ANNAMACHARYA UNIVERSITY, RAJAMPET

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016
RAJAMPET, Annamayya District, AP, INDIA

Course : **Linear Control Systems**

Course Code : **24AEEE51T**

Branch(s) : **EEE**

Prepared by : **Dr. M.Padma Lalitha**

Designation : **Professor**

Department : **Electrical and Electronics Engineering**

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Title of the Course: **Linear Control Systems**
Category: PCC
Semester: V
Course Code: 24AEEE51T
Branch/es: EEE

Lecture Hours	Tutorial Hours	Practice Hours	Credits
2	1	0	3

Course Objectives:

1. Merits and demerits of open loop and closed loop systems; the effect of feedback
2. The use of block diagram algebra and Mason's gain formula to find the overall transfer function
3. Transient and steady state response, time domain specifications and the concept of Root loci
4. Frequency domain specifications, Bode diagrams and Nyquist plots
5. State space modelling of Control system

Course Outcomes:

At the end of the course, the student will be able to …

1. Apply control concepts for modeling, block diagram/Signal flow graph analysis, and servo motor operation.

2. Analyze time response, steady-state errors, and controller performance.

3. Analyze system stability using Routh's criterion and root locus techniques.

4. Analyze frequency response stability using Bode, Polar, and Nyquist plots and design compensators for stability.

5. Analyze state-space models, system response, controllability, and observability of continuous systems.

Unit 1 Introduction to Control Systems Concepts **12**

Open loop and closed loop control systems and their differences- Examples of control systems, Feedback characteristics; Mathematical models – Differential equations of translational and rotational mechanical systems, Analogous Systems, problems; Block diagram reduction Rules, problems; Signal flow graphs using Mason's gain formula, problems; DC and AC Servo motors, Transfer function of DC servo motor & AC servo motor, Synchros.

Unit 2 Time Response Analysis **8**

Standard test signals - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems, problems- Time domain specifications, problems – Steady state response - Steady state errors and Static error co-efficients, problems, P, PI, PID Controllers.

Unit 3 Stability Analysis in Time Domain **10**

The concept of stability – Routh's stability criterion, problems-absolute stability and conditional stability – limitations of Routh's stability criterion. The Root locus concept, problems –effects of adding poles and zeros to G(S)H(S) on the root loci.

Unit 4 Frequency Response Analysis **10**

Frequency domain specifications-Definitions, Bode Diagrams-Stability Analysis from Bode Plots, Problems. Polar Plots-Stability Analysis from polar plots, problems-Nyquist stability criterion-Nyquist Plots, problems. Compensation techniques – Lag, Lead, and Lag-Lead Compensator design using Bode plots.

Unit 5 State Space Analysis of Continuous Systems **10**

Concept of state, state variables and state model, state models – differential equations & Transfer function

models, problems, Transfer function from state model, solving the linear time invariant state equations – State transition Matrix and it's properties. System response through state space models. The concepts of controllability and observability, problems.

Textbooks:

1. I.J.Nagrath and M. Gopal "Control Systems Engineering" New Age International (P) Limited, Publishers, 6th edition, September 2018

Reference Books:

1. Automatic Control Systems by B. C. Kuo and Farid Golnaraghi, John wiley and sons, 8th edition, 2003.
2. Modern Control Engineering by Katsuhiko Ogata, Prentice Hall of India Pvt. Ltd., 5th edition, 2010.
3. Control System Design by Graham C. Goodwin, Stefan F. Graebe and Mario E. Salgado, Pearson, 2000.

Web Resources:

1. https://nptel.ac.in/courses/107106081
2. https://www.youtube.com/watch?v=ziu1OTwUrbw
3. https://www.youtube.com/watch?v=jji35E9hyAs
4. https://www.youtube.com/watch?v=C4-auv8J_7E
5. https://www.youtube.com/watch?v=pv3sqbzgTrI

CO-PO-PSO Mapping:

Course Outcomes	Engineering Knowledge	Problem Analysis	Design/Development of solutions	Conduct investigations of complex problems	Engineering Tool Usage	The Engineer and The World	Ethics	Individual and Collaborative Team work	Communication	Project management and finance	Life-long learning	PSO1	PSO2
24AEEE51T.1	3	3	3	-	2	1	-	-	-	-	3	3	3
24AEEE51T.2	3	3	2	-	3	1	-	-	-	-	3	3	3
24AEEE51T.3	3	2	3	-	3	3	1	-	-	-	3	3	3
24AEEE51T.4	3	3	3	-	3	3	3	-	-	-	3	3	3
24AEEE51T.5	3	3	2	-	3	3	3	-	-	-	3	3	3

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RAJAMPET, Annamayya District, AP, INDIA

Department of Electrical & Electronics Engineering

III B.Tech I Semester

Linear Control Systems (24AEEE51T)
(Electrical and Electronics Engineering)

Unit-I (Introduction to Control Systems Concepts)

Syllabus:

Open loop and closed loop control systems and their differences- Examples of control systems, Feedback characteristics; Mathematical models – Differential equations of translational and rotational mechanical systems, Analogous Systems, problems; Block diagram reduction Rules, problems; Signal flow graphs using Mason's gain formula, problems; DC and AC Servo motors, Transfer function of DC servo motor & AC servo motor, Synchros.

Types Of Control Systems

There are two types of control systems namely:

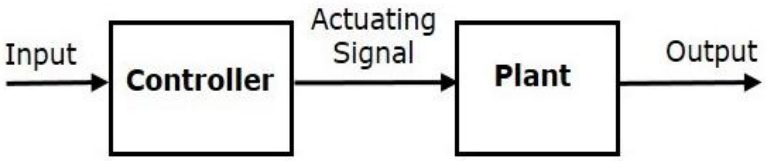
1. Open loop control systems (Non-feedback control systems)
2. Closed loop control systems (Feedback control systems)

Open Loop Control System

Def:

1. The systems in which the output has no effect on control action (or Input) are called as open loop control systems.

2. Any physical system which does not automatically corrects the variation in its output is called an open loop control system.



In these systems,

- The output is varied by varying the input.
- The control action (or input) is independent of output.
- The output is neither measured nor fed back to input for comparison.
- The accuracy of the system depends on calibration of input.
- Though the input is correct it may not produce desired output in the presence of disturbances and parameter variations.
- The changes in output can be corrected only by changing the input manually.
- These systems are useful when disturbances are known ahead of time and no parameter variation occur in the system

Advantages:

- These systems are simple in construction
- These systems are stable
- These systems are economical.
- Convenient when it is difficult to measure output
- Easy maintenance

Disadvantages:

- These systems are inaccurate
- These systems are unreliable

- These systems are sensitive to disturbances
- These systems are sensitive to parameter variations
- To make system accurate recalibration of controller is necessary at regular time intervals.

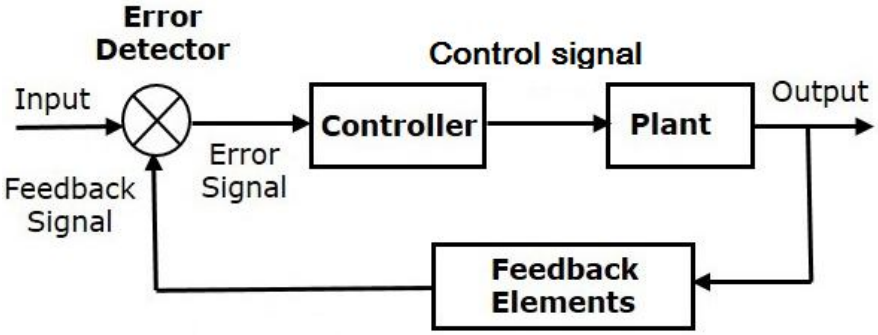
Closed Loop Control System

Def:

1. The systems in which the output has an effect on control action (or Input) in such a manner as to maintain the desired output are called as closed loop control systems.

2. Any physical system which automatically corrects the variation in its output is called closed loop control system.

An open loop control system becomes a closed loop control system by including a feedback. This feedback will automatically correct the change in output due to disturbances. This is why a closed loop control system is called as an automatic control system. The block diagram of a closed loop control system is shown in figure.



In these systems,

- The output is varied by varying the input.
- The control action (or input) is dependent of output.
- The output is measured and fed back to input for comparison.
- They initiate corrective actions automatically if senses any variation in its output.
- It produces desired output even in the presence of disturbances and parameter variations.

Advantages:

- These systems are accurate.
- These systems are reliable.
- These systems are insensitive to disturbances and changes in input.
- These systems are insensitive to parameter variations.

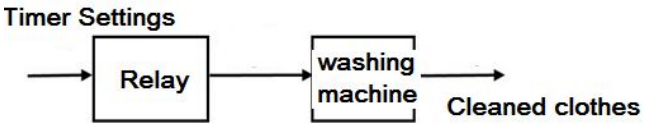
Disadvantages:

- These systems are complex in construction
- These systems may become unstable
- These systems are costly.
- These systems may results oscillatory response
- Difficulty in maintenance

Examples of open loop control systems:

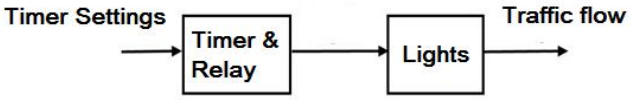
1. Automatic washing machine:

Soaking, washing and rinsing in the washer operate on time basis. The machine does not measure the output (i.e. cleanliness of the clothes). The machine automatically stops after the time without knowing whether clothes are cleaned or not.



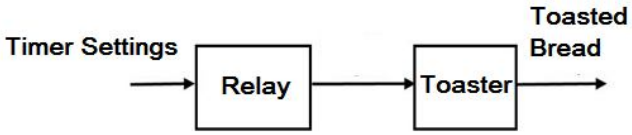
2. Traffic signal system (Open Loop):

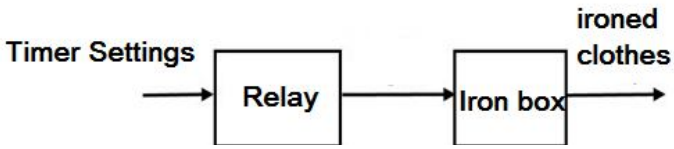
The traffic lights will glow based on pre determined time sequence (for example green-60 sec, orange-10 sec, red-20 sec). The rate of actual traffic flow is not measured and based on that the time slot are not given to lights. For example in a particular direction traffic is heavy so need to give more time to green light in that direction.



3. Bread Toaster, Iron Box, Fan etc.

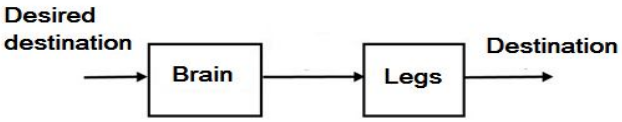
All these systems usually has a timer which instructs the system to switch on for sometime and then switch it off. The output is neither measured nor fed back to get desired output. Accuracy cannot be achieved as the system does not switch on/off based on the actual output but it does as per the pre-set value of time.





4. Walking with eyes closed:

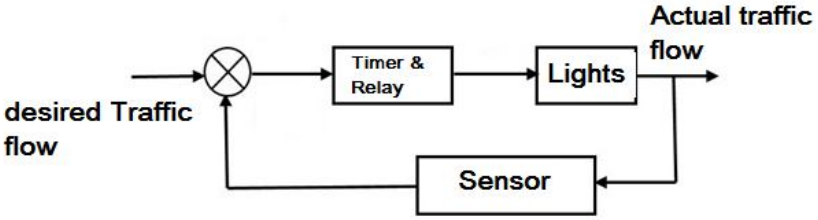
The input to brain is set of instructions to reach the desired location. If the instructions given are accurate we may reach our destination if no disturbances occur in the path. If any disturbances happen then we may not reach our destination.



Examples of Closed loop control systems:

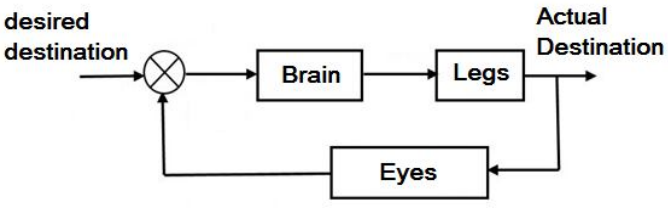
1. Traffic signal system (Closed Loop):

If the system consists a sensor which measures the rate of traffic flow in each direction and the traffic lights will glow based on this information then it becomes closed loop control system.

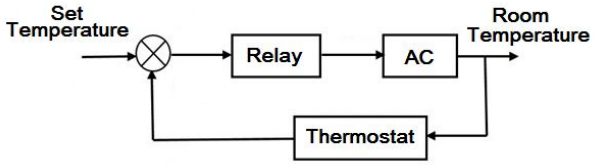


2. **Walking with eyes open:**

Our eyes are continuously monitoring the actual location The input to brain is set of instructions to reach the desired location. If the instructions given are accurate we may reach our destination if no disturbances occur in the path. If any disturbances happen then we may not reach our destination.



3. **Air Conditioner:** The thermostat in AC is continuously measuring the room temperature. If the room temperature falls below the required temperature the power supply is cut off to AC. Similarly if room temperature is more than the required temperature then power is supplied to Ac automatically.



Mathematical Modeling & Transfer Function

To analyze any systems there are two approaches viz.

- Experimental Approach
- Mathematical Modeling (Analytical Approach)

Mathematical Modeling : Each element and whole system is modeled by mathematical equations. The steps involved in this model are:

- Obtaining the physical model
- Obtaining mathematical model
- Solving mathematical model

1. Obtaining the physical model:

An idealized real system which consist all the salient features of the real system is known as physical model. Some ideal assumptions are made to simplify analysis.

Ex:

- Transformer is a real system and its equivalent circuit is physical model. Here the winding is assumed as series combination of R&L.
- Transmission line is a real system. It is a distributed parameter system. The system is assumed as a lumped parameter system and the physical model is either π or T model.
- Modeling of neuron

2. Obtaining mathematical model

Mathematical modeling is a set of mathematical equations which completely describe the elements and their interrelations, by solving which output can be determined for any given input.

For a system many mathematical models exist. For example for an electric circuit mathematical model can be obtained by using either loop analysis or nodal analysis.

3. Solving mathematical model

The mathematical equations can solved in time domain or in frequency domain.

In this course steps 2 & 3 are going to be discussed.

Obtaining mathematical model:

The mathematical equations of a system depend on type of System.

Static systems - Algebraic equations

Dynamic systems :

Discrete time systems: Difference equations

Continuous time system:

Distributed parameter systems - Partial Differential Equations

Lumped parameter systems - Differential Equations (DE)

Time varying systems - Coefficients of DE are variables of time

Time invariant systems - Coefficients of DE are constants

Linear systems - Linear differential equations

Non linear system - Non linear differential equations

In this course **continuous time lumped parameter time invariant linear dynamic systems** are to be studied so the equations are **Linear differential equations with constant coefficients**.

Linear differential equations:

A differential equation is said to be linear if there is only one independent variable and neither the dependant variable nor any of its derivatives is raised to a power greater than one and terms should not consist a product of two or more derivatives of dependent variables.

Ex:

Linear DE: $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = 3x$

Non-Linear DE: $2\left(\frac{d^2y}{dt^2}\right)^2 + 3\frac{dy}{dt} + 4y = 3x$

$2\frac{d^2y}{dt^2} + 3y\frac{dy}{dt} + 4y = 3x$

$2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = 3x^2$

Mathematicals Models of control systems:

There are two mathematical models that are used in control systems viz.

1. **Transfer function model or Input-Output model**
   Only terminal behavior of the system is important. System is represented by One equation which relates input and output using which output can be determined for any given input. An nth order system is represented by **one n^{th} order differential equation** in terms of input and output.
2. **State variable model**
   Not only external or terminal behavior of the system but internal state of the system is also determined with this model by using a minimum set of variables. These variables are known as state variables. In this an nth order system is represented by **n number of 1st order differential equations in terms of state variables**. State variables can be determined by solving the differential equations.Output can be determined from state variables.

Transfer Function:

For any linear time invariant system Transfer Function is defined as the ratio of laplace transform of the output to the lapace transform of the input under zero initial conditions.

$$\text{Transfer Function} = \frac{\mathcal{L}(O/P)}{\mathcal{L}(I/P)} \text{ under zero initial conditions}$$

- The Transfer function of a system is a mathematical model in that it is an operational method of expressing he differential equation that relates the O/P to the I/P.
- It is defined only for linear time invariant systems
- It is applicable for systems under zero initial conditions.
- It is the property of the system independent of the magnitude and nature of the I/P.
- It includes units necessary to relate the I/P to O/P.
- It does not provide any information regarding physical structure of the system.
- If transfer function is known the O/P can be determined for any I/P to understand the nature of the system.
- If transfer function of a system is unknown it may be established experimentally by introducing known I/P and studying the corresponding O/P of the system. Once established a transfer function gives a full description of the dynamics characteristics of the system.

Transfer Function is represented as G(s).
Let x(t) is the input and y(t) is the output.

$$\therefore \quad G(s) = Y(s)/X(s)$$

For any given input x(t)

$$\mathcal{L}(x(t)) = X(s)$$

&

$$Y(s) = G(s) * X(s)$$

$\therefore$ Output $y(t) = \mathcal{L}^{-1}Y(s)$

The general differential equation of an **n^{th} order** system is

$$a_0\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + a_2\frac{d^{n-2}y}{dt^{n-2}} + \dots\dots + a_{n-1}\frac{dy}{dt} + a_n y = b_0\frac{d^m x}{dt^m} + b_1\frac{d^{m-1}x}{dt^{m-1}} + \dots\dots + b_{m-1}\frac{dx}{dt} + b_m x$$

By applying laplace on both sides of above equation

$$a_0 S^n Y(s) + a_1 S^{n-1}Y(s) + \dots + a_{n-1} SY(s) + a_n Y(s) = b_0 S^m X(s) + b_1 S^{m-1}X(s) + \dots + b_{m-1} SX(s) + b_m X(s)$$

By taking Y(S) common in LHS and X(S) common in RHS the equation becomes

$$(a_0 S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n) Y(s) = (b_0 S^m + b_1 S^{m-1} + \dots + b_{m-1} S + b_m) X(s)$$

$\therefore$ The Transfer Function of the system

$$G(S) = Y(S)/X(S)$$

$$G(S) = \frac{b_0 S^m + b_1 S^{m-1} + \dots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n}$$

Transfer Function of electrical systems

The transformed impedances and admittances of the three basic elements of electrical system are given below.

Sl. No.	Element	Z(S)	Y(S)
1	R	R	1/R
2	L	LS	1/LS
3	C	1/CS	CS

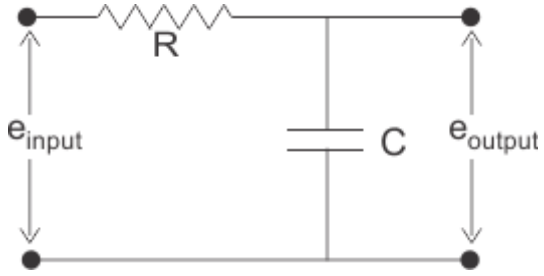
Procedure to find the transfer functions of electric networks:

1. First draw the given electrical network in the s domain with each inductance L replaced by Ls and each capacitance replaced by 1/Cs.
2. Replace all sources and time variables with their Laplace transforms so that $v_i(t)$ is replaced by $V_i(s)$ and $i_i(t)$ by $I_i(s)$ respectively.

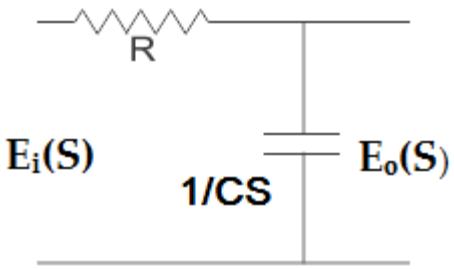
3. Use mesh analysis or node analysis to write the network equations.
4. Eliminate the unwanted variables and get an equation which relates input and output variables.
5. Determine the transfer function as the ratio of (O/P)/(I/P).

Problems:

1. Find transfer function of the given system.



Solution: The transformed network of the given circuit is



By voltage division rule

$V_1=V_T*(R_1/(R_1+R_2))$

$\therefore \quad E_o(s)=E_i(s)*(Z_2/(Z_1+Z_2))$

Where $Z_1 = R$ &

$Z_2 = 1/CS$

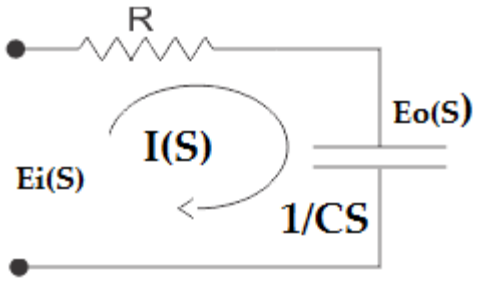
$\therefore \quad E_o(s)=E_i(s)*(1/CS /(R+1/CS))$

$\Longrightarrow \quad \frac{Eo(s)}{Ei(s)} = \frac{1/CS}{R+1/CS}$

$$\Longrightarrow \frac{Eo(s)}{Ei(s)} = \frac{1}{RCS+1}$$

$$\therefore Transfer\ Function = \frac{Eo(s)}{Ei(s)} = \frac{1}{RCS+1}$$

Method 2: Assume Loop Current is I(S)



So,

$E_i(S)= (R+1/CS)I(S)$ …(1) (since R & 1/CS are in series)

&

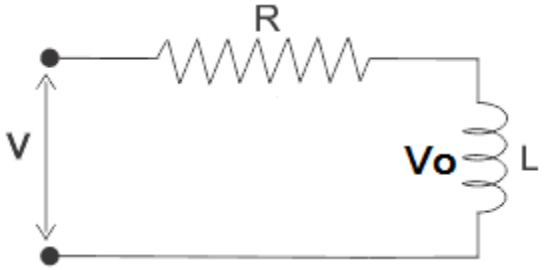
$E_o(S)= (1/CS)\ I(S)$ …(2)

By dividing eqn (2) with eqn(1)

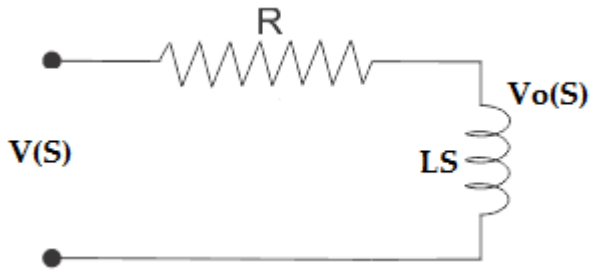
$$(\boldsymbol{E_o(s)})/(\boldsymbol{E_i(s)}) = \frac{\mathbf{1/CS}}{\mathbf{R+1/CS}}$$

$$\therefore \quad Transfer\ Function = \frac{E_o(s)}{E_i(s)} = \frac{1}{RCS+1}$$

2. **Find transfer function of the given system**.



