x-at)\right]$$

$$\Rightarrow \boxed{t = a^2 \cdot r} \quad \langle \text{by } (4) \rangle$$

(iv) $z = (x+y)\,\phi(x^2-y^2) \longrightarrow (1)$

$$p = \frac{\partial z}{\partial x} = \cancel{(x+y)}\,\cancel{\phi'(x^2-y^2)(2x)} \quad \phi(x^2-y^2) = \frac{z}{x+y} \longrightarrow (2)$$

$q = \cancel{(x+y)}$

Diff. partially w.r.t x o.b.s, we get

$$\phi'(x^2-y^2)(2x) = \frac{(x+y)\frac{\partial z}{\partial x} - z(1)}{(x+y)^2}$$

$$\Rightarrow \phi'(x^2-y^2)\cdot(2x) = \frac{p(x+y)-z}{(x+y)^2} \longrightarrow (3)$$

Diff partially w.r.t "y" of eqn (2), we get

$$\phi'(x^2-y^2)(-2y) = \frac{(x+y)\frac{\partial z}{\partial y} - z(1)}{(x+y)^2}$$

$$\Rightarrow \phi'(x^2-y^2)(-2y) = \frac{(x+y)q - z}{(x+y)^2} \longrightarrow (4)$$

$$\frac{(3)}{(4)} \Rightarrow \frac{\phi'\cancel{(x^2-y^2)}(\cancel{2}x)}{\phi'\cancel{(x^2-y^2)}(-\cancel{2}y)} = \frac{\dfrac{p(x+y)-z}{\cancel{(x+y)^2}}}{\dfrac{q(x+y)-z}{\cancel{(x+y)^2}}}$$

$$\Rightarrow \frac{x}{-y} = \frac{p(x+y)-z}{q(x+y)-z} \Rightarrow qx(x+y) - xz = -py(x+y) + yz$$

$$\cancel{\Rightarrow qx^2 + qxy - xz + pxy + py^2 - yz = 0}$$

$$\cancel{\Rightarrow xy(p+q) - z(\quad)}$$

$$\Rightarrow qx(x+y) + py(x+y) = yz + zx$$

$$\Rightarrow \cancel{(x+y)}(qx+py) = z\cancel{(x+y)}$$

$\Rightarrow Py + qx = z$

which is the required PDE.

Formation of PDE by eliminating the arbitrary function from the relation of the form $\phi(u,v)=0$ (or) $u=\phi(v)$ (or) $v=\phi(u)$:-

The given relation is $\phi(u,v)=0$ (or) $u=\phi(v)$ (or) $v=\phi(u)$

where $u = u(x,y,z)$, $v = v(x,y,z)$.

$\therefore$ The equation of the required PDE is $Pp+Qq=R$

where $P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}$, $Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}$, $R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$

Problems

1) Form a PDE by eliminating the arbitrary function from the relation $\phi(x+y+z,\ x^2+y^2-z^2)=0$

Sol: Given $\phi(x+y+z,\ x^2+y^2-z^2)=0$

$\therefore u = x+y+z$, $v = x^2+y^2-z^2$

$u_x=1$, $u_y=1$, $u_z=1$, $v_x=2x$, $v_y=2y$, $v_z=-2z$

$\therefore P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} = \begin{vmatrix} 1 & 2y \\ 1 & -2z \end{vmatrix} = -2z-2y = -2(z+y)$

$Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix} = \begin{vmatrix} 1 & -2z \\ 1 & 2x \end{vmatrix} = 2x+2z = 2(x+z)$

$$R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} 1 & 2x \\ 1 & 2y \end{vmatrix} = 2y - 2x = 2(y-x)$$

$\therefore$ The required PDE is $pP + qQ = R$

$$\Rightarrow p\left(-2(z+y)\right) + q\left(2(x+z)\right) = 2(y-x)$$

$$\Rightarrow -(z+y)p + (x+z)q = y - x$$

H.W (ii) $xyz = f(x^2+y^2+z^2)$, H.W (iv) $f(x^2+y^2, z-xy)=0$

Ans $xq - yp = x^2 - y^2$

(iii) $\phi(x^2+y^2+z^2,\ z^2-2xy) = 0$

Let $u = x^2+y^2+z^2$, $\quad v = z^2 - 2xy$

$u_x = 2x \qquad v_x = 0 - 2y = -2y$

$u_y = 2y \qquad v_y = 0 - 2x = -2x$

$u_z = 2z \qquad v_z = 2z - 0 = 2z$

$$\therefore P = \begin{vmatrix} 2y & -2x \\ 2z & 2z \end{vmatrix} = 4yz + 4xz = 4z(x+y)$$

$$Q = \begin{vmatrix} 2z & 2z \\ 2x & -2y \end{vmatrix} = -4yz - 4zx = -4z(y+x)$$

$$R = \begin{vmatrix} 2x & -2y \\ 2y & -2x \end{vmatrix} = -4x^2 + 4y^2 = -4(x^2-y^2)$$

$\therefore$ The required PDE is $pP + qQ = R$

$$\Rightarrow 4z(x+y)p - 4z(y+x)q = -4(x^2-y^2)$$

$$\Rightarrow \cancel{4}z\left[(x+y)(p) - (y+x)q\right] = -\cancel{4}(x^2-y^2)$$

$$\cancel{\Rightarrow z(x+y) - z(y+x) + (x^2-y^2) = 0}$$

$$\Rightarrow z\cancel{(x+y)}(p-q) = -\cancel{(x+y)}(x-y)$$

$$\Rightarrow z(p-q) + (x-y) = 0$$

First order Linear PDE : A D.E consists of First order partial derivatives p and q which occurs First degree only and not multiplying together, is Called First order Linear PDE.

The standard form of first order Linear PDE is $Pp + Q.q = R \longrightarrow (1)$. This equation is Called Lagrange's Linear equation. where P, Q, R are functions of x, y and z.

Solution:- The complete solution of first order PDE is $\phi(u, v) = 0$. (or) $u = \phi(v)$ (or) $v = \phi(u)$

The solution of First order Linear PDE by the following Methods (i) By Method of grouping
(ii) By Method of Multipliers.

i) **Procedure to find the solution of First order Linear PDE by Method of grouping**

Step 1: consider the First order Linear PDE is $Pp + Qq = R \rightarrow (1)$

Step 2: First we write the Auxiliary (subsidiary) equations of (1) are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step 3: Take $\frac{dx}{P} = \frac{dy}{Q}$ (or) $\frac{dy}{Q} = \frac{dz}{R}$ (or) $\frac{dz}{R} = \frac{dx}{P}$

and solve them, we get $u = a$ (or) $v = b$ (or) $w = c$

where a, b, c are arbitrary constants.

step 4: Take any two of these solutions

step 5: The complete solution of given eqn (1) is

$$\phi(u, v) = 0 \text{ (or) } \phi(v, w) = 0 \text{ (or) } \phi(w, u) = 0$$

$\Rightarrow \quad \Rightarrow \quad =$

Problems

1) Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Given that $p\sqrt{x} + q\cdot\sqrt{y} = \sqrt{z} \rightarrow (1)$, which is in the form of $pP + q\cdot Q = R$.

$\therefore P = \sqrt{x}, \; Q = \sqrt{y}, \; R = \sqrt{z}$.

The Auxiliary equations of eqn (1) are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Take $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \Rightarrow \int x^{-\frac{1}{2}}dx = \int y^{-\frac{1}{2}}dy$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + a \Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + a \Rightarrow 2\sqrt{x} = 2\sqrt{y} + a$$

$$\Rightarrow 2\sqrt{x} - 2\sqrt{y} = a$$

Also Take $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \Rightarrow \int y^{-\frac{1}{2}}dy = \int z^{-\frac{1}{2}}dz$

$$\Rightarrow \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + b \Rightarrow \frac{y^{1/2}}{\frac{1}{2}} = \frac{z^{1/2}}{\frac{1}{2}} + b \Rightarrow 2\sqrt{y} = 2\sqrt{z} + b$$

$\Rightarrow 2\sqrt{y} - 2\sqrt{z} = b$

The complete solution of eqn (1) is $\phi(a, b) = 0$

$\Rightarrow \phi(2\sqrt{x} - 2\sqrt{y},\ 2\sqrt{y} - 2\sqrt{z}) = 0$

2) Solve $px + qy = z$

Sol: Given that $px + qy = z \rightarrow (1)$ which is a first order Linear PDE. $\therefore P = x,\ Q = y,\ R = z.$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Take $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \int \frac{1}{x}\,dx = \int \frac{1}{y}\,dy \Rightarrow \log x = \log y + \log a$

$\Rightarrow \log x - \log y = \log a \Rightarrow \log\left(\frac{x}{y}\right) = \log a \Rightarrow \boxed{\frac{x}{y} = a}$

Also take $\frac{dy}{y} = \frac{dz}{z} \Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz \Rightarrow \log y = \log z + \log b$

$\Rightarrow \log y - \log z = \log b \Rightarrow \log\left(\frac{y}{z}\right) = \log b \Rightarrow \boxed{\frac{y}{z} = b}$

$\therefore$ The complete solution of eqn (1) is $\phi(a, b) = 0$

$\Rightarrow \phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

3) Solve $p - q = \log(x+y)$.

Sol: which is a first order Linear PDE.

$\therefore P = 1,\ Q = -1,\ R = \log(x+y)$

$\therefore$ The auxiliary eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Take $\frac{dx}{1} = \frac{dy}{-1} \Rightarrow x = -y + a \Rightarrow \boxed{x+y = a}$

Take $\frac{dy}{-1} = \frac{dz}{\log(x+y)} = -\int dy = \frac{1}{\log(x+y)} \int dz.$

$\Rightarrow -y = \frac{1}{\log c} \cdot z + b$ $\langle \because$ Take $x+y = c \rangle$

$\Rightarrow y + \frac{z}{\log c} = -b \Rightarrow y + \frac{z}{\log(x+y)} = b^1 \ (\because -b = b')$

$\therefore$ The solution is $\phi(a, b') = 0 \Rightarrow \phi\left(x+y,\ y + \frac{z}{\log(x+y)}\right) = 0$

4) Solve $y^2 p - yxq = x(z - 2y)$

A: It is a First order Linear PDE.

$\therefore P = y^2, \ Q = -yx, \ R = x(z-2y)$

The Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-yx} = \frac{dz}{x(z-2y)}$.

Take $\frac{dx}{\cancel{y}y} = \frac{dy}{-\cancel{y}x} \Rightarrow \frac{dx}{y} = \frac{dy}{-x} \Rightarrow x\,dx = -y\,dy$

$\therefore \int x\,dx = -\int y\,dy \Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + a \Rightarrow \boxed{\frac{x^2}{2} + \frac{y^2}{2} = a}$

Also $\Rightarrow$ Take $\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)} \Rightarrow z\,dy - 2y\,dy = -y\,dz$

$\Rightarrow y\,dz + z\,dy = 2y\,dy \Rightarrow d(yz) = 2y\,dy$

$\int d(yz) = 2\int y\,dy \Rightarrow yz = \cancel{2}\cdot\frac{y^2}{\cancel{2}} + b$

$\Rightarrow \boxed{yz - y^2 = b} \Rightarrow y(z-y) = b$

$\therefore$ The Complete solution is $\phi(a, b) = 0$

$$\Rightarrow \phi\left(\frac{x^2+y^2}{2},\ y(z-y)\right) = 0$$

5) Solve $x^2 p + y^2 q = z^2$ A: $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

6) Solve $yzp + zxq = xy$ A: $\phi(x^2 - y^2, y^2 - z^2) = 0$

7) Solve $\frac{y^2 z}{x} p + xzq = y^2$

A: given that $\frac{y^2 z}{x} p + xzq = y^2$

$\Rightarrow y^2 z p + x^2 z q = xy^2$. This is a First order PDE.

$\therefore P = y^2 z,\ Q = x^2 z,\ R = xy^2$

$\therefore$ The Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Take $\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}} \Rightarrow x^2 dx = y^2 dy \Rightarrow \int x^2 dx = \int y^2 dy$

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + a \Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = a \Rightarrow \frac{x^3 - y^3}{3} = a \Rightarrow x^3 - y^3 = 3a$

$\Rightarrow \boxed{x^3 - y^3 = a'}$ $\langle \because a' = 3a \rangle$

Also Take $\frac{dx}{\cancel{y^2} z} = \frac{dz}{x\cancel{y^2}} \Rightarrow x\,dx = z\,dz \Rightarrow \int x\,dx = \int z\,dz$

$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + b$

$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = b \Rightarrow \frac{x^2 - z^2}{2} = b \Rightarrow x^2 - z^2 = 2b$

$\Rightarrow \boxed{x^2 - z^2 = b'} \quad \langle \because b' = 2b \rangle$

$\therefore$ The Complete solution is $\phi(a', b') = 0$

$$\Rightarrow \phi(x^3 - y^3, x^2 - z^2) = 0$$

ii) Solution of First order Linear PDE by The Method of Multipliers: By a proper choice of multipliers l, m, n, which are not necessary constants, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ such that $lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$, which can be solved. i.e $\frac{l\,dx + m\,dy + n\,dz}{0}$ = each fraction

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$. Now Take integration o.b.s, we get $\phi(x, y, z) = a \rightarrow (1)$

Again search for another set of multipliers l', m', n' such that $l'P + m'Q + n'R = 0$ then $l'dx + m'dy + n'dz = 0$. i.e $\frac{l'dx + m'dy + n'dz}{0}$ = each fraction

$\Rightarrow l'dx + m'dy + n'dz = 0$. Now Take integration o.b.s, we get $\psi(x, y, z) = b \rightarrow (2)$

$\therefore$ The complete solution is $f(a, b) = 0$

Problems

1) Solve $(z-y)p + (x-z)q = y - x$

Sol: It is a First order Linear PDE.

$\therefore P = z - y, \quad Q = x - z, \quad R = y - x.$

$\therefore$ The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\therefore \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \rightarrow (1)$$

Case (i): Take The multipliers l, m, n are $1, 1, 1$

$$\therefore (1) \Rightarrow \frac{l\,dx + m\,dy + n\,dz}{l(z-y) + m(x-z) + n(y-x)} = \text{each fraction}$$

$$\Rightarrow \frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{\cancel{z} - \cancel{y} + \cancel{x} - \cancel{z} + \cancel{y} - \cancel{x}} = \text{each fraction}$$

$$\Rightarrow \frac{dx + dy + dz}{0} = \text{each fraction}$$

$\Rightarrow dx + dy + dz = 0$, Integrating o.b.s, we get

$\therefore \int dx + \int dy + \int dz = \int 0 \Rightarrow x + y + z = 0 + a$

$\Rightarrow \boxed{x + y + z = a}$

Case (ii): Take l, m, n are x, y, z Then

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \text{each fraction}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x(z-y)+y(x-z)+z(y-x)} = \text{each fraction}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} = \text{each fraction}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$. Integrating o.b.s, we get

$\therefore \int x\,dx + \int y\,dy + \int z\,dz = \int 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$

$\Rightarrow \frac{x^2+y^2+z^2}{2} = c \Rightarrow x^2+y^2+z^2 = 2c$

$\Rightarrow \boxed{x^2+y^2+z^2 = b}$ ⟨Take $2c = b$⟩

$\therefore$ The Complete solution is $f(a, b) = 0$

$\Rightarrow f(x+y+z,\ x^2+y^2+z^2) = 0$

2) solve $px(y-z) + y(z-x)q = z(x-y)$

A It is a First order PDE.

$\therefore P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$

$\therefore$ The subsidiary eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ → (1)

Take

Case (i): l, m, n are $1, 1, 1$.

(1) $\Rightarrow \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \text{each fraction}$

$$\Rightarrow \frac{1\cdot dx + 1\cdot dy + 1\cdot dz}{1\cdot x(y-z) + 1\cdot y(z-x) + 1\cdot z(x-y)} = \text{each fraction}$$

$$\Rightarrow \frac{dx + dy + dz}{0} = \text{each fraction}$$

$\Rightarrow dx + dy + dz = 0 \Rightarrow \int dx + \int dy + \int dz = \int 0$

$\Rightarrow x + y + z = 0 + a \Rightarrow \boxed{x + y + z = a}$

Case (ii): Take l, m, n are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$\therefore ① \Rightarrow \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \text{each fraction}$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y-z) + \frac{1}{y}y(z-x) + \frac{1}{z}z(x-y)} = \text{each fraction}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = \text{each fraction}$$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\therefore \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$

$\Rightarrow \log x + \log y + \log z = 0 + \log b$

$\Rightarrow \log(xyz) = \log b \Rightarrow \boxed{xyz = b}$

$\therefore$ The complete solution is $f(a, b) = 0$

$\Rightarrow f(x + y + z, xyz) = 0$

3) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol: This is a First Order Linear PDE.

$\therefore P = x^2(y-z),\ Q = y^2(z-x),\ R = z^2(x-y)$

$\therefore$ The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \longrightarrow ①$$

Case (i): Take l, m, n are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \text{each fraction} \longrightarrow (1')$$

$$(1') \Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\cdot x^2(y-z) + \frac{1}{y}\cdot y^2(z-x) + \frac{1}{z^2}\cdot z^2(x-y)} = \text{each fraction.}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = \text{each fraction}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0,$$ Integrate o.b.s, we get

$$\therefore \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0$$

$$\Rightarrow \log x + \log y + \log z = 0 + \log a \Rightarrow \log(xyz) = \log a$$

$$\Rightarrow \boxed{xyz = a}$$

Case (ii): Take l, m, n are $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, we get

$$(1) \Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x^2}x^2(y-z) + \frac{1}{y^2}\cdot y^2(z-x) + \frac{1}{z^2}\cdot z^2(x-y)} = \text{each fraction}$$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0} = \text{each fraction}$$

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$\therefore$ Integrate o.b.s, we get

$$\therefore \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = \int 0$$

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0 + b$$

$$\Rightarrow -\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=b \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-b$$

$$\Rightarrow \boxed{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c} \quad \langle \because c=-b \rangle$$

$\therefore$ The complete solution is $f(a, c)=0$

$$\Rightarrow f\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$$

4) Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$

Sol: It is a first order Linear PDE.

$\therefore P = x^2-y^2-z^2, \; Q = 2xy, \; R = 2xz$

$\therefore$ The Auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR} \longrightarrow (1)$$

<u>case i)</u>: <u>Grouping</u>: $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

Take $\frac{dy}{\cancel{2x}y}=\frac{dz}{\cancel{2x}z} \Rightarrow \frac{dy}{y}=\frac{dz}{z}$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz \Rightarrow \log y = \log z + \log a$$

$$\Rightarrow \log y = \log za \Rightarrow y = za \Rightarrow \boxed{\frac{y}{z}=a}$$

<u>case(ii)</u>: Take l, m, n are x, y, z Then we get

$$(1) \Rightarrow \frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR} = \text{each fraction.}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2)+y(2xy)+z(2xz)} = \text{each fraction} = \frac{dz}{2xz}$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{dz}{2xz}$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2} = \frac{dz}{2xz} \Rightarrow \frac{x\,dx+y\,dy+z\,dz}{\cancel{x}(x^2+y^2+z^2)} = \frac{dz}{2\cancel{x}z}$$

$$\Rightarrow \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\Rightarrow d\left[\log(x^2+y^2+z^2)\right] = \frac{1}{z}dz$$

Now integrate o.b.s, we get

$$\therefore \int d\left[\log(x^2+y^2+z^2)\right] = \int \frac{1}{z}\,dz$$

$$\Rightarrow \log(x^2+y^2+z^2) = \log z + \log b$$

$$\Rightarrow \log(x^2+y^2+z^2) = \log(zb) \Rightarrow x^2+y^2+z^2 = zb$$

$$\Rightarrow \boxed{\frac{x^2+y^2+z^2}{z} = b}$$

∴ The complete solution is $f(a, b) = 0$

$$\Rightarrow f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

First order Non-Linear PDE :- A PDE consists of first order partial derivatives p & q, which occurs other than in the first degree and multiplying together, is called first order Non-Linear PDE.

Solution of First order Non-Linear PDE by Charpit's Method

step ① :- consider the first order non-linear PDE is of the form $f(x, y, z, p, q) = 0 \rightarrow$ ①

step 2 :- since z depends on x & y i.e $z = z(x,y)$

$\therefore$ By total derivative, $dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$

$\Rightarrow dz = p\,dx + q\,dy \rightarrow$ ②

step ③ :- The subsidiary equations of ① are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\cdot\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\cdot\frac{\partial f}{\partial z}} \rightarrow ③$$

An integral of these equations involving p or q or both, can be taken as relation ④, which substitute in eqn ①.

which gives the values of p and q. Substitute p and q values in eqn (2) and solve it, then we get the required solution.

Problems

1) Solve $2xz - px^2 - 2qxy + pq = 0$.

A: It's a first order non-linear PDE.

Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0 \quad \rightarrow (1)$

$\therefore \frac{\partial f}{\partial x} = 2z - 2xp - 2qy, \quad \frac{\partial f}{\partial y} = 0 - 0 - 2qx = -2qx$

$\frac{\partial f}{\partial z} = 2x - 0 - 0 + 0 = 2x, \quad \frac{\partial f}{\partial p} = 0 - x^2 - 0 + q = -x^2 + q$

$\frac{\partial f}{\partial q} = 0 - 0 - 2xy + p = -2xy + p$

$\therefore$ The subsidiary equations of eqn (1) are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0} \quad \rightarrow (2)$$

Take $\frac{dp}{2z - 2qy} = \frac{dq}{0} \Rightarrow dq = \frac{0}{2z - 2qy} = 0$

$\Rightarrow dq = 0 \Rightarrow \int dq = \int 0 \Rightarrow q = 0 + a \Rightarrow \boxed{q = a} \quad \rightarrow (3)$

Substitute q in (1), we get (1) $\Rightarrow 2xz - px^2 - 2axy + ap = 0$

$\Rightarrow 2xz - p(x^2 - a) - 2axy = 0$

$\Rightarrow 2x(z - ay) = p(x^2 - a) \Rightarrow \boxed{p = \frac{2x(z - ay)}{x^2 - a}}$

Then By total derivative, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{2x(z-ay)}{x^2-a}dx + a\,dy$$

$$\Rightarrow dz - a\,dy = \frac{2x(z-ay)}{x^2-a}dx \Rightarrow \frac{dz - a\,dy}{z-ay} = \frac{2x}{x^2-a}dx$$

$$\Rightarrow d[\log(z-ay)] = d[\log(x^2-a)]$$

on Integrating o.b.s, we get

$$\int d[\log(z-ay)] = \int d[\log(x^2-a)]$$

$$\Rightarrow \log(z-ay) = \log(x^2-a) + \log b$$

$$\Rightarrow \log(z-ay) = \log[(x^2-a)\cdot b]$$

$$\Rightarrow z-ay = (x^2-a)b$$

$$\Rightarrow z = ay + b(x^2-a)$$ which is the required solution.

2) Solve $(p^2+q^2)y = qz$

A: It's a first order Non-Linear PDE.

Let $f = (p^2+q^2)y - qz = 0 \rightarrow (1)$

$$\frac{\partial f}{\partial x} = 0-0=0,\quad \frac{\partial f}{\partial y} = p^2+q^2-0 = p^2+q^2$$

$$\frac{\partial f}{\partial z} = 0-q = -q,\quad \frac{\partial f}{\partial p} = (2p+0)y - 0 = 2py$$

$$\frac{\partial f}{\partial q} = (0+2q)y - z = 2qy - z$$

$\therefore$ The subsidiary equations are:

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} \rightarrow (2)$$

$$\Rightarrow \frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

Take $\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p\,dp = -q\,dq$

$$\Rightarrow \int p\,dp = -\int q\,dq \Rightarrow \frac{p^2}{2} = -\frac{q^2}{2} + a$$

$$\Rightarrow \frac{p^2}{2} + \frac{q^2}{2} = a \Rightarrow p^2 + q^2 = 2a$$

$$\Rightarrow p^2 + q^2 = c^2 \quad \langle \text{where } 2a = c^2 \rangle$$
$\rightarrow (3)$

Substitute eqn (3) in (1), we get, (1) $\Rightarrow c^2 y - qz = 0$

$$\Rightarrow qz = c^2 y \Rightarrow \boxed{q = \frac{c^2 y}{z}}$$

Substitute q value in eqn (3), we get

$$(3) \Rightarrow p^2 + \left(\frac{c^2 y}{z}\right)^2 = c^2 \Rightarrow p^2 = c^2 - \frac{c^4 y^2}{z^2}$$

$$\Rightarrow p^2 = \frac{c^2 z^2 - c^4 y^2}{z^2} \Rightarrow p^2 = \frac{c^2(z^2 - c^2 y^2)}{z^2}$$

$$\Rightarrow p = \frac{c}{z}\sqrt{z^2 - c^2 y^2}$$

By Total derivative, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{c}{z}\sqrt{z^2 - c^2y^2}\,dx + \frac{c^2y}{z}\,dy$$

$$\Rightarrow dz = \frac{c\sqrt{z^2-c^2y^2}\,dx + c^2y\,dy}{z}$$

$$\Rightarrow z\,dz = c\sqrt{z^2-c^2y^2}\,dx + c^2y\,dy$$

$$\Rightarrow z\,dz - c^2y\,dy = c\sqrt{z^2-c^2y^2}\,dx$$

$$\Rightarrow \frac{2z\,dz - c^2(2y)\,dy}{2} = c\sqrt{z^2-c^2y^2}\,dx$$

$$\Rightarrow \frac{1}{2}d(z^2 - c^2y^2) = c\sqrt{z^2-c^2y^2}\,dx$$

$$\Rightarrow \frac{\frac{1}{2}\cdot d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = c\,dx \Rightarrow d\left[\sqrt{z^2-c^2y^2}\right] = c\,dx$$

Integrate o.b.s. we get

$$\int d\left[\sqrt{z^2-c^2y^2}\right] = c\int dx \Rightarrow \sqrt{z^2-c^2y^2} = cx+a$$

$$\Rightarrow z^2 - c^2y^2 = (cx+a)^2$$

$$\Rightarrow z^2 = (cx+a)^2 + c^2y^2$$

which is the required solution.

3) solve $2z + p^2 + qy + 2y^2 = 0$.

Sol: It is a First order Non-Linear PDE

Let $f(x,y,z,p,q) = 2z + p^2 + qy + 2y^2 = 0 \longrightarrow (1)$

$\therefore \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 + 0 + q + 4y \Rightarrow \frac{\partial f}{\partial y} = q + 4y$

$\frac{\partial f}{\partial z} = 2 + 0 + 0 + 0 = 2, \quad \frac{\partial f}{\partial p} = 0 + 2p + 0 + 0 = 2p$

$\frac{\partial f}{\partial q} = 0 + 0 + y + 0 = y.$

The subsidiary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} \longrightarrow (2)$$

$$\Rightarrow \frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first & fourth ratio, $\frac{dx}{-p} = \frac{dp}{2p} \Rightarrow dp = -dx$

$\Rightarrow \int dp = -\int dx \Rightarrow \boxed{p = -x + a}$

Substitute $p = -x + a$ in (1), we get

$\therefore (1) \Rightarrow 2z + (-x+a)^2 + qy + 2y^2 = 0$

$\Rightarrow qy = -2z - 2y^2 - (a-x)^2$

$\Rightarrow \boxed{q = -\frac{1}{y}\left[2z + 2y^2 + (a-x)^2\right]}$

By total derivative, $dz = p\,dx + q\,dy \longrightarrow (3)$

substitute p & q values in eqn (3), we get

$$\therefore (3) \Rightarrow dz = (a-x)\,dx - \frac{1}{y}\left[2z + 2y^2 + (a-x)^2\right]dy$$

Multiplying both sides by "$2y^2$".

$$\therefore 2y^2\,dz = 2y^2(a-x)\,dx - 2y^2 \cdot \frac{1}{y}\left[2z + 2y^2 + (a-x)^2\right]dy$$

$$\Rightarrow 2y^2\,dz + 4yz\,dy = 2y^2(a-x)\,dx - 4y^3\,dy - 2y(a-x)^2\,dy$$

$$\Rightarrow d\left[2zy^2\right] = d\left[-\left(y^2(a-x)^2 + y^4\right)\right]$$

$$\therefore \int d\left[2zy^2\right] = \int d\left[-\left(y^2(a-x)^2 + y^4\right)\right]$$

$$\Rightarrow 2zy^2 = -\left[y^2(a-x)^2 + y^4\right] + b$$

$$\Rightarrow y^2\left[(a-x)^2 + 2z + y^2\right] = b$$

which is the desired solution.

4) Solve $q + xp = p^2$

Method of separation of variables :- This method is suitable to solve the boundary value problems. In this method the PDE can be converted into ODEs.

Procedure :- 1) Let us assume that $U = X(x) \cdot Y(y)$ (i.e $U = X \cdot Y$) is the complete solution of the given PDE.

2) Find the first & second order partial derivatives of $U = X \cdot Y$ and substitute these partial derivatives in the given PDE.

3) The PDE obtained in step (2) split into two ODEs w.r.t independent variables that are involved.

4) Solve the ODEs for X & Y and write the complete solution of the form $U = X \cdot Y$.

Problems

1) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ with $u(x,0) = 6e^{-3x}$ by the method of separation of variables

S Let $u = X(x) \cdot T(t)$ be the solution of given PDE.

Given that $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow (1)$

$\therefore u = X(x)\, T(t)$

$\frac{\partial u}{\partial x} = X^1 \cdot T, \quad \frac{\partial u}{\partial t} = X T^1, \quad u = X \cdot T$

$(1) \Rightarrow X^1 \cdot T = 2 X T^1 + X T = X(2T^1 + T)$

$\Rightarrow X^1 T = X(2T^1 + T) \Rightarrow \frac{X^1}{X} = \frac{2T^1 + T}{T}$

$\therefore \frac{X^1}{X} = \frac{2T^1 + T}{T} = \lambda \text{ (say)}$

Take $\frac{X^1}{X} = \lambda \longrightarrow (2)$

$\Rightarrow X^1 = \lambda X$

$\Rightarrow X^1 - \lambda X = 0 \longrightarrow (2)$

$D - \lambda = 0$

$\therefore$ A.E is $m - \lambda = 0$

$\Rightarrow \boxed{m = \lambda}$

$\therefore C.F = c_1 e^{mx}$

$C.F = c_1 e^{\lambda x}$

$\therefore$ The solution of (2) is

$X = C.F$

$\Rightarrow \boxed{X = c_1 e^{\lambda x}}$

Take $\frac{2T^1 + T}{T} = \lambda$

$\Rightarrow 2T^1 + T = \lambda T$

$\Rightarrow 2T^1 + (1-\lambda) T = 0 \longrightarrow (3)$

A.E is $2D + (1-\lambda) = 0$

$\therefore 2m + (1-\lambda) = 0$

$\Rightarrow 2m = \lambda - 1$

$\boxed{m = \frac{\lambda - 1}{2}}$

$C.F = c_2 e^{mt}$

$\Rightarrow C.F = c_2 e^{\left(\frac{\lambda-1}{2}\right)t}$

$\therefore$ The solution of eqn (3) is

$T = C.F$

$\Rightarrow \boxed{T = c_2 e^{\left(\frac{\lambda-1}{2}\right)t}}$

The solution of eqn (1) is

$u = XT$

$\therefore u = X \cdot T = C_1 e^{\lambda x} \cdot C_2 e^{\left(\frac{\lambda-1}{2}\right)t}$

$\Rightarrow \boxed{u = A \cdot e^{\lambda x} \cdot e^{\left(\frac{\lambda-1}{2}\right)t}} \longrightarrow (4)$ $\langle$ where $C_1 \cdot C_2 = A \rangle$

Given that $u(x, 0) = 6 \cdot e^{-3x}$ i.e $u = 6 \cdot e^{-3x}$ at $t = 0$

$\therefore (4) \Rightarrow 6 e^{-3x} = A \cdot e^{\lambda x}$

on comparing, we get $\boxed{A = 6}$ & $\boxed{\lambda = -3}$

Substitute A & λ values in eqn-(4), we get

$(4) \Rightarrow u = 6 \cdot e^{-3x} \cdot e^{-2t} \Rightarrow \boxed{u = 6 \cdot e^{-(3x + 2t)}}$

which is the required solution.

(2) By the Method of Separation of Variables -

solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$ (OR)

Solve $y^3 u_x + x^2 u_y = 0$

Given $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Let $z = X(x) \cdot Y(y)$ be the solution of eqn (1),

$\therefore \frac{\partial z}{\partial x} = X' \cdot Y$, $\frac{\partial z}{\partial y} = X \cdot Y'$

$(1) \Rightarrow y^3 X' \cdot Y + x^2 X \cdot Y' = 0$

$\Rightarrow y^3 \cdot X Y' = x^2 \cdot X Y'$

$\Rightarrow \dfrac{X'}{x^2 \cdot X} = \dfrac{-Y'}{y^3 Y} = \lambda$ (say)

Take $\dfrac{X'}{x^2 \cdot X} = \lambda$

$\Rightarrow X' = \lambda x^2 \cdot X$

$\Rightarrow X' - \lambda x^2 \cdot X = 0 \longrightarrow$ (2)

A.E is $m - \lambda x^2 = 0$

$\Rightarrow m = \lambda x^2$, C.F $= c_1 e$

$\therefore$ The solution of eqn (2) is

$X = c_1 e^{\lambda x^2 \cdot x}$

$\Rightarrow \boxed{X = c_1 e^{\frac{\lambda x^3}{3}}}$

Take $\dfrac{Y'}{-y^3 \cdot Y} = \lambda$

$\Rightarrow Y' = -y^3 \cdot Y \lambda$

$\Rightarrow Y' + \lambda y^3 \cdot Y = 0 \longrightarrow$ (3)

A.E is $m + \lambda y^3 = 0$

$m = -\lambda y^3$

$\Rightarrow$ C.F $= c_2 e^{m \cdot y}$

$= c_2 e^{-\lambda y^3 \cdot y}$

$\Rightarrow$ C.F $= c_2 e$

C.F $= c_2 e^{-\lambda y^4}$

The solution of eqn (3) is

$\boxed{Y = c_2 e^{-\lambda y^4}}$

$\therefore$ The solution of eqn (1) is $u = X \cdot Y$

$u = c_1 e^{\lambda x^3} \cdot c_2 e^{-\lambda y^4}$

$\Rightarrow u = A \cdot e^{\lambda(x^3 - y^4)}$ $\langle c_1 c_2 = A \rangle$

which is the required solution.

(3) Method of Seperation of variables solve :-

$u_x - 4 \cdot u_y = 0$ & $u(0, y) = 8 \cdot e^{-3y}$

A: $u = 8\, e^{-3(4x+y)}$

(4) Solve $4 \cdot u_x + u_y = 3u$ with $u(0, y) = 3e^{-5y} - e^{-5y}$

A: $u = A \cdot e^{\frac{\lambda+3}{4}x - \lambda y}$

(5) using Method of Seperation of variables solve

$u_{xt} = e^{-t} \cdot \cos x$ with $u(x, 0) = 0$, $u(0, t) = 0$

A: Given that $u_{xt} = e^{-t} \cos x \Rightarrow$

$$\Rightarrow \frac{\partial^2 u}{\partial x \cdot \partial t} = e^{-t} \cos x \longrightarrow (1)$$

Let $u = X \cdot T$ be the solution of eqn (1)

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(X T'\right) = X' \cdot T'$$

$$(1) \Rightarrow X' \cdot T' = e^{-t} \cos x \Rightarrow \frac{X'}{\cos x} = \frac{e^{-t}}{T'} = \lambda$$

Take $\frac{X'}{\cos x} = \lambda$

$X' = \lambda \cos x$

$\Rightarrow X = \int \lambda \cdot \cos x\, dx$

$\boxed{X = \lambda \cdot \sin x + c_1}$ → (2)

$\frac{e^{-t}}{T'} = \lambda \Rightarrow \frac{T'}{e^{-t}} = \frac{1}{\lambda} = K$

$\Rightarrow T' = e^{-t} \cdot K \quad \left(\because K = \frac{1}{\lambda}\right)$

$\int T = \int e^{-t} \cdot K\, dt$

$$\Rightarrow \boxed{T = -k \cdot e^{-t} + c_2} \rightarrow (3)$$

The solution of eqn (1) is

$$u = (\lambda \sin x + c_1)(-k \cdot e^{-t} + c_2)$$

$$\Rightarrow u = (\lambda \sin x + c_1)\left(-\frac{1}{\lambda} e^{-t} + c_2\right) \rightarrow (4)$$

Put $u(x,0) = 0$ in (4), we get

$$(4) \Rightarrow 0 = (\lambda \sin x + c_1)\left(-\frac{1}{\lambda} + c_2\right) = 0$$

$$\Rightarrow -\frac{1}{\lambda} + c_2 = 0 \quad \because \lambda \sin x + c_1 \neq 0$$

$$\Rightarrow \boxed{c_2 = \frac{1}{\lambda}}$$

Put $u(0,t) = 0$ in (4), we get $(4) \Rightarrow (0 + c_1)\left(-\frac{1}{\lambda} e^{-t} + c_2\right) = 0$

$$\Rightarrow c_1 \left(-\frac{1}{\lambda} e^{-t} + \frac{1}{\lambda}\right) = 0 \Rightarrow c_1 = 0 \quad \because e^{-t} \neq 0$$

$$\Rightarrow \boxed{c_1 = 0}$$

substitute c_1 & c_2 values in eqn (4), we get

$$(4) \Rightarrow u = (\lambda \sin x)\left(-\frac{1}{\lambda} e^{-t} + \frac{1}{\lambda}\right)$$

$$\Rightarrow u = \cancel{\lambda}(\sin x) \cdot \frac{1}{\cancel{\lambda}}(-e^{-t} + 1)$$

$$u = \sin x \cdot (1 - e^{-t})$$

which is the required solution.

6) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form

$u = f(x) \cdot g(y)$, obtain the solution satisfying

$u = 0$, $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$ for all values of x.

(OR)

Solve $u_{xx} = u_y + 2u$ with $u(0, y) = 0$,

$\frac{\partial}{\partial x}[u(0, y)] = 1 + e^{-3y}$ using Method of Separation variables.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \longrightarrow (1)$

Let $u = X(x) \cdot Y(y)$ is the solution of eqn (1)

$\frac{\partial u}{\partial x} = X' \cdot Y$, $\frac{\partial^2 u}{\partial x^2} = X'' \cdot Y$, $\frac{\partial u}{\partial y} = X Y'$

$(1) \Rightarrow X'' Y = X Y' + 2 \cdot X Y$

$\Rightarrow [X'' - 2X] Y = X Y'$

$\Rightarrow \frac{X'' - 2X}{X} = \frac{Y'}{Y}$

$\Rightarrow \frac{X'' - 2X}{X} = \frac{Y'}{Y} = \lambda$ (say)

Take $\frac{X'' - 2X}{X} = \lambda$

$\Rightarrow X'' - 2X = \lambda X$

$\Rightarrow X'' - 2X - \lambda X = 0$

$\Rightarrow X'' - (\lambda + 2) X = 0 \rightarrow (2)$

A-E is $m^2 - (\lambda + 2) = 0$

$m = \pm \sqrt{\lambda + 2}$

$\therefore C \cdot F = A e^{m_1 x} + B e^{m_2 x}$

$\Rightarrow C \cdot F = A \cdot e^{(\sqrt{\lambda+2}) x} + B e^{-(\sqrt{\lambda+2}) x}$

Solution of (1) is $X = C \cdot F$

$$\boxed{X = A \cdot e^{(\sqrt{\lambda+2}) x} + B \cdot e^{-(\sqrt{\lambda+2}) x}}$$

Also Take $\frac{Y'}{Y} = \lambda$

$\Rightarrow Y' = \lambda Y$

$Y' - \lambda Y = 0 \rightarrow (3)$

A-E is $m - \lambda = 0$

$m = \lambda$

The solution of (3) is

$Y = C \cdot F = c\, e^{my}$

$\Rightarrow \boxed{Y = c\, e^{\lambda y}}$

$\therefore$ The solution of eqn (1) is

$u = X \cdot Y$

$u = \left(A \cdot e^{(\sqrt{\lambda+2}) x} + B \cdot e^{-(\sqrt{\lambda+2}) x}\right) \cdot \left(C e^{\lambda y}\right)$

$\Rightarrow u = \left(A_1 e^{\sqrt{\lambda+2}\, x} + B_1 e^{-(\sqrt{\lambda+2}) x}\right) e^{\lambda y} \rightarrow (4)$

where $\left.\begin{array}{l} A \cdot c = A_1 \\ B \cdot c = B_1 \end{array}\right]$

put $u = 0$ when $x = 0$ in (4)

(4) $\Rightarrow 0 = (A_1 + B_1) e^{\lambda y}$

$\Rightarrow \boxed{A_1 + B_1 = 0} \rightarrow (5)$ (if $e^{\lambda y} \neq 0$)

$\frac{\partial u}{\partial x} = e^{\lambda y}\left[A_1 \cdot \sqrt{\lambda+2}\, e^{\sqrt{\lambda+2}\, x} - B_1 \cdot \sqrt{\lambda+2} \cdot e^{-\sqrt{\lambda+2}\, x}\right] \rightarrow (6)$

Put $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$

in eqn (5), we get

(6) $\Rightarrow 1 + e^{-3y} = \sqrt{\lambda+2}\,(A_1 - B_1) \cdot e^{\lambda y}$

$\Rightarrow e^{0\cdot y} + e^{-3y} = \sqrt{\lambda+2}\,(A_1 - B_1)\, e^{\lambda y}$

Case (i): If $\lambda = -3$ Then $\sqrt{\lambda+2}\,(A_1 - B_1) = 1$

$\Rightarrow \sqrt{-3+2}\,(A_1 - B_1) = 1 \Rightarrow A_1 - B_1 = \frac{1}{i}$

$\Rightarrow \boxed{A_1 - B_1 = -i}$, which is imaginary $\rightarrow \boxed{6A}$

Case (ii): If $\lambda = 0$ Then $\sqrt{\lambda+2}\,(A_1 - B_1) = 1$

$\Rightarrow \sqrt{0+2}\,(A_1 - B_1) = 1 \Rightarrow (A_1 - B_1)\sqrt{2} = 1$

$\Rightarrow \boxed{A_1 - B_1 = \frac{1}{\sqrt{2}}} \rightarrow (7)$

solve (5) & (7), we get

(5) $\Rightarrow A_1 + B_1 = 0$

(7) $\Rightarrow A_1 - B_1 = \frac{1}{\sqrt{2}}$

$2A_1 = \frac{1}{\sqrt{2}}$

$\Rightarrow \boxed{A_1 = \frac{1}{2\sqrt{2}}}$

$\therefore$ (5) $\Rightarrow A_1 + B_1 = 0$

$\Rightarrow B_1 = -A_1$

$\Rightarrow \boxed{B_1 = -\frac{1}{2\sqrt{2}}}$

(4) $\Rightarrow u = \frac{1}{2\sqrt{2}}\left[e^{\sqrt{\lambda+2}\,x} - e^{-\sqrt{\lambda+2}\,x}\right] e^{\lambda y}$

~~which is the Requir~~ Put $\lambda = 0$

$\Rightarrow u = \frac{1}{\sqrt{2}}\left[\frac{e^{\sqrt{2}x} - e^{-\sqrt{2}x}}{2}\right] (1)$

$\Rightarrow \boxed{u = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x}$

which is the required solution when $\lambda = 0$

Case (i): solve the eqn (6A) & (5), we get

(6A) $\Rightarrow A_1 - B_1 = -i$

(5) $\Rightarrow A_1 + B_1 = 0$

$2A = -i$

$\Rightarrow \boxed{A_1 = -\frac{i}{2}}$

(5) $\Rightarrow A_1 + B_1 = 0$

$\Rightarrow B_1 = -A$

$\Rightarrow \boxed{B_1 = \frac{i}{2}}$

$\therefore$ (4) $\Rightarrow u = -\frac{i}{2}\left[e^{ix} - e^{-ix}\right] \cdot e^{-3y}$ (By putting $\lambda = -3$)

$\Rightarrow u = -i\left(\frac{e^{ix} - e^{-ix}}{2}\right) \cdot e^{-3y}$

$\Rightarrow u = \left(-i \sinh hx\right) \cdot e^{-3y}$ $\sin x$

$\Rightarrow \boxed{u = \sin x \cdot e^{-3y}}$

which is the required solution when $\lambda = -3$

$\because \sinh hx = i \sin x$

$\Rightarrow \frac{1}{i}\sinh hx = \sin x$

$\Rightarrow -i \sinh hx = \sin x$

7) By the method of separation of variables, solve

$$\frac{\partial^2 z}{\partial x^2} - 2 \cdot \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

A: Given that $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \rightarrow (1)$

Let $z = X(x) \cdot Y(y)$ is the solution of eqn (1)

$\therefore \frac{\partial z}{\partial x} = X^1 \cdot Y$, $\frac{\partial z}{\partial y} = X \cdot Y^1$, $\frac{\partial^2 z}{\partial x^2} = X^{11} \cdot Y$.

Substitute these values in eqn (1), we get

$(1) \Rightarrow X^{11} \cdot Y - 2 \cdot X^1 Y + X Y^1 = 0$

$\Rightarrow Y(X^{11} - 2X^1) = -XY^1 \Rightarrow \frac{X^{11} - 2X^1}{X} = -\frac{Y^1}{Y}$

$\Rightarrow \frac{X^{11} - 2X^1}{X} = -\frac{Y^1}{Y} = \lambda \text{ (say)} \rightarrow (2)$

from (2), Take $\frac{X^{11} - 2X^1}{X} = \lambda \Rightarrow X^{11} - 2X^1 = \lambda X \Rightarrow X^{11} - 2X^1 - \lambda X = 0$ $\rightarrow (3)$

A-E is $m^2 - 2m - \lambda = 0$

$\therefore m = \frac{2 \pm \sqrt{4 - 4(1)(-\lambda)}}{2(1)} = \frac{2 \pm \sqrt{4(1+\lambda)}}{2}$

$\Rightarrow m = \frac{2 \pm 2\sqrt{1+\lambda}}{2} \Rightarrow m = 1 \pm \sqrt{1+\lambda}$

$\therefore C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{(1+\sqrt{1+\lambda})x} + c_2 e^{(1-\sqrt{1+\lambda})x}$

$\therefore$ The solution of eqn (3) is $X = C.F$

$$\Rightarrow X = c_1 e^{(1+\sqrt{1+\lambda})x} + c_2 \cdot e^{(1-\sqrt{1+\lambda})x} \longrightarrow (4)$$

from (2), Take $-\frac{Y^1}{Y} = \lambda \Rightarrow -Y^1 = \lambda Y \Rightarrow Y^1 + \lambda Y = 0 \rightarrow (5)$

$\therefore$ A.E is $m + \lambda = 0 \Rightarrow m = -\lambda$.

$\therefore C.F = c_3 e^{-\lambda y}$

$\therefore$ The solution of eqn (5) is $Y = C.F$

$$\Rightarrow Y = c_3 e^{-\lambda y} \rightarrow (6)$$

substitute X & Y values in $Z = X \cdot Y$, we get

$$Z = \left[c_1 e^{(1+\sqrt{1+\lambda})x} + c_2 e^{(1-\sqrt{1+\lambda})x}\right] \cdot c_3 e^{-\lambda y}$$

$$\Rightarrow Z = \left[A e^{(1+\sqrt{1+\lambda})x} + B \cdot e^{(1+\sqrt{1+\lambda})x}\right] \cdot e^{-\lambda y} \quad \left(\text{where } c_1 c_3 = A,\ c_2 \cdot c_3 = B \right)$$

which is the required Complete solution.

8) solve the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2}$ by method of separation of variable

A: Given that $\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$

Let $y = X(x) \cdot T(t) \rightarrow (2)$ is the solution of eqn (1)

$\therefore \frac{\partial y}{\partial x} = X^1 \cdot T, \quad \frac{\partial y}{\partial t} = X T^1$

$\frac{\partial^2 y}{\partial x^2} = X^{11} T, \quad \frac{\partial^2 y}{\partial t^2} = X T^{11}$

Substitute these values in eqn (1), we get

(1) $\Rightarrow X T'' = c^2 \cdot X'' T \Rightarrow X'' \cdot T = \frac{1}{c^2} \cdot X \cdot T''$

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T}$.

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} = \lambda \rightarrow (3)$

from (3), Take $\frac{X''}{X} = \lambda \Rightarrow \frac{X''}{X} = \lambda \Rightarrow X'' = \lambda X$

$\Rightarrow X'' - \lambda X = 0 \rightarrow (4)$

$\therefore$ A.E is $f(m) = 0 \Rightarrow m^2 - \lambda = 0 \Rightarrow m^2 = \lambda$

$\therefore m = \pm\sqrt{\lambda} \Rightarrow m = -\sqrt{\lambda}, \sqrt{\lambda}$.

$\therefore C.F = c_1 e^{-\sqrt{\lambda} x} + c_2 e^{\sqrt{\lambda} \cdot x}$

$\therefore$ The solution of eqn (4) is $X = C.F$

$\Rightarrow X = c_1 e^{-\sqrt{\lambda} x} + c_2 e^{\sqrt{\lambda} \cdot x} \longrightarrow (5)$

from (3) also Take $\frac{1}{c^2} \cdot \frac{T''}{T} = \lambda \Rightarrow T'' = c^2 \cdot T \cdot \lambda$

$\Rightarrow T'' - c^2 \lambda \cdot T = 0 \longrightarrow (6)$

$\therefore$ A.E is $m^2 - c^2\lambda = 0 \Rightarrow m^2 = c^2 \cdot \lambda$

$\Rightarrow m = \pm\sqrt{c^2\lambda} \Rightarrow m = \pm c\sqrt{\lambda} \Rightarrow m = -c\sqrt{\lambda}, c\sqrt{\lambda}$

$\therefore C.F = c_3 e^{-c\sqrt{\lambda} t} + c_4 e^{c\sqrt{\lambda} t}$

$\therefore$ The solution of eqn (6) is $T = C.F$

$$\therefore T = c_3 e^{-c\sqrt{\lambda} t} + c_4 e^{c\sqrt{\lambda} t} \longrightarrow (7)$$

Substitute eqns (5) & (7) in eqn (2), we get

$$(2) \Rightarrow y = \left(c_1 e^{-\sqrt{\lambda} x} + c_2 e^{\sqrt{\lambda} x}\right)\left(c_3 e^{-c\sqrt{\lambda} t} + c_4 e^{c\sqrt{\lambda} t}\right)$$

which is the required solution.