Solution: The transformed network of the given circuit is



By voltage division rule

$V_1=V_T*(R_1/(R_1+R_2))$

$\therefore \quad V_o(s)=V_i(s)*(Z_2/(Z_1+Z_2))$

Where $Z_1 = R$ &

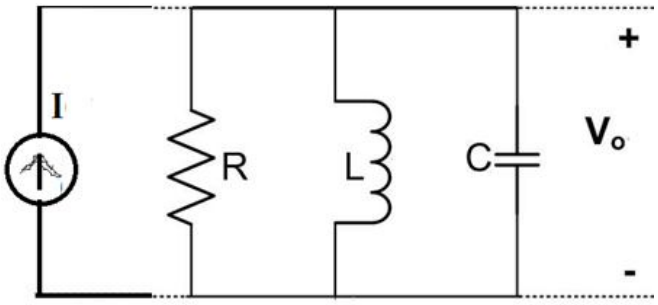
$Z_2 = LS$

$\therefore \quad V_o(s)=V_i(s)*(LS /(R+LS))$

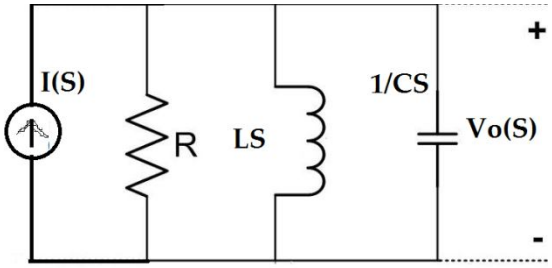
$$\Longrightarrow \quad \frac{V_o(s)}{V_i(s)} = \frac{LS}{R+LS}$$

$$\therefore Transfer\ Function = \frac{V_o(s)}{V_i(s)} = \frac{LS}{R+LS}$$

3. Find transfer function of the given system.



Solution: The transformed network of the given circuit is



By Nodal analysis

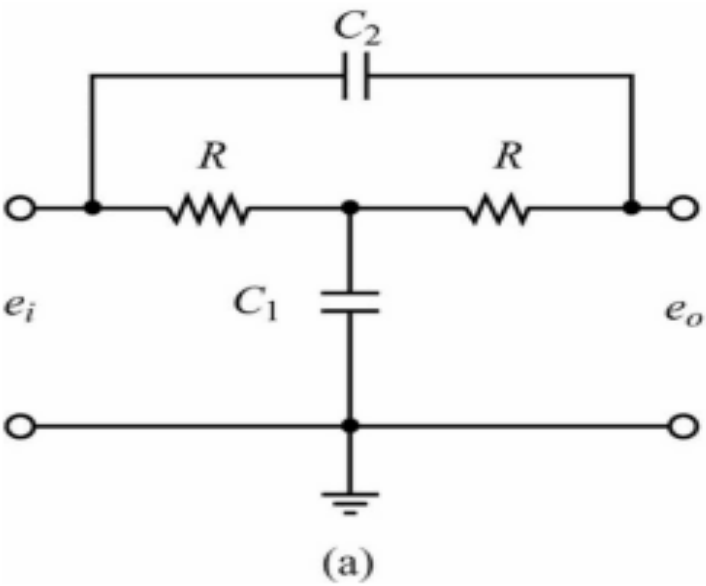
$$I(S) = \frac{V_o(S)}{R} + \frac{V_o(S)}{LS} + \frac{V_o(S)}{1/CS}$$

$$\therefore \quad I(S) = \left(\frac{1}{R} + \frac{1}{LS} + CS\right) V_o(S)$$

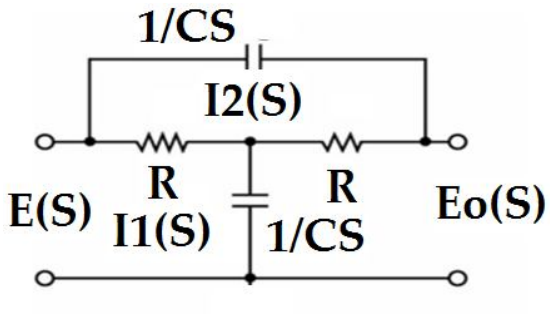
$$\therefore \quad I(S) = \frac{(RLC\,S^2 + LS + R)}{RLS} V_o(S)$$

$$\therefore Transfer\ Function = \frac{V_o(s)}{I(s)} = \frac{RLS}{RLCS^2 + LS + R}$$

4. **Find transfer function of the given system.**



Solution: The transformed network of the given circuit is



Let $I_1(S)$ and $I_2(S)$ are loop currents in clock wise direction then

By Loop analysis

$E(S) = (R+1/CS)(I_1(S)) - R\, I_2(S)$ ….(1)

$0 = -R\, I_1(S) + (R+R+1/CS))\, I_2(S)$ …..(2)

And output

$E_o(S) = (1/CS)\, I_1(S) + R\, I_2(S)$ …..(3)

From eqn (2)

$$I_2(S) = \frac{R}{2R+1/CS} I_1(S) \quad \text{…..(4)}$$

Substitute eqn(4) in eqn (1) & eqn(3)

$$E(S) = (R+1/CS)\, I_1(S) - R \left(\frac{R}{2R+\frac{1}{CS}}\right) I_1(S)$$

$$E(S) = \left(\frac{(R+1/CS)\,(2R+\frac{1}{CS}) - R^2}{2R+\frac{1}{CS}}\right) I_1(S) \quad \text{…} \quad (5)$$

$E_o(S) = (1/CS)\, I_1(S) + R \frac{R}{2R+1/CS} I_1(S)$

$$E_o(S) = \left(\frac{\left(\frac{1}{CS}\right)\left(2R+\frac{1}{CS}\right)+R^2}{2R+\frac{1}{CS}} \right) I_1(S) \qquad \ldots.(6)$$

By dividing eqn (6) with eqn (5)

$$\frac{E_o(S)}{E(S)} = \frac{\left(\frac{\left(\frac{1}{CS}\right)\left(2R+\frac{1}{CS}\right)+R^2}{2R+\frac{1}{CS}} \right)}{\left(\frac{(R+1/CS)\,(2R+\frac{1}{CS})-R^2}{2R+\frac{1}{CS}} \right)}$$

$$\therefore \quad \frac{E_o(S)}{E(S)} = \frac{\left(\left(\frac{1}{CS}\right)\left(2R+\frac{1}{CS}\right)+R^2\right)}{(R+1/CS)\,(2R+\frac{1}{CS})-R^2}$$

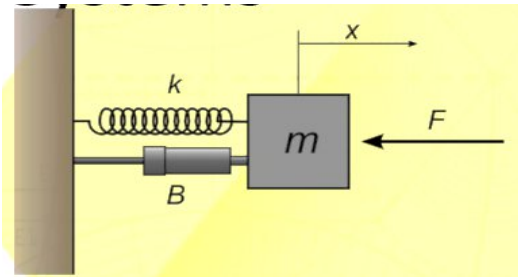
After simplification

$$Transfer\ Function = \frac{E_o(s)}{E(s)} = \frac{R^2C^2S^2+2RCS+1}{R^2C^2S^2+3RCS+1}$$

In this module, let us discuss the **differential equation modeling** of mechanical systems. There are two types of mechanical systems based on the type of motion.

- Translational mechanical systems
- Rotational mechanical systems

Modeling of Translational Mechanical Systems



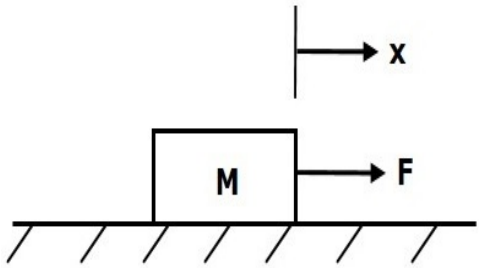
Translational mechanical systems move along a **straight line**. These systems mainly consist of three basic elements. Those are

- **mass,**
- **spring and**
- **dashpot or damper.**

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Let us now see the force opposed by these three elements individually.

Mass

Mass is the property of a body, which stores **kinetic energy**. If a force is applied on a body having mass **M**, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.



$$F_m \propto a$$

$$F_m = Ma = M\frac{d^2x}{dt^2}$$

$$F = F_m = Ma = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$

Where,

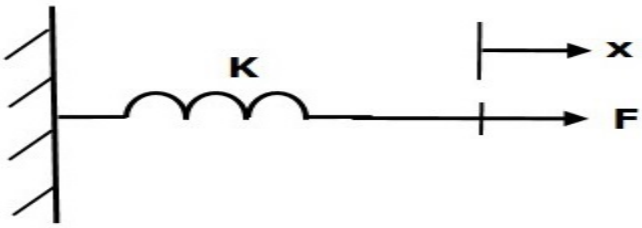
- **F** is the applied force
- $\mathbf{F_m}$ is the opposing force due to mass
- **M** is mass
- **a** is acceleration, v velocity &
- **x** is displacement

NOTE: There is only one displacement, velocity & acceleration for Mass.

Spring



Spring is an element, which stores **potential energy**. If a force is applied on spring **K**, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.



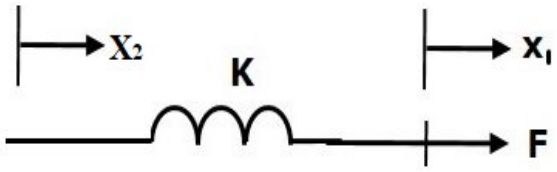
$F \propto x$

$$F = F_k = Kx = K\int v dt$$

Where,

- **F** is the applied force
- **F_k** is the opposing force due to elasticity of spring
- **K** is spring constant
- **x** is displacement

When both ends are free to move:



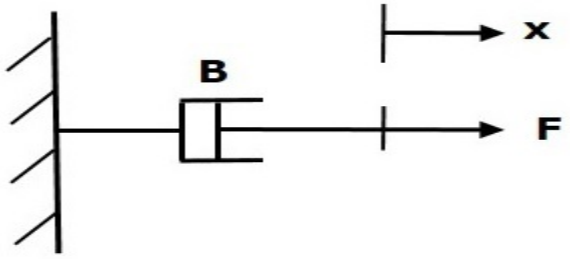
$$F_k = K(X_1 \sim X_2) = K\int (v_1 \sim v_2)dt$$

$F=F_k$

Dashpot



If a force is applied on dashpot **B**, then it is opposed by an opposing force due to **friction** of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible.



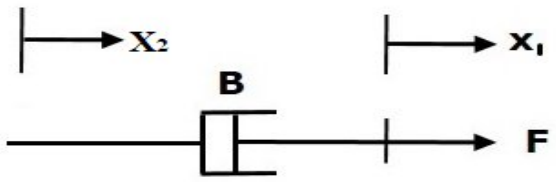
$F_b \propto v$

$$F = F_b = Bv = B\frac{dx}{dt}$$

Where,

- **F_b** is the opposing force due to friction of dashpot
- **B** is the frictional coefficient
- **v** is velocity
- **x** is displacement

When both ends are free to move:



$$F_b = B(v_1 - v_2) = B\frac{d(x_1 - x_2)}{dt}$$

$$F = F_b$$

D'Alembert Principal:

The algebraic sum of the forces acting on the system in any given direction is zero.

SF =0

Please click on below link to a demo on translational mechanical system.

https://lpsa.swarthmore.edu/Systems/MechTranslating/TransMechSysAll.html

Transfer function of Translational Mechanical Sytsem:

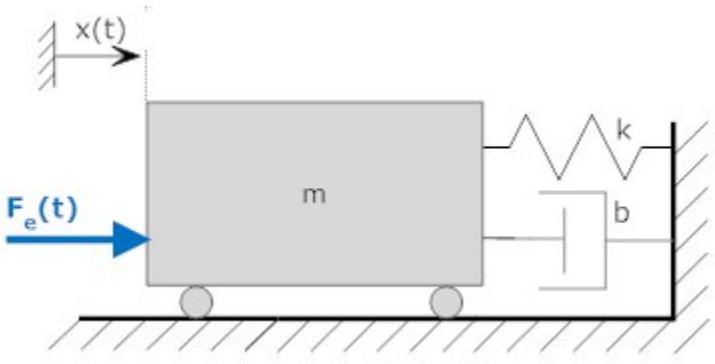
We can apply D'Alembert's law to develop equations of motion for translating mechanical systems through the use of free body diagrams. To do this we draw a free body diagram for each displacement in a system.

Procedure:

1. identify nodes in the given system. Generally Mass elements are the nodes. In addition to mass elements, meeting points of Spring & Dashpot / Spring &force / Dashpot & force are considered as nodes.
2. Assign displacement to each node.
3. Draw free body diagrams for each node. The free body diagram is obtained by drawing each node separately and then marking all the forces acting on that node.
4. For each free body diagram write one differential eqn. by using D'Alembert law.

Consider a simple system with a mass that is separated from a wall by a spring and a dashpot. The mass could represent a car, with the spring and dashpot representing the car's bumper and tires.

1. Find the transfer function of the system.

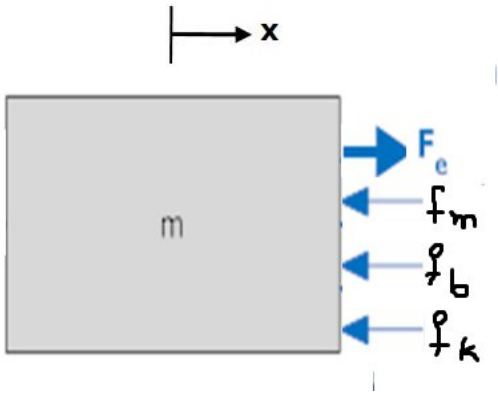


Sol: Input = F_e, output = x

There is only one node. There are four forces acting on that node:

1. An external force (Fe)
2. A force from the spring. To determine the direction consider that the position "x" is defined positive to the right. If the mass moves in the positive "x" direction, the force from the spring, k·x,will appear to the left.
3. A force from the dashpot. By an argument similar to that for the spring there will be a force from the dashpot, b·v, to the left.
4. Finally, there is the inertia force which is defined to be opposed to the defined direction of motion. This is represented by m·a to the left. Don't forget this force, it is easy to do so.

The completed free body diagram is shown below



We sum all of these forces to zero and get.

$F_e=f_m+f_b+f_k$

$$F_e = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

By applying Laplace Tr

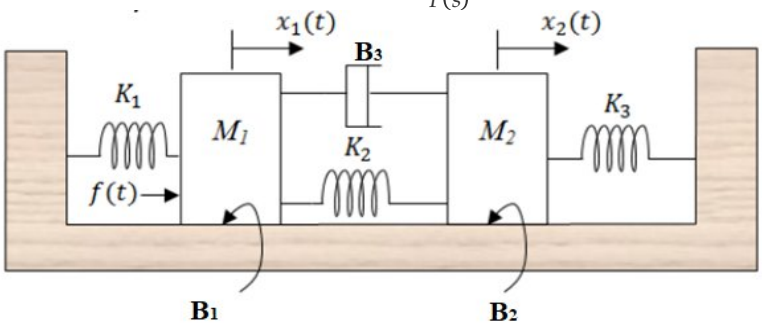
$F_e(s)=Ms^2X(s)+BsX(s)+KX(s)$

$=(Ms^2+Bs+K)X(s)$

Therefore transfer function of the system is

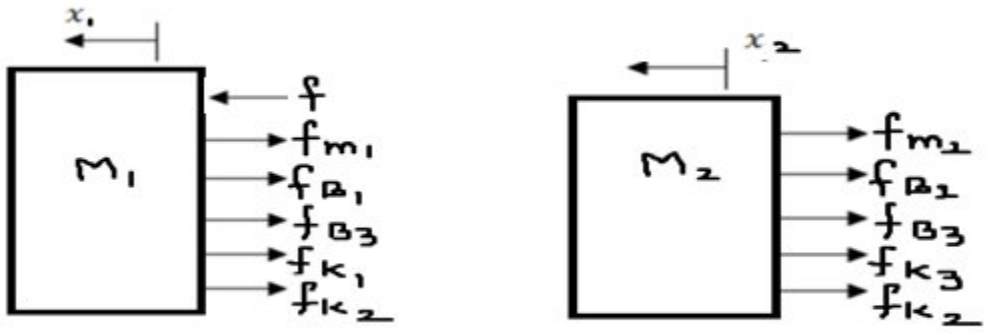
$$\frac{X(s)}{F_e(s)} = \frac{1}{Ms^2 + Bs + K}$$

2. Find the transfer function $\frac{X_1(s)}{F(s)}$ of the system.



Sol: Input = f(t), output = $x_1(t)$

There are two nodes(2 Masses). Freebody diagrams of the nodes are as shown below.



Node1:

$f=f_{m1}+f_{b1}+f_{b3}+f_{k1}+f_{k2}$

$$f = M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_3\frac{d(x_1-x_2)}{dt} + K_1x_1 + K_2(x_1-x_2)\ldots(1)$$

Node2:

$0=f_{m2}+f_{b2}+f_{b3}+f_{k3}+f_{k2}$

$$0 = M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + B_3\frac{d(x_2-x_1)}{dt} + K_3x_2 + K_2(x_2-x_1)\ldots(2)$$

After applying laplace eqn 1 becomes

$F(s)= M_1s^2\ X_1(s) + B_1s\ X_1(s) + B_3s[X_1(s)- X_2(s)]+ K_1\ X_1(s)+ K_3[X_1(s)- X_2(s)]$

$F(s)= [M_1s^2 + B_1s + B_3s+ K_1 + K_3]\ X_1(s) - [B_3s+ K_3]\ X_2(s)$ ….(3)

After applying laplace eqn 2 becomes

$0= M_2s^2\ X_2(s) + B_2s\ X_2(s) + B_3s[X_2(s)- X_1(s)]+ K_3\ X_2(s)+ K_2[X_2(s)- X_1(s)]$

$0= [M_2s^2 + B_2s + B_3s + K_3 + K_2] X_2(s) - [B_3s+ K_3] X_1(s)$

$[M_2s^2 + B_2s + B_3s + K_3 + K_2] X_2(s) = [B_3s+ K_3] X_1(s)$

We need $X_1(s)$. So need to eliminate $X_2(s)$.

Therefore **$X_2(s) = \{(B_3s+ K_3)/ (M_2s^2 + B_2s + B_3s + K_3 + K_2)\} X_1(s)$**

Substitute this in eqn 3

$F(s)= (M_1s^2 + B_1s + B_3s+ K_1 + K_3) X_1(s) – (B_3s+ K_3) \{ (B_3s+ K_3)/ (M_2s^2 + B_2s + B_3s + K_3 + K_2) \} X_1(s)$

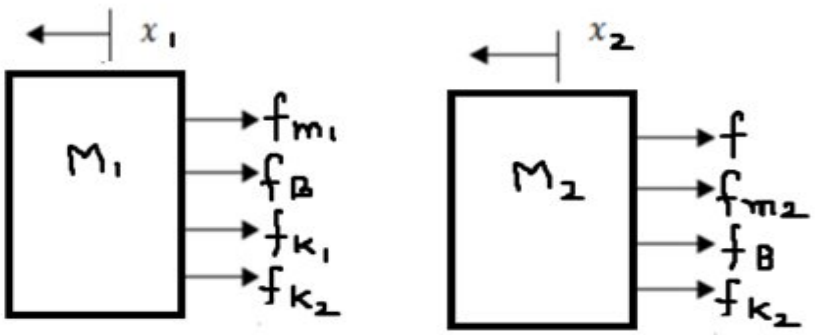
$F(s)= [(M_1s^2 + B_1s + B_3s+ K_1 + K_3) – (B_3s+ K_3) \{ (B_3s+ K_3)/ (M_2s^2 + B_2s + B_3s + K_3 + K_2) \}] X_1(s)$

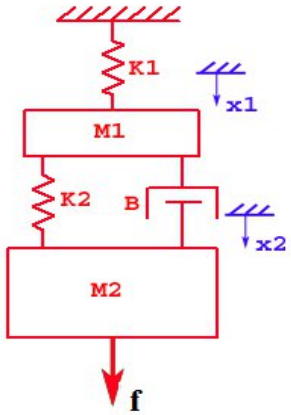
Therefore transfer function

$$\frac{X_1(s)}{F(s)} = \frac{(M_2s^2 + (B_2 + B_3)s + K_2 + K_3)}{[(M_1s^2 + (B_1 + B_3)s + K_1 + K_3)(M_2s^2 + (B_2 + B_3)s + K_2 + K_3)] - (B_3s + K_3)^2}$$

3. Find the transfer function $\frac{X_2(s)}{F(s)}$ of the system.

Sol: This system has two nodes(2 Mass). Freebody diagrams of these nodes are as shown in figure.





Node1:

$0=f_{m1}+f_b+ f_{k1}+ f_{k2}$

$$f = M_1\frac{d^2x_1}{dt^2} + B\frac{d(x_1-x_2)}{dt} + K_1x_1 + K_2(x_1-x_2)\ldots(1)$$

Node2:

$f=f_{m2}+f_b + f_{k2}$

$$f = M_2\frac{d^2x_2}{dt^2} + B\frac{d(x_2-x_1)}{dt} + K_2(x_2-x_1)\ldots(2)$$

After applying laplace eqn 1 becomes

$0= M_1s^2 X_1(s) + Bs[X_1(s)- X_2(s)]+ K_1 X_1(s)+ K_2[X_1(s)- X_2(s)]$

$0= [M_1s^2 + Bs + K_1 + K_2] X_1(s) - [Bs+ K_2] X_2(s)$ ….(3)

After applying laplace eqn 2 becomes

$F(s)= M_2s^2 X_2(s) + Bs[X_2(s)- X_1(s)]+ K_2[X_2(s)- X_1(s)]$

$F(s)= [M_2s^2 + Bs + K_2] X_2(s) - [Bs+ K_2] X_1(s)$ ….(4)

From eqn 3

$[M_1s^2 + Bs + K_1 + K_2] X_1(s) = [Bs+ K_2] X_2(s)$

Therefore

$X_1(s) = \{[Bs+ K_2]/ [M_1s^2 + Bs + K_1 + K_2] \} X_2(s)$ …(5)

(We need $X_2(s)$. So need to eliminate $X_1(s)$.)

Substitute this in eqn 4

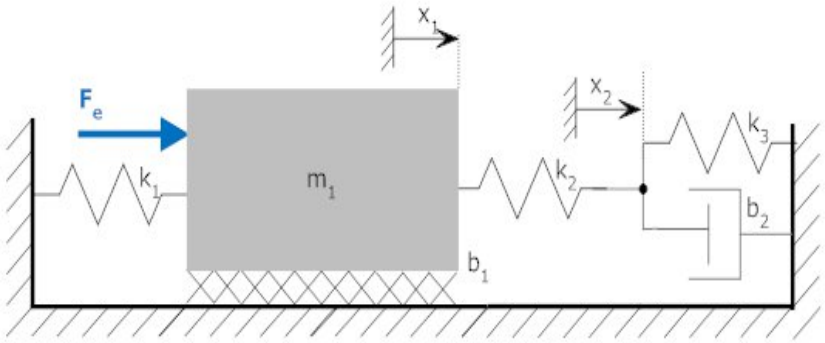
$F(s)= [M_2s^2 + Bs + K_2] X_2(s) - [Bs+ K_2]$ **$\{[Bs+ K_2]/ [M_1s^2 + Bs + K_1 + K_2] \} X_2(s)$**

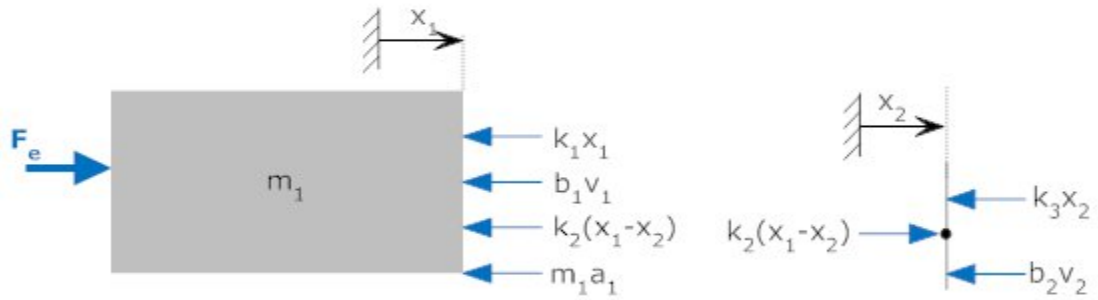
$F(s)= \{(M_2s^2 + Bs + K_2) – \{(Bs+ K_2)^2 /$ **$(M_1s^2 + Bs + K_1 + K_2) \}\} X_2(s)$**

Therefore transfer function

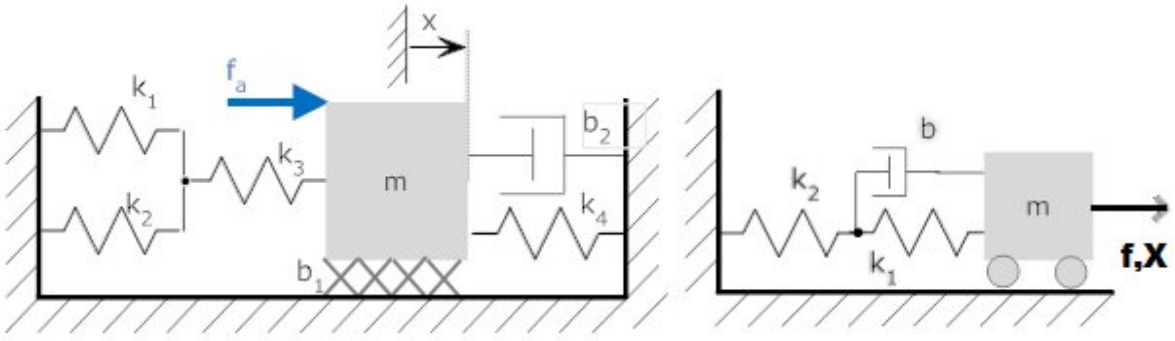
$$\frac{X_2(s)}{F(s)} = \frac{M_1s^2 + Bs + K_1 + K_2}{[\left(M_1s^2 + Bs + K_1 + K_2\right)(M_2s^2 + Bs + K_2] - (Bs + K_2)^2}$$

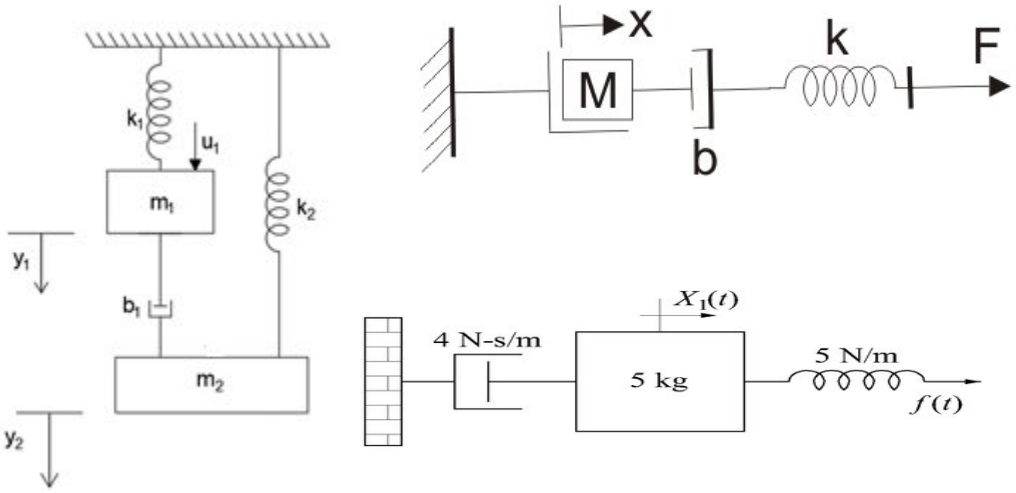
4.