Unit 3

LAPLACE TRANSFORMS

The main application of Laplace Transforms is that we can solve Ordinary Differential equations without finding complementary function (C.F) and Particular Integral (P.I).

$$
\begin{tikzcd}
\text{O.D.E with initial conditions} \arrow[d] & \text{Solution of O.D.E} \\
\text{L.T} \arrow[d] & \text{Inverse L.T.} \arrow[u] \\
\text{Algebraic (subsidiary) equation} \arrow[r] & \text{Solution of A.E} \arrow[u]
\end{tikzcd}
$$

Transformation :- It is a Mathematical device which converts (changes) (transforms) one function into another function.

Eg: $D(\tan x) = \sec^2 x$

$\int \cos x \, dx = \sin x.$

Here $D, \int$ are transformations.

Integral Transform :-

$$I[f(t)] = \int_a^b k(s,t) \, f(t) \, dt.$$

where $k(s,t)$ is called kernal of I.T. and a function of s, t.

(i). If $k(s,t) = e^{-st}$ and $a = 0, b \to \infty$, We get Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} . f(t) \, dt.$$

(ii). If $k(s,t) = e^{ist}$ and $a \to -\infty, b \to \infty$, We get Fourier transform

$$F[f(t)] = \int_{-\infty}^{\infty} e^{ist} . f(t) \, dt.$$

Definition :

Laplace Transform :-

Let $f(t)$ be a function defined for all $+^{ve}$ values of t.

The Laplace transform of $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s).$$

Properties :-

(i). $L[c \cdot f(t)] = c \cdot L[f(t)]$, c = constant.

(ii). $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$.

Laplace Transforms of Standard functions:

(1). $L(1) = \frac{1}{s}$.

(2). $L(t) = \frac{1}{s^2}$.

(3). $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$.

(4). $L[e^{at}] = \frac{1}{s-a}$.

(5). $L[e^{-at}] = \frac{1}{s+a}$.

(6). $L[\sin at] = \frac{a}{s^2+a^2}$.

(7). $L[\cos at] = \frac{s}{s^2+a^2}$.

(8). $L[\sinh at] = \frac{a}{s^2-a^2}$.

(9). $L[\cosh at] = \frac{s}{s^2-a^2}$.

Gamma function :-

If $n>0$, the gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx.$$

Properties :-

(1). $\Gamma(1) = 1$

(2). $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(3). $\Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi}$

(4) $\Gamma(0), \Gamma(-1), \Gamma(-2), \ldots$ are not defined.

We have $\quad L\left[t^n\right] = \dfrac{\Gamma(n+1)}{s^{n+1}}$.

$\Gamma(n+1) = n!$ if n is a +Ve integer

$\Gamma(n+1) = n \cdot \Gamma(n)$ if n is a +Ve fraction

$\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ if n is a -Ve fraction.

Trigonometry formulae :

$$\sinh x = \frac{e^{x} - e^{-x}}{2}, \quad \cosh x = \frac{e^{x} + e^{-x}}{2}.$$

$$\sin^2 x + \cos^2 x = 1.$$

$$\sin 2x = 2 \sin x \cdot \cos x.$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1.$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

$$\sin A \cdot \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}.$$

$$\cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}.$$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$$\cos 3t = 4\cos^3 t - 3\cos t$$

$$\Rightarrow \cos^3 t = \frac{\cos 3t + 3\cos t}{4}$$

$$\sin 3t = 3\sin t - 4\sin^3 t.$$

$$\Rightarrow \sin^3 t = \frac{3\sin t - \sin 3t}{4}.$$

$$\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right].$$

$(t^2+1)^2$

$t^3+5\cos t$

$2e^{3t} - e^{-3t}$

$\sin^2 t$

$\cos 5t.\cos 2t$

$\frac{e^{-at}-1}{a}$

$\sin(\omega t+\alpha)$

Problems

Find the Laplace transforms of the following functions.

1). $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$.

Sol:- $L\left[e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\right]$

$$= L\left[e^{2t}\right] + 4L\left[t^3\right] - 2L(\sin 3t) + 3L(\cos 3t)$$

$$= \frac{1}{s-2} + 4\cdot\frac{3!}{s^4} - 2\cdot\frac{3}{s^2+9} + 3\cdot\frac{s}{s^2+9}$$

2). $(\sin t - \cos t)^2$

Sol:- $L\left[(\sin t - \cos t)\right]^2$

$$= L\left[\sin^2 t + \cos^2 t - 2\sin t\cdot\cos t\right].$$

$$= L\left[1 - \sin 2t\right]$$

$$= L(1) - L(\sin 2t)$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

3). $3\cosh 5t - 4\sinh 5t$

Sol:- $L\left[3\cosh 5t - 4\sinh 5t\right]$

$$= 3 \cdot L(\cosh 5t) - 4 \cdot L(\sinh 5t)$$

$$= 3 \cdot \frac{s}{s^2-25} - 4 \cdot \frac{5}{s^2-25}$$

$$= \frac{3s-20}{s^2-25}.$$

4). $\cos(at+b)$

Sol:- $L\left[\cos(at+b)\right]$

$$= L\left[\cos at \cdot \cos b - \sin at \cdot \sin b\right]$$

$$= \cos b \cdot L(\cos at) - \sin b \cdot L(\sin at)$$

$$= \cos b \cdot \frac{s}{s^2+a^2} - \sin b \cdot \frac{a}{s^2+a^2}.$$

5). $\cos^3 2t$ $\qquad \left\{\cos^3 t = \frac{\cos 3t + 3\cos t}{4}\right\}$

Sol:- $L\left[\cos^3 2t\right] = L\left[\frac{\cos 6t - 3\cos 2t}{4}\right]$

$$= \frac{1}{4} \cdot L(\cos 6t) - \frac{3}{4} \cdot L(\cos 2t)$$

$$= \frac{1}{4} \cdot \frac{s}{s^2+36} - \frac{3}{4} \cdot \frac{s}{s^2+4}.$$

6). $\sin 2t \cdot \cos 3t$

Sol:- $L\{\sin 2t \cdot \cos 3t\}$

$$= L\left[\frac{\sin 5t + \sin(-t)}{2}\right]$$

$$= \frac{1}{2} L(\sin 5t) - \frac{1}{2} \cdot L(\sin t)$$

$$= \frac{1}{2} \cdot \frac{5}{s^2+25} - \frac{1}{2} \cdot \frac{1}{s^2+1}.$$

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7). $1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$.

Sol:- $L\left[1 + 2\sqrt{t} + 3 \cdot t^{-1/2}\right]$

$$= L(1) + 2L\left[t^{1/2}\right] + 3 \cdot L\left[t^{-1/2}\right]$$

$$= \frac{1}{s} + 2 \cdot \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} + 3 \frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}}.$$

$$= \frac{1}{s} + 2 \cdot \frac{\frac{1}{2} \cdot \Gamma(\frac{1}{2})}{s^{3/2}} + 3 \frac{\Gamma(\frac{1}{2})}{s^{1/2}}.$$

$$= \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + 3 \cdot \frac{\sqrt{\pi}}{s^{1/2}}.$$

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(8). $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$.

$$\left\{\begin{array}{l}\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{array}\right\}.$$

Sol:- $\cos\sqrt{t} = 1 - \dfrac{(\sqrt{t})^2}{2!} + \dfrac{(\sqrt{t})^4}{4!} - \dfrac{(\sqrt{t})^6}{6!} + \cdots$

$$= 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \frac{1}{2!}\frac{t}{\sqrt{t}} + \frac{1}{4!}\frac{t^2}{\sqrt{t}} - \frac{1}{6!}\frac{t^3}{\sqrt{t}}$$

$$= t^{-1/2} - \frac{1}{2!}\cdot t^{1/2} + \frac{1}{4!}t^{3/2} - \frac{1}{6!}t^{5/2}.$$

Now $L\left(t^{-1/2}\right) = \dfrac{\Gamma\left(\frac{-1}{2}+1\right)}{s^{1/2}} = \dfrac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$.

$$L\left(t^{1/2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{3/2}}.$$

$$L\left(t^{3/2}\right) = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{3/2+1}} = \frac{\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} = \frac{\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}} = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{5/2}}.$$

$$L\left[t^{5/2}\right] = \frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{5/2+1}} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{7/2}}.$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} - \frac{1}{4}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3}{96}\cdot\frac{\sqrt{\pi}}{s^{5/2}} - \frac{1}{720}\times\frac{15}{8}\cdot\frac{\sqrt{\pi}}{s^{7/2}}.$$

(1). $\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3$. $\left\{(a-b)^3 = a^3-3a^2b+3ab^2-b^3\right\}$.

Sol:- $= (\sqrt{t})^3 - 3(\sqrt{t})^2\left(\frac{1}{\sqrt{t}}\right) + 3\sqrt{t}\left(\frac{1}{\sqrt{t}}\right)^2 - \left(\frac{1}{\sqrt{t}}\right)^3$

$$= t^{\frac{3}{2}} - 3.t^{\frac{1}{2}} + 3t^{-1/2} - t^{\frac{-3}{2}}.$$

Now $L\left[t^{\frac{3}{2}}\right] = \dfrac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} = \dfrac{\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} = \dfrac{\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{5/2}} = \frac{3}{4}\cdot\frac{\sqrt{\pi}}{s^{5/2}}.$$

$$L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{3/2}}.$$

$$L\left[t^{\frac{-1}{2}}\right] = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{s^{-1/2+1}} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

$$L\left[t^{\frac{-3}{2}}\right] = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{s^{\frac{-3}{2}+1}} = \frac{\Gamma\left(\frac{-1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{1/\sqrt{s}} = -2\sqrt{\pi}\sqrt{s}.$$

$$\therefore L\left[\sqrt{t}-\frac{1}{\sqrt{t}}\right]^3 = \frac{3}{4}\cdot\frac{\sqrt{\pi}}{s^{5/2}} - 3\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{3/2}} + 3\frac{\sqrt{\pi}}{\sqrt{s}} + 2\sqrt{\pi}\sqrt{s}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[\frac{3}{4}\cdot\frac{1}{s^2} - \frac{3}{2}\cdot\frac{1}{s} + 3 + 2s\right].$$

10). 2^t.

Sol:- $L\left[2^t\right] = L\left[e^{\log 2^t}\right] = L\left[e^{t(\log 2)}\right]$

$$= L\left[e^{(\log 2)t}\right] = \frac{1}{s-\log 2}.$$

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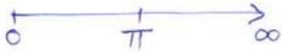
11). $f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$



Sol: By the defn. of Laplace transform

$$L\left[f(t)\right] = \int_0^\infty e^{-st}.\, f(t)\, dt$$

$$= \int_0^\pi e^{-st}.\,\sin t\, dt + \int_\pi^\infty e^{-st}.(0)\, dt$$

$$= \int_0^\pi e^{-st}.\,\sin t\, dt$$

$$\boxed{\int e^{at}.\sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)}$$

$\downarrow$

$$\boxed{\text{Put } a=-s,\ b=1}$$

$$= \left[\frac{e^{-st}}{(-s)^2+1}\left[(-s).\sin t - 1.\cos t\right]\right]_0^\pi$$

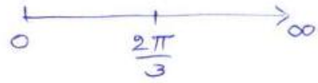
$$= \left[\frac{e^{-st}}{s^2+1}\left(-s.\sin t - \cos t\right)\right]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1}(0+1) - \frac{1}{s^2+1}(0-1) = \frac{1}{s^2+1}\left[e^{-\pi s}+1\right]$$

12). $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3}. \end{cases}$

Sol: By the defn. of L.T

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$



$$= \int_0^{2\pi/3} e^{-st} \cdot (0)\, dt + \int_{\frac{2\pi}{3}}^\infty e^{-st} \cdot \cos\left(t - \frac{2\pi}{3}\right) dt$$

$$= \int_{\frac{2\pi}{3}}^\infty e^{-st} \cdot \cos\left(t - \frac{2\pi}{3}\right) dt.$$

$$\boxed{\int e^{at} \cdot \cos bt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)}$$

$$\boxed{\text{Put } a = -s,\ b = 1}$$

$$= \left[\frac{e^{-st}}{(-s)^2+1}\left[(-s)\cos\left(t - \frac{2\pi}{3}\right) + 1 \cdot \sin\left(t - \frac{2\pi}{3}\right)\right]\right]_{\frac{2\pi}{3}}^{\infty}$$

$$\boxed{e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0}$$

$$= 0 - \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}\left((-s)(1) + 0\right)$$

$$= \frac{s}{s^2+1} e^{-\frac{2\pi}{3} \cdot s}.$$

$$\boxed{\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots}$$

where $u', u'', u''', \ldots$ are derivatives.
$v_1, v_2, v_3, \ldots$ are integrals.

13). $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3. \end{cases}$

Sol: By the defn. of L.T.

$0 \quad 2 \quad 3 \quad \infty$

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^2 \underset{v}{e^{-st}} \cdot \underset{u}{t^2}\, dt + \int_2^3 \underset{v}{e^{-st}} \cdot \underset{u}{(t-1)}\, dt + \int_3^\infty e^{-st} \cdot 7\, dt$$

$$= \left[(t^2)\left(\frac{e^{-st}}{-s}\right) - (2t)\left(\frac{e^{-st}}{s^2}\right) + (2)\left(\frac{e^{-st}}{s^2(-s)}\right)\right]_0^2$$

$$+ \left[(t-1)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_2^3 + 7 \cdot \left(\frac{e^{-st}}{-s}\right)_3^\infty$$

$$= \left(\frac{-4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s}\right) - \left(0 - 0 - \frac{2}{s^3}\right)$$

$$+ \left(\frac{-2}{s} e^{-3s} - \frac{1}{s^2} e^{-3s}\right) - \left(\frac{-1}{s} e^{-2s} + \frac{1}{s^2} e^{-2s}\right) - \frac{7}{s}\left(0 - e^{-3s}\right)$$

$$= \frac{-4}{s^3}\left(e^{-2s}\right)\left[s^2 + s + \frac{1}{2}\right] + \frac{2}{s^3} + \frac{e^{-3s}}{s^2}\left(-2s-1\right)$$

$$+ \frac{e^{-2s}}{s^2}\left(s-1\right) + \frac{7}{s} e^{-3s}.$$

14). $f(t) = \begin{cases} 4, & 0 \le t \le 1 \\ 3, & t > 1. \end{cases}$

15). $f(x) = \begin{cases} \sin\left(x - \frac{\pi}{3}\right), & x > \frac{\pi}{3} \\ 0, & x < \frac{\pi}{3} \end{cases}$

16). $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2. \end{cases}$

17). show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

18). $(t^2 + 5)^2 + \sin^2 t$.

First shifting theorem

If $L[f(t)] = \bar{F}(s)$, then

$$L\left[e^{at} \cdot f(t)\right] = \bar{F}(s-a).$$

Applications

(1). $L\left(e^{at}\right) = \dfrac{1}{s-a}$ $\left\{L(1) = \dfrac{1}{s}\right\}$.

(2). $L\left[e^{at} \cdot t^n\right] = \dfrac{n!}{(s-a)^{n+1}}$ $\left\{L(t^n) = \dfrac{n!}{s^{n+1}}\right\}$.

(3). $L\left[e^{at} \sin bt\right] = \dfrac{b}{(s-a)^2 + b^2}$ $\left\{\because L(\sin bt) = \dfrac{b}{s^2+b^2}\right\}$.

(4). $L\left[e^{at} \cdot \cos bt\right] = \dfrac{s-a}{(s-a)^2 + b^2}$ $\left\{\because L(\cos bt) = \dfrac{s}{s^2+b^2}\right\}$.

(5). $L\left[e^{at} \sinh bt\right] = \dfrac{b}{(s-a)^2 - b^2}$ $\left\{\because L(\sinh bt) = \dfrac{b}{s^2-b^2}\right\}$.

(6). $L\left\{e^{at} \cdot \cosh bt\right] = \dfrac{s-a}{(s-a)^2 - b^2}$ $\left\{\because L(\cosh bt) = \dfrac{s}{s^2-b^2}\right\}$.

Problems : Find the L.Ts to the following

1). $e^{2t}(3t^5 - \cos 4t)$

Sol:- $L[3t^5 - \cos 4t] = 3L(t^5) - L(\cos 4t)$

$$= 3 \cdot \frac{5!}{s^6} - \frac{s}{s^2+16} = \bar{f}(s).$$

By First shifting theorem

$$L[e^{2t}(3t^5 - \cos 4t)] = \frac{3 \times 5!}{(s-2)^6} - \frac{s-2}{(s-2)^2+16}.$$

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2). $e^{-3t} \sin 5t \cdot \sin 3t$.

Sol:- $L[\sin 5t \cdot \sin 3t] = L\left[\frac{\cos 2t - \cos 8t}{2}\right]$

$$= \frac{1}{2} L(\cos 2t) - \frac{1}{2} L(\cos 8t)$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{s}{s^2+64} = \bar{f}(s).$$

By First shifting property

$$L[e^{-3t} \sin 5t \cdot \sin 3t] = \frac{1}{2} \frac{s+3}{(s+3)^2+4} - \frac{1}{2} \cdot \frac{s+3}{(s+3)^2+64}.$$

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3). $\sinh 3t \cdot \cos^2 t$.

Sol:- $$= \left[\frac{e^{3t} - e^{-3t}}{2}\right]\left[\frac{1+\cos 2t}{2}\right]$$

$$= \frac{1}{4}e^{3t} + \frac{1}{4}e^{3t}\cdot\cos 2t - \frac{1}{4}e^{-3t} - \frac{1}{4}e^{-3t}\cdot\cos 2t.$$

$$L\left[\sinh 3t \cdot \cos^2 t\right]$$

$$= \frac{1}{4}L\left(e^{3t}\right) + \frac{1}{4}L\left(e^{3t}\cdot\cos 2t\right) - \frac{1}{4}L\left(e^{-3t}\right) - \frac{1}{4}L\left(e^{-3t}\cdot\cos 2t\right)$$

$$= \frac{1}{4}\cdot\frac{1}{s-3} + \frac{1}{4}\cdot\frac{s-3}{(s-3)^2+4} - \frac{1}{4}\cdot\frac{1}{s+3} - \frac{1}{4}\cdot\frac{s+3}{(s+3)^2+4}.$$

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4). $e^{2t} \cdot \sin^4 t$.

Sol:- $$L(\sin^4 t) = L\left[(\sin^2 t)^2\right] = L\left[\left(\frac{1-\cos 2t}{2}\right)^2\right]$$

$$= L\left[\frac{1+\cos^2 2t - 2\cos 2t}{4}\right].$$

$$= \frac{1}{4}\cdot L(1) + \frac{1}{4}\cdot L\left[\frac{1+\cos 4t}{2}\right] - \frac{2}{4}\cdot L(\cos 2t).$$

$$= \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{8}\cdot\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+16} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$$= \frac{3}{8}\cdot\frac{1}{s} - \frac{3}{8}\cdot\frac{s}{s^2+4} = \frac{3}{8}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \bar{F}(s).$$

By F.S. theorem

$$L\left[e^{2t}\cdot\sin^4 t\right] = \frac{3}{8}\left[\frac{1}{s-2} - \frac{s-2}{(s-2)^2+4}\right].$$

(5). $\cosh 3t \cdot \cos 2t$.

(6) $e^{2t} \cdot \cos^2 t$.

(7) $t^{\frac{7}{2}} \cdot e^{3t}$.

(8). Show that $L\left[\sinh at \cdot \sin at\right] = \dfrac{2a^2 s}{s^4 + 4a^4}$.

(9). $e^{-3t} \cdot \sin^2 t$.

(10). $e^{t}\left[\cos 2t + \frac{1}{2}\sinh 2t\right]$.

(11). If $L[f(t)] = \overline{f}(s)$. show that

$$L\left[(\sinh at)\, f(t)\right] = \frac{1}{2}\left[\overline{f}(s-a) - \overline{f}(s+a)\right].$$

Hence evaluate L.T. of $\sinh 2t \cdot \sin 3t$.

(12). $(1 + t\, e^{-t})^3$.

Change of Scale property:

If $L[f(t)] = \bar{f}(s)$, then

$$L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

$$L\left[f\left(\frac{t}{a}\right)\right] = a.\bar{f}(as).$$

Problems:-

(1). If $L[f(t)] = \frac{9s^2-12s+5}{(s-1)^3}$. Find $L[f(3t)]$.

Sol: Given $L[f(t)] = \frac{9s^2-12s+5}{(s-1)^3} = \bar{f}(s)$.

By change of Scale property

$$L[f(3t)] = \frac{1}{3}.\bar{f}\left(\frac{s}{3}\right)$$

$$= \frac{1}{3}.\frac{9\left(\frac{s}{3}\right)^2-12\left(\frac{s}{3}\right)+5}{\left(\frac{s}{3}-1\right)^3}.$$

$$= \frac{1}{3}\times 27\ \frac{s^2-4s+5}{(s-3)^3}$$

$$= \frac{9(s^2-4s+5)}{(s-3)^3}.$$

(2). If $L[f(t)] = \frac{1}{s} e^{\frac{-1}{s}}$, Prove that $L\left[e^{-t}. f(3t)\right] = \frac{e^{\frac{-3}{s+1}}}{s+1}$.

Sol:- Given $L[f(t)] = \frac{1}{s} \cdot e^{\frac{-1}{s}} = \bar{f}(s)$.

By change of scale property

$$L[f(3t)] = \frac{1}{3} \cdot \bar{f}\left(\frac{s}{3}\right) = \frac{1}{3} \cdot \frac{1}{s/3} e^{\frac{-1}{s/3}}$$

$$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{s} \cdot e^{\frac{-3}{s}} = \frac{1}{s} \cdot e^{\frac{-3}{s}} = \bar{f}(s).$$

Now, By First shifting property

$$L\left[e^{-t}. f(3t)\right] = \bar{f}(s+1) = \frac{1}{s+1} \cdot e^{\frac{-3}{s+1}}.$$

---

(3). If $L[f(t)] = \frac{1}{s(s^2+1)}$. Find $L\left[e^{-t}. f(2t)\right]$.

Sol:- Given $L[f(t)] = \frac{1}{s(s^2+1)} = \bar{f}(s)$.

$$L[f(2t)] = \frac{1}{2} \cdot \bar{f}\left(\frac{s}{2}\right) = \frac{1}{2} \cdot \frac{1}{s/2\left(\left(\frac{s}{2}\right)^2+1\right)}$$

$$= \frac{1}{2} \cdot \frac{2}{s} \cdot \frac{4}{s^2+4} = \frac{1}{s} \cdot \frac{4}{s^2+4} = \bar{f}(s).$$

By First shifting property.

$$L\left[e^{-t}. f(2t)\right] = \bar{f}(s+1) = \frac{1}{s+1} \cdot \frac{4}{(s+1)^2+4}.$$

---

4). Find $L\left[\frac{\sin at}{at}\right]$, given that $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$.

<u>Sol</u>:- Given $L\left(\frac{\sin t}{t}\right) = \tan^{-1}\left(\frac{1}{s}\right) = \bar{f}(s)$.

By change of Scale property

$$L\left[\frac{\sin at}{at}\right] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) = \frac{1}{a}\tan^{-1}\left(\frac{1}{s/a}\right)$$

$$= \frac{1}{a}\cdot\tan^{-1}\left(\frac{a}{s}\right).$$

$$\Rightarrow L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right).$$

Multiplication by t^n :-

If $L[f(t)] = \bar{F}(s)$, then

$$L\left[t^n \cdot f(t)\right] = (-1)^n \cdot \frac{d^n}{ds^n} \bar{F}(s), \quad n = 1, 2, \ldots\ldots$$

If $n = 1$, $\boxed{L\left[t \cdot f(t)\right] = (-1) \frac{d}{ds} \bar{F}(s)}$

If $n = 2$, $\boxed{L\left[t^2 \cdot f(t)\right] = (-1)^2 \frac{d^2}{ds^2} \left[\bar{F}(s)\right]}$.

Problems : Find L.T.s to the following functions.

1). $t \cdot \sin^2 t$.