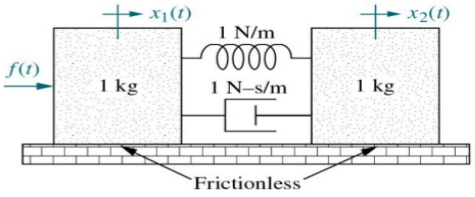


Practice Problems:





Solve for G(s) = $X_2(s)/F(s)$



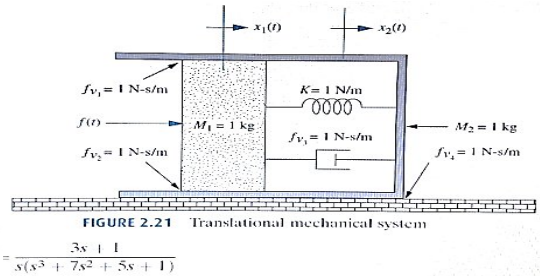
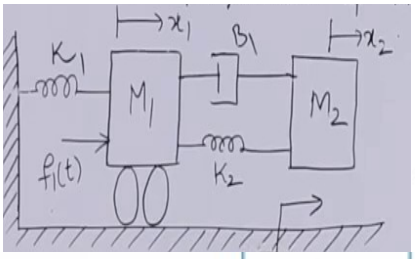
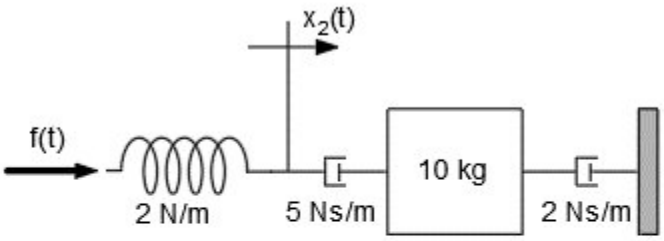
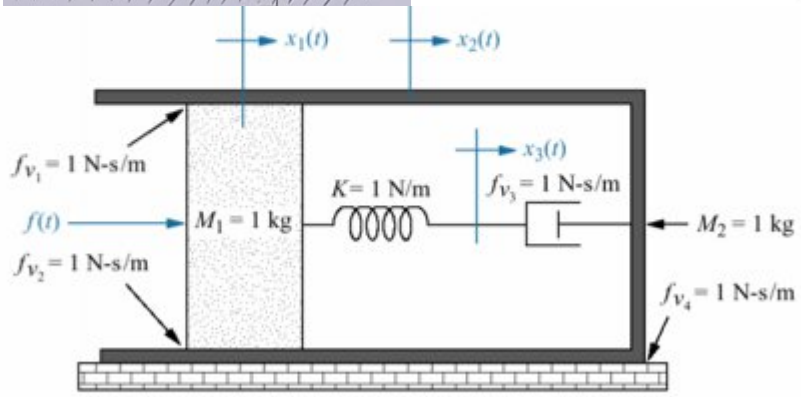


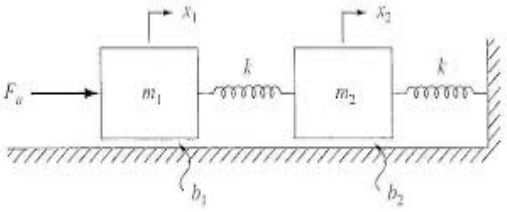
FIGURE 2.21 Translational mechanical system

$$= \frac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$$

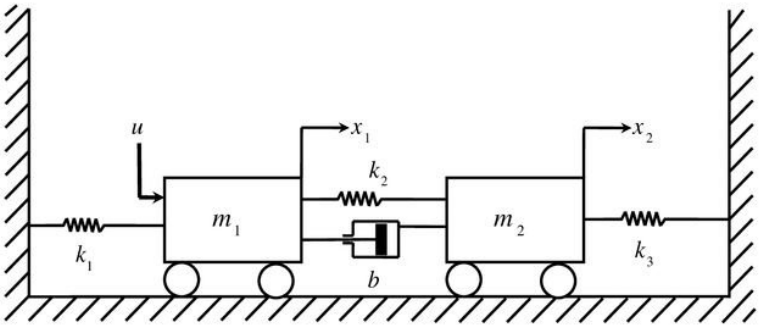


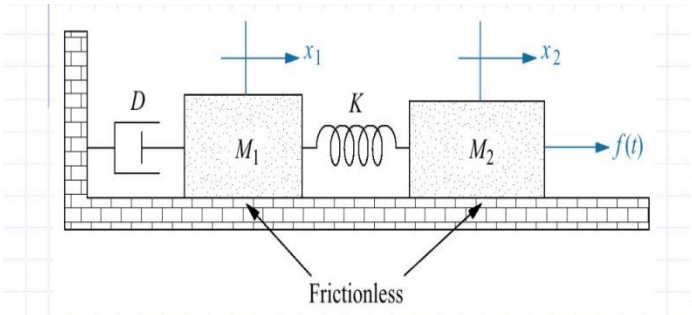


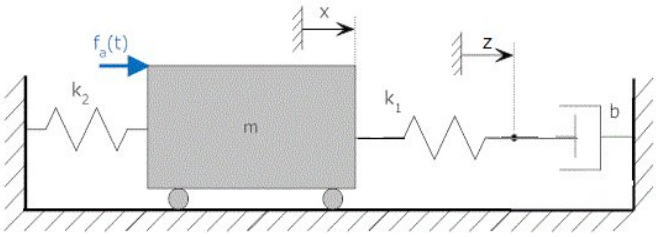


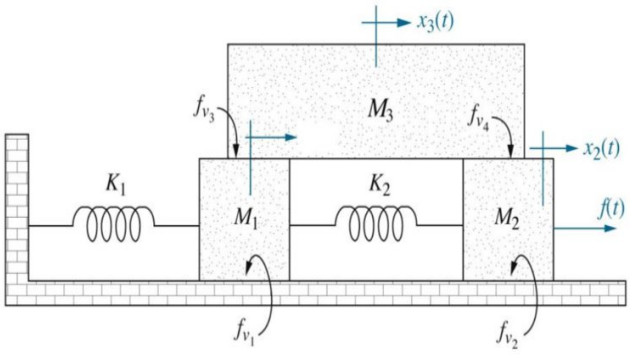


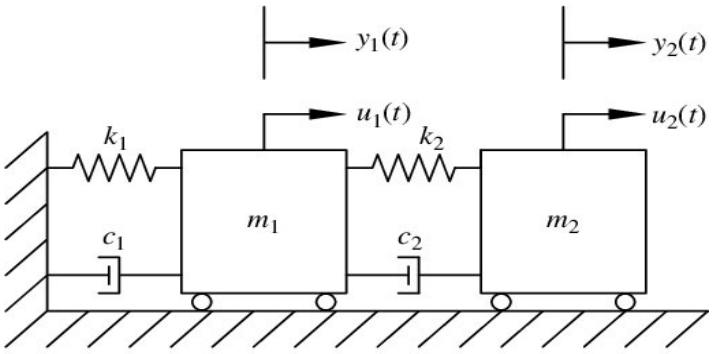
Example. Mechanical system is shown below. Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, where u denotes the force, and x_1 and x_2 denote the displacements of the two carts, respectively.

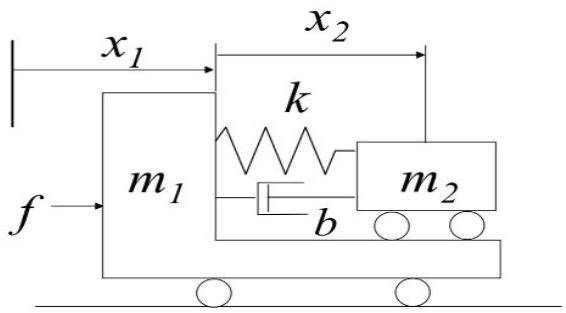


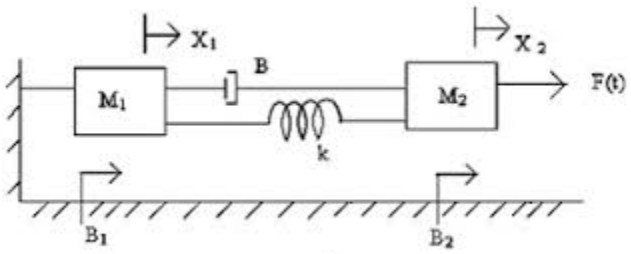




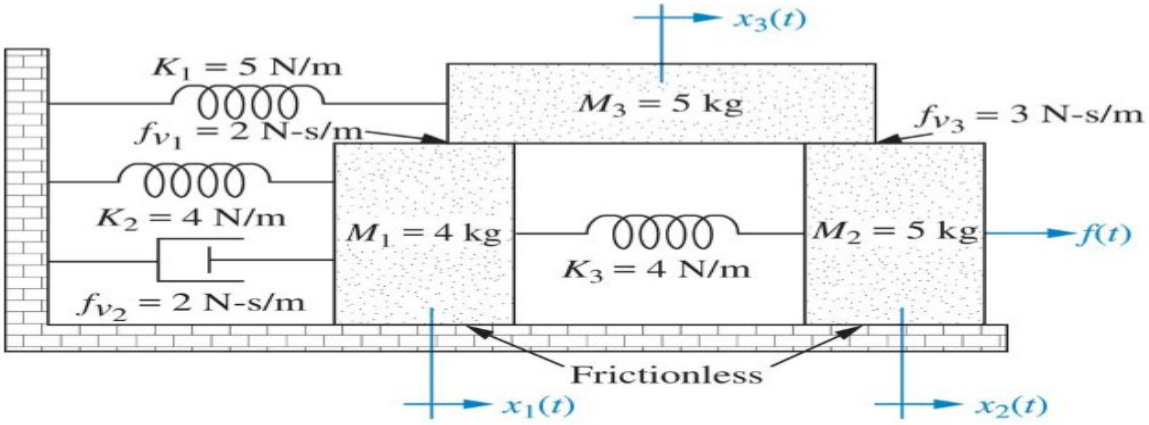








Q3: Find the transfer function $G(s) = \frac{X_2(s)}{F(s)}$ for the translational mechanical system in Figure 2. (10 Points)



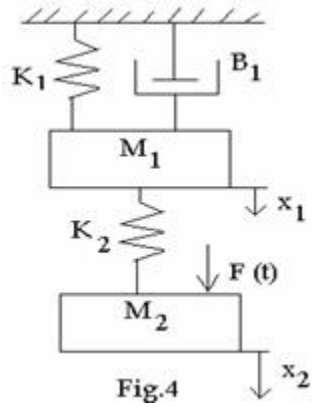
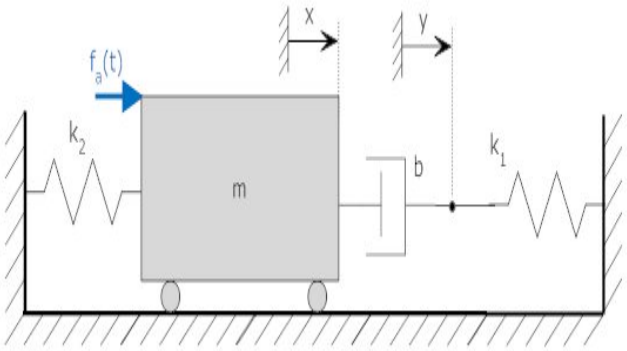
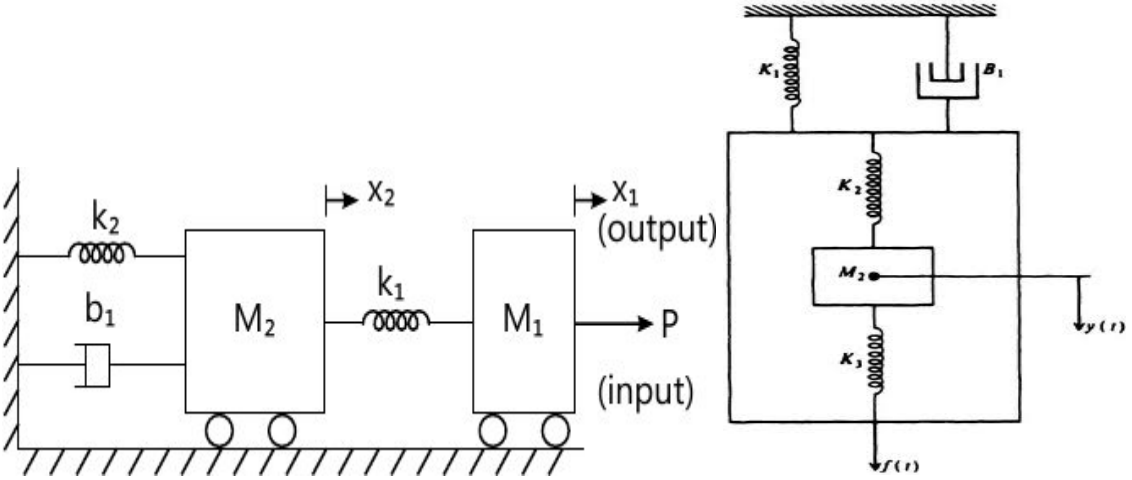


Fig.4





In some of the problems B is represented by f_v and input is represented by u or P.

Analogous Systems:

Two systems are said to be **analogous** to each other if Differential equations modelling of these two systems are same.

Electrical systems and mechanical systems are two physically different systems. There are two types of electrical analogies of translational mechanical systems. Those are force voltage analogy and force current analogy.

Translational MS	Force-voltage analogy	Force-current analogy
Force(F)	Voltage source(Vs)	Current Source(Is)
Velocity(v)	Loop Current(i)	Node Voltage(v)
Mass(M)	(L)	Capacitance(C)
Frictional Coefficient(B)	Resistance(R)	Conductance(1/R)
Spring Constant(K)	Reciprocal of Capacitance (1/C)	Reciprocal of Inductance(1/L)
D'Alembert Principle	Loop Analysis	Nodal Analysis

Force Voltage Analogy

In force voltage analogy, force is equated to Voltage source and the mathematical equations of **translational mechanical system** are compared with mesh equations of the electrical system.

Procedure to obtain f-v analogous circuit:

- Identify the nodes in the mechanical system.
- Each node in the mechanical system corresponds to a loop in f-v electrical circuit.

  Number of loops in f-v circuit = number of nodes in MS
- The elements that are connected to only one node in mechanical system are represented by equivalent electrical elements connected in series in the corresponding loop.
- The elements that are connected between two nodes in mechanical system are represented by equivalent electrical elements connected in series in the common branch of two corresponding loops.
- The voltage source polarity is always such that the loop current enter at –ve terminal of source.(All loop currents are in clock wise direction only)
- Apply loop analysis to write the differential equations.

Force Current Analogy

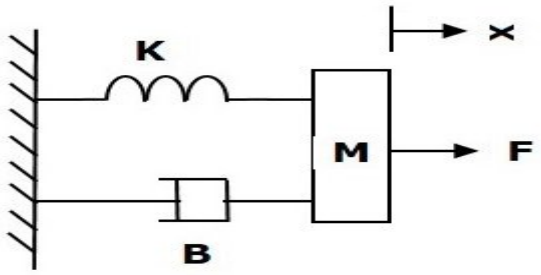
In force current analogy, force is equated to current source and the mathematical equations of **translational mechanical system** are compared with nodal equations of the electrical system.

Procedure to obtain f-i analogous circuit:

- Identify the nodes in the mechanical system.
- Each node in the mechanical system corresponds to a node in f-i electrical circuit.

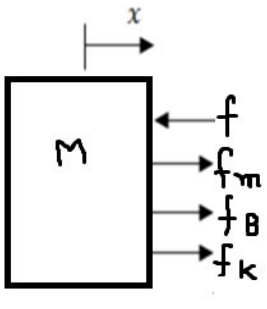
  Number of nodes in f-i circuit = number of nodes in MS +1 (reference or ground)
- The elements that are connected to only one node in mechanical system are represented by equivalent electrical elements connected in parallel between the corresponding node and reference.
- The elements that are connected between two nodes in mechanical system are represented by equivalent electrical elements connected in parallel between two corresponding nodes.
- The current source direction is always such that it enters the corresponding node.
- Apply nodal analysis to write the differential equations.

Consider the following translational mechanical system as shown in the following figure.



Sol: there is only one node in the given mechanical system.

The **force balanced equation** for this node is



$$F = F_m + F_b + F_k$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

$F(s) = Ms^2X(s) + BsX(s) + KX(s) = (Ms^2 + Bs + K)\,X(s)$

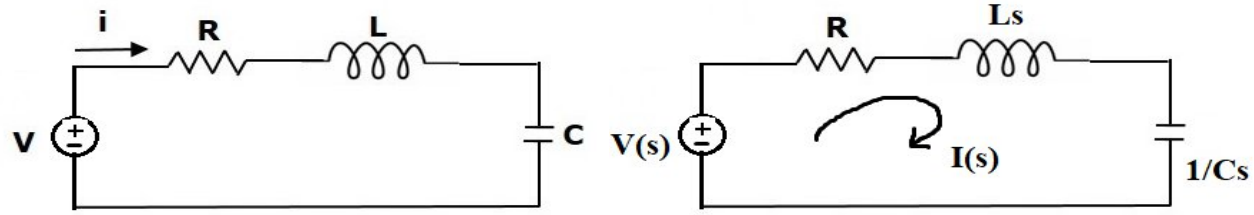
Therefore transfer function of the system is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

f-v analogous circuit:

There is only one node in the given mechanical system.

Translational MS	Force-voltage analogy
Force(F)	Voltage source(Vs)
Velocity(v)	Loop Current(i)
B	R
M	L
1/K	C
D'Alembert Principle	Loop Analysis



Loop equation for this circuit is $V(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s)$

$$\frac{I(s)}{V(s)} = \frac{1}{R + Ls + \frac{1}{Cs}}$$

Replace with corresponding mechanical quantities $\frac{v(s)}{F(s)} = \frac{1}{B+Ms+\frac{K}{s}} = \frac{s}{Ms^2+Bs+K}$

But v(t)= d(x(t))/dt

V(s) = s X(s)

Therefore

$$\frac{sX(s)}{F(s)} = \frac{s}{Ms^2 + Bs + K}$$

Therefore transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

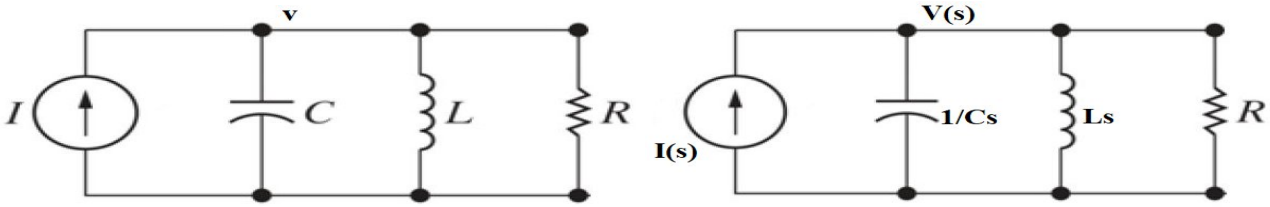
f-i analogous circuit:

Translational MS	Force-current analogy
Force(F)	Current source(Is)
Velocity(v)	Node voltage (v)
1/B	R
M	C
1/K	L
D'Alembert Principle	nodal Analysis

There is only one node in the given mechanical system.

Number of nodes in f-i analogous circuit = 1+ref

M,K & B are connected to only node 1 in mechanical system so the equivalent electrical elements R,L &C will be connected in parallel between node 1 & ref. So the circuit is as shown below.



nodal equation for this circuit is

$$I(s) = \frac{V(s)}{R} + \frac{V(s)}{Ls} + Cs\,V(s) = (\frac{1}{R} + \frac{1}{Ls} + Cs\,)V(s)$$

$$\frac{V(s)}{I(s)} = \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs}$$

Replace with corresponding mechanical quantities

$$\frac{v(s)}{F(s)} = \frac{1}{B + \frac{K}{s} + Ms} = \frac{s}{Ms^2 + Bs + K}$$

But v(t)= d(x(t))/dt

V(s) = s X(s)

Therefore $$\frac{sX(s)}{F(s)} = \frac{s}{Ms^2+Bs+K}$$

Therefore transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Modeling of Rotational Mechanical Systems



If the movement of the elements of the mechanical system is circular about an axis then it is called as Rotarional Mechanical System. These systems mainly consist of three basic elements. Those are

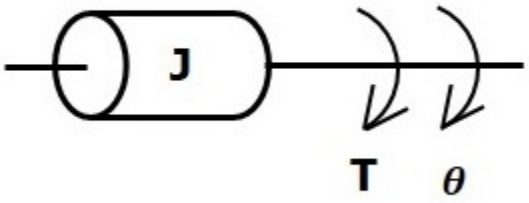
- **Moment of Inertia,**
- **Rotational Spring (or) Torsional Spring and**
- **Rotational dashpot or damper.**

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

Moment of Inertia

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores **kinetic energy**.

If a torque is applied on a body having moment of inertia **J**, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.



$$T_J \propto \alpha$$

$$T = T_J = J\propto = M\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$

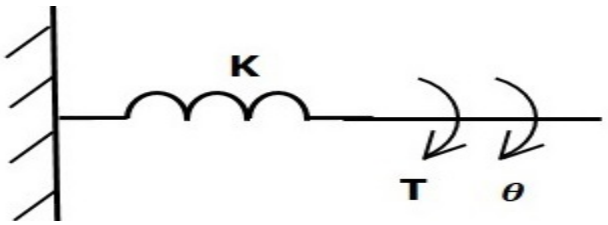
Where,

- **T** is the applied torque
- $\mathbf{T_j}$ is the opposing torque due to moment of inertia
- **J** is moment of inertia
- **α** is angular acceleration
- **θ** is angular displacement

Torsional Spring

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores **potential energy**.

If a torque is applied on torsional spring **K**, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.



$T \propto \Theta$

$T = T_k = K\Theta = K\int \mu g \, dt$

If element is connected between two nodes

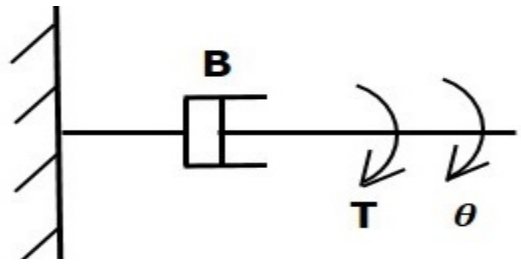
$T = T_k = K(\Theta 1 \sim \Theta 2) = K\int(\mu g 1 - \mu g 2)\, dt$

Where,

- **T** is the applied torque
- **T_k** is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- **θ** is angular displacement

Rotational Dashpot

If a torque is applied on dashpot **B**, then it is opposed by an opposing torque due to the **rotational friction** of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



$T_b \propto \omega$

$\Rightarrow T = T_b = B\omega = Bd\theta/dt$

If element is connected between two nodes

$T = T_B = B\frac{d}{dt}(\Theta 1 \sim \Theta 2) = B(\mu g 1 - \mu g 2)$

Where,

- **T_b** is the opposing torque due to the rotational friction of the dashpot
- **B** is the rotational friction coefficient
- **ω** is the angular velocity
- **θ** is the angular displacement

Analogous circuits for RMS:

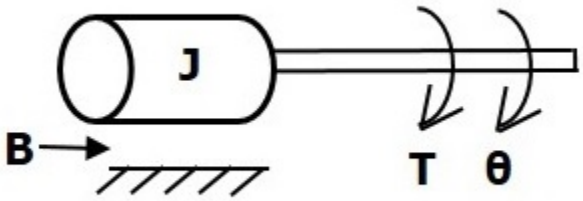
Similar to translational mechanical systems rotational mechanical systems also have two type of analogous systems namely

- Torque voltage analogy(T-v)
- Torque current analogy (T-i)

The below table presents the equivalent elements of TMS and electrical analogous systems.