Sol:- $L(\sin^2 t) = L\left[\frac{1 - \cos 2t}{2}\right] = \frac{1}{2} \cdot L(1) - \frac{1}{2} L(\cos 2t)$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2 + 4} = \bar{F}(s).$$

$$L\left[t \cdot \sin^2 t\right] = (-1) \frac{d}{ds} \bar{F}(s)$$

$$= (-1) \cdot \frac{d}{ds} \left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]$$

$$= \frac{-1}{2} \cdot \frac{d}{ds}\left(\frac{1}{s}\right) + \frac{1}{2} \cdot \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{-1}{2}\cdot\left(\frac{-1}{s^2}\right) + \frac{1}{2}\cdot\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{s^2+4-2s^2}{(s^2+4)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right].$$

2). $t^2\cdot\cos at$

Sol:- $L(\cos at) = \dfrac{s}{s^2+a^2} = \bar{F}(s).$

$$L\left[t^2\cdot\cos at\right] = (-1)^2\frac{d^2}{ds^2}\cdot\bar{F}(s).$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right].$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)(1) - s\cdot(2s)}{(s^2+a^2)^2}\right].$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\,2(s^2+a^2)\cdot(2s)}{(s^2+a^2)^4}$$

$$= \frac{\cancel{(s^2+a^2)}\left[-2s^3-2a^2s-4a^2s+4s^3\right]}{\cancel{(s^2+a^2)}(s^2+a^2)^3} = \frac{2s^3-6a^2s}{(s^2+a^2)^3}.$$

3). $t\,\bar{e}^{2t}.\sin 4t.$

Sol:- $L(\sin 4t) = \dfrac{4}{s^2+16} = \bar{f}(s).$

$$L\left[t.\sin 4t\right] = (-1)\frac{d}{ds}\bar{f}(s) = (-1)\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$$

$$= (-4).\frac{d}{ds}\left(\frac{1}{s^2+16}\right) = (-4)\frac{-1}{(s^2+16)^2}.(2s)$$

$$= \frac{8s}{(s^2+16)^2} = \bar{f}(s).$$

By first shifting property

$$L\left[\bar{e}^{2t}.t.\sin 4t\right] = \bar{f}(s+2) = \frac{8(s+2)}{\left((s+2)^2+16\right)^2}$$

$$= \frac{8(s+2)}{(s^2+2s+20)^2}.$$

Problems

4). $t \cos at$

5). $t e^{2t} \cdot \sin 3t$

6). $t e^{-t} \sin 2t$

7). $t^2 \sin 2t$

8). $t \sinh t \, e^{-t}$

9). $t \sin 3t \cos 2t$

10). $\sin 2t - 2t \cos 2t$

11). $t \sinh at$

12). $t^2 e^{-3t} \cdot \sin 2t$

Division by t

If $L[f(t)] = \bar{F}(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{F}(s)\,ds$.

Find the L·Ts to the following functions:

1). $\dfrac{e^{at} - \cos bt}{t}$

Sol:- $L[e^{at} - \cos bt] = L(e^{at}) - L(\cos bt)$

$$= \frac{1}{s-a} - \frac{s}{s^2+b^2} = \bar{F}(s).$$

$$L\left[\frac{e^{at} - \cos bt}{t}\right] = \int_s^\infty \bar{F}(s)\,ds$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right) ds.$$

$$= \int_s^\infty \frac{1}{s-a}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+b^2}\,ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)\right]_s^\infty = \lim_{s\to\infty} \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right) - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)$$

$$= \lim_{s\to\infty} \log\left[\frac{\cancel{s}\left(1-\frac{a}{s}\right)}{\cancel{s}\sqrt{1+\frac{b^2}{s^2}}}\right] - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)$$

$$= \log\left(\frac{1-0}{\sqrt{1+0}}\right) - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)$$

$$= 0 - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right) = \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right).$$

---

2). $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t.$

Sol: $L(2^t) = L\left[e^{\log 2^t}\right] = L\left[e^{t(\log 2)}\right] = L\left[e^{(\log 2)t}\right]$

$$= \frac{1}{s-\log 2}.$$

$$L\left[\cos 2t - \cos 3t\right] = L(\cos 2t) - L(\cos 3t)$$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9} = \bar{F}(s)$$

$$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\, ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+9}\, ds$$

$$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \left[\log\sqrt{s^2+4} - \log\sqrt{s^2+9}\right]_s^\infty$$

$$= \left[\log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)\right]_s^{\infty}$$

$$= \lim_{s\to\infty} \log\left[\frac{\cancel{s}\sqrt{1+4/s^2}}{\cancel{s}\sqrt{1+9/s^2}}\right] - \log\left(\frac{\sqrt{s^2+4}}{\sqrt{s^2+9}}\right)$$

$$= \log\left[\frac{\sqrt{1+0}}{\sqrt{1+0}}\right] - \log\left(\sqrt{\frac{s^2+4}{s^2+9}}\right)$$

$$= 0 - \log\left(\sqrt{\frac{s^2+4}{s^2+9}}\right) = \log\sqrt{\frac{s^2+9}{s^2+4}}.$$

$$L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s).$$

$$L(t\cdot\sin t) = (-1)\frac{d}{ds}\bar{f}(s) = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= (-1)\cdot\frac{-1}{(s^2+1)^2}\cdot(2s) = \frac{2s}{(s^2+1)^2}$$

$$\therefore L\left[2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t\right]$$

$$= \frac{1}{s-\log 2} + \log\sqrt{\frac{s^2+9}{s^2+4}} + \frac{2s}{(s^2+1)^2}.$$

---

(3). $\dfrac{e^{-3t} \cdot \sin 2t}{t}$.

$$\boxed{\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$$\boxed{\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x}$$

Sol:- $L(\sin 2t) = \dfrac{2}{s^2+4} = \bar{f}(s)$.

$$L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \bar{f}(s)\,ds = 2\int_s^\infty \frac{1}{s^2+4}\,ds.$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right) = \cot^{-1}\left(\frac{s}{2}\right). = \bar{f}(s).$$

By First shifting property

$$L\left[\frac{e^{-3t} \cdot \sin 2t}{t}\right] = \bar{f}(s+3) = \cot^{-1}\left(\frac{s+3}{2}\right).$$

---

4). $\dfrac{\sin t}{t}$.

5). $\dfrac{1-e^{t}}{t}$

6). $\dfrac{\cos at - \cos bt}{t} + t\sin at$.

7). $\dfrac{\sin t \cdot \sin 5t}{t}$

8). $\dfrac{e^{-at} - e^{-bt}}{t}$

9). $\dfrac{1-\cos 3t}{t}$

Laplace Transforms of Derivatives:-

If $L[f(t)] = \bar{f}(s)$, then

$$L[f'(t)] = s.\bar{f}(s) - f(0).$$

$$L[f''(t)] = s^2\bar{f}(s) - s.f(0) - f'(0).$$

Laplace Transforms of Integrals:-

If $L[f(t)] = \bar{f}(s)$, then

$$\boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\bar{f}(s).}$$

Find the L.T.s to the following integrals:-

(1). $\int_0^t \bar{e}^{t} \cos t \, dt$

Sol:- $L(\cos t) = \dfrac{s}{s^2+1} = \bar{f}(s)$.

$$L[\bar{e}^{t} \cdot \cos t] = \bar{f}(s+1) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}.$$

$$= \bar{f}(s).$$

$$L\left[\int_0^t \bar{e}^{t} \cdot \cos t \, dt\right] = \frac{1}{s}\bar{f}(s) = \frac{1}{s} \cdot \frac{s+1}{s^2+2s+2}.$$

---

(2). $\int_0^t \int_0^t \int_0^t (t \sin t) \, dt \, dt \, dt.$

Sol:- $L(\sin t) = \dfrac{1}{s^2+1} = \bar{f}(s)$.

$$L[t \cdot \sin t] = (-1)\frac{d}{ds}\bar{f}(s) = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= (-1)\frac{-1}{(s^2+1)^2} \cdot 2s = \frac{2s}{(s^2+1)^2} = \bar{f}(s).$$

$$L\left[\int_0^t (t \sin t) \, dt\right] = \frac{1}{s}\bar{f}(s) = \frac{1}{\cancel{s}} \cdot \frac{2\cancel{s}}{(s^2+1)^2} = \frac{2}{(s^2+1)^2} = \bar{f}(s).$$

$$L\left[\int_0^t \int_0^t (t \sin t) \, dt\right] = \frac{1}{s}\bar{f}(s) = \frac{1}{s} \cdot \frac{2}{(s^2+1)^2} = \bar{f}(s).$$

$$L\left[\int_0^t \int_0^t \int_0^t (t \sin t) \, dt\right] = \frac{1}{s}\bar{f}(s) = \frac{1}{s^2} \cdot \frac{2}{(s^2+1)^2}.$$

(3). Evaluate $L\left[t\int_0^t \frac{\bar{e}^{t}.\sin t}{t}\,dt\right]$.

Sol:- $L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s).$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \left(\tan^{-1}s\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s = \bar{f}(s).$$

$$L\left[\bar{e}^{t}.\frac{\sin t}{t}\right] = \bar{f}(s+1) = \cot^{-1}(s+1) = \bar{f}(s).$$

$$L\left[\int_0^t \frac{\bar{e}^{t}.\sin t}{t}\,dt\right] = \frac{1}{s}\bar{f}(s) = \frac{1}{s}.\cot^{-1}(s+1) = \bar{f}(s).$$

$$L\left[t\int_0^t \frac{\bar{e}^{t}.\sin t}{t}\,dt\right] = (-1)\frac{d}{ds}\bar{f}(s) = (-1)\frac{d}{ds}\left[\frac{\cot^{-1}(s+1)}{s}\right]$$

$$= (-1).\frac{s.\frac{-1}{(s+1)^2+1} - \cot^{-1}(s+1).(1)}{s^2}$$

$$= \frac{\frac{s}{s^2+2s+2} + \cot^{-1}(s+1)}{s^2}$$

$$= \frac{s + (s^2+2s+2).\cot^{-1}(s+1)}{s^2(s^2+2s+2)}.$$

(4). Evaluate $L\left[\int_0^t \frac{e^{-t}\cdot \sin t}{t}\,dt\right]$.

(5). Evaluate $L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right]$.

(6). Evaluate $L\left[\int_0^t \frac{1-e^{-2t}}{t}\,dt\right]$.

Unit step function :-

It is defined as

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a. \end{cases} \quad \text{where } a>0.$$

L.T. of unit step function :-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[u(t-a)] = \int_0^a e^{-st} \cdot (0)\, dt + \int_a^\infty e^{-st} \cdot (1)\, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}.$$

Second shifting theorem :-

If $L[f(t)] = \bar{f}(s)$ and $g(t) = \begin{cases} 0, & t<a \\ f(t-a), & t \geq a. \end{cases}$

Then $L[g(t)] = e^{-as} \cdot \bar{f}(s)$.

Problems:-

(1). Find the L.T. of $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3}. \end{cases}$

Sol:- Let $u(t) = \cos t$.

Then $L[u(t)] = \frac{s}{s^2+1} = \bar{u}(s)$.

Let $f(t) = \begin{cases} u(t-a), & t > a \\ 0, & t < a. \end{cases}$

$\left\{ \text{Here } a = \frac{2\pi}{3} \right\}$,

Then $L[f(t)] = e^{-as} \cdot \bar{u}(s)$

$$= e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}.$$

(2). Find the L.T. of $f(x) = \begin{cases} \sin\left(x - \frac{\pi}{3}\right), & x \geq \frac{\pi}{3} \\ 0, & x < \frac{\pi}{3}. \end{cases}$

Periodic functions :-

A function $f(t)$ is said to be a periodic function if $f(t) = f(a+t) = f(2a+t) = \dots\dots$

Then the period of $f(t)$ is a. $\left\{ \begin{array}{l} \text{smallest +ve value} \\ \text{from } a, 2a, 3a, \dots\dots \end{array} \right\}$

Eg :- (1). $\sin x = \sin(2\pi + x) = \sin(4\pi + x) = \dots\dots$

$\therefore$ The period of $\sin x$ is 2π.

(2). $\tan x = \tan(\pi + x) = \tan(2\pi + x) = \tan(3\pi + x) = \dots\dots$

$\therefore$ The period of $\tan x$ is π.

(3). $\sin 3x = \sin 3\left(\frac{2\pi}{3} + x\right) = \sin 3\left(\frac{4\pi}{3} + x\right) = \dots\dots$

$\therefore$ The period of $\sin 3x$ is $\frac{2\pi}{3}$.

Laplace Transforms of Periodic functions :-

If $f(t)$ is a periodic function with period T, then

$$L\left[f(t)\right] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt.$$

Problems:

1). Find the L.T. of full-wave Rectifier

$$f(t) = E\cdot\sin\omega t, \quad 0 < t < \frac{\pi}{\omega}$$
$$= 0 \quad , \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}.$$

Sol:- $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\cdot f(t)\,dt \quad \left\{T = \frac{2\pi}{\omega}\right\}.$$

$$= \frac{1}{1-e^{-s\left(\frac{2\pi}{\omega}\right)}}\int_0^{2\pi/\omega} e^{-st}\cdot f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\cdot E\cdot\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot(0)\,dt\right].$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\cdot E\left[\frac{e^{-st}}{(-s)^2+\omega^2}\left[(-s)\cdot\sin\omega t - \omega\cdot\cos\omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{E}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-\pi s}{\omega}}}{s^2+\omega^2}[0+\omega] - \frac{1}{s^2+\omega^2}(0-\omega)\right].$$

$$= \frac{E}{(1)^2-\left(e^{\frac{-\pi s}{\omega}}\right)^2}\cdot\frac{\omega}{s^2+\omega^2}\left[e^{\frac{-\pi s}{\omega}}+1\right].$$

$$= \frac{E.\omega.\cancel{\left(1+e^{\frac{-\pi s}{\omega}}\right)}}{\cancel{\left(1+e^{\frac{-\pi s}{\omega}}\right)}\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)} = \frac{E\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}.$$

2). Find the L.T. of the saw-toothed wave of period T, given that $f(t) = \frac{t}{T}$ for $0<t<T$.

Sol: $$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot \frac{t}{T}\, dt$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{1}{T} \int_0^T \underset{u}{t}\, \underset{v}{e^{-st}} \cdot dt$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{1}{T} \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^T$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{1}{T} \left[\left(\frac{T e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{e^{-sT}}{-s} + \frac{1}{\cancel{1-e^{-sT}}} \cdot \frac{1}{T} \cdot \frac{1}{s^2} \left[\cancel{1-e^{-sT}} \right]$$

$$= \frac{1}{T} \cdot \frac{1}{s^2} - \frac{e^{-sT}}{s(1-e^{-sT})}$$

(3). Find the L.T. of the square-wave function of period 2a defined as $f(t) = k,\ 0<t<a$

$= -k,\ a<t<2a.$

Sol:- $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \qquad \{T = 2a\}$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot k\,dt + \int_a^{2a} e^{-st} \cdot (-k)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[k\left(\frac{e^{-st}}{-s}\right)_0^a - k\left(\frac{e^{-st}}{-s}\right)_a^{2a} \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\left(\frac{e^{-as}}{-s} - \frac{1}{-s}\right) + \left(\frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right) \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\frac{-2}{s} e^{-as} + \frac{1}{s}\left(1+e^{-2as}\right) \right]$$

(4). Compute $L[f(t)]$, if $f(t) = t^2,\ 0<t<2$ and $f(t+2) = f(t)$.

Evaluation of Integrals by using Laplace Transforms:—

(1) Evaluate $\int_0^\infty t\, e^{-3t} \sin t \, dt$.

Sol:— $L(\sin t) = \frac{1}{s^2+1} = \bar{f}(s)$.

$$L(t \cdot \sin t) = (-1)\frac{d}{ds}\bar{f}(s) = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= (-1)\frac{-1}{(s^2+1)^2} \cdot 2s = \frac{2s}{(s^2+1)^2}$$

By the definition of Laplace Transform

$$\int_0^\infty e^{-st} f(t)\, dt = L[f(t)].$$

Take $f(t) = t \sin t$ and $s = 3$.

$$\int_0^\infty e^{-3t}\, t \cdot \sin t \, dt = L[t \sin t]$$

$$= \frac{2s}{(s^2+1)^2} \text{ at } s = 3.$$

$$= \frac{6}{100} = \frac{3}{50}.$$

(2). Evaluate $\int_0^\infty \frac{e^{-\sqrt{2}t} \cdot \sinh t \cdot \sin t}{t}\, dt$ by using Laplace Transforms.

Sol:- $L[\sinh t \cdot \sin t] = L\left[\left(\frac{e^t - e^{-t}}{2}\right) \cdot \sin t\right]$

$$= \frac{1}{2}\left[L(e^t \cdot \sin t) - L(e^{-t} \cdot \sin t)\right].$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right] = \bar{F}(s).$$

$$L\left[\frac{\sinh t \cdot \sin t}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{(s-1)^2+1}\, ds - \frac{1}{2}\int_s^\infty \frac{1}{(s+1)^2+1}\, ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1)\right]_s^\infty - \frac{1}{2}\left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1)\right] - \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right] = \bar{F}(s).$$

By the defn. of Laplace Transform

$$\int_0^\infty e^{-st} \cdot f(t)\, dt = L\{f(t)\}.$$

Take $f(t) = \frac{\sinh t \cdot \sin t}{t}$ and $s = \sqrt{2}$.

$$\int_0^\infty \frac{e^{-\sqrt{2}t} \cdot \sinh t \cdot \sin t}{t}\, dt$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1)\right] - \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right] \text{ at } s = \sqrt{2}.$$

$$= \frac{\pi}{4} - \frac{1}{2}\tan^{-1}(\sqrt{2}-1) - \frac{\pi}{4} + \frac{1}{2}\tan^{-1}(\sqrt{2}+1)$$

$$= \frac{-1}{2}\cdot\frac{\pi}{8} + \frac{1}{2}\cdot\frac{3\pi}{8}$$

$$= \frac{3\pi}{16} - \frac{\pi}{16} = \frac{2\pi}{16} = \frac{\pi}{8}.$$

---

(3). Evaluate $\int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right) dt$ $\left\{\frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)\right\}$.

(4). Evaluate $\int_0^\infty \frac{\cos 5t - \cos 3t}{t}\, dt$ $\left\{\log \frac{3}{5}\right\}$,

(5). $\int_0^\infty t\, e^{-2t} \cdot \sin 3t\, dt.$ $\left\{\frac{12}{169}\right\}$

(6). Evaluate $\int_0^\infty t\, e^{-t} \cdot \sin^4 t\, dt$ $\left\{\frac{8(s+1)}{s(s^2+2s+17)}\right\}$.

(7). Prove that $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}$.

(8). Prove that $\displaystyle\int_0^\infty \frac{e^{-2t} \cdot \sinh t}{t}\, dt = \frac{1}{2} \log 3$.

(9). Prove that $\displaystyle\int_0^\infty \frac{e^{-t} \cdot \sin t}{t}\, dt = \frac{\pi}{4}$.

Unit - 4

INVERSE LAPLACE TRANSFORMS

If $L[f(t)] = \bar{f}(s)$, then $L^{-1}[\bar{f}(s)] = f(t)$.

This means $\bar{f}(s)$ is the L.T. of $f(t)$. and $f(t)$ is the Inverse L.T. of $\bar{f}(s)$.

L is the Laplace transform operator and L^{-1} is the Inverse L.T. operator.

Also $L L^{-1} = L^{-1} L = 1$.

Inverse L.T.s of standard functions

L.T.s	Inverse L.T.s
1). $L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1.$
2). $L(t) = \frac{1}{s^2}.$	$L^{-1}\left(\frac{1}{s^2}\right) = t.$
3). $L\left[t^n\right] = \frac{n!}{s^{n+1}}$	$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
4). $L\left(e^{at}\right) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$
5). $L\left(e^{-at}\right) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}.$
6). $L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$
7). $L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
8). $L(\sinh at) = \frac{a}{s^2-a^2}$	$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at.$
9). $L(\cosh at) = \frac{s}{s^2-a^2}$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

Properties :

Linearity Property :-

If $\bar{f}(s), \bar{g}(s)$ are two Inverse L.T.s of $f(t), g(t)$ respectively, then

$$L^{-1}\left[c_1 \bar{f}(s) \pm c_2 \bar{g}(s)\right] = c_1 L^{-1}\left[\bar{f}(s)\right] \pm c_2 L^{-1}\left[\bar{g}(s)\right].$$

First shifting theorem :-

$$\left\{\begin{array}{l} \text{If } L[f(t)] = \bar{f}(s), \text{ then } L\left(e^{at} f(t)\right) = \bar{f}(s-a) \\ \qquad\qquad\qquad\qquad\qquad\quad L\left(e^{-at} f(t)\right) = \bar{f}(s+a) \end{array}\right\}.$$

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$, then $L^{-1}\left[\bar{f}(s-a)\right] = e^{at} \cdot f(t)$

$$L^{-1}\left[\bar{f}(s+a)\right] = e^{-at} \cdot f(t).$$

Results :

(i). $L^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin bt \Rightarrow L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \cdot \sin bt.$

(ii). $L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt \Rightarrow L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cdot \cos bt.$

(iii). $L^{-1}\left(\frac{1}{s^2-b^2}\right) = \frac{1}{b}\sinh bt \Rightarrow L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \cdot \sinh bt.$

(iv). $L^{-1}\left(\frac{s}{s^2-b^2}\right) = \cosh bt \Rightarrow L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at} \cdot \cosh bt.$

Change of Scale property:-

$$\left\{\begin{array}{l} \text{If } L[f(t)] = \bar{f}(s), \text{ then } L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) \\ \qquad\qquad\qquad\qquad L\left[f\left(\frac{t}{a}\right)\right] = a.\bar{f}(as) \end{array}\right\}.$$

If $L^{-1}[\bar{f}(s)] = f(t)$, then $L^{-1}\left[\bar{f}\left(\frac{s}{a}\right)\right] = a.f(at)$.

$$L^{-1}[\bar{f}(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Formulae:-

1). $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}[\sin at - at\cos at]$.

2). $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$.

---

3). $L^{-1}[\bar{f}(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right]$.

Problems:

Find Inverse Laplace Transforms to the following functions:

1. (a). $\dfrac{s^2-3s+4}{s^3}$ (b). $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-8}{9-s^2}$.

(c). $\dfrac{3s-8}{4s^2+25}$ (d). $\dfrac{s^2+9s-9}{s^3-9s}$.

Sol: (a). $L^{-1}\left[\dfrac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\dfrac{1}{s} - \dfrac{3}{s^2} + \dfrac{4}{s^3}\right]$

$$= L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right)$$

$$= 1 - 3t + 4\cdot\frac{t^2}{2!} = 1-3t+2t^2.$$

(b). $L^{-1}\left[\dfrac{2s-5}{4s^2+25} + \dfrac{4s-8}{9-s^2}\right] = L^{-1}\left[\dfrac{2s-5}{4\left(s^2+\frac{25}{4}\right)} - \dfrac{4s-8}{s^2-9}\right]$.

$$= L^{-1}\left[\frac{2s}{4\left(s^2+\frac{25}{4}\right)} - \frac{5}{4\left(s^2+\frac{25}{4}\right)} - \frac{4s}{s^2-9} + \frac{8}{s^2-9}\right].$$

$$= \frac{1}{2}\cdot L^{-1}\left[\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right] - \frac{5}{4}L^{-1}\left[\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right] - 4L^{-1}\left(\frac{s}{s^2-9}\right) + 8L^{-1}\left(\frac{1}{s^2-9}\right)$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{5}{4}\cdot\frac{1}{5/2}\sin\frac{5}{2}t - 4\cosh 3t + 8\cdot\frac{1}{3}\sinh 3t$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t - 4\cosh 3t + \frac{8}{3}\sinh 3t$$

(2). Find Inverse L.T.s to the following functions.
(Partial fractions):

(a). $\frac{s}{(2s-1)(3s-1)}$

Sol: $= \frac{s}{2\left(s-\frac{1}{2}\right)3\left(s-\frac{1}{3}\right)} = \frac{1}{6}\left[\frac{s}{\left(s-\frac{1}{2}\right)\left(s-\frac{1}{3}\right)}\right]$.

Let $\frac{s}{\left(s-\frac{1}{2}\right)\left(s-\frac{1}{3}\right)} = \frac{A}{s-\frac{1}{2}} + \frac{B}{s-\frac{1}{3}} = \frac{A\left(s-\frac{1}{3}\right)+B\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)\left(s-\frac{1}{3}\right)}$

Comparing the numerators

$$s = A\left(s-\frac{1}{3}\right) + B\left(s-\frac{1}{2}\right)$$

Put $s=\frac{1}{3} \Rightarrow \frac{1}{3} = 0 + B\left(\frac{1}{3}-\frac{1}{2}\right) \Rightarrow \frac{1}{3} = B\left(\frac{-1}{6}\right)$

$\Rightarrow \boxed{B=-2}$

Put $s=\frac{1}{2} \Rightarrow \frac{1}{2} = A\left(\frac{1}{2}-\frac{1}{3}\right)+0 \Rightarrow \frac{1}{2} = A\left(\frac{1}{6}\right) \Rightarrow \boxed{A=3}$

$\therefore L^{-1}\left[\frac{s}{(2s-1)(3s-1)}\right] = \frac{1}{6} L^{-1}\left[\frac{3}{s-\frac{1}{2}} + \frac{-2}{s-\frac{1}{3}}\right]$.

$$= \frac{3}{6} L^{-1}\left(\frac{1}{s-\frac{1}{2}}\right) - \frac{2}{6} L^{-1}\left(\frac{1}{s-\frac{1}{3}}\right)$$

$$= \frac{1}{2} e^{\frac{1}{2}t} - \frac{1}{3} e^{\frac{1}{3}t}.$$

(b). $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$

Sol: $= \dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)} = \dfrac{A}{(s-7)} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$$= \frac{A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)}{(s-7)(s-2)(s-3)}.$$

$\Rightarrow s^2-10s+13 = A(s-2)(s-3)+B(s-7)(s-3)+C(s-7)(s-2)$

Put $s=7 \Rightarrow -8 = A(5)(4)+0+0 \Rightarrow 20A=-8 \Rightarrow \boxed{A=\frac{-2}{5}}$

Put $s=2 \Rightarrow -3 = 0+B(-5)(-1)+0 \Rightarrow -3=5B \Rightarrow \boxed{B=\frac{-3}{5}}$

Put $s=3 \Rightarrow -8 = 0+0+C(-4)(1) \Rightarrow -4C=-8 \Rightarrow \boxed{C=2}$

$\therefore L^{-1}\left[\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right] = L^{-1}\left[\dfrac{-2/5}{s-7} + \dfrac{-3/5}{s-2} + \dfrac{2}{s-3}\right]$

$= -\dfrac{2}{5}L^{-1}\left(\dfrac{1}{s-7}\right) - \dfrac{3}{5}L^{-1}\left(\dfrac{1}{s-2}\right) + 2L^{-1}\left(\dfrac{1}{s-3}\right)$

$= -\dfrac{2}{5}e^{7t} - \dfrac{3}{5}e^{2t} + 2e^{3t}.$

(c). $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$

(d). $\dfrac{s}{(s+1)(s+2)}$

(e). $\dfrac{3s+2}{s^2-s-2}$

(f). $\dfrac{1}{s(s^2-1)}$.