Rotational MS	Translational MS	T-v or F-v analogy	T-i or F-i analogy
Torque(T)	Force(F)	Voltage source(Vs)	Current source(Is)
Angular Velocity(ω)	Velocity(v)	Loop Current(i)	Node voltage (v)
B	B	R	1/ R
J	M	L	C
1/K	1/K	C	L
D'Alembert Principle	D'Alembert Principle	Loop Analysis	nodal Analysis

Pb:Consider a Rotational mechanical system is shown in the following figure. Obtain transfer function for the system and draw analogous circuits and verify the result by applying loop analysis and nodal analysis.



Sol:The torque balanced equation is

$$T = T_j + T_b + T_k \qquad \Rightarrow T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + k\theta$$

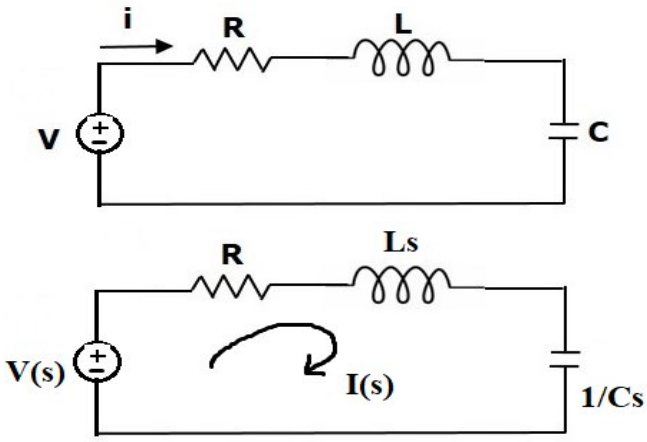
$T(s) = Js^2\Theta(s) + Bs\,\Theta(s) + K\,\Theta(s) = (Js^2 + Bs + K)\,\Theta(s)$

Therefore transfer function of the system is

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

T-v analogous circuit:

Rotational MS	Torque-voltage analogy
Torque(T)	Voltage source(Vs)
Angular Velocity(μg)	Loop Current(i)
B	R
J	L
1/K	C



Loop equation for this circuit is $V(s) = \left(R + Ls + \frac{1}{Cs}\right) I(s)$

$$\frac{I(s)}{V(s)} = \frac{1}{R + Ls + \frac{1}{Cs}}$$

Replace with corresponding mechanical quantitites

$$\frac{(s)}{T(s)} = \frac{1}{B + Js + \frac{K}{s}} = \frac{s}{Js^2 + Bs + K}$$

But $\mu g(t) = d\Theta/dt$

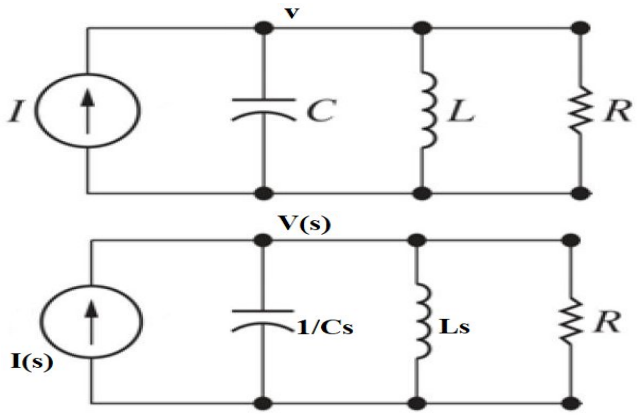
$\mu g\ (s) = s\ \Theta\ (s)$

Therefore $\frac{s\Theta(s)}{T(s)} = \frac{s}{Js^2+Bs+K}$

Therefore transfer function is $\frac{\Theta(s)}{F(s)} = \frac{1}{Js^2+Bs+K}$

T-i analogous circuit:

Rotational MS	Force-current analogy
Torque(T)	Current source(Is)
Angular Velocity(μg)	Node voltage (v)
1/B	R
J	C
1/K	L



nodal equation for this circuit is

$$I(s) = \frac{V(s)}{R} + \frac{V(s)}{Ls} + Cs\,V(s) = (\frac{1}{R} + \frac{1}{Ls} + Cs\,)V(s)$$

$$\frac{V(s)}{I(s)} = \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs}$$

Replace with corresponding mechanical quantitites

$$\frac{(s)}{T(s)} = \frac{1}{B + Js + \frac{K}{s}} = \frac{s}{Js^2 + Bs + K}$$

But $\mu g(t) = d\Theta/dt$

$\mu g\ (s) = s\ \Theta\ (s)$

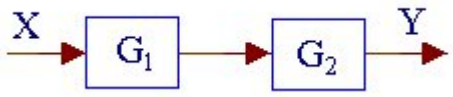
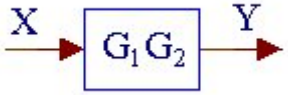
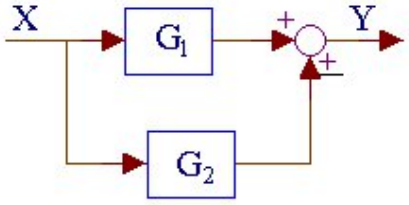
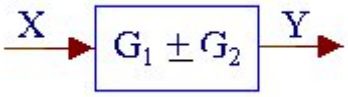
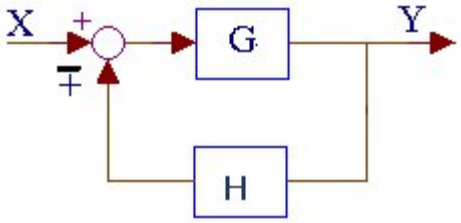
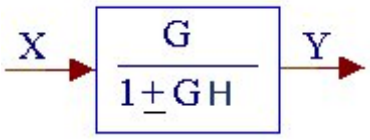
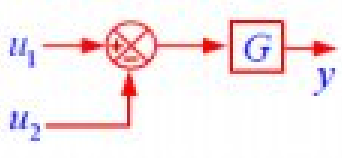
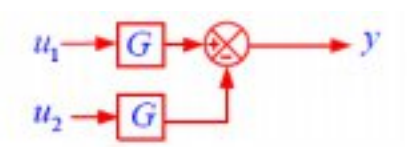
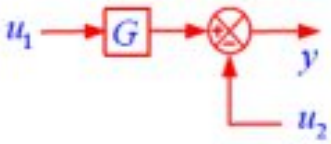
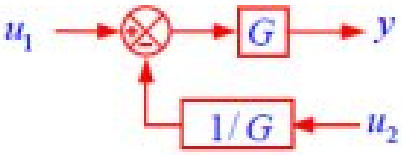
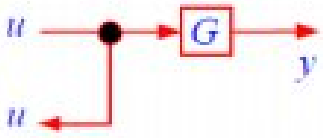
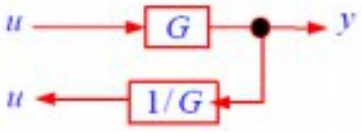
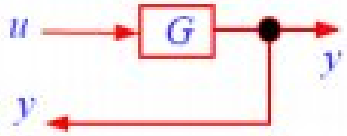
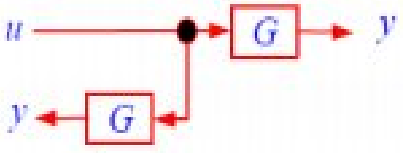
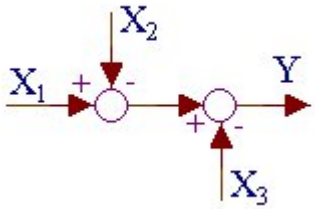
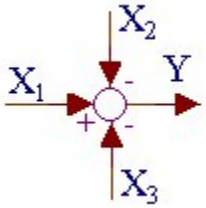
Therefore

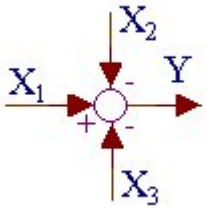
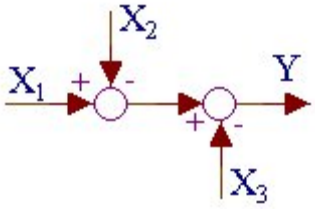
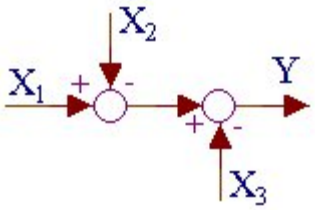
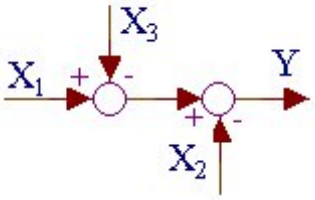
$$\frac{s\Theta(s)}{T(s)} = \frac{s}{Js^2 + Bs + K}$$

Therefore transfer function is

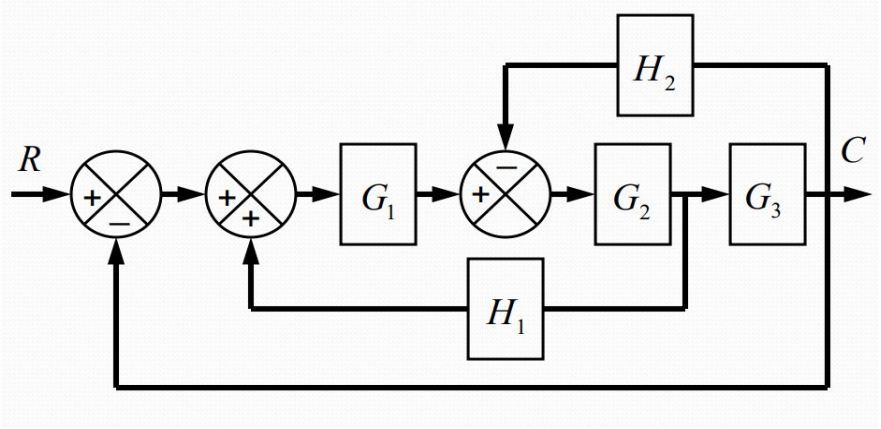
$$\frac{\Theta(s)}{F(s)} = \frac{1}{Js^2 + Bs + K}$$

Block Diagram Reduction Rules (or) Block Diagram Algebra

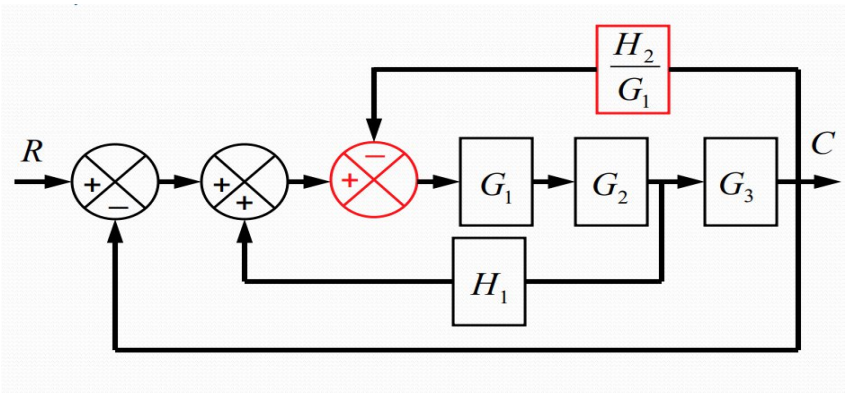
S. No.	Rule	BD before	BD after modification
1	Blocks in series		
2	Blocks in Parallel		
3	Eliminating feedback **loo**p		
4	Moving summing point in same direction Or after a block		
5	Moving summing point in opposite direction before a block		
6	Moving branch point in same direction after a block		
7	Moving branch point in opposite direction before a block		
8	Combining summing points		

9	Splitting summing points		
10	Interchanging summing points		
11	When there is no other element in between branch points they can also merged or split or interchanged		

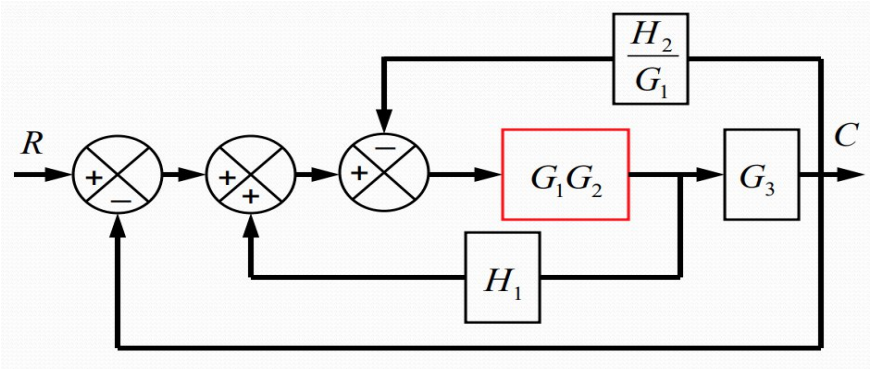
Q: Reduce Block diagram and determine transfer function.



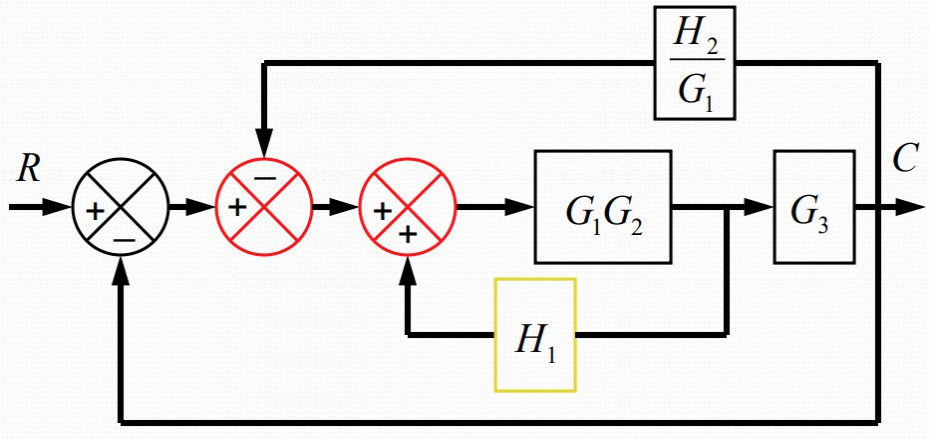
Sol: Shifting summing in opposite direction



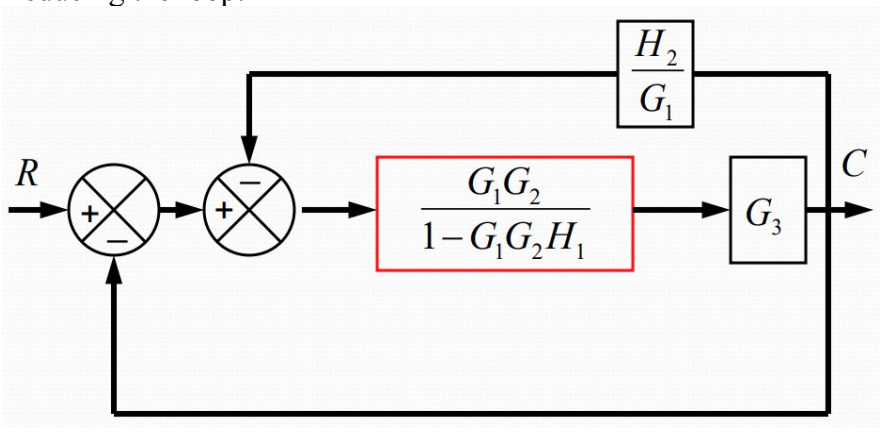
Blocks in series



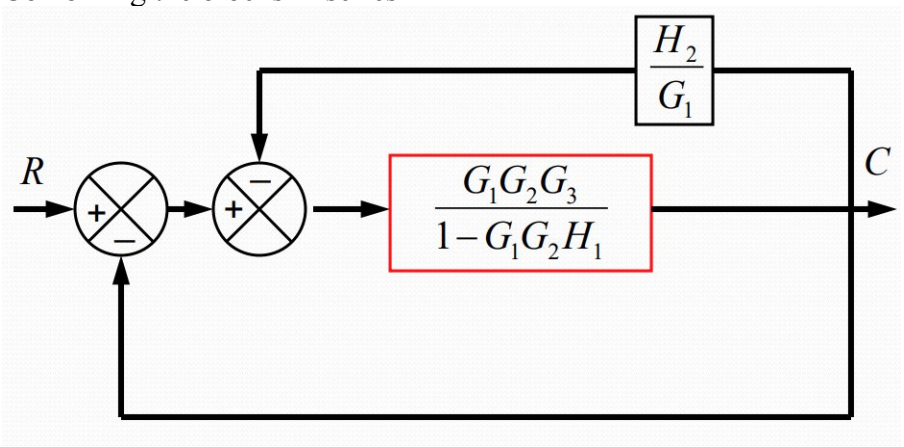
Interchanging the summing points:



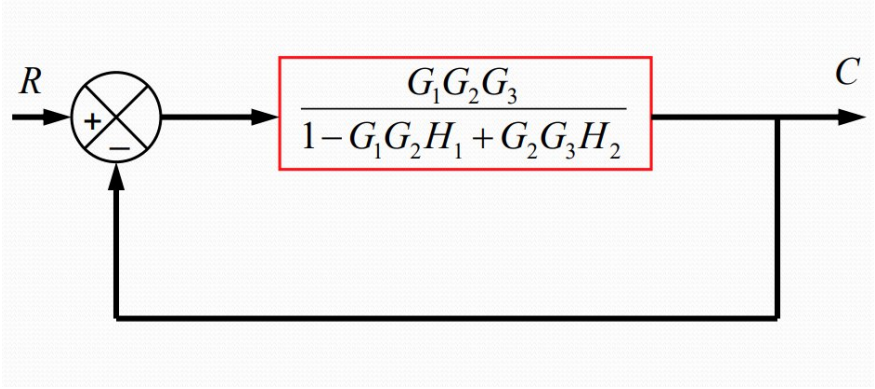
Reducing the loop:



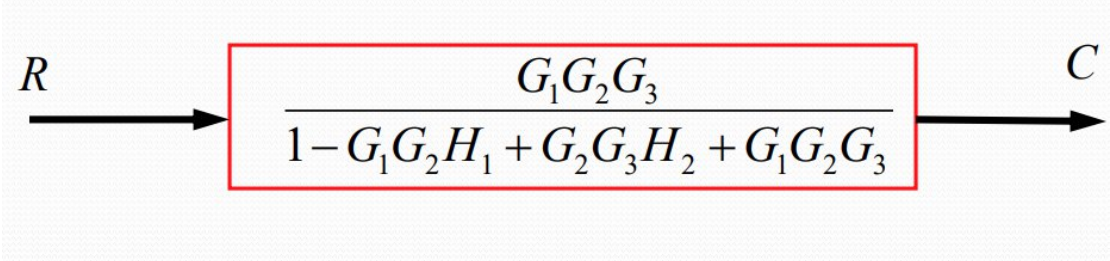
Combining the blocks in series



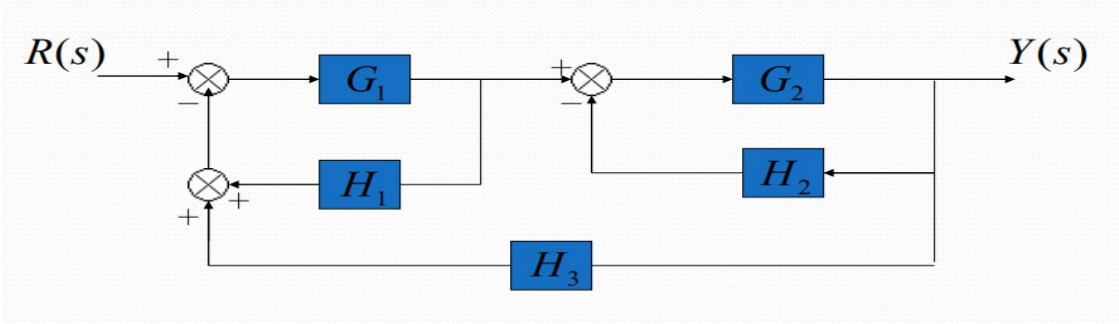
Reducing the loop:



Reducing the loop:

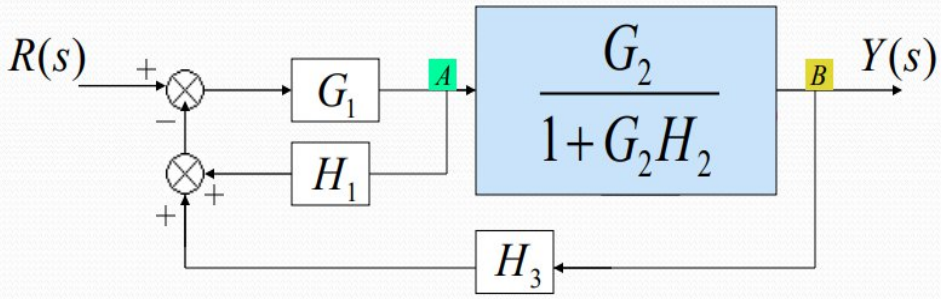


Q. Determine the transfer function of the given block diagram.

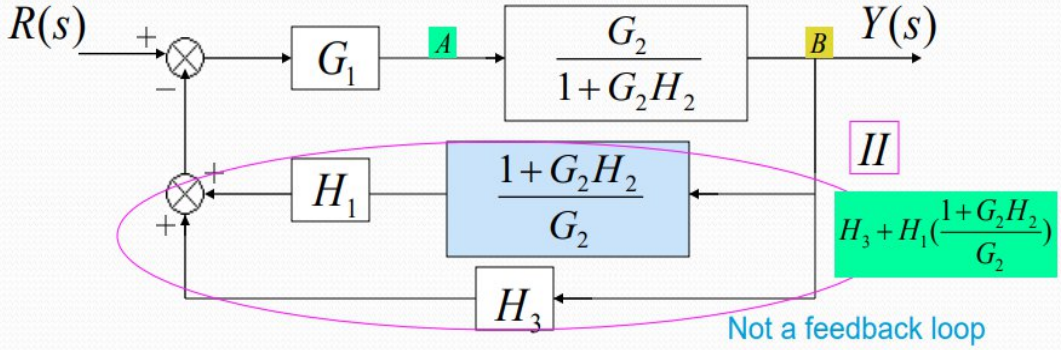


Sol:

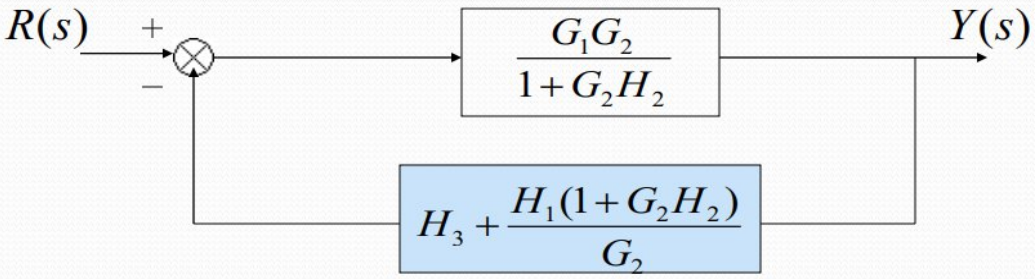
1. Eliminate loop I



2. Moving pickoff point A behind block $\frac{G_2}{1+G_2H_2}$

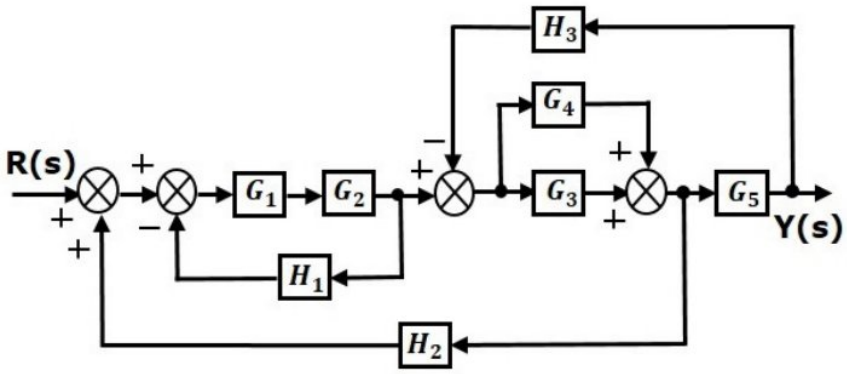


3. Eliminate loop II

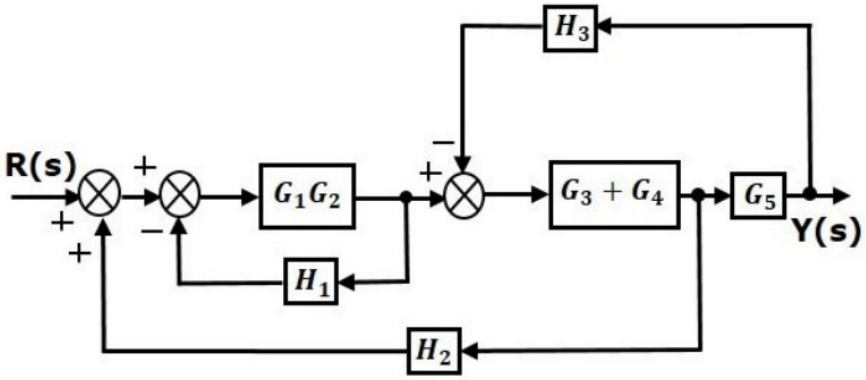


$$\frac{Y(s)}{R(s)} = \frac{G_1G_2}{1+G_2H_2+G_1G_2H_3+G_1H_1+G_1G_2H_1H_2}$$

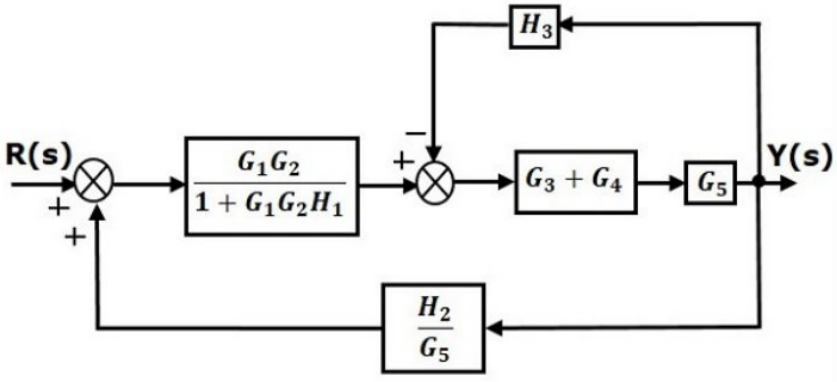
Q.Determine the transfer function of the given block diagram.



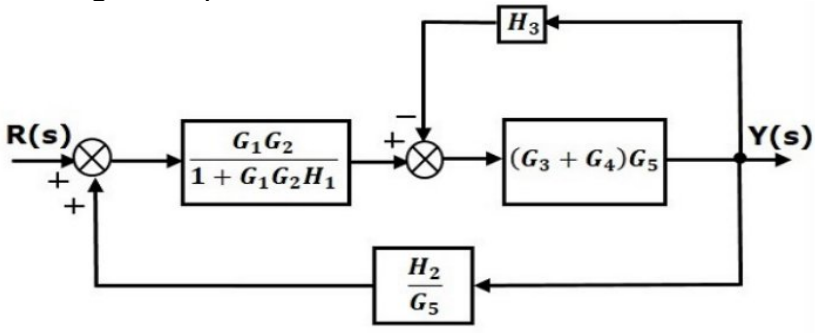
Sol:
Reducing series blocks and parallel blocks



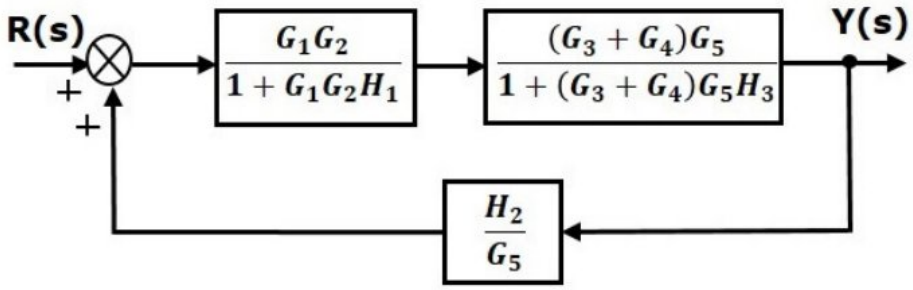
Reducing the loop and shifting branch points in same direction



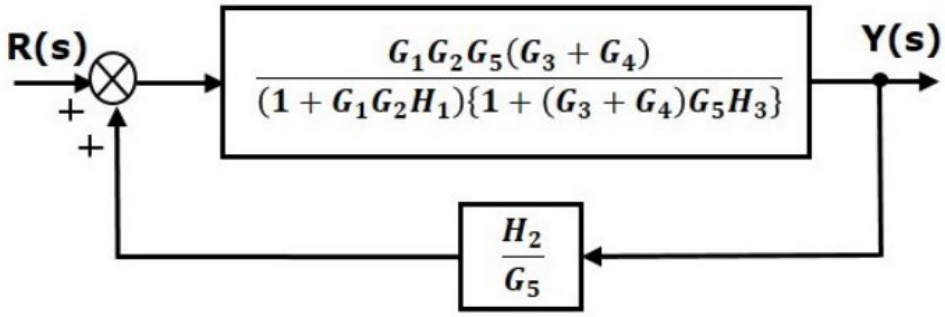
Reducing the loop



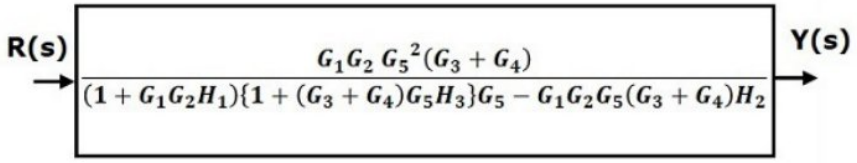
Reducing the loop:



Combining series blocks :



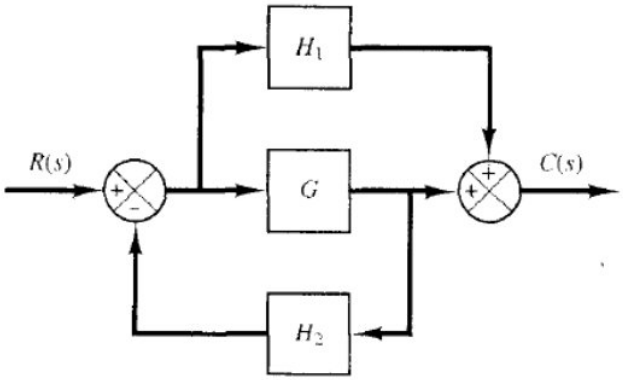
Reducing the loop:



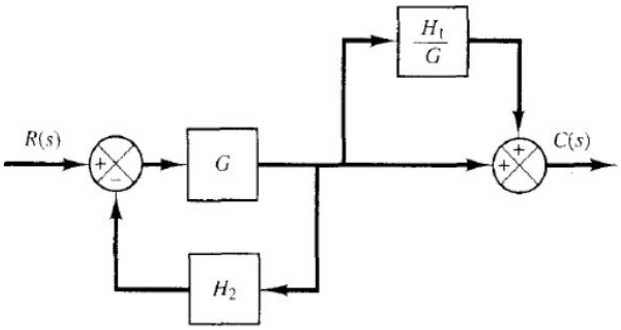
Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

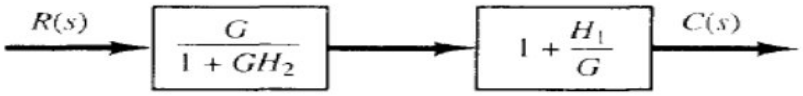
Q.Determine the transfer function of the given block diagram.

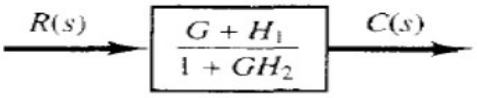


Sol:
Shifting the branch point in same direction:

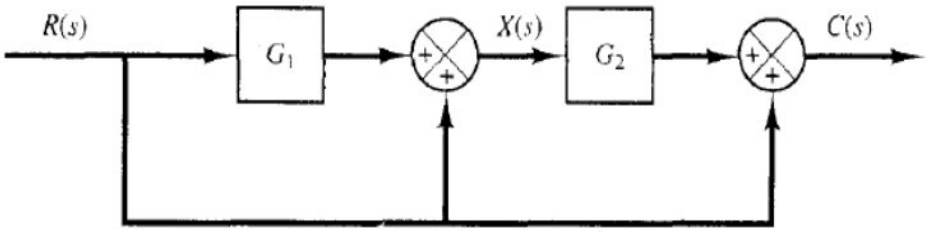


Reducing parallel blocks and loop:

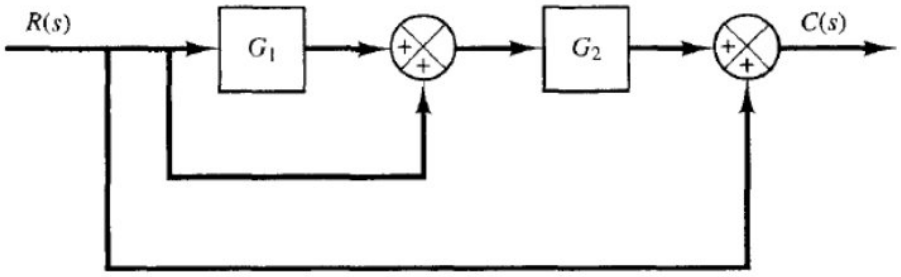




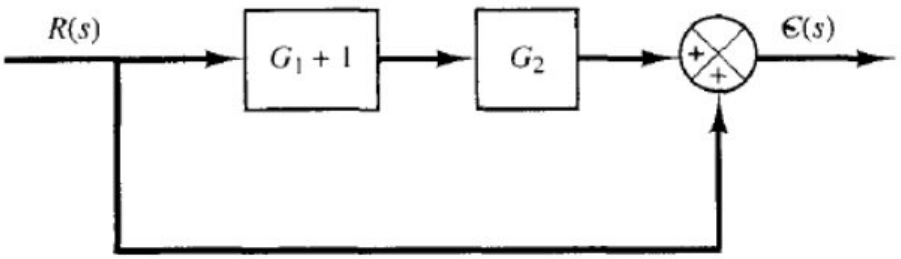
Q.Determine the transfer function of the given block diagram.



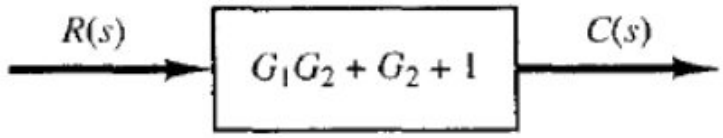
Sol:
Splitting the branch point:

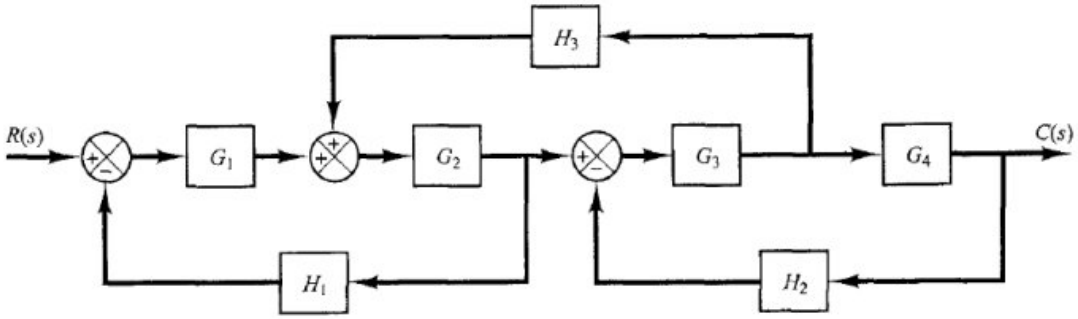


Reducing the loop:

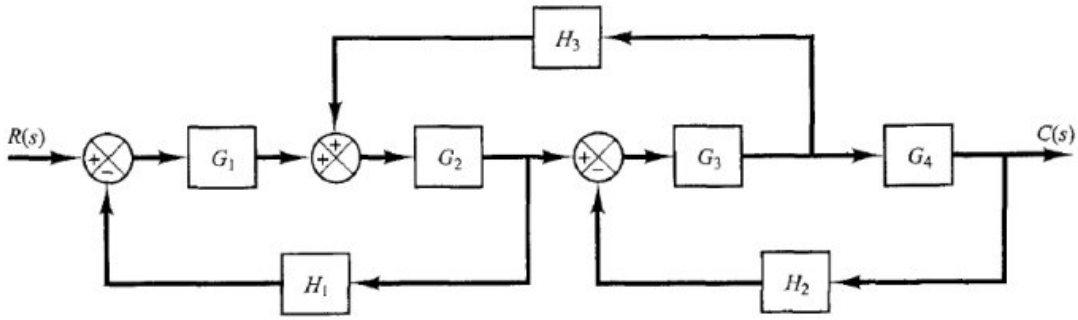


Reducing the loop:



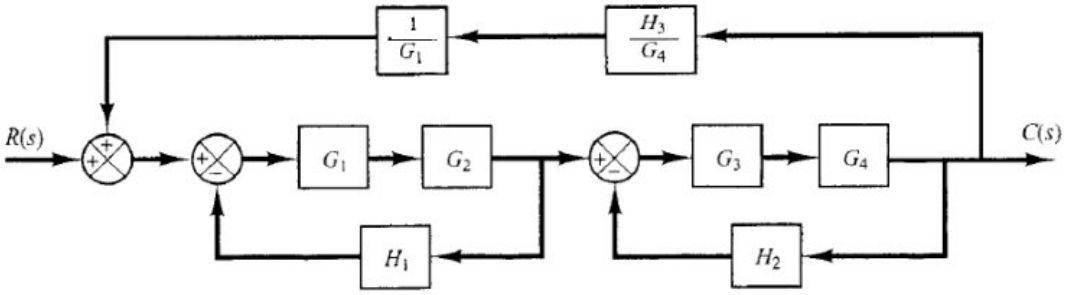


Q.Determine the transfer function of the given block diagram.

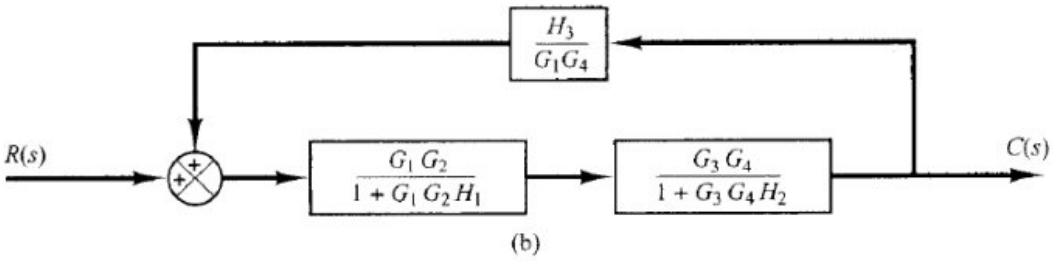


Sol:

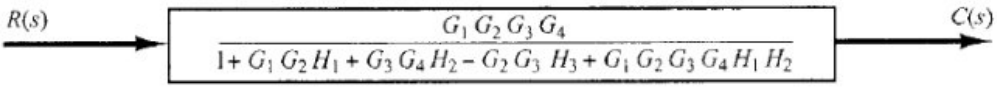
Shifting the summing point in opposite direction and branch point in same direction



Reducing the loops:


$\frac{H_3}{G_1 G_4}$
$R(s)$
$\frac{G_1 G_2}{1 + G_1 G_2 H_1}$
$\frac{G_3 G_4}{1 + G_3 G_4 H_2}$
$C(s)$

(b)


$R(s)$
$\frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_1 + G_3 G_4 H_2 - G_2 G_3 H_3 + G_1 G_2 G_3 G_4 H_1 H_2}$
$C(s)$

Signal Flow Graph

Def: Signal flow graph is pictorial representation of cause and effect relations of the equations governing a system.

Elements of SFG:

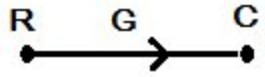
Node: **Node** is a point which represents a variable or a signal of a system.

Branch: Branch is a directed line segment which joins two nodes. It has both gain and direction.

Transmittance: The gain of the branch is known as Transmittance.

Properties of SFG:

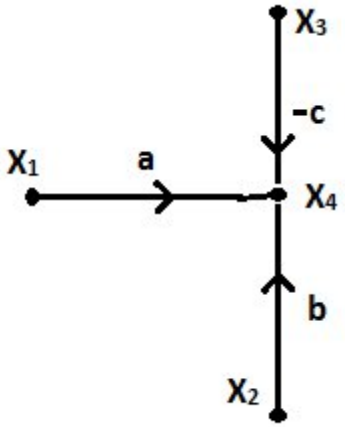
1. The signal flow graph is applicable only for linear time invariant systems.
2. The signal in the system flows along the branches only in the direction of arrows.
3. The signal is multiplied by the gain of the branch when it travels along it.



The signal R gets multiplied by G when it travels along the branch. Therefore the value of C is G times R.

i.e. **C=GR**

4. The value of the variable at any node is algebraic sum of all the incoming signals at that node.



$X_4 = a X_1 + b X_2 - C X_3$

5. The value of the variable represented by a node is available to all the branches leaving that node.
6. For a given system SFG is not unique.

Terminology used in SFG:

Input Node / Source − It is a node, which has only outgoing branches.

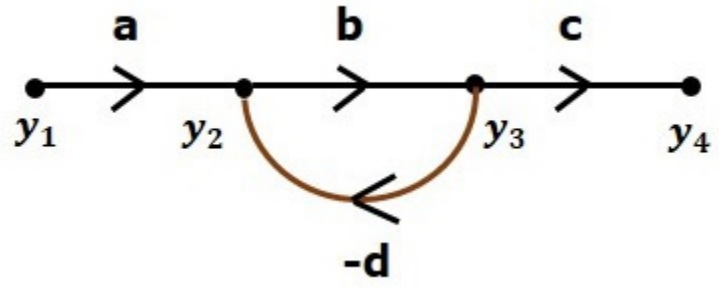
Output Node / Sink − It is a node, which has only incoming branches.

Mixed Node / Chain Node − It is a node, which has both incoming and outgoing branches.

Dummy Node: If both incoming & outgoing branches exist at Input or Output nodes they cannot be called as Input or Output nodes as per definition. In such case a separate Input or Output node can be created by adding branch with gain '1' to those nodes. Such nodes are called Dummy Nodes.

Example

Let us consider the following signal flow graph.



- The **nodes** present in this signal flow graph are y_1, y_2, y_3 and y_4.`
- y_1 and y_4 are the **input node** and **output node** respectively.
- y_2 and y_3 are **mixed nodes**.

There are four branches in the above signal flow graph. These branches have **gains** of **a, b, c** and **-d**.

Path: It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Closed Path/Loop: The path that starts from one node and ends at the same node is known as loop. Hence, it is a closed path.

Self Loop: A loop consisting of only one node is called self loop.

Forward Path: The path that exists from the input node to the output node is known as forward path.

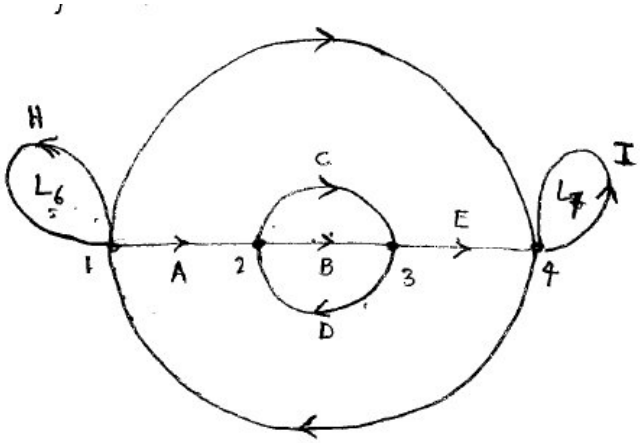
Forward Path Gain: It is the product of all branch gains of the forward path.

Loop Gain: It is the product of all branch gains of a loop.

Non-touching Loops: Two or more loops are said to be non-touching if they do not have common nodes.

Non-touching Loop to forward path: If there is no common node between a forward path & a loop then that loop is said to be non-touching to that forward path.

Example:



Forward Paths- 3 P1=ABE P2=ACE P3=F

Loops- 5 + 2 (self loops)

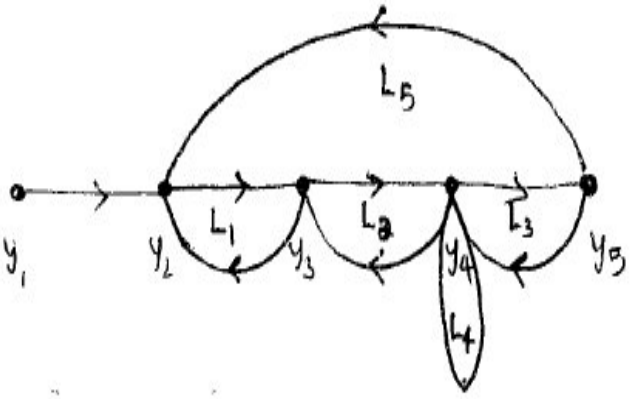
Loop gains: L1=BD L2=CD L3=ABEH L4=ACEH L5=FG L6= H L7=I

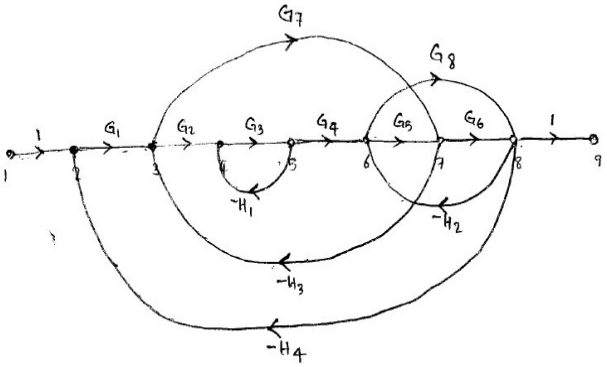
2 Non-touching loops: 7 L1L5, L1L6, L1L7, L2L5, L2L6, L2L7, L6L7

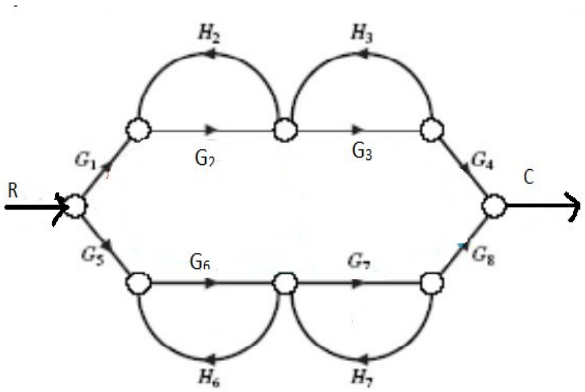
3 Non-touching loops:2 L1L6L7, L2L6L7

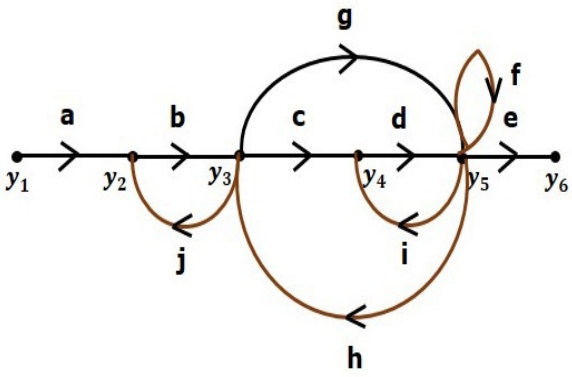
Non-touching Loops to forward paths: P1- nil P2-Nil P3- L1,L2

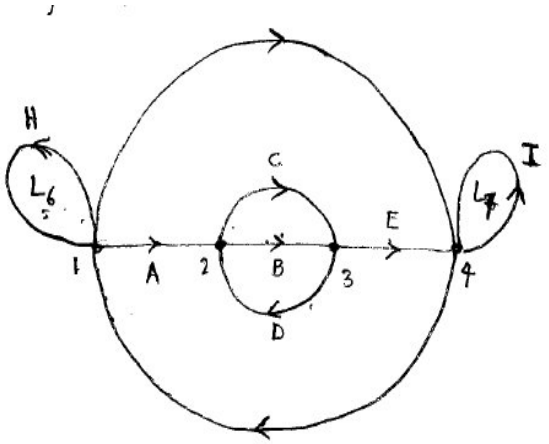
Problems: Find forward path gains, loop gains, non-touching loops and non-touching loops to forward paths.