(g). $\dfrac{1}{s^2-5s+6}$.

(h). $\dfrac{s^2+2s-4}{(s^2+9)(s-5)}$.

(i). $\dfrac{2s+3}{s^3-6s^2+11s-6}$

(3). Find Inverse L.T.s to the following functions (First shifting theorem):

(a). $\dfrac{s+2}{s^2-4s+13}$

Sol: $= \dfrac{s+2}{(s-2)^2-4+13} = \dfrac{s+2}{(s-2)^2+9} = \dfrac{s-2+4}{(s-2)^2+9}$.

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+9}\right] + 4\,L^{-1}\left[\frac{1}{(s-2)^2+9}\right].$$

$$= e^{2t}\cdot L^{-1}\left[\frac{s}{s^2+9}\right] + 4\,e^{2t}\cdot L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{2t}\cdot \cos 3t + 4\cdot e^{2t}\cdot \frac{1}{3}\cdot \sin 3t.$$

$$= e^{2t}\left[\cos 3t + \frac{4}{3}\sin 3t\right].$$

(b). $\dfrac{3s+2}{s^2-s-2}$ {Partial fractions}

Sol: $= \dfrac{3s+2}{s^2-2s\left(\frac{1}{2}\right)+\frac{1}{4}-\frac{1}{4}-2} = \dfrac{3s+2}{\left(s-\frac{1}{2}\right)^2-\left(\frac{1}{4}+2\right)}$

$$= \frac{3\left(s-\frac{1}{2}\right)+\frac{3}{2}+2}{\left(s-\frac{1}{2}\right)^2-\frac{9}{4}} = \frac{3\left(s-\frac{1}{2}\right)+\frac{7}{2}}{\left(s-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}.$$

$$\therefore \bar{L}^1\left[\frac{3s+2}{s^2-s-2}\right] = 3\,\bar{L}^1\left[\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2-(\frac{3}{2})^2}\right]+\frac{7}{2}\bar{L}^1\left[\frac{1}{(s-\frac{1}{2})^2-(\frac{3}{2})^2}\right]$$

$$= 3\,e^{\frac{1}{2}t}\,\bar{L}^1\left[\frac{s}{s^2-(\frac{3}{2})^2}\right]+\frac{7}{2}\,e^{\frac{1}{2}t}\,\bar{L}^1\left[\frac{1}{s^2-(\frac{3}{2})^2}\right].$$

$$= 3\,e^{\frac{t}{2}}\cosh\frac{3}{2}t+\frac{7}{2}\,e^{\frac{t}{2}}\cdot\frac{1}{3/2}\sinh\frac{3}{2}t$$

$$= e^{\frac{t}{2}}\left[3\cosh\frac{3}{2}t+\frac{7}{3}\sinh\frac{3}{2}t\right].$$

(c). $\dfrac{s^2}{(s-2)^3}$.

<u>Sol</u>: $$= \frac{(s-2)^2+4s-4}{(s-2)^3} = \frac{(s-2)^2+4(s-2)+4}{(s-2)^3}$$

$$= \frac{1}{s-2}+\frac{4}{(s-2)^2}+\frac{4}{(s-2)^3}$$

$$\therefore \bar{L}^1\left[\frac{s^2}{(s-2)^3}\right] = \bar{L}^1\left[\frac{1}{s-2}\right]+4\,\bar{L}^1\left[\frac{1}{(s-2)^2}\right]+4\,\bar{L}^1\left[\frac{1}{(s-2)^3}\right].$$

$$= e^{2t}+4\,e^{2t}\cdot\bar{L}^1\left[\frac{1}{s^2}\right]+4\,e^{2t}\,\bar{L}^1\left[\frac{1}{s^3}\right].$$

$$= e^{2t}+4e^{2t}\cdot t+4e^{2t}\cdot\frac{t^2}{2!}$$

$$= e^{2t}\left[1+4t+2t^2\right].$$

(d). $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Sol: $= \dfrac{(s+2)^2}{[(s+2)^2+4]^2}$

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = L^{-1}\left[\frac{(s+2)^2}{[(s+2)^2+4]^2}\right].$$

$$= e^{-2t} \cdot L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = e^{-2t} \cdot L^{-1}\left[\frac{s^2+4-4}{(s^2+4)^2}\right]$$

$$= e^{-2t} \cdot L^{-1}\left[\frac{1}{s^2+4}\right] - 4e^{-2t} \cdot L^{-1}\left[\frac{1}{(s^2+4)^2}\right].$$

$$= e^{-2t} \cdot \frac{1}{2}\sin 2t - 4e^{-2t} \cdot \frac{1}{2\times 8}\left[\sin 2t - 2t\cos 2t\right].$$

$$\left\{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}\left[\sin at - at\cos at\right]\right\}.$$

$$= e^{-2t}\left[\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{2t\cos 2t}{4}\right].$$

$$= \frac{e^{-2t}}{4}\left[\sin 2t + 2t\cos 2t\right].$$

(e). $\dfrac{3s-2}{s^2-4s+20}$

(f). $\dfrac{s+2}{(s^2+4s+5)^2}$

(g). $\dfrac{3s}{s^2+2s-8}$.

(h). $\dfrac{4s+5}{(s-1)^2(s+2)}$.

(i). $\dfrac{s}{s^4+4a^4}$.

(4). Find Inverse L.T.s to the following functions:

$$\boxed{\bar{L}^{1}\left[\bar{F}(s)\right] = \frac{-1}{t}\,\bar{L}^{1}\left[\frac{d}{ds}\,\bar{f}(s)\right]}$$

(a). $\log\left(\frac{s^2+1}{(s-1)^2}\right)$.

Sol: $\bar{F}(s) = \log(s^2+1) - \log\left((s-1)^2\right)$.

$$= \log(s^2+1) - 2\cdot\log(s-1).$$

$$\frac{d}{ds}\bar{F}(s) = \frac{1}{s^2+1}\cdot 2s - 2\cdot\frac{1}{s-1}$$

$$\bar{L}^{1}\left[\bar{f}(s)\right] = \frac{-1}{t}\,\bar{L}^{1}\left[\frac{d}{ds}\,\bar{f}(s)\right]$$

$$= \frac{-1}{t}\,\bar{L}^{1}\left[\frac{2s}{s^2+1} - \frac{2}{s-1}\right].$$

$$= \frac{-2}{t}\left[\bar{L}^{1}\left(\frac{s}{s^2+1}\right) - \bar{L}^{1}\left(\frac{1}{s-1}\right)\right].$$

$$= \frac{-2}{t}\left[\cos t - e^{t}\right].$$

$$= \frac{2}{t}\left(e^{t} - \cos t\right).$$

(b). $\cot^{-1}\left(\frac{s}{2}\right)$.

Sol: $\bar{F}(s) = \cot^{-1}\left(\frac{s}{2}\right)$.

$$\frac{d}{ds}\bar{F}(s) = \frac{-1}{1+\frac{s^2}{4}}\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{-4}{s^2+4} = \frac{-2}{s^2+4}.$$

$$L^{-1}\left[\bar{F}(s)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\bar{F}(s)\right]$$

$$= \frac{-1}{t}\cdot L^{-1}\left[\frac{-2}{s^2+4}\right] = \frac{2}{t}\cdot L^{-1}\left[\frac{1}{s^2+4}\right].$$

$$= \frac{2}{t}\cdot\frac{1}{2}\cdot \sin 2t = \frac{1}{t}\sin 2t.$$

(c). $\log\left(\frac{s+1}{s-1}\right)$

(d). $\log\left(\frac{s+3}{s+4}\right)$.

(e). $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$.

(f). $\log\left(\frac{s^2+1}{s(s+1)}\right)$.

(g). $\cot^{-1}\left(\frac{s+a}{b}\right)$.

(h). $\tan^{-1}(s+1)$

(i). $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Convolution theorem

Convolution product (or) Generalised product:-

Let $f(t)$, $g(t)$ are two functions defined for $t>0$. Then the convolution product of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du.$$

Convolution theorem:-

If $L[f(t)] = \bar{f}(s)$ and $L[g(t)] = \bar{g}(s)$,

then $L[f(t) * g(t)] = \bar{f}(s) \cdot \bar{g}(s)$.

Convolution theorem of Inverse L.Ts:-

If $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$,

then $L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t)$

$$= \int_0^t f(u) \cdot g(t-u)\, du.$$

Apply convolution theorem to find Inverse Laplace Transforms:—

1). $\frac{1}{s^3(s^2+1)}$.

Sol: Let $\bar{f}(s) = \frac{1}{s^3}$ and $\bar{g}(s) = \frac{1}{s^2+1}$.

$$L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2!} = \frac{t^2}{2} = f(t).$$

$$L^{-1}\left[\bar{g}(s)\right] = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = g(t).$$

By convolution theorem

$$L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = f(t) * g(t).$$

$$\Rightarrow L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t f(u).g(t-u)\,du.$$

$$= \int_0^t \frac{u^2}{2}.\sin(t-u)\,du.$$

$$= \frac{1}{2}\left[u^2.\left(\frac{-\cos(t-u)}{-1}\right) - (2u)\left(\frac{\sin(t-u)}{-1}\right) + (2)\left(\frac{-\cos(t-u)}{1}\right)\right]_0^t$$

$$= \frac{1}{2}\left[u^2(\cos(t-u)) + (2u)(\sin(t-u)) - 2\cos(t-u)\right]_0^t$$

$$= \frac{1}{2}\left[(t^2+0-2) - (0+0-2\cos t)\right]$$

$$= \frac{1}{2}\left[t^2-2+2\cos t\right] = \frac{t^2}{2} - 1 + \cos t.$$

(2). $\dfrac{s}{(s+2)(s^2+9)}$.

Sol: Let $\bar{f}(s) = \dfrac{1}{s+2}$ and $\bar{g}(s) = \dfrac{s}{s^2+9}$.

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t} = f(t).$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t = g(t).$$

By convolution theorem

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t).$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s+2)(s^2+9)}\right] = \int_0^t f(u)\cdot g(t-u)\,du.$$

$$= \int_0^t e^{-2u}\cdot\cos 3(t-u)\,du.$$

$$\boxed{\int e^{ax}\cdot\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cdot\cos(bx+c) + b\cdot\sin(bx+c)\right]}$$

$\boxed{a=-2,\ b=-3}$

$$= \int_0^t e^{-2u}\cdot\cos(-3u+3t)\,du$$

$$= \left[\frac{e^{-2u}}{4+9}\left[(-2)\cos(-3u+3t) + (-3)\cdot\sin(-3u+3t)\right]\right]_0^t$$

$$= \frac{e^{-2t}}{13}(-2+0) - \frac{1}{13}(-2\cos 3t - 3\sin 3t).$$

$$= \frac{1}{13}\left(2\cos 3t + 3\sin 3t - 2e^{-2t}\right).$$

(3). Find $\bar{L}^{1}\left[\dfrac{1}{(s^2+4s+13)^2}\right]$.

Sol:- Let $\bar{f}(s) = \dfrac{1}{s^2+4s+13}$ and $\bar{g}(s) = \dfrac{1}{s^2+4s+13}$.

$$\bar{L}^{1}[\bar{f}(s)] = \bar{L}^{1}\left[\frac{1}{s^2+4s+13}\right] = \bar{L}^{1}\left[\frac{1}{(s+2)^2+9}\right].$$

$$= e^{-2t}\cdot \bar{L}^{1}\left[\frac{1}{s^2+9}\right] = e^{-2t}\cdot\frac{1}{3}\sin 3t = f(t).$$

$$\bar{L}^{1}[\bar{g}(s)] = \bar{L}^{1}\left[\frac{1}{s^2+4s+13}\right] = e^{-2t}\cdot\frac{1}{3}\sin 3t = g(t).$$

By convolution theorem

$$\bar{L}^{1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t).$$

$$\Rightarrow \bar{L}^{1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \int_0^t f(u)\cdot g(t-u)\,du \qquad \left\{\begin{aligned} &e^{-2u}\cdot e^{2u} \\ &= e^{-2u+2u} = e^{0} = 1 \end{aligned}\right\}.$$

$$= \int_0^t e^{-2u}\cdot\frac{1}{3}\sin 3u\cdot e^{-2(t-u)}\cdot\frac{1}{3}\sin 3(t-u)\,du$$

$$= \frac{1}{9}\int_0^t e^{-2t}\cdot\sin 3u\cdot\sin(3t-3u)\,du \qquad \boxed{\begin{aligned} &\sin A\sin B \\ &= \frac{\cos(A-B)-\cos(A+B)}{2} \\ &A = 3u \quad B = 3t-3u \end{aligned}}$$

$$= \frac{1}{9}\int_0^t e^{-2t}\cdot\left(\frac{\cos(6u-3t)-\cos(3t)}{2}\right)du$$

$$= \frac{1}{18}\int_0^t e^{-2t}\cdot\cos(6u-3t)\,du - \frac{1}{18}\int_0^t e^{-2t}\cdot\cos 3t\,du.$$

$$= \frac{e^{-2t}}{18}\left[\frac{\sin(6u-3t)}{6}\right]_0^t - \frac{e^{-2t}}{18}\cdot\cos 3t\cdot(u)_0^t$$

$$= \frac{e^{-2t}}{108}\left[\sin 3t - 6t\cos 3t\right].$$

(+). $\dfrac{s^2}{(s^2+4)(s^2+9)}$

Sol: Let $\bar{f}(s) = \dfrac{s}{s^2+4}$ and $\bar{g}(s) = \dfrac{s}{s^2+9}$

$$\bar{L}^1\left[\bar{F}(s)\right] = \bar{L}^1\left[\frac{s}{s^2+4}\right] = \cos 2t = f(t).$$

$$\bar{L}^1\left[\bar{g}(s)\right] = \bar{L}^1\left[\frac{s}{s^2+9}\right] = \cos 3t = g(t).$$

By convolution theorem,

$$\bar{L}^1\left[\bar{F}(s)\cdot\bar{g}(s)\right] = f(t) * g(t).$$

$$\Rightarrow \bar{L}^1\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \cos 2u\cdot\cos 3(t-u)\,du.$$

$$\boxed{\cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}}$$

$A = 2u \quad B = 3t-3u$

$A+B = 3t-u$

$A-B = 5u-3t$

$$= \int_0^t \cos 2u\cdot\cos(3t-3u)\,du.$$

$$= \int_0^t \frac{\cos(3t-u)+\cos(5u-3t)}{2}\,du$$

$$= \frac{1}{2}\int_0^t \cos(-u+3t)\,du + \frac{1}{2}\int_0^t \cos(5u-3t)\,du$$

$$= \frac{1}{2}\cdot\left(\frac{\sin(-u+3t)}{-1}\right)_0^t + \frac{1}{2}\left(\frac{\sin(5u-3t)}{5}\right)_0^t$$

$$= \frac{-1}{2}(\sin 2t - \sin 3t) + \frac{1}{10}(\sin 2t + \sin 3t)$$

$$= (\sin 2t)\left[\frac{-1}{2}+\frac{1}{10}\right] + (\sin 3t)\left[\frac{1}{2}+\frac{1}{10}\right] = \underline{\frac{-2}{5}\sin 2t + \frac{3}{5}\sin 3t}.$$

(5). $\dfrac{1}{s^2(s^2+a^2)}$.

(6). $\dfrac{1}{s^2(s+1)^2}$.

(7). $\dfrac{1}{(s+a)(s+b)}$.

(8). $\dfrac{s}{(s^2+a^2)^2}$.

(9). Evaluate $L^{-1}\left[\dfrac{1}{(s^2+1)(s^2+9)}\right]$.

Also find $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}\right]$.

(10). $\dfrac{1}{s^2(s+2)}$.

(11). $\dfrac{s+3}{(s^2+6s+13)^2}$.

Applications of Laplace transforms to Ordinary Differential equations of first and second order with constant coefficients:-

The main application of L.T.s is that we ~~eats~~ can solve O.D.E.s with constant coefficients without finding complementary function (C.F) and Particular Integral (P.I).

$$
\begin{tikzcd}
\text{O.D.E with initial conditions} \arrow[r, bend left] \arrow[d] & \text{Solution of O.D.E.} \\
\text{L.T.s} \arrow[d] & \text{Inverse L.T.s} \arrow[u] \\
\text{Algebraic equation} \arrow[r] & \text{Solution of A.E} \arrow[u]
\end{tikzcd}
$$

L.T.s of Derivatives:—

If $L[f(t)] = \bar{F}(s)$, then

$$L[f'(t)] = s.\bar{F}(s) - f(0).$$

$$L[f'(t)] = L[y']$$
$$L[f(t)] = L[y]$$
$$f(0) = y(0)$$

$$\Rightarrow \boxed{L[y'] = s.L(y) - y(0).}$$

Illy

$$\boxed{L[y''] = s^2.L(y) - s.y(0) - y'(0)}$$

$$\boxed{L[x'] = s.L(x) - x(0)}$$

$$\boxed{L[x''] = s^2 L(x) - s.x(0) - x'(0)}$$

Problems

Solve the D.E. $\frac{d^2x}{dt^2} + 9x = \sin t$ using L.T.s

given that $x(0) = 1$, $x\left(\frac{\pi}{2}\right) = 1$.

Sol: Given D.E. is $x'' + 9x = \sin t$, given $x(0) = 1$.

Let $x'(0) = a$.

Taking Laplace Transforms on both sides

$$L(x'') + 9L(x) = L(\sin t).$$

$$\Rightarrow s^2 . L(x) - s.x(0) - x'(0) + 9L(x) = L(\sin t)$$

$$\Rightarrow (s^2+9). L(x) - s(1) - a = \frac{1}{s^2+1}.$$

$$\Rightarrow (s^2+9).L(x) = \frac{1}{s^2+1} + s + a.$$

$$\Rightarrow L(x) = \frac{1}{(s^2+1)(s^2+9)} + \frac{s}{s^2+9} + \frac{a}{s^2+9}.$$

Applying Inverse L.T.s on both sides,

$$x = L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + a.L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$\Rightarrow x = \frac{1}{8} L^{-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] + \cos 3t + a.\frac{1}{3}\sin 3t.$$

$$\Rightarrow x = \frac{1}{8}\sin t - \frac{1}{8}.\frac{1}{3}\sin 3t + \cos 3t + \frac{a}{3}\sin 3t.$$

Given $x\left(\frac{\pi}{2}\right)=1$ means $x=1$ when $t=\frac{\pi}{2}$.

$$\therefore\ 1=\frac{1}{8}(1)-\frac{1}{24}(-1)+0+\frac{a}{3}(-1).$$

$$\Rightarrow\ 1=\frac{1}{6}-\frac{a}{3}\ \Rightarrow\ \frac{a}{3}=\frac{1}{6}-1=\frac{-5}{6}.$$

$$\Rightarrow\ \boxed{a=\frac{-5}{2}}.$$

$\therefore$ The solution is given by

$$x=\frac{1}{8}\sin t-\frac{1}{24}\sin 3t+\cos 3t+\frac{1}{3}\left(\frac{-5}{2}\right)\sin 3t.$$

$$\Rightarrow\ x=\frac{1}{8}\sin t-\frac{7}{8}\sin 3t+\cos 3t.$$

Applications of L.Ts :—

Solve $(D^2+4D+3)y = e^{-t}$ given $y=1$, $\frac{dy}{dt}=1$ at $t=0$,

Sol:- Given D.E is

$$y''+4y'+3y = e^{-t}.$$

Taking L.T. on both sides.

$$L[y'']+4L[y']+3L[y] = L[e^{-t}].$$

$$s^2L(y) - s\cdot y(0) - y'(0) + 4sL(y) - 4y(0) + 3L(y) = \frac{1}{s+1}.$$

$$(s^2+4s+3)L(y) - s(1) - 1 - 4(1) = \frac{1}{s+1}.$$

$$(s^2+4s+3)\cdot L(y) = \frac{1}{s+1} + s + 5.$$

$$= \frac{1+s^2+s+5s+5}{(s+1)}$$

$$L(y) = \frac{s^2+6s+6}{(s+1)(s^2+4s+3)} = \frac{s^2+6s+6}{(s+1)(s+1)(s+3)}$$

$$= \frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s+3}.$$

$$= \frac{7/4}{s+1} + \frac{1/2}{(s+1)^2} + \frac{-3/4}{s+3}.$$

Taking Inverse L.T. on both sides.

$$y = \frac{7}{4}\cdot L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^2}\right) - \frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}\cdot e^{-t}L^{-1}\left(\frac{1}{s^2}\right) - \frac{3}{4}\cdot e^{-3t}$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}t\,e^{-t} - \frac{3}{4}e^{-3t}.$$

(2). $y'' - 3y' + 2y = 4t + e^{3t}$, when $y(0) = 1$, $y'(0) = -1$

~~(3).~~ $\cancel{(D^2 - 3D + 2)y = 4e^{2t}}$ ~~with~~ $\cancel{y(0) = -3,\ y'(0) = 5}$

(3). $(D^2+1)y = 6\cos 2t,\ t > 0$, if $y = 3,\ Dy = 1$ when $t = 0$.

Sol:— Given $y'' + y = 6\cos 2t$.

$$L(y'') + L(y) = 6L(\cos 2t).$$

$$s^2 L(y) - s\,y(0) - y'(0) + L(y) = 6 \cdot \frac{s}{s^2+4}.$$

$$(s^2+1)\,L(y) - s(3) - 1 = 6 \cdot \frac{s}{s^2+4}.$$

$$(s^2+1)\,L(y) = \frac{6s}{s^2+4} + 3s + 1.$$

$$L(y) = \frac{6s}{(s^2+4)(s^2+1)} + 3\,\frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

$$= 2\left[\frac{-s}{s^2+4} + \frac{s}{s^2+1}\right] + 3\,\frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Taking Inverse L.T.

$$y = -2L^{-1}\left(\frac{s}{s^2+4}\right) + 2L^{-1}\left(\frac{s}{s^2+1}\right) + 3L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right).$$

$$\Rightarrow y = -2\cos 2t + 2\cos t + 3\cos t + \sin t.$$

$$\Rightarrow y = 5\cos t - 2\cos 2t + \sin t.$$

NOTE:— $$\frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right].$$

$$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right].$$

S-07.

(4). $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \bar{e}^{t}\sin t$, given

$y = 0,\ \frac{dy}{dt} = 1$, when $t = 0$.

Sol: Given $y'' + 2y' + 5y = \bar{e}^{t}\sin t$.

Taking L.T.

$$L(y'') + 2L(y') + 5L(y) = L(\bar{e}^{t}\sin t)$$

$$s^2L(y) - sy(0) - y'(0) + 2sL(y) - 2y(0) + 5L(y) = \frac{1}{(s+1)^2+1}.$$

$$(s^2+2s+5)\cdot L(y) - s(0) - 1 - 2(0) = \frac{1}{s^2+2s+2}.$$

$$(s^2+2s+5)L(y) = \frac{1}{s^2+2s+2} + 1.$$

$$L(y) = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

Taking Inverse L.Ts.

$$y = \frac{1}{3}\bar{L}^{1}\left(\frac{1}{s^2+2s+2}\right) - \frac{1}{3}\bar{L}^{1}\left(\frac{1}{s^2+2s+5}\right) + \bar{L}^{1}\left(\frac{1}{s^2+2s+5}\right)$$

$$= \frac{1}{3}\cdot\bar{L}^{1}\left(\frac{1}{(s+1)^2+1}\right) - \frac{1}{3}\bar{L}^{1}\left(\frac{1}{(s+1)^2+4}\right) + \bar{L}^{1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= \frac{1}{3}\cdot\bar{e}^{t}\bar{L}^{1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\bar{e}^{t}\bar{L}^{1}\left(\frac{1}{s^2+4}\right) + \bar{e}^{t}\bar{L}^{1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{\bar{e}^{t}}{3}\sin t - \frac{\bar{e}^{t}}{3}\frac{1}{2}\sin 2t + \bar{e}^{t}\frac{1}{2}\sin 2t.$$

$$= \frac{\bar{e}^{t}}{3}(\sin t) + \frac{2}{3}\bar{e}^{t}\sin 2t$$

$$= \frac{\bar{e}^{t}}{3}(\sin t + \sin 2t).$$

1). Use Transform method to solve

Refer $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x = 2, \frac{dx}{dt} = -1$ at $t=0$.