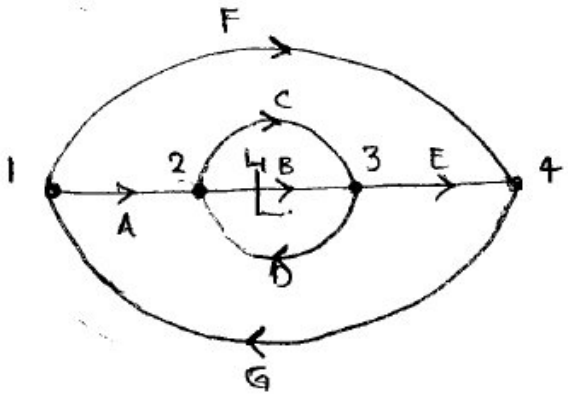


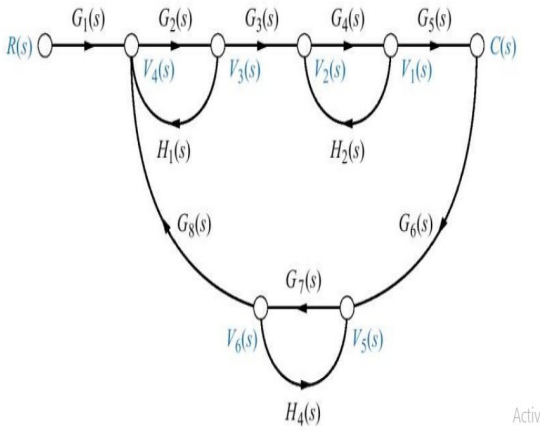


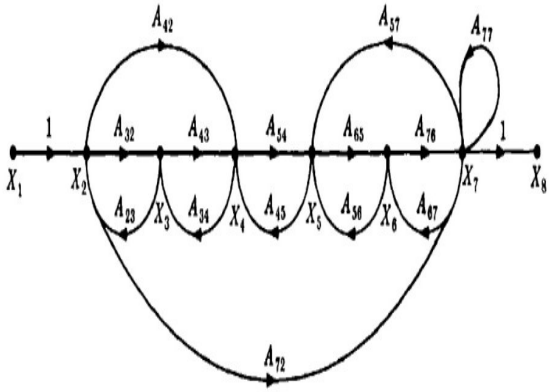





H
L6
C
I
L4
E
1
A
2
B
3
4
D


F
C
1
2
L4
B
3
E
4
A
D
G


R(s)
C(s)
G1(s)
G2(s)
G3(s)
G4(s)
G5(s)
V4(s)
V3(s)
V2(s)
V1(s)
H1(s)
H2(s)
G8(s)
G6(s)
G7(s)
V6(s)
V5(s)
H4(s)


A42
A57
A77
1
A32
A43
A54
A65
A76
1
X1
X2
X7
X8
A23
X3
A34
X4
A45
X5
A56
X6
A67
A72

Mason's gain formula:

Overall Gain

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

Where,

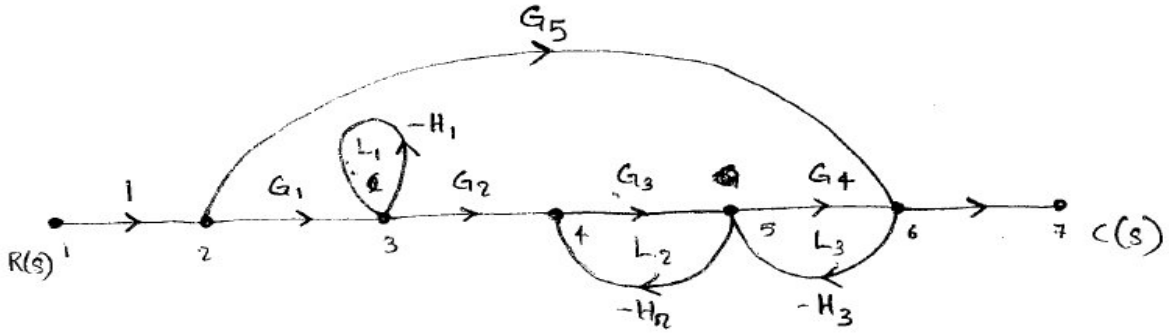
- **C(s)** is the output node
- **R(s)** is the input node
- **T** is the transfer function or gain between $R(s)$ and $C(s)$
- $\mathbf{P_i}$ is the ith forward path gain

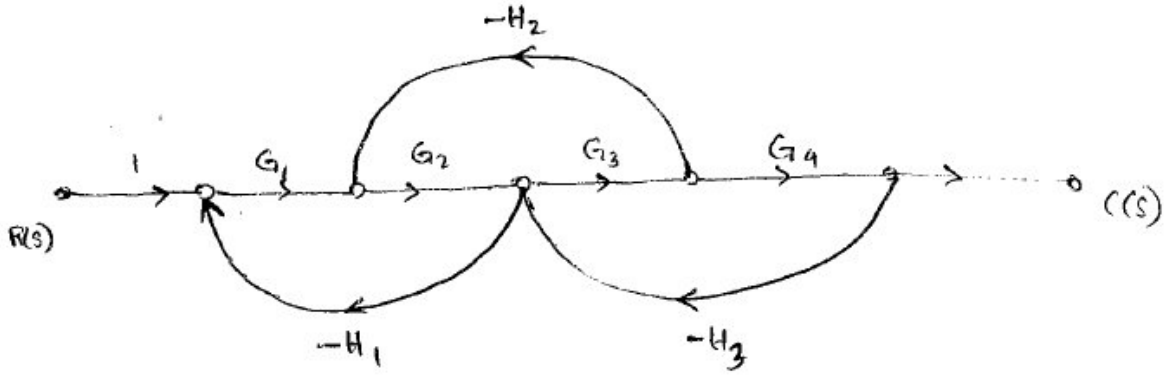
$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$
$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$
$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops)+\ldots$$

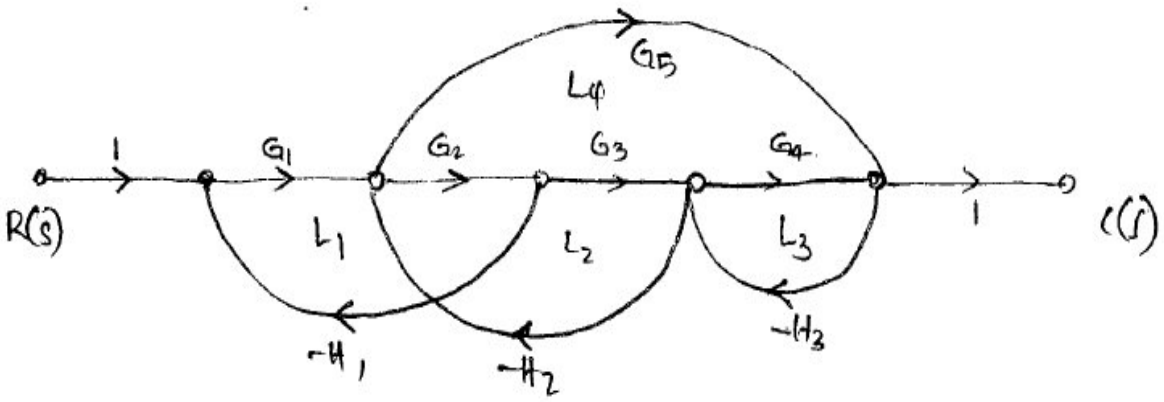
Δ_i = Δ after removing the loops which are touching the ith forward path.

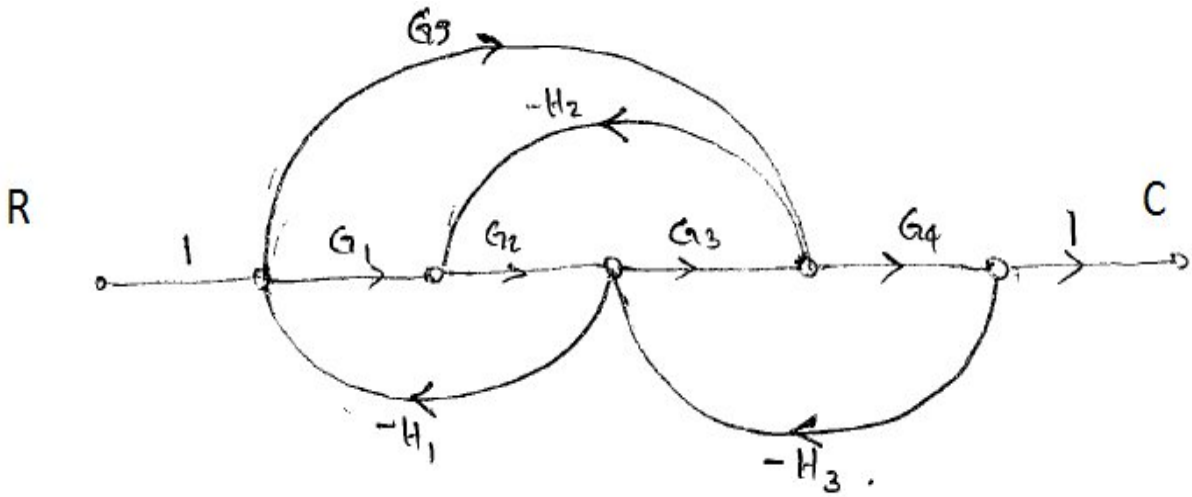
= Δ only for non touching loops for that forward path

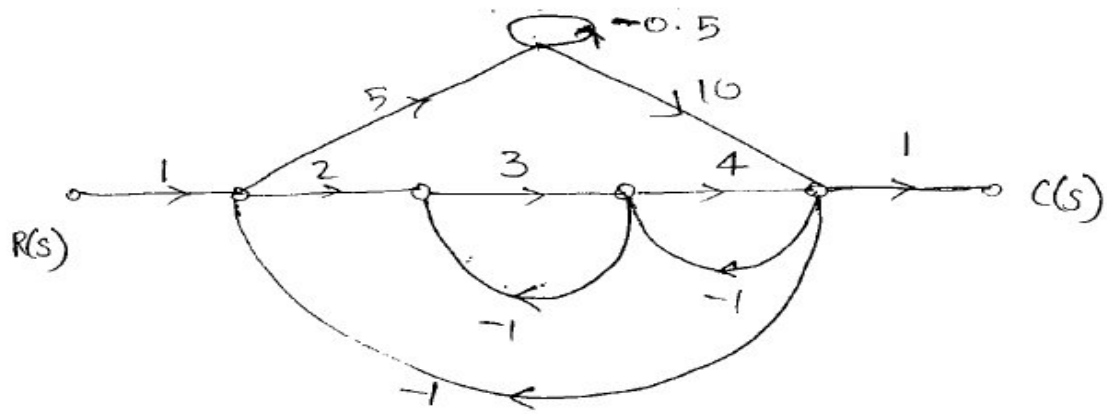
Problems:




-H2
1
G1
G2
G3
G4
R(s)
C(s)
-H1
-H3


G5
L4
1
G1
G2
G3
G4
R(s)
L1
L2
L3
1
C(s)
-H1
-H2
-H3


G5
-H2
R
1
G1
G2
G3
G4
1
C
-H1
-H3



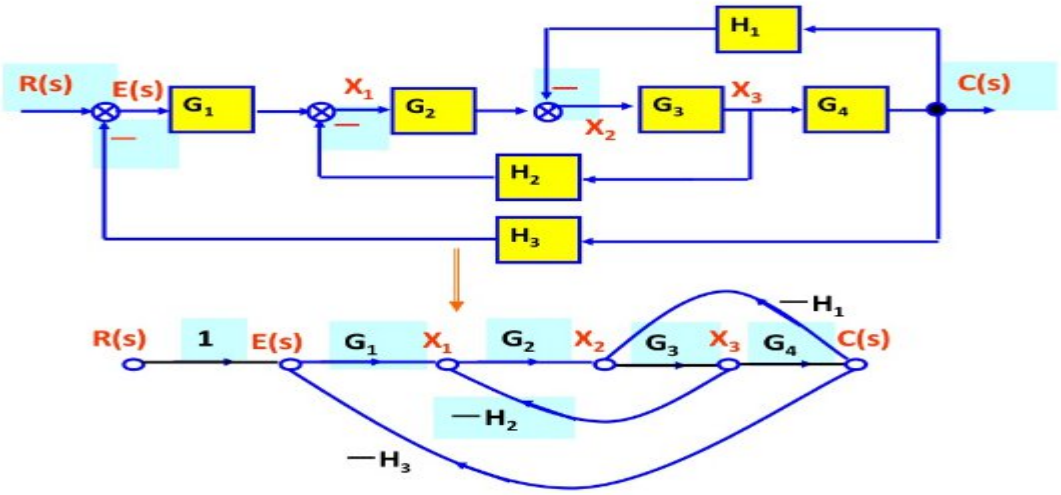
Conversion of Block Diagrams into Signal Flow Graphs:

Follow these steps for converting a block diagram into its equivalent signal flow graph.

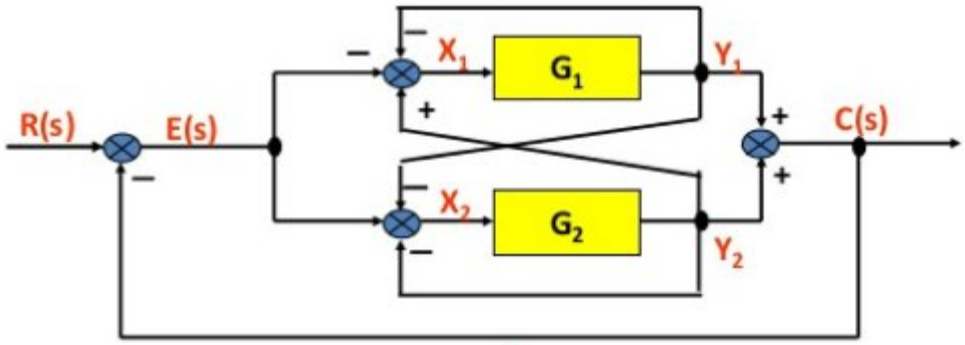
- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Represent the blocks of block diagram as **branches** in signal flow graph.
- Represent the transfer functions inside the blocks of block diagram as **gains** of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is a connection between two nodes (but there is no block in between), then represent the gain of the branch as one. **For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

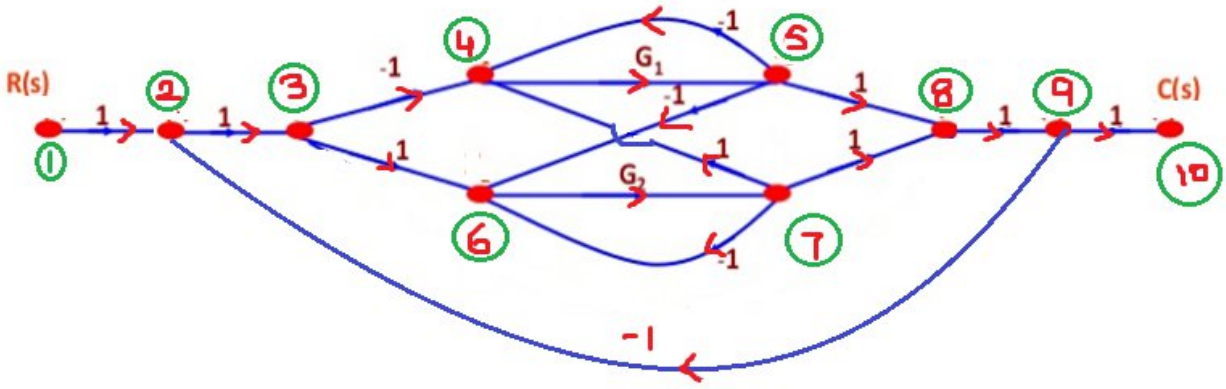
From Block Diagram to Signal-Flow Graph Models

Example# 1



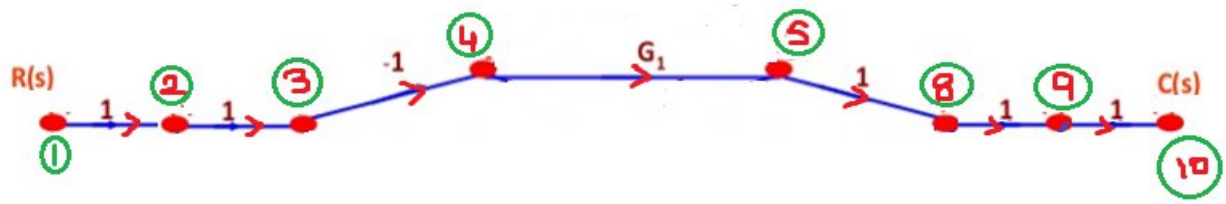
Example# 2





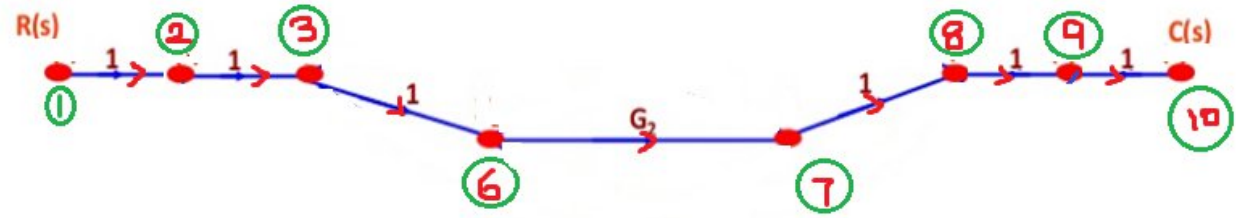
Forward Paths:

FP1:



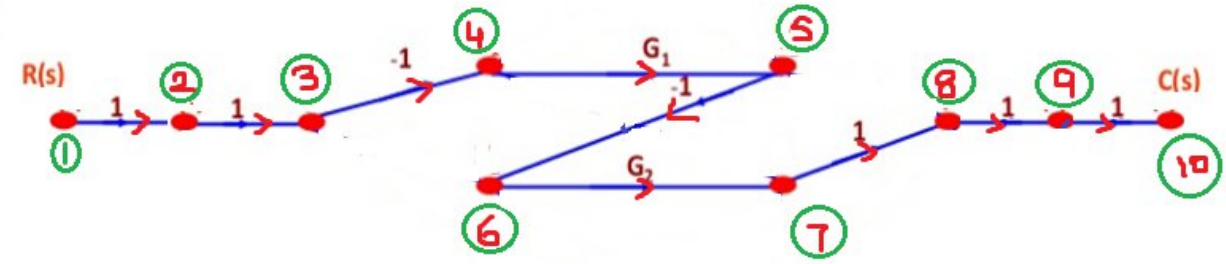
P1=- G1

FP2:



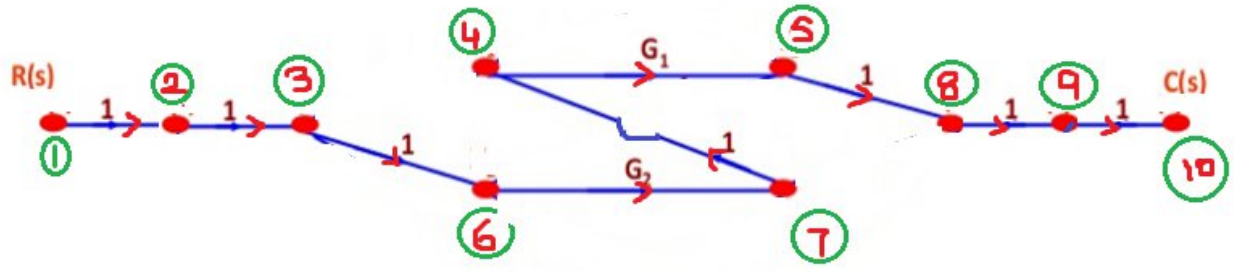
P2= G2

FP3:



P3= G1G2

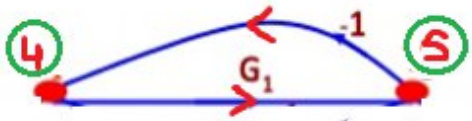
FP4:



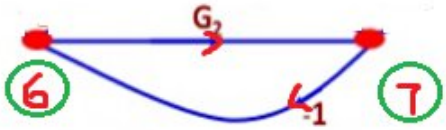
P4=G1G2

LOOPS:

L1=-G1

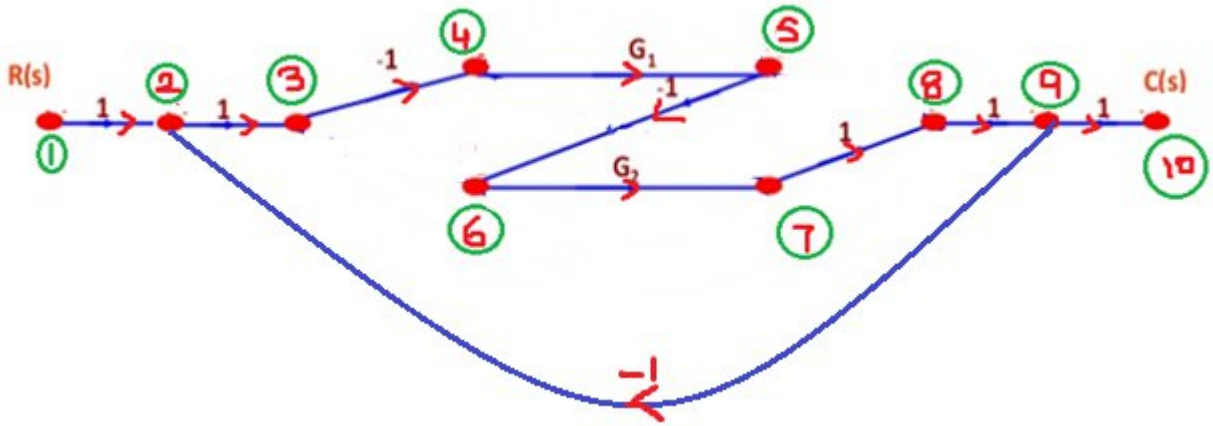


L2=-G2

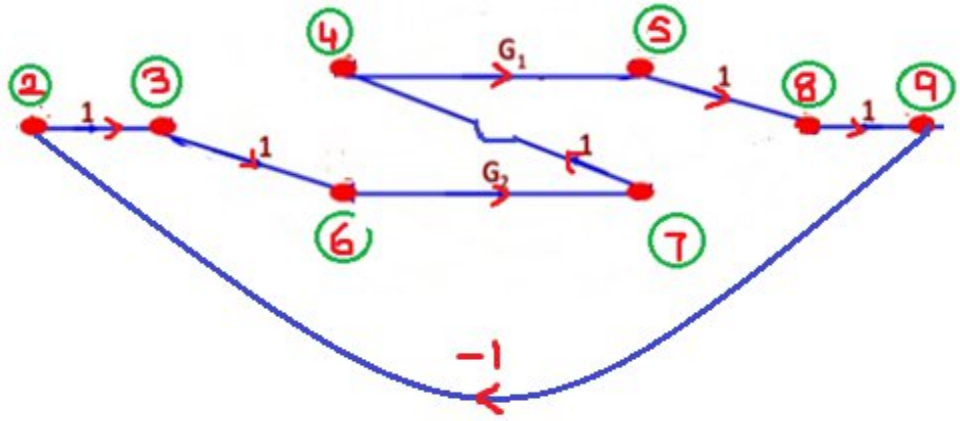


L3=-G1G2

L6=G1G2



L7=G1G2

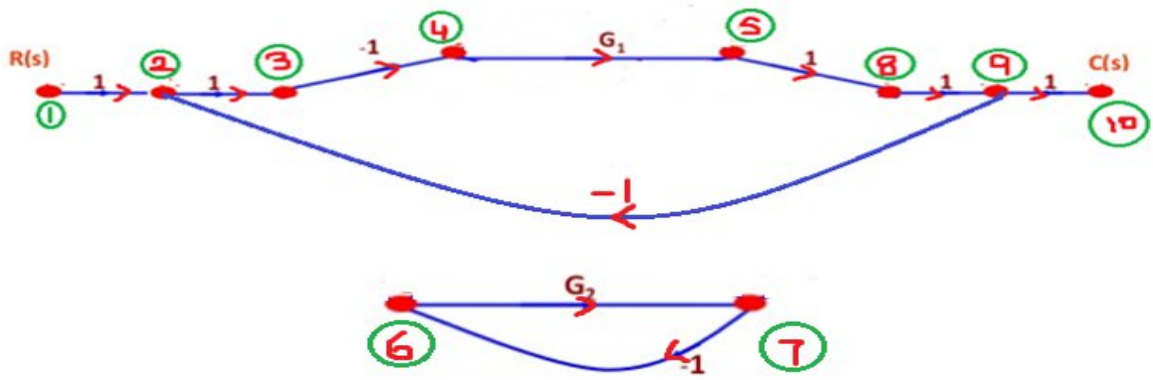


Two non-touching loops:

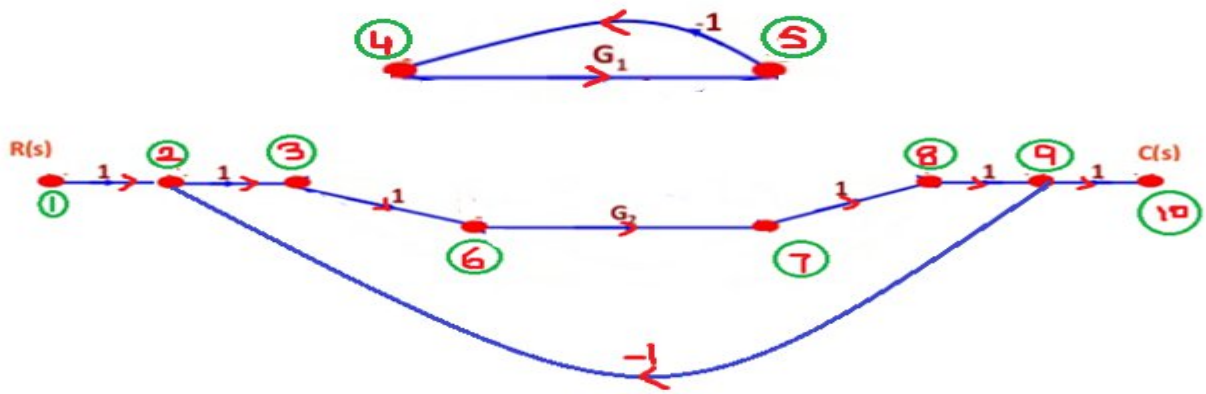
L1L2=G1G2



L2L4=-G1G2

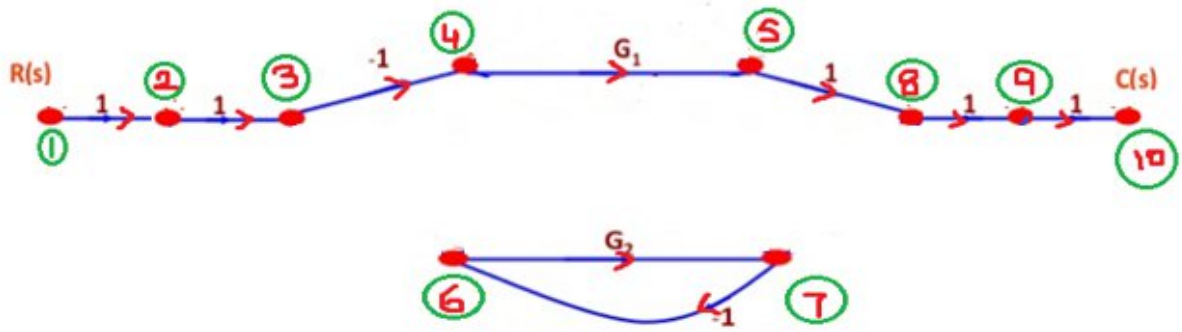


L1L5=G1G2

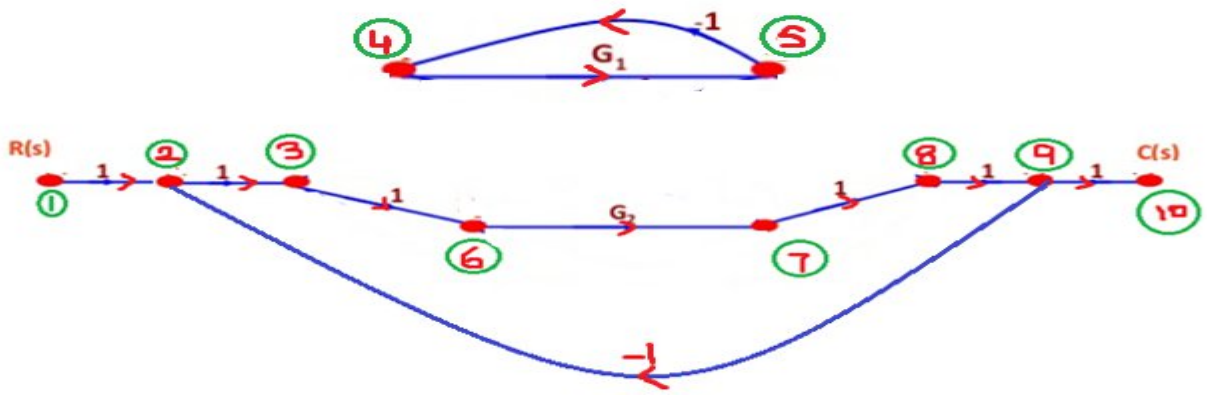


Non-touching loops to forward paths:

FP1: L2 is non touching to FP1. Δ1 = 1-L2 = 1+G2



FP2: L1 is non touching to FP2. Δ2 = 1-L1 = 1+G1



$$T = \frac{\Sigma_K P_K \Delta_K}{\Delta}$$

There are 4 forward paths $K = 4$

P1=- G1

P2= G2

P3= G1G2

P4= G1G2

L1=-G1

L2=-G2

L3=-G1G2

L4=G1

Δ1 = 1+G2

Δ2 = 1+G1

Δ3 = 1

Δ4 = 1

L5=-G2

L6=G1G2

L7=G1G2

Δ = 1-{L1+L2+L3+L4+L5+L6+L7} + {L1L2+L1L5 + L2L4}

Δ = 1+G1+G2+G1G2-G1+G2-G1G2-G1G2+ G1G2+ G1G2 –G1G2 = 1+2G2

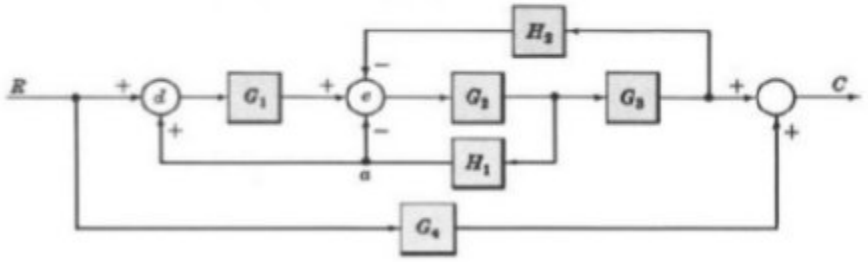
$$T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$= \frac{-G_1(1+G_2) + G_2(1+G_1) + G_1G_2 + G_1G_2}{1+2G_2}$$

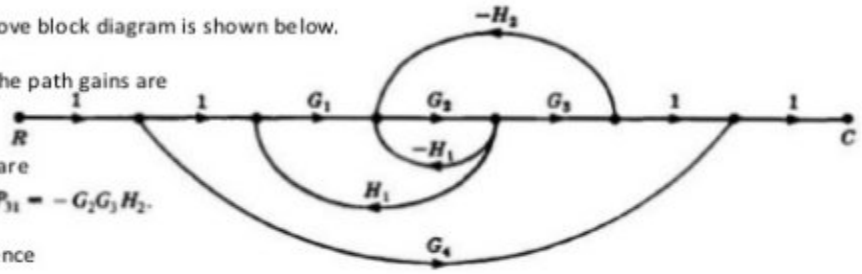
$$T = \frac{-G_1 - G_1G_2 + G_2 + G_1G_2 + G_1G_2 + G_1G_2}{1 + 2G_2}$$

$$T = \frac{-G_1 + G_2 + 2G_1G_2}{1 + 2G_2}$$

Example 3: Determine the transfer function C/R for the block diagram below by signal flow graph techniques.



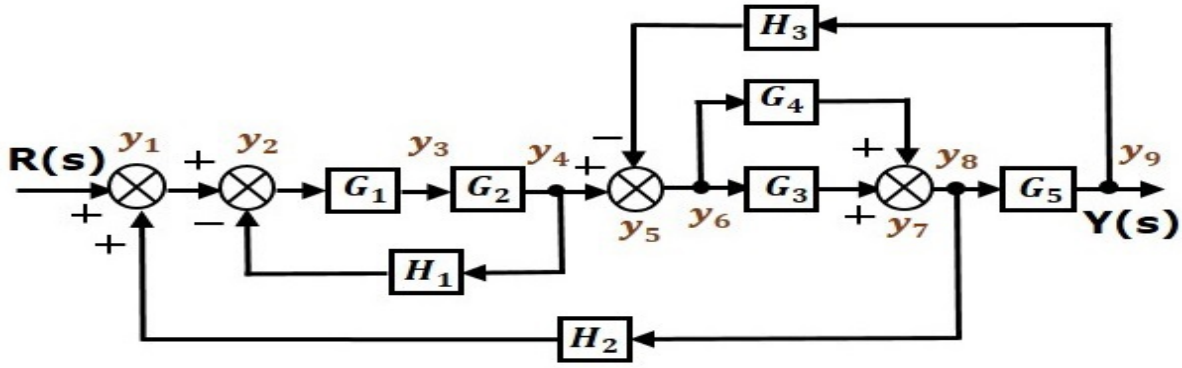
- The signal flow graph of the above block diagram is shown below.
- There are two forward paths. The path gains are $P_1 = G_1G_2G_3$ and $P_2 = G_4$
- The three feedback loop gains are $P_{11} = -G_2H_1$, $P_{21} = G_1G_2H_1$, $P_{31} = -G_2G_3H_2$.
- No loops are non-touching, hence $\Delta = 1 - (P_{11} + P_{21} + P_{31})$
- Because the loops touch the nodes of P1, hence $\Delta_1 = 1$
- Since no loops touch the nodes of P2, therefore $\Delta_2 = \Delta$.

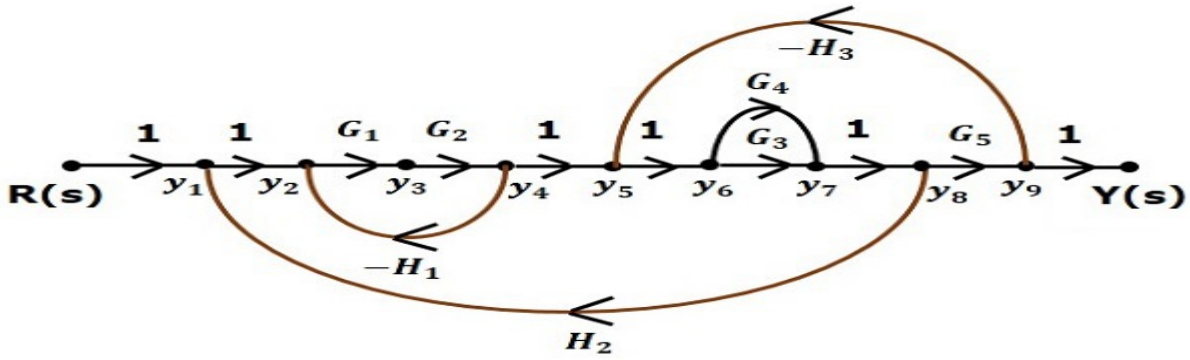


- Hence the control ratio T = C/R is

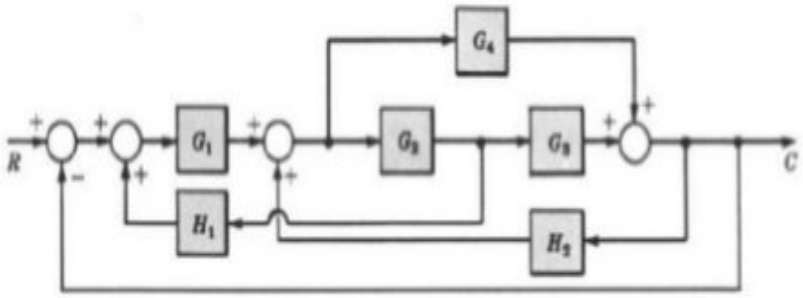
$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4 + G_2G_4H_1 - G_1G_2G_4H_1 + G_2G_3G_4H_2}{1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2}$$

Example-4:

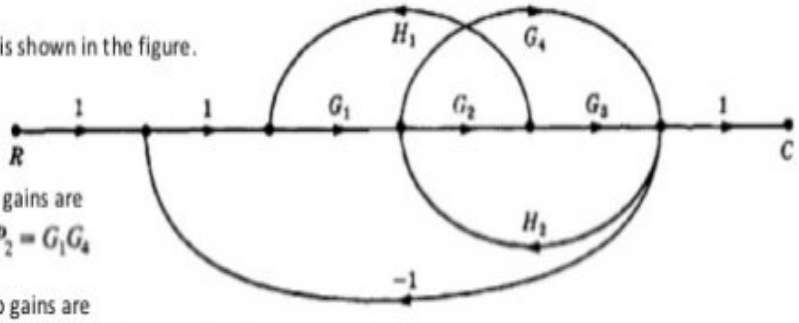




Example 5: Find the control ratio C/R for the system given below.



- The signal flow graph is shown in the figure.



- The two forward path gains are
  $P_1 = G_1G_2G_3$ and $P_2 = G_1G_4$
- The five feedback loop gains are
  $P_{11} = G_1G_2H_1$, $P_{21} = G_2G_3H_2$, $P_{31} = -G_1G_2G_3$,
  $P_{41} = G_4H_2$, and $P_{51} = -G_1G_4$.
- All feedback loops touches the two forward paths, hence $\Delta_1 = \Delta_2 = 1$
- There are no non-touching loops, hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51})$$
$$= 1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4$$

- Hence the control ratio T = $\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4}$

SFG from equations:

Draw SFG for the system represented by the following equations and find transfer function of the system.

x2 =a x1 + f x2

x3 =b x2 + e x4

x4 =c x3 + h x5

x5 =d x4 + g x2

Sol:

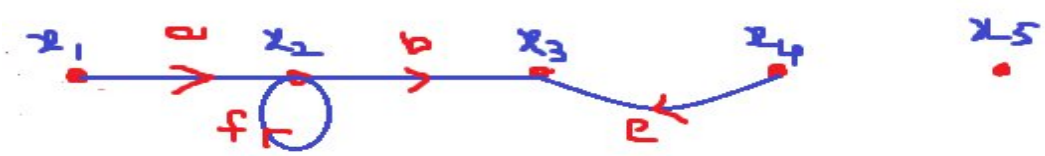
There are 5 variables – x1, x2, x3, x4, x5. So there will be 5 nodes in SFG.



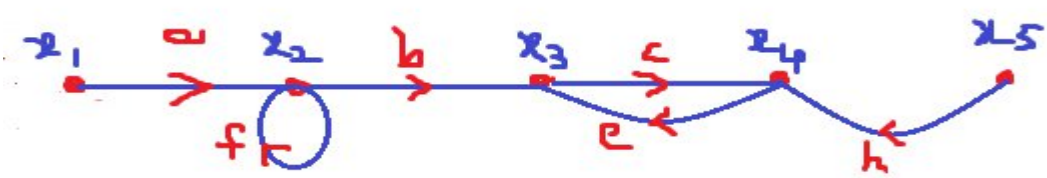
x2 =a x1 + f x2



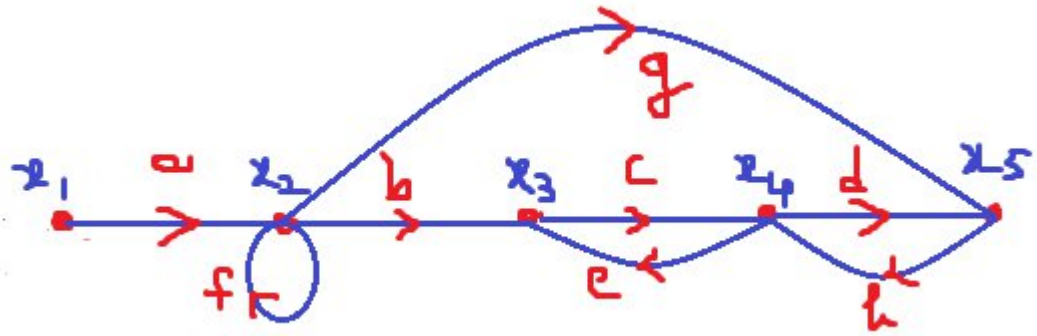
x3 =b x2 + e x4



x4 =c x3 + h x5



x5 =d x4 + g x2



2. Draw SFG for the system represented by the following equations and find transfer function of the system.

$$x2 = a12\ x1 + a22\ x2 + a32\ x3 - a42\ x4 + a52\ x5$$

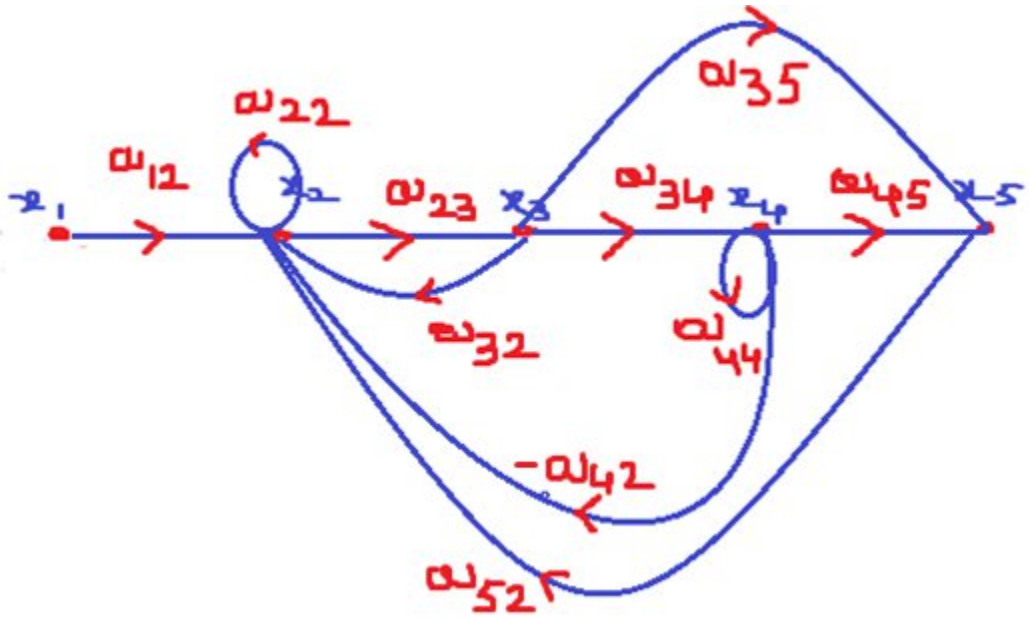
$$x3 = a23\ x2$$

$$x4 = a34\ x3 + a44\ x4$$

$$x5 = a35\ x3 + a45\ x4$$

Sol:

There are 5 variables – x1, x2, x3, x4, x5. So there will be 5 nodes in SFG.



3.**Draw SFG for the system represented by the following equations and obtain expression for the output.**

$x1 = a11\ x1 + a12\ x2 + a13\ x3 + b1\ u1$

$x2 = a21\ x1 + a22\ x2 + a23\ x3 + b2\ u2$

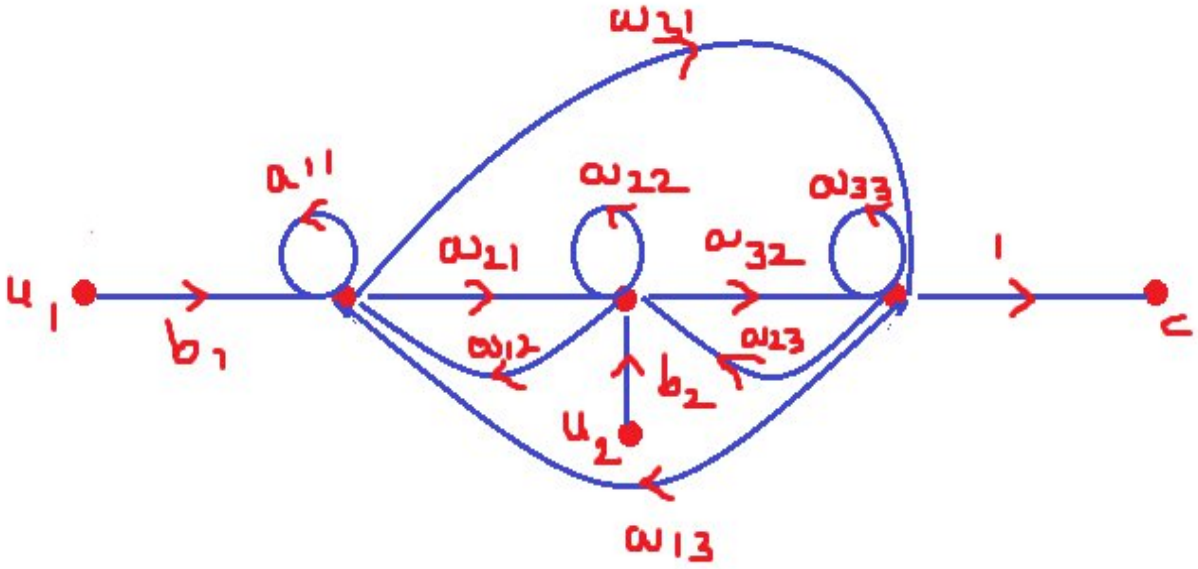
$x3 = a31\ x1 + a32\ x2 + a33\ x3$

$C = x3$

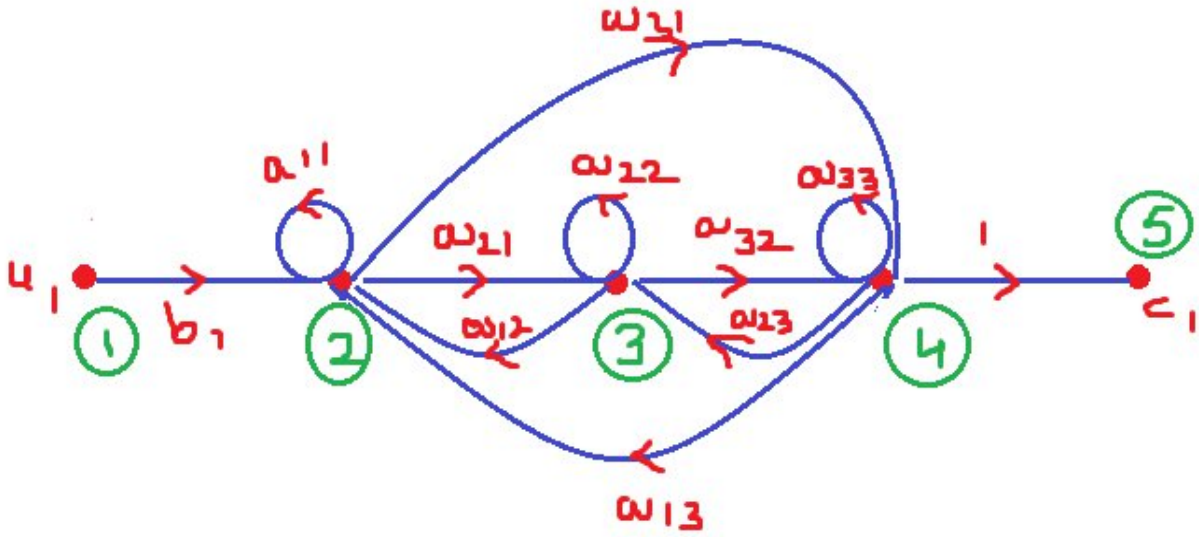
Where u1, u2 are inputs and c is output.

Sol:

There are 5 variables – u1, u2, x1, x2, x3, c. So there will be 6 nodes in SFG.



Case1: u1 is the only input. So u2=0



forward paths:

FP1: 1-2-3-4-5 **P1=b1 a21 a32**

FP2: 1-2-4-5 **P2= b1 a31**

Loops:

1. **L1 : 2-3-2** **L1=a21 a12**
2. **L2 : 3-4-3** **L2=a23 a32**

3. **L3 : 2-4-2 L3=a31 a13**
4. **L4 : 2-2 L4=a11**
5. **L5 : 3-3 L5=a22**
6. **L6 : 4-4 L6=a33**
7. **L7 : 2-3-4-2 L7=a21 a32 a13**
8. **L8 : 2-4-3-2 L8=a12 a23 a31**

Two non-touching loops:

1. **L1L6 = a21 a12 a33**
2. **L2L4 = a11 a23 a32**
3. **L3L5 = a22 a31 a13**
4. **L4L5 = a11 a22**
5. **L4L6 = a11 a33**
6. **L5L6 = a22 a33**

Three non-touching loops:

1. **L4L5L6 = a11 a22 a33**

Non-touching loops to forward paths:

FP1: all nodes are involved. Δ1= 1

FP2: L5 is non-touching loop. Δ2= 1-L5 = 1 – a22

$$T = \frac{\sum_K P_K \Delta_K}{\Delta}$$

Here K= 2 since there are 2 forward paths.