Ans:- $x = 2e^t - 3te^t + \frac{t^2}{2}e^t \Rightarrow x = e^t\left[\frac{t^2}{2} - 3t + 2\right]$.

---

2). Solve $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$. by using L-Ts.

Ans:- $\bar{y}(s) = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$.

$\Rightarrow y = \cos t + t - \sin t$.

---

3). Solve $y'' - 3y' + 2y = e^{3t}$ when $y(0) = 1$ and $y'(0) = 0$ using L.Ts.

Ans:- $\bar{y}(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$.

$\Rightarrow y = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$.

4). $y'' + 25y = 10\cos 5t$ given that $y(0) = 2$, $y'(0) = 0$.

Sol:- $y = 2\cos 5t + t\sin 5t$

5). $(D^2-1)x = a\cosh t$, $x(0) = x'(0) = 0$ by using L-Ts.

Sol:- $x = \frac{at}{2}\sinh t$.

$\left\{ L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{t}{2a}\sinh at \right\}$.

6). Solve $y'' + 2y' + 5y = 5(t-2)$, $y(0) = 0$, $y'(0) = 0$.

Ans:- $\bar{y}(s) = \dfrac{5 - 10s}{s^2(s^2+2s+5)}$

$$\Rightarrow y = \frac{-12}{5} + \frac{12}{5} e^{-t} \cos 2t + \frac{7}{10} e^{-t} \sin 2t + t$$

7). Solve $(D^2+\omega^2)y = \cos \omega t$, $t > 0$ given that

$y = 0$ and $Dy = 0$ at $t = 0$.

Ans.- $y = \dfrac{1}{2\omega} \sin \omega t$.

8). $(D^2+1)x = t\cos 2t$, $x = Dx = 0$ at $t = 0$.

Ans:- $x = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3} t\cos 2t$.

Repeated.

9). $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t$, $y = \dfrac{dy}{dt} = 0$ when $t = 0$.

Sol:- $y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$.

10). Solve $y'' - 4y' + 4y = 0$; $y(0) = 0$, $y'(0) = 3$

using Laplace Transforms. $\{y = 3te^{2t}\}$.

11). Solve $(D^2+4D+5)y = 5$. $y(0) = 0$, $y'(0) = 0$.

12). $(D^2+1)y = \sin t \cdot \sin 2t$, $y = 1$, $Dy = 0$ when $t = 0$.

~~Unit - III~~ CO5

Fourier Series: Determination of fourier co-efficients - fourier series of Even and Odd functions - fourier series in an arbitrary interval - Half-range fourier sine and Cosine expansions.

Euler's formulae:- The Fourier series for the function $f(x)$ in the interval $l \le x \le l+2\pi$ can be expressed in fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

where a_0, a_n, b_n are constants and are given by $a_0 = \frac{1}{\pi}\int_l^{l+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_l^{l+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_l^{l+2\pi} f(x)\sin nx\,dx.$$

Proof:- Let $f(x)$ be represented in the interval $[l, l+2\pi)$ by fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -①$$

Let us assume that the series is uniformly cgt in the interval $[l, l+2\pi]$. Then the series can be integrated ~~bu~~ term by term.

To evaluate a_0: Integrate ① w.r.t 'x' from $x=l$ to $x=l+2\pi$.

$$\therefore \int_l^{l+2\pi} f(x)\,dx = \frac{a_0}{2}\int_l^{l+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_l^{l+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_l^{l+2\pi} \sin nx\,dx$$

$$= \frac{a_0}{2}\cdot[x]_l^{l+2\pi} + \sum_{n=1}^{\infty} a_n \left[\frac{\sin nx}{n}\right]_l^{l+2\pi} + \sum_{n=1}^{\infty} b_n\left[-\frac{\cos nx}{n}\right]_l^{l+2\pi}$$

$$= \frac{a_0}{\cancel{2}}(\cancel{l}+2\pi-\cancel{l}) + \sum_{n=1}^{\infty} a_n\left(\frac{\sin n(l+2\pi)}{n} - \frac{\sin nl}{n}\right) + \sum_{n=1}^{\infty} b_n\left(-\frac{\cos n(l+2\pi)}{n} + \frac{\cos nl}{n}\right)$$

$$= a_0\pi + \sum_{n=1}^{\infty} a_n\left(\cancel{\frac{\sin nl}{n}} - \cancel{\frac{\sin nl}{n}}\right) + \sum_{n=1}^{\infty} b_n\left(-\cancel{\frac{\cos nl}{n}} + \cancel{\frac{\cos nl}{n}}\right)$$

$$= a_0\pi$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_l^{l+2\pi} f(x)\,dx$$

To evaluate a_n: Multiply ① with $\cos mx$ and then integrate w.r.t 'x' from l to $l+2\pi$.

$$\therefore \int_l^{l+2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_l^{l+2\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_l^{l+2\pi}\cos nx\cos mx\,dx + \sum_{n=1}^{\infty} b_n\int_l^{l+2\pi}\sin nx\cos mx\,dx$$

$$\int_l^{l+2\pi} f(x)\cos nx\,dx = 0 + \sum_{n=1}^{\infty} a_n\int_l^{l+2\pi}\cos^2 nx\,dx + 0 \qquad \because \int_l^{l+2\pi}\cos nx\,dx = 0$$

$$\& \int_l^{l+2\pi}\sin nx\cos mx\,dx = 0 \text{ if } m=n$$

$$= a_n \int_l^{l+2\pi} \frac{1+\cos 2nx}{2}\, dx$$

$$= \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_l^{l+2\pi}$$

$$= \frac{a_n}{2}\left(\cancel{l}+2\pi + \frac{\sin 2n(l+2\pi)}{2n} - \cancel{l} - \frac{\sin 2nl}{2n}\right)$$

$$= \frac{a_n}{2}\left(2\pi + \cancel{\frac{\sin 2nl}{2n}} - \cancel{\frac{\sin 2nl}{2n}}\right)$$

$$= a_n \pi$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_l^{l+2\pi} f(x) \cos nx\, dx.$$

<u>To evaluate b_n:</u> Multiply ① by $\sin mx$ and then integrate w.r.t 'x' from l to $l+2\pi$

$$\therefore \int_l^{l+2\pi} f(x) \sin mx\, dx = \frac{a_0}{2}\int_l^{l+2\pi} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_l^{l+2\pi} \cos nx \sin mx\, dx + \sum_{n=1}^{\infty} b_n \int_l^{l+2\pi} \sin nx \sin nx\, dx$$

$$= 0 + 0 + \sum_{n=1}^{\infty} b_n \int_l^{l+2\pi} \sin^2 nx\, dx$$

$$= \sum_{n=1}^{\infty} b_n \int_l^{l+2\pi} \frac{1-\cos 2nx}{2}\, dx$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{2}\left[x - \frac{\sin 2nx}{2n}\right]_l^{l+2\pi}$$

$$= \sum_{n=1}^{\infty} b_n \cdot \pi$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_l^{l+2\pi} f(x) \sin nx\, dx.$$

<u>Dirichlet Conditions:</u>- A function $f(x)$ can be expanded in fourier series provided in the interval i.) $f(x)$ and its integrals are finite and single valued.

ii.) $f(x)$ has finite no. of discontinuities.

iii.) $f(x)$ has finite no. of maxima and minima.

These conditions are known as Dirichlet conditions.

<u>Convergence of fourier series:</u>

1.) If $x=a$ is a point of continuity of the function $f(x)$, then the fourier series at $x=a$ is convergent to $f(a)$

2.) If $x=a$ is a point of discontinuity of the function $f(x)$, then the fourier series at $x=a$ is convergent to $\frac{1}{2}\left(f(a+) + f(a-)\right)$

3.) If the function $f(x)$ is defined in the interval $-\pi < x < \pi$ and if the function $f(x)$ has discontinuities at $x=-\pi$ and $x=\pi$, then the fourier series is convergent to $\frac{1}{2}\left(f(-\pi+) + f(\pi+)\right)$.

Formulae:

1.) $\int_{l}^{l+2\pi} \sin nx\,dx = \int_{l}^{l+2\pi} \cos nx\,dx = 0 \quad, n \neq 0$

2.) $\int_{l}^{l+2\pi} \sin mx \cos nx\,dx = 0 \quad, m \neq n$

3.) $\int_{l}^{l+2\pi} \cos mx \cos nx\,dx = \int_{l}^{l+2\pi} \sin mx \sin nx\,dx = 0, \quad m \neq n$

4.) $\int_{l}^{l+2\pi} \sin^2 nx\,dx = \int_{l}^{l+2\pi} \cos^2 nx\,dx = \pi, \quad n \neq 0.$

5.) $\int_{l}^{l+2\pi} \sin nx \cos nx\,dx = 0, \quad n \neq 0.$

6.) $\sin n\pi = 0, \ \forall n$

7.) $\cos n\pi = (-1)^n, \ \forall n$

8.) $\sin (2n+1)\frac{\pi}{2} = (-1)^n, \ \forall n$

9.) $\cos (2n+1)\frac{\pi}{2} = 0, \ \forall n$

10.) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right)$

11.) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a \cos bx + b \sin bx\right).$

1.) Find the fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Solution: The fourier series expansion of $f(x)$ in the interval $(0,2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

where $f(x) = x^2$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}\left(8\pi^3\right) = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(0 + 2(2\pi)\cdot\frac{1}{n^2} + 0 - 0\right)$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(4\pi^2\left(\frac{-1}{n}\right) + 0 + \cancel{\frac{2}{n^3}} - 0 - 0 - \cancel{\frac{2}{n^3}}\right)$$

$$= -\frac{4\pi}{n}$$

Substituting these values in (1), we get

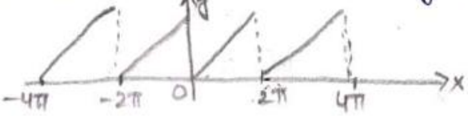
$$x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx$$

$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

$$= \frac{4\pi^2}{3} + 4\left(\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\cdots\right) - 4\pi\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\cdots\right)$$

2.) find the fourier series representing $f(x) = x,\ 0<x<2\pi$. Sketch the graph of $f(x)$ from -4π to 4π

Ans: $x = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx$.



3) obtain the fourier series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$. Hence show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \quad \text{or} \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}.$$

solution: The fourier series of $f(x)$ in the interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right) = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,dx = \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(0 + (1-2\pi)\frac{\cos n\pi}{n^2} - 0 - 0 - (1+2\pi)\frac{\cos n\pi}{n^2} - 0\right)$$

$$= \frac{1}{\pi}\left(\cancel{1} - 2\pi - \cancel{1} - 2\pi\right)\frac{\cos n\pi}{n^2}$$

$$= \frac{-4}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx = \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left((\pi - \pi^2)\left(\frac{-(-1)^n}{n}\right) - 0 - \cancel{\frac{2(-1)^n}{n^3}} - (-\pi - \pi^2)\left(\frac{-(-1)^n}{n}\right) - 0 + \cancel{\frac{2(-1)^n}{n^3}}\right)$$

$$= \frac{1}{\pi}\left(\pi - \cancel{\pi^2} + \pi + \cancel{\pi^2}\right)\frac{(-1)^{n+1}}{n}$$

$$= \frac{2(-1)^{n+1}}{n}$$

By substituting these values in (1), we get

$$(1) \Rightarrow x - x^2 = \frac{-\frac{\cancel{2}\pi^2}{3}}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

$$= -\frac{\pi^2}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots\cdots\right) \quad (2)$$

Put $x = 0$ in (2)

Then $(2) \Rightarrow 0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right) + 2(0)$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}.$$

4.) Obtain the fourier series expansion of $f(x) = \frac{1}{4}(\pi - x^2)$ in $0 < x < 2\pi$ and hence deduce $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

Ans: $\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$.

5.) Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a fourier series.

Solution: The Fourier series expansion of $f(x)$ in the interval $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{—} (1)$$

where $f(x) = x \sin x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx = \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}(-2\pi) = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x + \sin(1-n)x\right]dx \qquad \because 2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n+1)x - \sin(n-1)x\right)dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - 1\left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(2\pi\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - 0\right)$$

$$= \frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1} = \frac{2}{n^2-1}, \quad n \neq 1$$

If $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos x\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(\frac{-\sin 2x}{2^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{-2\pi}{2}\right) = \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(1-n)x - \cos(1+n)x\right]dx, \quad n \neq 1 \qquad \because 2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(n-1)x - \cos(n+1)x\right)dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right) - 1\cdot\left(\frac{-\cos(n-1)x}{(n-1)^2} - \frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\ \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} = 0, \quad n \neq 1$$

If $n=1$, then $b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \sin x\, dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \frac{(1-\cos 2x)}{2}\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} - \frac{-\cos 2x}{2^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(2\pi(2\pi) - \frac{(2\pi)^2}{2} - \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^2}}\right)$$

$$= \frac{4\pi^2}{4\pi} = \pi.$$

By substituting these values in (1), we get

$$x\sin x = \frac{-2}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= -1 + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx.$$

$$= -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x + 0$$

$$= -1 - \frac{\cos x}{2} + 2\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} + \cdots\right) + \pi\sin x.$$

6) Obtain the fourier series expansion of $f(x)$ given that $f(x) = kx(\pi - x)$ in $0<x<2\pi$, where k is a constant.

Ans: $kx(\pi-x) = -\frac{k\pi^2}{3} - 4k\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 2k\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$.

7) Expand $f(x) = e^{ax}$ in a fourier series in $0<x<2\pi$

Ans: $e^{ax} = \frac{e^{2a\pi}-1}{2a\pi} + \frac{a(e^{2a\pi}-1)}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\cos nx - \frac{e^{2a\pi}-1}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\sin nx.$

8) Find the fourier expansion of $f(x) = x\cos x$, $0<x<2\pi$.

Ans: $x\cos x = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=2,3,\ldots}^{\infty}\frac{n}{n^2-1}\sin nx.$

9) Find the fourier series of period 2π for the function $f(x) = x^2 - x$ in $(-\pi, \pi)$ and hence deduce the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

10.) The intensity of an alternating current after passing through a rectifier is given by

$$i(x) = \begin{cases} I_0 \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$$

where I_0 is max. current and period is 2π. Express $i(x)$ as a fourier series.

1.) If $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$, then show that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{4m^2-1}$ and hence deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{1}{4}(\pi - 2)$ and

$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{1}{2}$.

Solution: The Fourier series expansion of $f(x)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -(1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right)$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos nx\,dx \quad , n \neq 1$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left(\sin(1+n)x - \sin(n-1)x\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(1+n)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{1}{2\pi}\left(\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right)$$

$$= \frac{1}{2\pi}\left((-1)^n\left(\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right) - \frac{2}{n^2-1}\right)$$

$$= \frac{1}{\cancel{2}\pi}\cdot\frac{\cancel{2}}{n^2-1}\left((-1)^{n+1}-1\right)$$

$$= \frac{1}{(n^2-1)\pi}\left((-1)^{n+1}-1\right) \quad , n \neq 1$$

$$= \begin{cases} 0 & , n \text{ is odd} \\ \frac{-2}{(n^2-1)\pi} & , n \text{ is even} \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos x\,dx + \int_{0}^{\pi} f(x)\cos x\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0 + \int_{0}^{\pi}\sin x\cos x\,dx\right)$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{1}{2}+\frac{1}{2}\right) = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{2}$$

By substituting these values in the Fourier series of $f(x)$, we get

$$f(x) = \frac{\frac{2}{\pi}}{2} + \sum_{n=1}^{\infty}\frac{((-1)^{n+1}-1)}{(n^2-1)\pi}\cos nx + \sum_{n=1}^{\infty} 0\cdot\sin nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=2,4,\ldots}\frac{1}{n^2-1}\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right) + \frac{1}{2}\sin x.$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right) + \frac{1}{2}\sin x. \quad — (2)$$

Put $x=0$, then $f(x)=0$.

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right) + 0$$

$$\Rightarrow \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$$

Put $x=\frac{\pi}{2}$, then $f\left(\frac{\pi}{2}\right) = 1$

$$\therefore (2) \Rightarrow 1 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\right) + \frac{1}{2}\cdot 1$$

$$\Rightarrow 1 - \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right)$$

$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

2.) Given $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$, show that $f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

Deduce from this that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

3) find the fourier series in $[-\pi, \pi]$ for the function $f(x) = \begin{cases} \frac{1}{2}(\pi+x), & -\pi < x < 0 \\ \frac{1}{2}(\pi - x), & 0 < x < \pi \end{cases}$

Fourier Series in the interval $(c, c+2l)$ of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

1.) Obtain fourier series for the function $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$. Deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Sol:- Here $(c, c+2l) = (0, 2) \Rightarrow c = 0$ & $c + 2l = 2$
$\Rightarrow l = 1$

$\therefore$ fourier series of $f(x)$ in $(0,2)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{—(1)}$$

where $a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \left[\frac{\pi x^2}{2}\right]_0^1 + \left[\pi\left(2x - \frac{x^2}{2}\right)\right]_1^2$$

$$= \frac{\pi}{2} - 0 + \pi\left(4 - \frac{4}{2} - 2 + \frac{1}{2}\right)$$

$$= \pi$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_0^1 f(x)\cos n\pi x + \int_1^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^1 \pi x\cos n\pi x + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x \cdot \frac{\sin n\pi x}{n\pi} - \pi\cdot\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - \pi(-1)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_1^2$$

$$= \frac{\sin n\pi}{n} + \pi\frac{\cos n\pi}{n^2\pi^2} - 0 - \pi\frac{\cos 0}{(n\pi)^2} + \pi\cdot 0 - \pi\frac{\cos 2n\pi}{n^2\pi^2} - \pi(1)\frac{\sin n\pi}{n\pi} + \pi\frac{\cos n\pi}{n^2\pi^2}$$

$$= 0 + \frac{(-1)^n}{n^2\pi} - \frac{1}{n^2\pi} + 0 - \frac{1}{n^2\pi} - 0 + \frac{(-1)^n}{n^2\pi}$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right) = \begin{cases} \frac{-4}{n^2\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \int_0^1 f(x)\sin n\pi x\,dx + \int_1^2 f(x)\sin n\pi x\,dx.$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x) - \frac{\cos n\pi x}{n\pi} - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= -\frac{\cos n\pi}{n} + 0 - 0 - 0 - 0 - 0 + \frac{\cos n\pi}{n} + 0$$

$$= 0$$

Put a_0, a_n, b_n values in ①

$$① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos n\pi x + \sum_{n=1}^{\infty} 0\cdot\sin n\pi x$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5\ldots} \frac{-4}{n^2\pi}\cos n\pi x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right) \quad ②$$

Put $x=2$ in ②

Then ② $\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \cdots\right)$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

2) Expand $f(x) = e^{-x}$ as a fourier series in the interval $(-l, l)$

Sol:- Here $(c, c+2l) = (-l, l) \Rightarrow c = -l$ & $l = l$

$\therefore$ The fourier series of $f(x)$ in the interval $(-l, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \quad ①$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l}$

$$= \frac{1}{l}\left(-e^{-l} + e^{l}\right)$$

$$= \frac{2}{l}\sinh l$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right)\right]_{-l}^{l} = \frac{2l(-1)^n\sinh l}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x} \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{(-1)^2+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{-l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(0 - \frac{n\pi}{l}\cos n\pi\right) - \frac{e^{l}}{1+\left(\frac{n\pi}{l}\right)^2}\left(0 - \frac{n\pi}{l}\cos(-n\pi)\right)\right]$$

$$= \frac{1}{l^2}\ \frac{n\pi}{1+\left(\frac{n\pi}{l}\right)^2}\cos n\pi\left(-e^{-l}+e^{l}\right)$$

$$= \frac{2n\pi(-1)^n}{l^2+n^2\pi^2}\sinh l$$

Substitute a_0, a_n, b_n values in ①

Then ① $\Rightarrow e^{-x} = \frac{1}{l}\sinh l + \sum_{n=1}^{\infty}\frac{2l(-1)^n \sinh l}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n}{l^2+n^2\pi^2}\sinh l\cdot\sin\frac{n\pi x}{l}$

Even function:- A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$

Odd function:- A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$

Fourier Series of an even function in the interval $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

Fourier Series of an odd function in the interval $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

Fourier Series of $f(x)$ when it is even function in the interval $(-l,l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}\,dx$

Fourier Series of an odd function in the interval $(-l,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$

1.) Express $f(x)=x$ as a fourier Series in $(-\pi,\pi)$

Sol:- Here $f(x)$ is odd function, since $f(-x)=-x=-f(x)$

So, fourier Series of an odd function in $(-\pi,\pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(1)}$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$

$= \frac{2}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left(\frac{-\pi}{n}\cos n\pi + 0 - 0 - 0\right)$

$= \frac{-2}{n}(-1)^n$

Put this value in (1)

Then (1) $\Rightarrow x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$

$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$

2.) Expand the function $f(x)=x^2$ as a fourier series in $(-\pi,\pi)$

Sol Here $f(x)=x^2$ is even function, since $f(-x)=(-x)^2=x^2=f(x)$

So, Fourier Series of an even function in $(-\pi,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—(1)}$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$

$= \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$

$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[0 + 2\pi\frac{(-1)^n}{n^2} - 0 - 0 - 0 - 0\right]$

$= \frac{4(-1)^n}{n^2}$

Substitute these values in ①

Then ① $\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$

$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$

$= \frac{\pi^2}{3} + 4\left(\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$

$= \frac{\pi^2}{3} - 4\left(\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right)$

3.) find the fourier series to represent $f(x) = x^2 - 2$, $-2 \le x \le 2$.

Sol:- Here $f(x) = x^2 - 2$ is an even function and $l = 2$

So, Fourier series of the given function is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ — ① $\because l = 2$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$= \frac{2}{2}\int_0^{2} x^2 - 2\,dx$

$= \left[\frac{x^3}{3} - 2x\right]_0^{2}$

$= \frac{8}{3} - 4 = \frac{-4}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[(x^2-2)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\cdot\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} + 2\cdot\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^3}\right]_0^2$$

$$= 0 + \frac{4}{\frac{n^2\pi^2}{4}}\cdot\cos n\pi - 0 - 0 + 0 + 0$$

$$= \frac{16}{n^2\pi^2}(-1)^n$$

Substitute these values in (1)

$$(1) \Rightarrow x^2 - 2 = \frac{-\frac{4}{3}}{2} + \sum_{n=1}^{\infty}\frac{16(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$= \frac{-2}{3} + \frac{16}{\pi^2}\left(\frac{-1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\pi x - \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right)$$

$$= -\frac{2}{3} - \frac{16}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right)$$

4) Obtain fourier series Expansion for $\sin ax$ in the interval $-l<x<l$

Sol:- Here $f(x) = \sin ax$ is odd function.

So, fourier series of $f(x)$ in the interval $-l<x<l$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{—(1)}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\int_0^l \sin ax \cdot \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \frac{1}{2}\left[\cos\left(a-\frac{n\pi}{l}\right)x - \cos\left(a+\frac{n\pi}{l}\right)x\right]dx$$

$$= \frac{1}{l}\left[\frac{\sin\left(a-\frac{n\pi}{l}\right)x}{a-\frac{n\pi}{l}} - \frac{\sin\left(a+\frac{n\pi}{l}\right)x}{a+\frac{n\pi}{l}}\right]_0^l$$

$$= \frac{\sin(al-n\pi)}{al-n\pi} - \frac{\sin(al+n\pi)}{al+n\pi}$$

$$= \frac{\sin al\cos n\pi - \cos al\sin n\pi}{al-n\pi} - \frac{\sin al\cos n\pi + \cos al\sin n\pi}{al+n\pi}$$

$$= \frac{\sin al\cos n\pi}{al-n\pi} - \frac{\sin al\cos n\pi}{al+n\pi} = \sin al(-1)^n\frac{2n\pi}{a^2l^2-n^2\pi^2}$$

Substitute this value in (1)

Then $(1) \Rightarrow \sin ax = \sum_{n=1}^{\infty}\sin al\,(-1)^n\frac{2n\pi}{a^2l^2-n^2\pi^2}\cdot\sin\frac{n\pi x}{l}$

Half Range fourier Series:- Sometimes it is necessary to expand a function $f(x)$ as a fourier series in a half range $(0,\pi)$ or $(0,l)$ not in full range $(-\pi,\pi)$ or $(-l,l)$. Such a series is known as Half range series.

Half range Sine Series expansion of $f(x)$ in $(0,l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

Half range Cosine Series expansion of $f(x)$ in $(0,l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\, dx$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

Note: To get Half range sine series and Half range Cosine series in the interval $(0,\pi)$, put $l=\pi$ in the above formula.

1.) Obtain the half range cosine and sine series for $f(x)=x$ in the interval $(0,l)$

Sol:- Half range cosine series of $f(x)$ in $(0,l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{— (1)}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\, dx$

$$= \frac{2}{l}\int_0^l x\, dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left(\frac{l^2}{2} - 0\right) = l$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[x \cdot \frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} - 1 \cdot \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[0 + \frac{l^2}{n^2\pi^2}(-1)^n - 0 - \frac{l^2}{n^2\pi^2}(1)\right] = \frac{2}{l}\frac{l^2}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$= \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-4l}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

Put a_0, a_n values in ①, then

$$① \Rightarrow x = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right) \cos \frac{n\pi x}{l}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1,3,5..} \frac{1}{n^2} \cos \frac{n\pi x}{l}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{1^2} \cos \frac{\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} + \cdots\right)$$ is the required Half range Cosine series

Half range Sine series in the interval $(0,l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad — ②$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \left[x \cdot \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} - 1 \cdot \frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right]_0^l$$

$$= \frac{2}{l}\left(-\frac{l^2}{n\pi}(-1)^n + 0 + 0 - 0\right)$$

$$= -\frac{2l}{n\pi}(-1)^n$$

$$= \frac{2l}{n\pi}(-1)^{n+1}$$

Put b_n value in ②

Then $② \Rightarrow x = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{l}$

$$= \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}$$

$$= \frac{2l}{\pi}\left(\sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \cdots\right)$$ is the required Half range Sine series.

2.) find the half range sine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$ and hence deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$

Sol:- Half range Sine series of $f(x)$ in $(0,\pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—} ① \qquad \because l = \pi$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$
$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left(0 - 0 - \frac{2\cos n\pi}{n^3} - 0 - 0 + \frac{2\cos 0}{n^3}\right)$$
$$= \frac{4}{n^3\pi}\left(-(-1)^n + 1\right)$$
$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8}{n^3\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Put b_n value in ①

Then ① $\Rightarrow$
$$x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}\left(1 - (-1)^n\right) \sin nx$$
$$= \sum_{n=1,3,5,\ldots} \frac{8}{n^3\pi} \sin nx$$
$$= \frac{8}{\pi}\left(\frac{1}{1^3}\sin x + \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \cdots\cdots\right) \quad \text{—} ②$$

Put $x = \frac{\pi}{2}$ in ②, then

$$② \Rightarrow \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\frac{1}{1^3} - \frac{1}{3^2} + \frac{1}{5^3} \cdots\right)$$
$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$$
$$\Rightarrow \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

3.) find the half range Cosine series of $f(x) = x$ in $(0,\pi)$ and hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol:- Half range Cosine series of $f(x)$ in $(0,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ① \qquad \because l = \pi$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right)$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{-4}{n^2\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Put these a_0, a_n values in ①

$$① \Rightarrow x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos nx$$

$$= \frac{\pi}{2} + \sum_{n=1,3,5,\dots} \frac{-4}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right) \quad ②$$

Put $x = 0$ in ②

$$② \Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}.$$

Fourier Transforms

Fourier Integral theorem: If $f(x)$ is a function satisfying the Dirichlet conditions defined in the interval $(-l, l)$, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$ is known

as fourier integral of $f(x)$.

Fourier Sine Integral:- $f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t\,dt\,d\lambda$

Fourier Cosine Integral:- $f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\lambda x\int_0^{\infty} f(t)\cos\lambda t\,dt\,d\lambda$

1.) Express the function $f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$ as a fourier Integral and hence evaluate $\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ & $\int_0^{\infty}\frac{\sin x}{x}\,dx$

Sol:- Given $f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Assume $x = t$

then $f(t) = \begin{cases} 1, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases} = \begin{cases} 1, & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$

w.k.t fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\boxed{\int_{-\infty}^{\infty}} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty}\right)$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{-1} f(t)\cos\lambda(t-x)\,dt + \int_{-1}^{1} f(t)\cos\lambda(t-x)\,dt + \int_{1}^{\infty} f(t)\cos\lambda(t-x)\,dt\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{-1} 0\cos\lambda(t-x)\,dt + \int_{-1}^{1} 1\cdot\cos\lambda(t-x)\,dt + \int_{1}^{\infty} 0\cdot\cos\lambda(t-x)\,dt\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(0 + \int_{-1}^{1}\cos\lambda(t-x)\,dt + 0\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-1}^{1}\cos(\lambda t-\lambda x)\,dt\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin(\lambda t-\lambda x)}{\lambda}\right]_{t=-1}^{1}d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin(\lambda-\lambda x)}{\lambda} - \frac{\sin(-\lambda-\lambda x)}{\lambda}\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{1}{\lambda}\left(\sin(\lambda-\lambda x) - \sin(-(\lambda+\lambda x))\right)d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{1}{\lambda}\left(\sin(\lambda-\lambda x) + \sin(\lambda+\lambda x)\right)d\lambda \qquad \sin(-\theta) = -\sin\theta)$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{1}{\lambda}\left(2\sin\lambda\cos\lambda x\right)d\lambda \qquad \left(\because \sin(A-B)+\sin(A+B) = 2\sin A\cos B\right)$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

$$\Rightarrow \int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}f(x)$$

$$= \frac{\pi}{2} \cdot \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

i.) $f(x)$ is discontinuous at $x = 1$

$$\therefore f(1) = \frac{1}{2}\left(f(1+) + f(1-)\right)$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

$\therefore$ At $x = 1$.

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(1)$$

$$= \frac{\pi}{2} \cdot \frac{1}{2}$$

$$= \frac{\pi}{4}$$

ii.) $|x| < 1$.

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2}$$

Put $x = 0$, $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$.

$$\Rightarrow \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad (\text{Assume } \lambda = x)$$

2.) Express $f(x) = \begin{cases} 1, & \text{for } 0 \le x \le \pi \\ 0, & \text{for } x > \pi \end{cases}$ as a fourier sine.

Integral and hence evaluate $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda} \sin\lambda x \, d\lambda$

Sol: Given $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

Assume $x = t$

then $f(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

w.k.t fourier Sine Integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^\infty f(t) \sin\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(\int_0^\pi f(t) \sin\lambda t \, dt + \int_\pi^\infty f(t) \sin\lambda t \, dt \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(\int_0^\pi 1 \cdot \sin\lambda t \, dt + \int_\pi^\infty 0 \cdot \sin\lambda t \, dt \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(\int_0^\pi \sin\lambda t \, dt + 0 \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda t}{\lambda} \right]_{t=0}^{\pi} d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(-\frac{\cos\lambda\pi}{\lambda} + \frac{\cos\lambda(0)}{\lambda} \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(-\frac{\cos\lambda\pi}{\lambda} + \frac{1}{\lambda} \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x . \left(\frac{1 - \cos\lambda\pi}{\lambda} \right) d\lambda$$

$$\Rightarrow \int_0^\infty \sin \lambda x \left(\frac{1-\cos\lambda\pi}{\lambda}\right) d\lambda = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} \cdot \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

At $x = \pi$, $f(x)$ is discontinuous

$\therefore$ The above integral value $= \frac{\pi}{2}\left(\frac{f(\pi-0)+f(\pi+0)}{2}\right)$

$$= \frac{\pi}{2}\left(\frac{1+0}{2}\right)$$

$$= \frac{\pi}{4}$$

3) Using fourier Integral, show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)} d\lambda, \text{ where } a, b > 0$$

Sol: Let $f(x) = e^{-ax} - e^{-bx}$, where $a, b > 0$

Then $f(t) = e^{-at} - e^{-bt}$

w.k.t fourier Sine Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda$$

$$\Rightarrow e^{-ax} - e^{-bx} = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left(\int_0^\infty (e^{-at} - e^{-bt}) \sin \lambda t \, dt \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \left(\int_0^\infty e^{-at} \sin \lambda t \, dt - \int_0^\infty e^{-bt} \sin \lambda t \, dt \right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left(\left[\frac{e^{-at}}{a^2+\lambda^2}(-a\sin\lambda t - \lambda\cos\lambda t) \right]_0^{\infty} + \left[\frac{e^{-bt}}{b^2+\lambda^2}(-b\sin\lambda t - \lambda\cos\lambda t) \right]_0^{\infty} \right) d\lambda$$

$$\left[\because \int e^{-ax}\sin bx\, dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx) \right]$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left(\left[0 - \frac{1}{a^2+\lambda^2}(-a(0) - \lambda(1)) \right] + \left[0 - \frac{1}{b^2+\lambda^2}(-b(0) - \lambda(1)) \right] \right) d\lambda$$

$$\because e^{-\infty} = 0, \quad e^{0} = 1$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left(\frac{-1}{a^2+\lambda^2}(-\lambda) + \frac{1}{b^2+\lambda^2}(-\lambda) \right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin\lambda x \left(\frac{\lambda}{a^2+\lambda^2} - \frac{\lambda}{b^2+\lambda^2} \right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \lambda \sin\lambda x \left(\frac{1}{a^2+\lambda^2} - \frac{1}{b^2+\lambda^2} \right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \lambda \sin\lambda x \; \frac{b^2+\lambda^2-(a^2+\lambda^2)}{(a^2+\lambda^2)(b^2+\lambda^2)}\, d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \lambda \sin\lambda x \; \frac{b^2+\cancel{\lambda^2}-a^2-\cancel{\lambda^2}}{(a^2+\lambda^2)(b^2+\lambda^2)}\, d\lambda$$

$$\Rightarrow e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^{\infty} \frac{\lambda \sin\lambda x}{(a^2+\lambda^2)(b^2+\lambda^2)}\, d\lambda$$

4) Using fourier Integral, P.T $\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2+p^2} d\lambda = \frac{\pi}{2} e^{-px}$

Hint: Take $f(x) = e^{-px}$

Use fourier Sine Integral formula

5) Using fourier Integral, show that

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{\lambda^2+a^2} d\lambda$$

Solution :- Let $f(x) = e^{-ax}$

Then $f(t) = e^{-at}$

w.k.t fourier Cosine Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos \lambda x \left(\int_0^\infty e^{-at} \cos \lambda t \, dt\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos \lambda x \left[\frac{e^{-at}}{(-a)^2+\lambda^2}(-a \cos \lambda t + \lambda \sin \lambda t)\right]_{t=0}^{\infty} d\lambda$$

$$\left(\because \int e^{-ax} \cos bx \, dx = \frac{e^{-ax}}{a^2+b^2}(-a \cos bx + b \sin bx)\right)$$

$$= \frac{2}{\pi}\int_0^\infty \cos \lambda x \left(0 - \frac{1}{a^2+\lambda^2}(-a+0)\right) d\lambda$$

$$\left(\because e^{-\infty} = 0 \; , \; e^0 = 1\right)$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2} d\lambda$$

Fourier Transform

Fourier Transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx = F(s), \text{ say}$$

Inverse Fourier Transform of $F(s)$ is given by

$$F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\,ds = f(x), \text{ say}$$

Fourier Cosine Transform of $f(x)$ is given by

$$F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx\,dx = F_c(s), \text{ say}$$

Inverse Fourier Cosine Transform of $F_c(s)$ is given by

$$F^{-1}\{F_c(s)\} = \frac{2}{\pi}\int_0^{\infty} F_c(s) \cos sx\,ds = f(x), \text{ say}$$

Fourier Sine Transform of $f(x)$ is given by

$$F_s\{f(x)\} = \int_0^{\infty} f(x) \sin sx\,dx = F_s(s), \text{ say}$$

Inverse Fourier Sine Transform is given by

$$F^{-1}\{F_s(s)\} = \frac{2}{\pi}\int_0^{\infty} F_s(s) \sin sx\,ds = f(x), \text{ say}$$

Properties of Fourier Transforms

1.) Linear property :- If $F\{f(x)\} = F(s)$ & $F\{g(x)\} = G(s)$ then

$$F\{af(x) + bg(x)\} = aF\{f(x)\} + bF\{g(x)\}$$
$$= aF(s) + bG(s)$$

2.) change of scale property :- If $F\{f(x)\} = F(s)$ then

$$F\{f(ax)\} = \frac{1}{a}F(s/a), \text{ where } a \neq 0.$$

Note i.) $F_c\{f(ax)\} = \frac{1}{a}F_c(s/a)$

ii.) $F_s\{f(ax)\} = \frac{1}{a}F_s(s/a)$

3.) shifting property :- If $F\{f(x)\} = F(s)$ then

$$F\{f(x-a)\} = e^{isa}F(s).$$

1.) Find the Fourier Transform of $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$.

Solution:- Given $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$

$$= \begin{cases} 1, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$$

By def, $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$

$$= \int_{-\infty}^{-1} e^{isx} f(x)\,dx + \int_{-1}^{1} e^{isx} f(x)\,dx + \int_{1}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-\infty}^{-1} e^{isx}\cdot 0\,dx + \int_{-1}^{1} e^{isx}\cdot 1\cdot dx + \int_{1}^{\infty} e^{isx}\cdot 0\,dx$$

$$= 0 + \left[\frac{e^{isx}}{is}\right]_{-1}^{1} + 0$$

$$= \frac{e^{is}}{is} - \frac{e^{-is}}{is}$$

$$= \frac{e^{is}-e^{-is}}{is}$$

$$= \frac{2}{s}\sin s \qquad \left(\because \sin x = \frac{e^{ix}-e^{-ix}}{2i}\right)$$

$= F(s)$, say

w.k.t $F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\,ds$

$$\Rightarrow F^{-1}\left\{\frac{2\sin s}{s}\right\} = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} e^{-isx}\,\frac{\cancel{2}\sin s}{s}\,ds$$

$$\Rightarrow \quad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-isx}\,\frac{\sin s}{s}\,ds \qquad \left(\because F^{-1}\{F(s)\} = f(x)\right)$$

Put $x=0$

$$\Rightarrow f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{is(0)} \frac{\sin s}{s}\, ds$$

$$\Rightarrow 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin s}{s}\, ds \qquad \left(\because \frac{\sin s}{s} \text{ is an even function}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

2) Find the Fourier Transform of $f(x)=\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$

and hence evaluate $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$.

Sol:- Given $f(x)=\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases} = \begin{cases} 1-x^2, & -1\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$

By Def, $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

$$= \int_{-\infty}^{-1} e^{isx} f(x)\, dx + \int_{-1}^{1} e^{isx} f(x)\, dx + \int_{1}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{-1} e^{isx}\cdot 0\, dx + \int_{-1}^{1} e^{isx}(1-x^2)\, dx + \int_{1}^{\infty} e^{isx}\cdot 0\, dx$$

$$= 0 + \int_{-1}^{1} e^{isx}(1-x^2)\, dx + 0$$

$$= \int_{-1}^{1} \underbrace{e^{isx}}_{v}\underbrace{(1-x^2)}_{u}\, dx$$

$$=\left[(1-x^2)\frac{e^{isx}}{is}-(-2x)\frac{e^{isx}}{(is)^2}+(-2)\frac{e^{isx}}{(is)^3}\right]_{x=-1}^{1}$$

$\because \int uv = uv_1 - u'v_2 + u''v_3 - \cdots$

where $v_1 = \int v$

$v_2 = \int v_1 = \iint v$

$----$

$$=(1-1)\frac{e^{is(1)}}{is}+2(1)\frac{e^{is(1)}}{i^2s^2}-2\frac{e^{is(1)}}{i^3s^3}-\left((1-1)\frac{e^{+is(-1)}}{is}+2(-1)\frac{e^{is(-1)}}{i^2s^2}-2\frac{e^{is(-1)}}{i^3s^3}\right)$$

$i^3 = i^2 \cdot i = (-1)\cdot i = -i$

$$=0+\frac{2e^{is}}{-s^2}-\frac{2e^{is}}{-is^3}-\left(0-2\frac{e^{-is}}{-s^2}-\frac{2e^{-is}}{-is^3}\right)$$

$$=\frac{-2e^{is}}{s^2}-\frac{2ie^{is}}{s^3}-\frac{2e^{-is}}{s^2}+\frac{2ie^{-is}}{s^3}$$

$\frac{1}{i}=\frac{1}{i}\cdot\frac{i}{i}=\frac{i}{i^2}=-i$

$$=\frac{-2}{s^2}\left(e^{is}+e^{-is}\right)-\frac{2i}{s^3}\left(e^{is}-e^{-is}\right)$$

$$=\frac{-2}{s^2}(2\cos s)-\frac{2i}{s^3}(2i\sin s)$$

$\left(\because \cos x = \frac{e^{ix}+e^{-ix}}{2} \Rightarrow e^{ix}+e^{-ix} = 2\cos x\right.$

$\left.\sin x = \frac{e^{ix}-e^{-ix}}{2i} \Rightarrow e^{ix}-e^{-ix} = 2i\sin x\right)$

$$=-\frac{4}{s^2}\cos s+\frac{4}{s^3}\sin s$$

$$=\frac{-4}{s^3}(s\cos s-\sin s)=F(s),\ \text{say}$$

w.k.t by Inverse fourier Transform,

$$F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\,ds = f(x), \text{ say}$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\cdot -4\left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

$$= \frac{-\cancel{4}^{2}}{\cancel{2}\pi}\int_{-\infty}^{\infty}(\cos sx - i\sin sx)\left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

$$(\because e^{-i\theta} = \cos\theta - i\sin\theta)$$

$$= \frac{-2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

Put $x=\frac{1}{2}$

$$\Rightarrow f\left(\tfrac{1}{2}\right) = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right) ds$$

$1-\frac{1}{4}$

$$\Rightarrow 1-\left(\tfrac{1}{2}\right)^2 = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right) ds$$

$$\Rightarrow \frac{3}{4} = \frac{-2}{\pi}\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right) ds$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right) ds = \frac{-3\pi}{8}$$

Equating real parts, we have

$$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\,ds = \frac{-3\pi}{8}$$

$$\Rightarrow 2\int_0^\infty \frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

$$\Rightarrow \int_0^\infty \frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}\,ds = -\frac{3\pi}{16}$$

Assume $s=x$

$$\Rightarrow \int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$$

3.) Find the fourier transform is $e^{-a^2x^2}$, $a<0$ Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of fourier transform.

Sol: Let $f(x)=e^{-a^2x^2}$, $a<0$

By def, $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx = F(s)$, say

$$\therefore F\{e^{-a^2x^2}\} = \int_{-\infty}^{\infty} e^{isx}\, e^{-a^2x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{isx-a^2x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2-\frac{isx}{a^2}\right)}\,dx$$

$a^2-2ab=(a-b)^2-b^2$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2-2\cdot x\cdot\frac{is}{2a^2}\right)}\,dx.$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(\left(x-\frac{is}{2a^2}\right)^2-\left(\frac{is}{2a^2}\right)^2\right)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2+\cancel{a^2}\frac{i^2s^2}{4a^{\cancel{4}2}}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2 - \frac{s^2}{4a^2}}\, dx$$

$e^{a+b} = e^a \cdot e^b$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2} \cdot e^{-\frac{s^2}{4a^2}}\, dx$$

$$= e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2} dx$$

Put $a\left(x-\frac{is}{2a^2}\right) = t$

$$a\,dx = dt \Rightarrow dx = \frac{dt}{a}$$

<u>L.L</u>

Put $x=-\infty$ in $a\left(x-\frac{is}{2a^2}\right)=t$

$\Rightarrow a\left(-\infty - \frac{is}{2a^2}\right) = t$

$\Rightarrow a(-\infty) = t$

$\Rightarrow t = -\infty$

<u>U.L</u>

Put $x=\infty$ in $a\left(x-\frac{is}{2a^2}\right)=t$

$\Rightarrow a\left(\infty - \frac{is}{2a^2}\right) = t$

$\Rightarrow a(\infty) = t$

$\Rightarrow t = \infty$

$$= e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \cdot \frac{dt}{a}$$

$$= \frac{1}{a} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{a} e^{-\frac{s^2}{4a^2}} \cdot 2\int_0^{\infty} e^{-t^2} dt$$

$(\because e^{-t^2}$ is even$)$

Put $t^2 = p \Rightarrow t = \sqrt{p}$

$\Rightarrow dt = \frac{1}{2\sqrt{p}}\, dp$

<u>L.L</u>

~~Put $t=\infty$, in $t^2=\sqrt{p}$~~

Put $t=0$ in $t=\sqrt{p}$

then $p=0$

<u>U.L</u>

Put $t=\infty$ in $t^2=p$

$\Rightarrow p=\infty$

$$= \frac{\cancel{2}}{a} e^{-\frac{s^2}{4a^2}} \int_0^{\infty} e^{-p} \frac{1}{\cancel{2}\sqrt{p}} dp$$

$$= \frac{1}{a} e^{-\frac{s^2}{4a^2}} \int_0^{\infty} e^{-p} p^{-1/2} dp$$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_0^{\infty} e^{-p} p^{1/2-1} dp$$

Assume $p = x$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1} dx$$

$$= \frac{1}{a} e^{-\frac{s^2}{4a^2}} \Gamma\frac{1}{2}$$

$\because$ by Gammafunction, $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$

$$= \frac{1}{a} e^{-s^2/4a^2} \sqrt{\pi}$$

$\because \Gamma\frac{1}{2} = \sqrt{\pi}$

$$\Rightarrow F\{e^{-a^2x^2}\} = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \quad \text{—— (1)}.$$

Put $a^2 = \frac{1}{2}$ in (1)

Then $F\{e^{-\frac{x^2}{2}}\} = \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}} e^{-s^2/\cancel{4}(\frac{1}{\cancel{2}})}$ $\quad \because a^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}}$

$$= \sqrt{2\pi}\, e^{-s^2/2}$$

$\Rightarrow F\{e^{-x^2/2}\}$ is $\sqrt{2\pi}$ times of $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self reciprocal under fourier transfrom.

(5)

4) Find the fourier Cosine transform of e^{-x^2}.

Sol: Let $f(x) = e^{-x^2}$

By def, $F_c\{f(x)\} = \int_0^\infty f(x) \cos sx \, dx = F_c(s)$, say

$\therefore F_c\{e^{-x^2}\} = \int_0^\infty e^{-x^2} \cos sx \, dx = I$, say — (1)

D (1) w.r.t 's'

$$\frac{dI}{ds} = \int_0^\infty e^{-x^2} (-\sin sx \cdot x) \, dx$$

$$= \int_0^\infty -x e^{-x^2} \sin sx \, dx$$

$$= \frac{1}{2} \int_0^\infty \underline{-2x e^{-x^2}} \sin sx \, \underline{dx}$$

$$= \frac{1}{2} \int_0^\infty \underset{\downarrow u}{\sin sx} \, d(\underset{\downarrow v}{e^{-x^2}})$$

$$= \frac{1}{2} \left(\left[\sin sx \cdot e^{-x^2} \right]_0^\infty \int_0^\infty e^{-x^2} \cos sx \cdot s \, dx \right)$$

$$\left(\because \int u \, dv = uv - \int v \, du \right)$$

$e^{-\infty} = 0$
$e^0 = 1$
$\sin 0 = 0$

$$= \frac{1}{2} \left(0 - 0 - s \int_0^\infty e^{-x^2} \cos sx \, dx \right)$$

$$= -\frac{s}{2} \int_0^\infty e^{-x^2} \cos sx \, dx$$

$$= \frac{-s}{2} I \quad (\because \text{from } (1))$$

$$\Rightarrow \frac{dI}{ds} = \frac{-s}{2} I$$

$$\Rightarrow \frac{dI}{I} = -\frac{s}{2} ds$$

On integrating b.s

$$\Rightarrow \int \frac{dI}{I} = -\frac{1}{2}\int s\, ds \qquad \because \int \frac{1}{x}dx = \log x$$

$$\Rightarrow \log I = -\frac{1}{2}\cdot\frac{s^2}{2} + \log c$$

$$= -\frac{s^2}{4} + \log c$$

$$= \log e^{-s^2/4} + \log c$$

$$= \log c e^{-s^2/4}$$

$$\Rightarrow I = c e^{-s^2/4} \quad \text{—} (2)$$

$$\Rightarrow \int_0^\infty e^{-x^2} \cos sx\, dx = c e^{-s^2/4} \quad (\because \text{from } (1))$$

Put $s=0$.

$$\Rightarrow \int_0^\infty e^{-x^2} \cdot 1\, dx = c(1)$$

$$\Rightarrow c = \int_0^\infty e^{-x^2} dx \qquad \because \int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2} \qquad \Rightarrow 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}$$

Put $c = \frac{\sqrt{\pi}}{2}$ in (2)

$$\therefore I = c e^{-s^2/4} = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

$$\Rightarrow F_c\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

5:) Find the fourier sine transform of $\frac{e^{-ax}}{x}$.

Sol:- Let $f(x) = \frac{e^{-ax}}{x}$

By def $F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = F_s(s)$

$\therefore F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I$, say

D. 'I' w.r.t 's'

$$\frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot \cos sx \cdot \cancel{x} \, dx$$

$$= \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2+s^2}(-a\cos sx + s\sin sx)\right]_{x=0}^{\infty}$$

$$\because \int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$\left(\because e^{-\infty}=0,\ e^0=1\right)$

$$= 0 - \frac{1}{a^2+s^2}(-a(1)+s(0))$$

$$= \frac{a}{a^2+s^2}$$

$$\Rightarrow \frac{dI}{ds} = \frac{a}{a^2+s^2}$$

$$\Rightarrow dI = \frac{a}{a^2+s^2} ds$$

by applying integration on bothsides

$$\int dI = \int \frac{a}{a^2+s^2} ds$$

$$\Rightarrow I = \tan^{-1}\left(\frac{s}{a}\right) + c \quad — (1)$$

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx = \tan^{-1}\left(\frac{s}{a}\right) + c$$

Put $s=0$.

$$\Rightarrow \int_0^{\infty} \frac{e^{-ax}}{x} \sin 0 \cdot dx = \tan^{-1}(0) + c$$

$$\Rightarrow \quad 0 = 0 + c$$

$$\Rightarrow \quad c = 0.$$

Put $c=0$ in (1)

$$\therefore (1) \Rightarrow I = \tan^{-1}\left(\frac{s}{a}\right) + 0$$

$$\Rightarrow I = \tan^{-1}\left(\frac{s}{a}\right)$$

$$\Rightarrow F_s\left\{\frac{e^{-ax}}{x}\right\} = \tan^{-1}\left(\frac{s}{a}\right).$$

(6)

6.) find the fourier sine transform of $e^{-|x|}$

Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m>0$

sol: Let $f(x) = e^{-|x|}$.

By def $F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$

$$\therefore F_s\{e^{-|x|}\} = \int_0^\infty e^{-|x|} \sin sx \, dx$$

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2+s^2}(-1 \cdot \sin sx - s \cos sx)\right]_{x=0}^{\infty}$$

$$\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$\because e^{-\infty} = 0$, $e^0 = 1$

$$= 0 - \frac{1}{1+s^2}(-0 - s(1))$$

$$= \frac{s}{1+s^2} = F_s(s), \text{ say}.$$

w.k.t by inverse fourier sine transform,

$$F^{-1}\{F_s(s)\} = \frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds = f(x), \text{ say}$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

$$\Rightarrow e^{-|x|} = \frac{2}{\pi} \int_0^\infty \frac{s \cdot \sin sx}{1+s^2} ds$$

$$\Rightarrow \int_0^\infty \frac{s \cdot \sin sx}{1+s^2} ds = \frac{\pi}{2} e^{-|x|}$$

$$= \frac{\pi}{2} e^{-x}$$

Assume $x = m$

$$\int_0^\infty \frac{s \sin sm}{1+s^2} ds = \frac{\pi}{2} e^{-m}$$

Assume $s = x$, then

$$\int_0^{\infty} \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$$

7) Find the fourier Cosine transform of

$$f(x) = \begin{cases} x, & \text{for } 0<x<1 \\ 2-x, & \text{for } 1<x<2 \\ 0, & \text{for } x \geq 2 \end{cases}$$

Sol: By def, $F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx\, dx$

$$= \int_0^1 f(x) \cos sx\, dx + \int_1^2 f(x) \cos sx\, dx + \int_2^{\infty} f(x) \cos sx\, dx$$

$$= \int_0^1 x \cdot \cos sx\, dx + \int_1^2 (2-x) \cos sx\, dx + \int_2^{\infty} 0 \cdot \cos sx\, dx$$

$$= \int_0^1 \underset{u}{x}\, \underset{v}{\cos sx}\, dx + \int_1^2 \underset{u}{(2-x)}\, \underset{v}{\cos sx}\, dx + 0$$

$$= \left[x \cdot \frac{\sin sx}{s} - 1 \cdot \left(-\frac{\cos sx}{s^2} \right) \right]_{x=0}^{1} + \left[(2-x) \frac{\sin sx}{s} - (-1) \cdot \left(-\frac{\cos sx}{s^2} \right) \right]_{x=1}^{2}$$

$$\because \int uv = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$v_1 = \int v, \quad v_2 = \int v_1 = \iint v, \cdots$$

$$= 1 \cdot \cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2} - 0 - \frac{1}{s^2} + 0 - \frac{\cos 2s}{s^2} - 1 \cdot \cancel{\frac{\sin s}{s}} + \frac{\cos s}{s^2}$$

$$= \frac{2\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2}$$

8.) Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} \sin x, & \text{for } 0 < x < a \\ 0, & \text{for } x > a \end{cases}$$

Sol:- By def $F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$

$$= \int_0^a f(x) \sin sx \, dx + \int_a^\infty f(x) \sin sx \, dx$$

$$= \int_0^a \sin x \cdot \sin sx \, dx + \int_a^\infty 0 \cdot \sin sx \, dx$$

$$= \int_0^a \sin x \cdot \sin sx \, dx + 0.$$

$$= \frac{1}{2} \int_0^a \left(\cos(1-s)x - \cos(1+s)x \right) dx$$

$$\because \sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$\cos(1-s)x$
$= \cos(-(s-1)x)$
$= \cos(s-1)x$

$$= \frac{1}{2} \int_0^a \left(\cos(s-1)x - \cos(s+1)x \right) dx$$

$$= \frac{1}{2} \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1} \right]_{x=0}^{a}$$

$$= \frac{1}{2} \left(\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} - 0 \right)$$

$$= \frac{1}{2} \left(\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right), \text{ for } s \neq 1$$

By def $F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$

$$= \int_0^a f(x)\cos sx\,dx + \int_a^\infty f(x)\cos sx\,dx$$

$$= \int_0^a \sin x\cos sx\,dx + \int_a^\infty 0\cdot\cos sx\,dx$$

$$= \frac{1}{2}\int_0^a 2\sin x\cos sx\,dx + 0$$

$$= \frac{1}{2}\int_0^a \left(\sin(1+s)x + \sin(1-s)x\right)dx$$

$$= \frac{1}{2}\left[-\frac{\cos(1+s)x}{1+s} - \frac{\cos(1-s)x}{1-s}\right]_{x=0}^{a}$$

$$= \frac{1}{2}\left(-\frac{\cos(s+1)a}{s+1} + \frac{\cos(s-1)a}{s-1} + \frac{1}{1+s} + \frac{1}{1-s}\right)$$

H.W

Q.) Find the Fourier sine and Cosine transforms of $2e^{-5x} + 5e^{-2x}$.

Ans: $F_s\{f(x)\} = \dfrac{2s}{s^2+25} + \dfrac{5s}{s^2+4}$

$$F_c\{2e^{-5x} + 5e^{-2x}\} = \frac{10}{s^2+25} + \frac{10}{s^2+4}.$$

10) Find the fourier Cosine transform of $f(x)=\frac{1}{1+x^2}$ and hence derive fourier sine transform of $\phi(x)=\frac{x}{1+x^2}$

Sol: Given $f(x)=\frac{1}{1+x^2}$

w.k.t by def $F_C\{f(x)\}=\int_0^{\infty} f(x)\cos sx\,dx = F_C(s)$, say

$$\therefore F_C\left\{\frac{1}{1+x^2}\right\}=\int_0^{\infty}\frac{1}{1+x^2}\cos sx\,dx = I, \text{say} \quad (1)$$

D. (1) w.r.t 's'

$$\frac{dI}{ds}=\int_0^{\infty}\frac{1}{1+x^2}(-\sin sx)\cdot x\,dx$$

$$=-\int_0^{\infty}\frac{x}{1+x^2}\sin sx\,dx. \quad \text{——— (a)}$$

$$=-\int_0^{\infty}\frac{x^2}{x(1+x^2)}\sin sx\,dx$$

$$=-\int_0^{\infty}\frac{(1+x^2)-1}{x(1+x^2)}\sin sx\,dx$$

$$=-\int_0^{\infty}\left(\frac{\cancel{1+x^2}}{x\cancel{(1+x^2)}}-\frac{1}{x(1+x^2)}\right)\sin sx\,dx$$

$$=-\int_0^{\infty}\frac{\sin sx}{x}\,dx+\int_0^{\infty}\frac{1}{x(1+x^2)}\sin sx\,dx$$

$\left(\because \int_0^{\infty}\frac{\sin mx}{x}dx=\frac{\pi}{2}\right)$ $$=-\frac{\pi}{2}+\int_0^{\infty}\frac{1}{x(1+x^2)}\sin sx\,dx$$

$$\Rightarrow \frac{dI}{ds}=-\frac{\pi}{2}+\int_0^{\infty}\frac{1}{x(1+x^2)}\sin sx\,dx \quad (2)$$

D ② w.r.t 's'

$$\frac{d^2I}{ds^2} = 0 + \int_0^{\infty} \frac{1}{\cancel{x}(1+x^2)} \cos sx \cdot \cancel{x}\, dx$$

$$= \int_0^{\infty} \frac{1}{1+x^2} \cos sx \, dx$$

$$= I \qquad (\because \text{from } ①)$$

$$\Rightarrow \frac{d^2I}{ds^2} = I$$

$$\Rightarrow \frac{d^2I}{ds^2} - I = 0.$$

$$\Rightarrow (D^2-1)I = 0.$$

$$\Rightarrow f(D)I = 0$$

A.E is $f(D) = 0 \Rightarrow D^2-1=0$

$$\Rightarrow D^2 = 1$$

$$\Rightarrow D = \pm 1$$

$$\Rightarrow D = -1, 1 \text{ real \& distinct}$$

$$C.F = c_1 e^{-s} + c_2 e^{s}$$

The solution is $I = C.F$

$$\Rightarrow I = c_1 e^{-s} + c_2 e^{s} \quad \text{--- ③}$$

D. ③ w.r.t 's'

$$\frac{dI}{ds} = -c_1 e^{-s} + c_2 e^{s} \quad \text{--- ④}$$

From ① & ③ $\Rightarrow \int_0^{\infty} \frac{1}{1+x^2} \cos sx \, dx = c_1 e^{-s} + c_2 e^{s}$

Put $s=0$.

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2} \cos 0 \, dx = c_1(1) + c_2(1)$$

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2} dx = c_1 + c_2$$

$$\Rightarrow \left[\tan^{-1} x\right]_0^\infty = c_1 + c_2$$

$$\Rightarrow \tan^{-1}\infty - \tan^{-1} 0 = c_1 + c_2$$

$$\Rightarrow \frac{\pi}{2} - 0 = c_1 + c_2$$

$$\Rightarrow c_1 + c_2 = \frac{\pi}{2} \quad \text{--- (5)}$$

From (2) & (4)

$$\Rightarrow -\frac{\pi}{2} + \int_0^\infty \frac{1}{x(1+x^2)} \sin sx \, dx = -c_1 e^{-s} + c_2 e^{s}$$

Put $s = 0$

$$\Rightarrow -\frac{\pi}{2} + \int_0^\infty \frac{1}{x(1+x^2)} \sin . 0 \, dx = -c_1(1) + c_2(1)$$

$$\Rightarrow -\frac{\pi}{2} + 0 = -c_1 + c_2$$

$$\Rightarrow -c_1 + c_2 = -\frac{\pi}{2} \quad \text{--- (6)}$$

$$(5) + (6) \Rightarrow \cancel{c_1} + c_2 - \cancel{c_1} + c_2 = \frac{\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow 2c_2 = 0$$

$$\Rightarrow c_2 = 0$$

Put $c_2 = 0$ in (5)

$$\therefore (5) \Rightarrow c_1 + 0 = \frac{\pi}{2}$$

$$\Rightarrow c_1 = \frac{\pi}{2}$$

$\therefore F_c\left\{\frac{1}{1+x^2}\right\} = I$

$= c_1 e^{-s} + c_2 e^{s}$ $\quad$ ($\because$ from (3))

$= \frac{\pi}{2} e^{-s} + 0.e^{s}$ $\quad$ ($\because$ Put c_1, c_2 values)

$= \frac{\pi}{2} e^{-s}$

By def $F_s\left\{\frac{x}{1+x^2}\right\} = \int_0^\infty \frac{x}{1+x^2} \sin sx \, dx$

$= -\frac{dI}{ds}$ $\quad$ ($\because$ from (a))

$= -\left(-c_1 e^{-s} + c_2 e^{s}\right)$ $\quad$ ($\because$ from (4))

$= c_1 e^{-s} - c_2 e^{s}$

$= \frac{\pi}{2} e^{-s} - 0.e^{s}$ $\quad$ ($\because$ Put c_1, c_2 values)

$= \frac{\pi}{2} e^{-s}$

11.) Find the fourier Sine and Cosine transform of x^{n-1}, $n>0$ and prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under fourier Sine and Cosine transform.

Sol: By def $F_s\{x^{n-1}\} = \int_0^\infty x^{n-1} \sin sx \, dx$

$|||^{ly}$ $F_c\{x^{n-1}\} = \int_0^\infty x^{n-1} \cos sx \, dx$

Now, $F_c\{x^{n-1}\} + i F_s\{x^{n-1}\} = \int_0^\infty x^{n-1} \cos sx \, dx + i \int_0^\infty x^{n-1} \sin sx \, dx$

$$= \int_0^\infty x^{n-1} \cancel{(\cos sx + i\sin sx)\, dx} \, (\cos sx + i \sin sx)\, dx$$

$$= \int_0^\infty x^{n-1} e^{isx} dx$$

$$(\because e^{i\theta} = \cos\theta + i\sin\theta)$$

Put $isx = -t$

$$\Rightarrow x = \frac{-t}{is}$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx.$$

$$\Rightarrow dx = \frac{-dt}{is}$$

If $x = 0$, then $t = 0$

If $x = \infty$, then $t = -\infty$

$$= \int_0^{-\infty} \left(\frac{-t}{is}\right)^{n-1} e^{-t} \left(\frac{-dt}{is}\right)$$

$$= \int_0^{-\infty} \frac{(-1)^{n-1}(-1)}{(is)^{n-1}(is)} t^{n-1} e^{-t} dt$$

$$= \int_0^{-\infty} \frac{(-1)^{n \cancel{-1} \cancel{+1}}}{(is)^{n \cancel{-1} \cancel{+1}}} t^{n-1} e^{-t} dt$$

$$= \int_0^{-\infty} \frac{(-1)^n t^{n-1} e^{-t}}{(is)^n} dt$$

$$= \frac{(-1)^n}{(is)^n} \int_0^{-\infty} e^{-t} t^{n-1} dt$$

$$-1 = i^2$$

$$(-1)^n = (i^2)^n = i^{2n}$$

$$= \frac{i^{2n}}{i^n s^n} \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{i^n}{s^n} \Gamma n$$

$$= \frac{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n}{s^n} \Gamma n$$

$$= \frac{\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma n}{s^n}$$

$$= \frac{\cos\frac{n\pi}{2}}{s^n} \cdot \Gamma n + i\frac{\sin\frac{n\pi}{2}}{s^n} \cdot \Gamma n$$

Equating real & imaginary parts

$$F_c\{x^{n-1}\} = \frac{\Gamma n}{s^n} \cdot \cos\frac{n\pi}{2} \quad \text{— (1)}$$

$$F_s\{x^{n-1}\} = \frac{\Gamma n}{s^n} \sin\frac{n\pi}{2} \quad \text{— (2)}.$$

Put $n = \frac{1}{2}$ in (1)

$$F_c\{x^{1/2-1}\} = \frac{\Gamma 1/2}{s^n} \cos\frac{(1/2)\pi}{2}$$

$$\Rightarrow F_c\{x^{-1/2}\} = \frac{\sqrt{\pi}}{s^{1/2}} \cos\frac{\pi}{4} \qquad \because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Rightarrow F_c\left\{\frac{1}{\sqrt{x}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} \cdot \frac{1}{\sqrt{2}}$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{s}}$$

$$= \sqrt{\frac{\pi}{2}} \text{ times of } \frac{1}{\sqrt{s}}$$

Put $n = 1/2$ in (2)

$$F_s\{x^{1/2-1}\} = \frac{\Gamma 1/2}{s^{1/2}} \sin\frac{1/2\,\pi}{2}$$

$$\Rightarrow F_s\{x^{-1/2}\} = \frac{\sqrt{\pi}}{\sqrt{s}} \sin\frac{\pi}{4}$$

$$\Rightarrow F_s\left\{\frac{1}{\sqrt{x}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} \cdot \frac{1}{\sqrt{2}}$$

$$= \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{s}}$$

$$= \sqrt{\frac{\pi}{2}} \text{ times of } \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self reciprocal under fourier Cosine and sine transforms.

12) Show that $F_s\{x f(x)\} = -\frac{d}{ds} F_c(s)$ and

$F_c\{x f(x)\} = \frac{d}{ds} F_s(s)$.

Sol:- By def $F_c\{f(x)\} = \int_0^{\infty} f(x) \cos sx \, dx = F_c(s)$, say

Now, $F_c(s) = \int_0^{\infty} f(x) \cos sx \, dx$.

D. w.r.t 's'

$$\frac{d}{ds} F_c(s) = \int_0^{\infty} f(x) (-\sin sx \cdot x) \, dx$$

$$= -\int_0^{\infty} x f(x) \sin sx \, dx$$

$$= -F_s\{x f(x)\}$$

$$\Rightarrow F_s\{x f(x)\} = -\frac{d}{ds} F_c(s).$$

Now, by def $F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = F_s(s)$.

D. w.r.t 's'

$$\frac{d}{ds} F_s(s) = \int_0^\infty f(x) \cos sx \cdot x \, dx$$

$$= \int_0^\infty x f(x) \cos sx \, dx$$

$$= F_c\{x f(x)\}$$

$$\Rightarrow F_c\{x f(x)\} = \frac{d}{ds} F_s(s).$$

13.) Find the fourier sine and cosine transform of $x\bar{e}^{ax}$.

Given $x\bar{e}^{ax} = x f(x)$, say

$$\Rightarrow f(x) = \bar{e}^{ax}.$$

By def $F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = F_s(s)$

$$\therefore F_s\{\bar{e}^{ax}\} = \int_0^\infty \bar{e}^{ax} \sin sx \, dx$$

$$= \left[\frac{\bar{e}^{ax}}{(-a)^2+s^2}\left(-a\sin sx - s\cos sx\right)\right]_{x=0}^{\infty}$$

$$= 0 - \frac{1}{a^2+s^2}\left(-a(0) - s(1)\right)$$

$$= \frac{s}{a^2+s^2} = F_s(s), \text{ say}$$

w.k.t $F_c\{x f(x)\} = \frac{d}{ds} F_s(s)$

$$\Rightarrow F_c\{x\bar{e}^{ax}\} = \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right)$$

(9)

$$= \frac{(a^2+s^2)(1) - s(2s)}{(a^2+s^2)^2}$$

$$= \frac{a^2+s^2-2s^2}{(a^2+s^2)^2}$$

$$\Rightarrow F_c\{xe^{-ax}\} = \frac{a^2-s^2}{(a^2+s^2)^2}$$

By def $F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$.

$$\therefore F_c\{e^{-ax}\} = \int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \left[\frac{e^{-ax}}{(a)^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_{x=0}^{\infty}$$

$$= 0 - \frac{1}{a^2+s^2}\left(-a(1)+s(0)\right)$$

$$= \frac{a}{a^2+s^2} = F_c(s), \text{ say.}$$

w.k.t $F_s\{xf(x)\} = -\frac{d}{ds}F_c(s)$

$$\therefore F_s\{xe^{-ax}\} = \frac{-d}{ds}\left(\frac{a}{a^2+s^2}\right)$$

$$= -\left(\frac{-a}{(a^2+s^2)^2}\cdot 2s\right)$$

$$\Rightarrow F_s\{xe^{-ax}\} = \frac{2as}{(a^2+s^2)^2}$$

14.) Solve the integral equation $\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0\le\alpha \\ 0, & \alpha>1 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$

Sol:- We know that $\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = F_C\{f(\theta)\}$ ($\because$ by def)

$= F_C(\alpha)$, say.

But given that $\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$

$$\Rightarrow F_C(\alpha) = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$$

By inverse Fourier Cosine transform,

$$F^{-1}\{F_C(\alpha)\} = \frac{2}{\pi}\int_0^\infty F_C(\alpha)\cos\alpha\theta\, d\alpha = f(\theta), \text{say}$$

$$\Rightarrow f(\theta) = \frac{2}{\pi}\int_0^\infty F_C(\alpha)\cos\alpha\theta\, d\alpha.$$

$$= \frac{2}{\pi}\left(\int_0^1 F_C(\alpha)\cos\alpha\theta\, d\alpha + \int_1^\infty F_C(\alpha)\cos\alpha\theta\, d\alpha\right)$$

$$= \frac{2}{\pi}\left(\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha + \int_1^\infty 0\cdot\cos\alpha\theta\, d\alpha\right)$$

$$= \frac{2}{\pi}\left(\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha + 0\right)$$

$$= \frac{2}{\pi}\int_0^1 \underset{u}{\underset{\downarrow}{(1-\alpha)}}\,\underset{v}{\underset{\downarrow}{\cos\alpha\theta}}\, d\alpha$$

≤ 1

$$= \frac{2}{\pi}\left[(1-\alpha)\ \frac{\sin\alpha\theta}{\theta} - (-1)\left(-\frac{\cos\alpha\theta}{\theta^2}\right) \right]_{\alpha=0}^{1}$$

$\because \int uv = uv_1 - u'v_2 + u''v_3 - - - -$

$$= \frac{2}{\pi}\left(0 - \frac{\cos\theta}{\theta^2} - 0 + \frac{1}{\theta^2}\right)$$

$$\Rightarrow f(\theta) = \frac{2}{\pi}\left(\frac{1-\cos\theta}{\theta^2}\right)$$

w.k.t $F_c\{f(\theta)\} = F_c(\alpha)$

$$\Rightarrow F_c\left\{\frac{2}{\pi}\left(\frac{1-\cos\theta}{\theta^2}\right)\right\} = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

$$\Rightarrow \int_0^\infty \frac{2}{\pi}\left(\frac{1-\cos\theta}{\theta^2}\right)\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, 0 \le \alpha \le 1 \\ 0, \alpha > 1 \end{cases}$$

Put $\alpha = 0$

$$\Rightarrow \frac{4}{\pi}\int_0^\infty \frac{\sin^2\theta/2}{\theta^2}\cdot\cos(0)\, d\theta = 1$$

$$\Rightarrow \frac{4}{\pi}\int_0^\infty \frac{\sin^2\theta/2}{\theta^2}\cdot d\theta = 1$$

$$\Rightarrow \int_0^\infty \frac{\sin^2\theta/2}{\theta^2}\, d\theta = \frac{\pi}{4}.$$

Put $\theta/2 = t \Rightarrow \theta = 2t$

$\Rightarrow d\theta = 2dt$

If $\theta = 0$, then $t = 0$

If $\theta = \infty$, then $t = \infty$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{(2t)^2} \cdot 2dt = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{\cancel{4}t^2} \cdot 2dt = \frac{\pi}{\cancel{4}}$$

$$\Rightarrow 2\int_0^\infty \frac{\sin^2 t}{t^2} dt = \pi$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

1) If the fourier sine transform of $f(x) = \frac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$ find $f(x)$

Sol:- We have $f(x)$ = Inverse finite fourier Sine transform of

$$F_s(n) = \frac{2}{l} \sum_1 F_s(n) \sin \frac{n\pi x}{l}$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi}$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1-\cos n\pi}{n^2\pi^2}\right) \sin nx$$

$$= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1-\cos n\pi}{n^2}\right) \sin nx .$$

2) Find the inverse finite cosine transform $f(x)$ if

$$F_c(n) = \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2}, \text{ where } 0<x<4$$

Sol:- Here $l=4$

$\therefore$ Inverse finite cosine transform,

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{4} \cdot 1 + \frac{2}{4} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\left(\frac{n\pi x}{4}\right)$$

3) Find the finite fourier sine and cosine transform of $f(x)$, defined by $f(x)=x$, $0<x<4$.

Sol:- Here $l=4$

$$F_s\{f(x)\} = \int_0^l f(x) \sin\frac{n\pi x}{l} dx$$

$$= \int_0^4 x \cdot \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \left[x \cdot \frac{-\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}} - 1 \cdot \frac{-\sin\left(\frac{n\pi x}{4}\right)}{\frac{n^2\pi^2}{16}} \right]_0^4$$

$$= \frac{-16\cos n\pi}{n\pi} + \frac{16}{n^2\pi^2} \sin n\pi - 0 - \frac{16}{n^2\pi^2} \sin 0$$

$$= \frac{-16}{n\pi}(-1)^n, \quad n = 1,2,3,\ldots$$

$$F_c\{f(x)\} = \int_0^l f(x) \cos\frac{n\pi x}{l} dx$$

$$= \int_0^4 x \cos\frac{n\pi x}{4} dx$$

$$= \left[x \frac{\sin\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}} - 1 \cdot \frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2} \right]_0^4$$

$$= \frac{16}{n\pi} \sin n\pi + \frac{16}{n^2\pi^2} \cos n\pi - 0 - \frac{16}{n^2\pi^2} \cos 0$$

$$= 0 + \frac{16}{n^2\pi^2} (-1)^n - \frac{16}{n^2\pi^2} (1)$$

$$= \frac{16}{n^2\pi^2} \left[(-1)^n - 1 \right] \quad , n > 0$$

At $n = 0$, $F_c\{F(x)\} = \int_0^4 x \, dx = \left[\frac{x^2}{2} \right]_0^4 = 8$