Δ = 1 – { L1 + L2 + L3+ L4+ L5+L6+L7+L8} + { L1L6 + L2L4 + L3L5+L4L5+L4L6+L5L6} - {L4L5L6}

Δ = 1 – a21 a12 - a23 a32 - a31 a13 - a11 - a22 - a33 - a21 a32 a13 - a12 a23 a31 + a21 a12 a33 + a11 a23 a32 + a22 a31 a13 + a11 a22 + a11 a33 + a22 a33 – a11 a22 a33

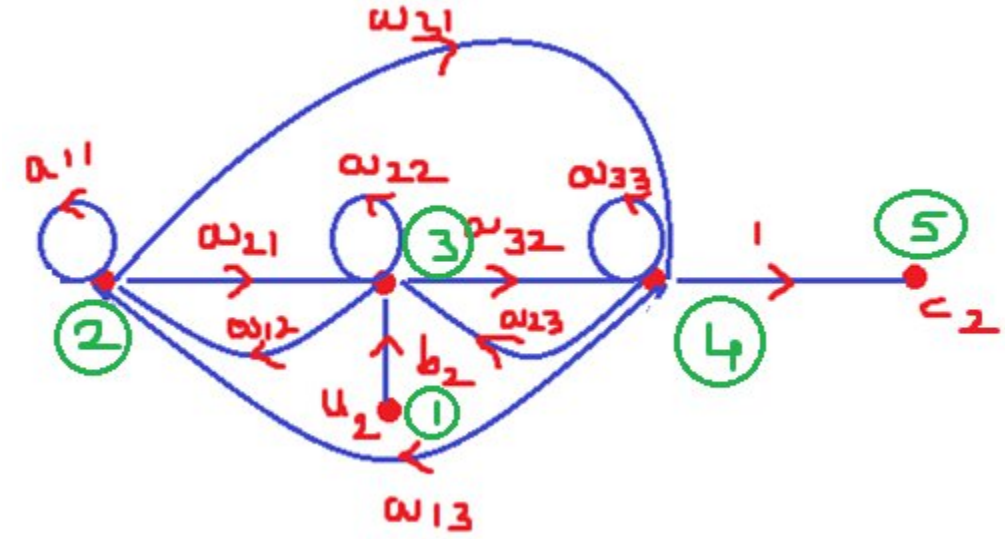
$$\frac{C1}{u1} = T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{b1\ a21\ a32 + b1\ a31\ (1 - a22)}{1 - a21\ a12 - a23\ a32 - a31\ a13 - a11 - a22 - a33 - a21\ a32\ a13 - a12\ a23\ a31 + a21\ a12\ a33 + a11\ a23\ a32 + a22\ a31\ a13 + a11\ a22 + a11\ a33 + a22\ a33 - a11\ a22\ a33}$$

$$= \left(\frac{c1}{\text{b1 a21 a32} + \text{b1 a31 (1} - a22)} \right) u1$$

$$1 - \text{a21 a12} - \text{a23 a32} - \text{a31 a13} - \text{a11} - \text{a22} - \text{a33} - \text{a21 a32 a13} - \text{a12 a23 a31} + \text{a21 a12 a33} + \text{a11 a23 a32} + \text{a22 a31 a13} + \text{a11 a22} + \text{a11 a33} + \text{a22 a33} - \text{a11 a22 a33}$$

Case2: u2 is the only input. So u1=0



Find

$$c2 = (\quad) u2$$

By superposition

C = C1 + C2

SERVO MOTORS

Servo mechanism or servo systems are used to do a task repeatedly.(origin servant).

Servo system is a system whose output is controlled position, velocity or acceleration.

The motors used in servo systems are called as servo motors.

They are used in feedback control systems as actuators and does not use for continuous energy conversion. The principle of the Servomotor is similar to that of the other electromagnetic motor, but the construction and the operation are different. Their power rating varies from a fraction of a watt to a few hundred watts.

Applications of the Servo Motor

The Applications of the Servomotor are as follows:-

- They are used in Radar system and process controller.
- Servomotors are used in computers and robotics.
- They are also used in automation & machine tools.
- Tracking and guidance systems.

Requirements of a good servo motor:

1. Linear relationship between electrical control signal (voltage) and rotor speed over a wide range.

2. Inertia of rotor should be as low as possible. (A servo motor must stop running when control signal is removed.)

   To get low inertia it is designed with large length to diameter ratio for rotors.

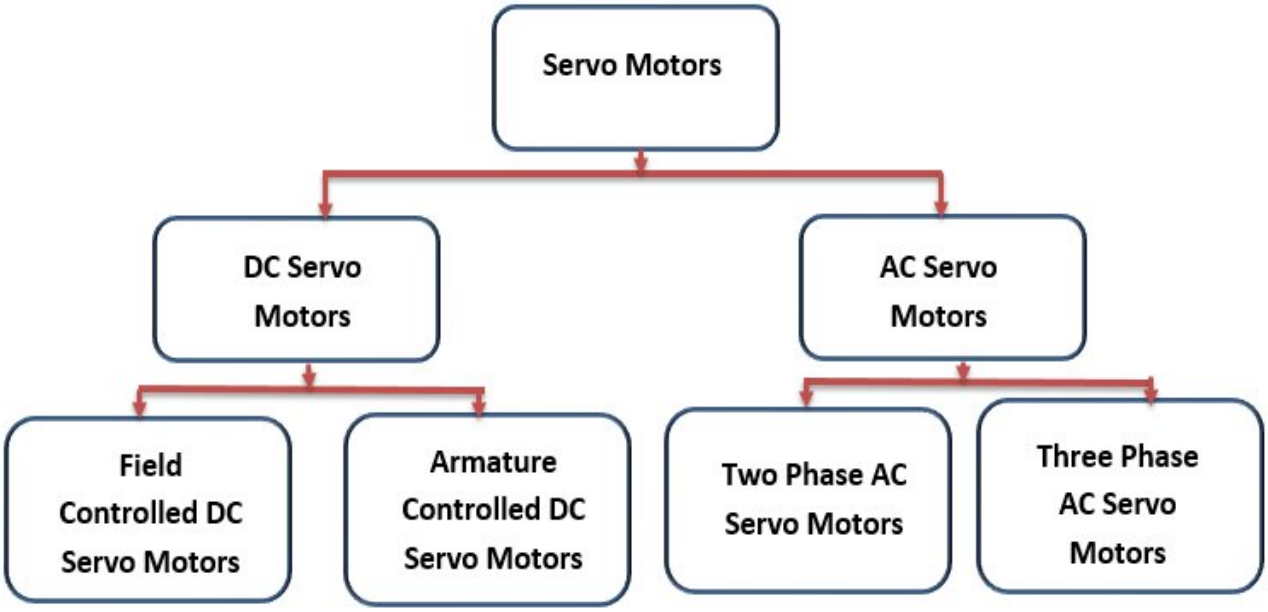
3. Its response should be as fast as possible. For quickly changing error signals it must react with good response.

   This is achieved by keeping torque to weight ratio high.

4. It should be easily reversible.
5. It should have linear torque-speed characteristics.
6. Operation should be stable without oscillations or overshoots.

Classification of Servo Motors

They are classified as AC and DC Servo Motor.



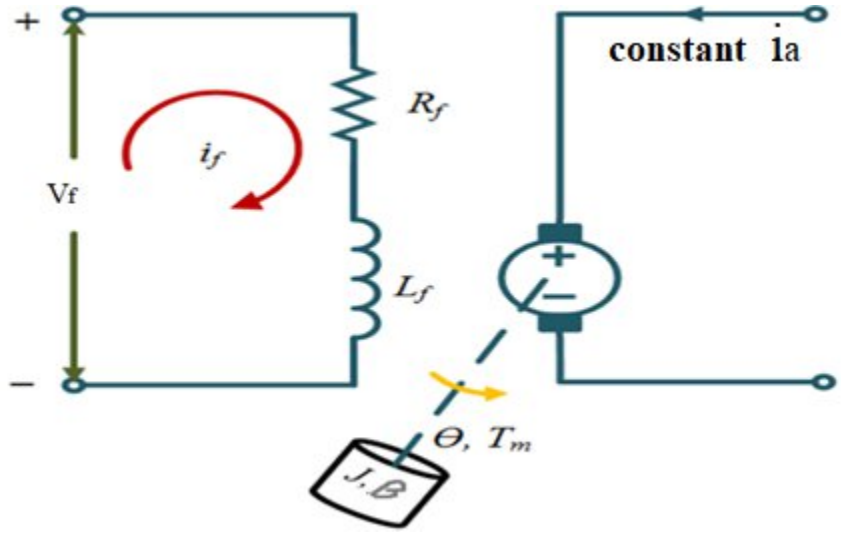
DC Servo Motors:

DC Servo Motors are separately excited DC motor(to get linear torque-speed characteristics) or permanent magnet DC motors. Based on controlling method they are divided in to two types.

Field Controlled DC Servo Motors &Armature Controlled DC Servo Motors

Field Controlled DC Servo Motor:

The figure below illustrates the schematic diagram for a field controlled DC servo motor. In this arrangement the field of DC motor is excited by the a field voltage(amplified error signal) and armature winding is energized by a constant current source.



R_f= Field Resistance
L_f = Field Inductance
I_f = Field Current
V_f = Field Voltage
I_a = Armature Current
μg = Angular Velocity
Θ= Angular displacement
J = Moment of Inertia
B= friction coefficient of rotor
Tm = Torque Produced

In field controlled DC servo motor armature current (i_a) is maintained constant.

When v_f is applied to field circuit a current i_f will flow through the field winding.

$$V_f = R_f\ i_f + L_f\ (di_f/dt) \ldots(1)$$

A flux ϕ is produced by the i_f.

$$\phi\ \alpha\ i_f$$

$$\phi = K_f\ i_f \quad (K_f = \text{field constant})$$

Because of the interaction between ϕ & i_a a torque will be developed. This torque is proportional to product of flux and current.

$$T_m\ \alpha\ \phi\, i_a$$

$$T_m = K \phi\, i_a$$

$$T_m = K K_f\ i_f\, i_a$$

$$T_m = K_m K_f\ i_f \ldots.(2) \quad (Km = \text{motor constant})$$

This torque is used to drive the load connected to shaft. So this torque is opposed by inertia(J) & friction(B).

$$T_m = J\ (d^2\theta/dt^2) + B\ (d\theta/dt)\ \ldots(3)$$

The input to the system is v_f and output is **θ.**

Apply laplace to above eqns.

$$V_f(s) = R_f\ I_f(s) + L_f\ s\ I_f(s)$$

$$V_f(s) = (R_f + s\ L_f)\ I_f(s) \rightarrow I_f(s) = [1\ /\ (R_f + s\ L_f)]\ V_f(s)\ \ldots(4)$$

$$T_m(s) = K_m\ K_f\ I_f(s)\ \ldots(5)$$

$$T_m(s) = J\ s^2\ \theta(s) + Bs\ \theta(s)$$

$$\theta(s) = [1/(J\ s^2 + Bs)]\ T_m(s)\ \ldots(6)$$



Therefore the transfer function is

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f K_m}{(R_f + sL_f)(Js^2 + Bs)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f}{(R_f + sL_f)} \frac{K_m}{(Js^2 + Bs)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f}{R_f(1 + sL_f/R_f)} \frac{K_m}{BS(1 + sJ/B)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_f}{R_f(1+sT_f)} \frac{K_m}{BS(1+sT_m)}$$

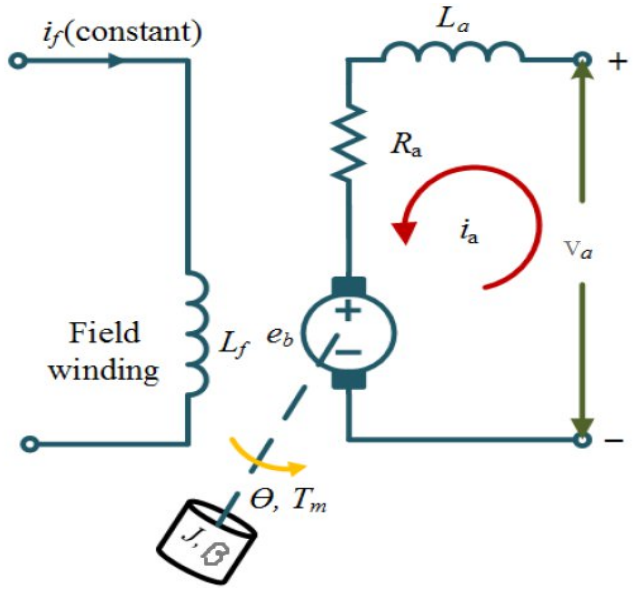
Where $T_f = L_f/R_f$ - field time constant

$T_m = J/B$ - motor time constant

- **Preferred for small ratings**
- **This motor has large time constant because of high field inductance.so it can not give rapid response to quick changing signal.**
- **It is open loop system.**
- **Simple to design & maintain.**

Armature controlled DC Servo Motors:

The figure below illustrates the schematic diagram for a armature controlled DC servo motor. In this arrangement the armature of DC motor is excited by the armature voltage(amplified error signal) and field winding is energized by a constant current source.



R_f= Field Resistance
L_f = Field Inductance
I_f = Field Current
R_a= Armature Resistance
L_a = Armature Inductance
V_a = Armature Voltage
I_a = Armature Current
eb= back emf
μg = Angular Velocity
Θ= Angular displacement
J = Moment of Inertia
B= friction coefficient of rotor
Tm = Torque Produced

In armature controlled DC servo motor field current (i_f) is maintained constant.

When v_a is applied to field circuit a current i_f will flow through the armature winding.

$$V_a = R_a\ i_a + L_a\ (di_a/dt) + e_b \ldots(1)$$

A flux ϕ is produced by the i_f.

$$\phi\ \alpha\ i_f$$

$$\phi = K_f\ i_f \quad \text{is constant.}$$

Because of the interaction between ϕ & i_a a torque will be developed. This torque is proportional to product of flux and current.

$$T_m\ \alpha\ \phi\, i_a$$

$$T_m = K \phi\, i_a$$

$$T_m = K_T\ i_a \ldots.(2)$$ (K_T= motor torque constant)

This torque is used to drive the load connected to shaft. So this torque is opposed by inertia(J) & friction(B).

$$T_m = J\ (d^2\theta/dt^2) + B\ (d\theta/dt)\ \ldots(3)$$

Back emf is proportional to angular velocity.

$$e_b = K_b\ \mu g = K_b(d\theta/dt)..(4)$$

K_b= back emf constant

The input to the system is v_a and output is **θ.**

Apply laplace to above eqns.

$$V_a(s) = R_a\ I_a(s) + L_a\ s\ I_a(s) + e_b(s)$$

$$V_a(s) - e_b(s) = (R_a + s\ L_a)\ I_a(s)$$

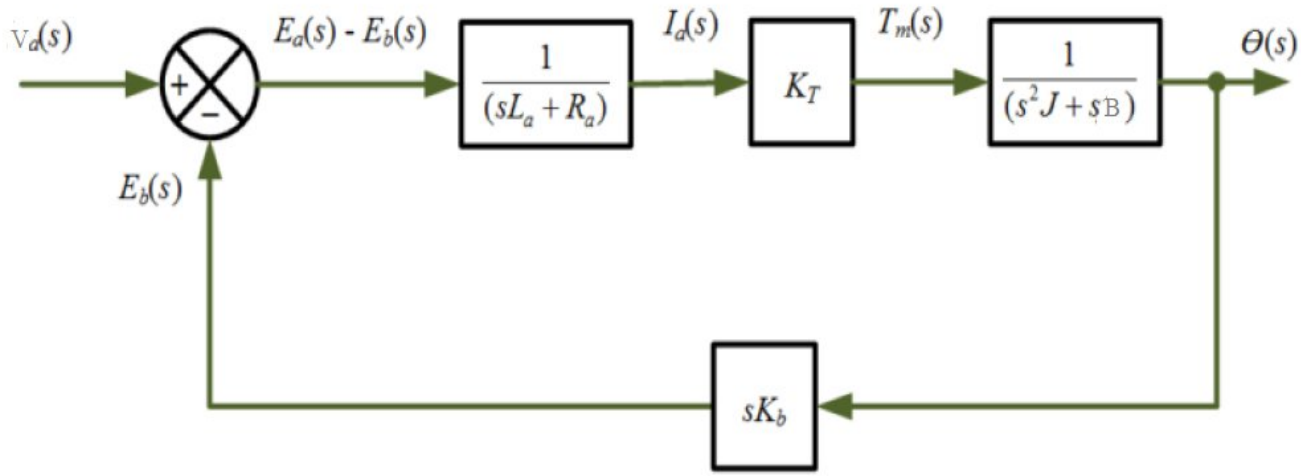
$I_a(s) = (V_a(s) - e_b(s)) / (R_a + s L_a)$ …(5)

$T_m(s) = K_T I_a(s)$ …(5)

$T_m(s) = J s^2 \theta(s) + Bs \theta(s)$

$\theta(s) = T_m(s) \, 1/(J s^2 + Bs)$ …(6)

$e_b(s) = K_b s \theta(s)$ …(7)



Therefore the transfer function is

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{(R_a + sL_a)\left(Js^2 + Bs\right) + K_T K_b s}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{s[\,(R_a + sL_a)\,(Js + B) + K_T K_b]}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T}{sR_aB[\left(1 + \frac{sL_a}{R_a}\right)\left(1 + \frac{sJ}{B}\right) + K_T K_b / R_a B]}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[\,(1 + sT_a)\,(1 + sT_m) + K_m K_b]}$$

Where K_m= motor constant

$T_a = L_a/R_a$ armature time constant

$T_m = J/B$ motor time constant

- **Preferred for large ratings**
- **This motor has small time constant because of high armature resistance.so it gives rapid response to quick changing signal.**
- **It is closed loop system.**
- **Back emf provides internal damping so motor operation is stable.**
- **Efficiency and overall performance is better than field controlled DC servo motor.**

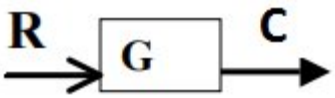
Feedback Characteristics of Control Systems or effect of feedback

In this section we will investigate the effects of negative feedback on various aspects of system performance. Following are some effects of feedback.

- Effect of feedback on gain of the system
- reduction of effect of parameter variation by feedback
- reduction of effect of disturbance by feedback
- Effect of feedback on bandwidth of the system
- Effect of feedback on system dynamics
- Linearization effect of feedback.

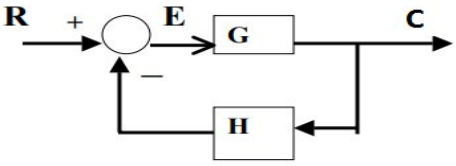
I. Effect of feedback on gain of the system

For Open loop system the gain is given by



$$\frac{C}{R} = G$$

For Closed loop system the gain is given by



$$\frac{C}{R} = \frac{G}{1 + GH}$$

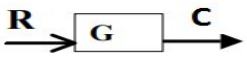
Therefore gain of the system is reduced by a factor 1/(1+ GH) by providing feedback.

II. Reduction of effect of parameter variation by feedback

Let ΔG(S) is the change in G(S) due to parameter variation. Let us investigate the effect of this on output. The output due to parameter variation is ΔC(S).

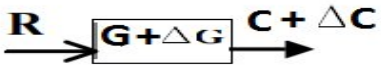
In Open loop system

Before variation



C = G R

After variation



C +ΔC = (G +ΔG) R = G R + ΔG R

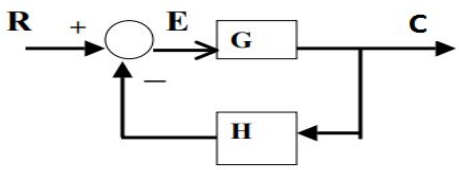
But C = G R

Therefore ΔC = ΔG R

The change in output is directly proportional to parameter variation in open loop system.

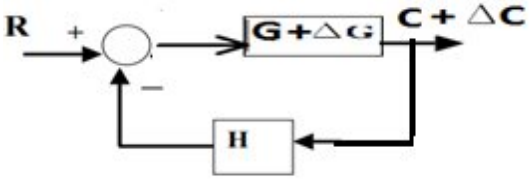
In closed loop system:

Before variation



$$C = \frac{G}{1+GH}R$$

After variation



$$C + \Delta C = \frac{G+\Delta G}{1+(G+\Delta G)H}R$$

But ΔG << G .Therefore (G+ΔG)H ≈ GH

Hence $C + \Delta C = \frac{G+\Delta G}{1+GH}R = \frac{G}{1+GH}R + \frac{\Delta G}{1+GH}R$

Therefore $\Delta C = \frac{\Delta G}{1+GH}R$

The change in output due to parameter variation is reduced by a factor1/ (1+ GH) in closed loop system.

The same can be explained by using **sensitivity.**

$$S_G^T = \frac{Ratio\ of\ \%\ change\ in\ T}{Ratio\ of\ \%\ change\ in\ G}$$

$$S_G^T = \frac{\Delta T/T}{\Delta G/G} = \frac{\Delta T}{\Delta G}\frac{G}{T} = \frac{\partial T}{\partial G}\frac{G}{T}(for\ small\ changes)$$

Open loop System:

T= G

Therefore $S_G^T = \frac{\partial G}{\partial G}\frac{G}{G} = 1$

Closed loop:

$$T = \frac{C}{R} = \frac{G}{1 + GH}$$

sensitivity of the system to the parameter variation in G

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T}$$

$$\frac{\partial T}{\partial G} = \frac{(1 + GH) * 1 - GH}{(1 + GH)^2} = \frac{1}{(1 + GH)^2}$$

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} = \frac{1}{(1+GH)^2}\frac{G}{G/(1+GH)}$$

$$S_G^T = \frac{1}{1 + GH}$$

Therefore sensitivity of the closed loop system is 1/(1+GH).

If G*H >>1.0 sensitivity of the system to the parameter variations in G is very less.

i.e. The sensitivity of the system to the parameter variation in G is reduced by providing feedback.

Sensitivity of the system to the parameter variation in H

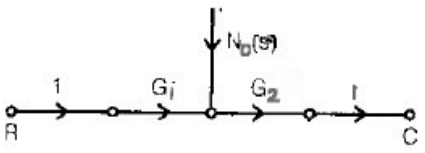
$$S_H^T = \frac{\partial T}{\partial H} \cdot \frac{H}{T} = \frac{-GH}{(1+GH)}$$

If G*H >>1.0 $S_H^T = -1$

i.e. The system is sensitive to the parameter variation in H by providing feedback. Feedback components should not be varied with environmental changes as this directly effects output.

III. Reduction of effect of Disturbance by feedback:

In Open loop system

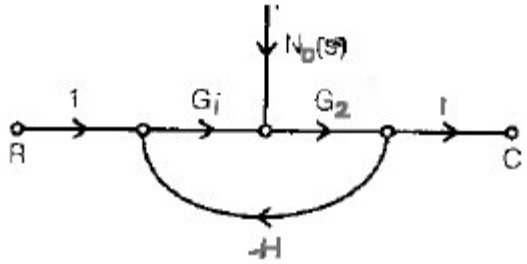


Output due to input R - $C_R = G_1 G_2 R$

Output due to disturbance - $C_N = G_2 N_D$

Closed loop system:

Disturbance in forward path



Output due to input R - $C_R = \frac{G1\,G2}{1+G1\,G2H}\,R$

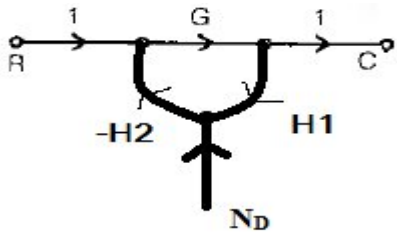
Output due to disturbance- $C_N = \frac{G2}{1+G1\,G2H}N_D$

The effect of disturbance in forward path is reduced by the factor $1/(1+G_1G_2H)$ by providing feedback.

Output due to disturbance- $C_N \approx \frac{G2}{G1\,G2H}N_D = \frac{1}{G1H}N_D$

By increasing G_1 disturbance effect can be reduced.

Disturbance in feedback path



Output due to input R - $C_R = \frac{G}{1+G\,H1H2}\,R$

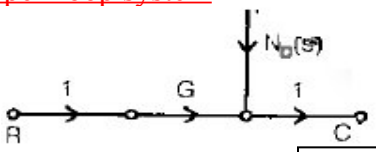
Output due to disturbance- $C_D = \frac{-\,GH_2}{1+G\,H_1H_2}N_D$

$\frac{C_D}{N_D} \approx \frac{-\,GH_2}{G\,H_1H_2} = \frac{-\,1}{H_1}$

The effect of disturbance in feedback path is reduced by the selecting proper value for H_1. By selecting an element which have high signal to noise ratio this H_1 can be made high.

Disturbance at Output:

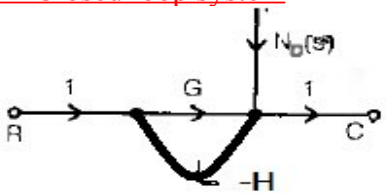
In Open loop system



Output due to disturbance $C_N = N_D$

$\frac{C_N}{N_D} = 1$

In Closed loop system



$\frac{C_N}{N_D} = \frac{1}{1+GH}$

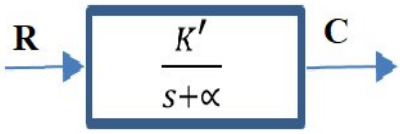
The effect of disturbance at output is reduced by the factor $1/(1+GH)$ by providing feedback

IV. Effect of feedback on System Dynamics (or dynamic response or transient response)

System response depends on location of the pole and time constant.

Let us consider open loop and closed loop of a first order system.

Open loop system:



$$G(s) = \frac{K'}{s+\alpha}$$

Roots of denominator are called as poles of the transfer function.

It has one pole at s = -α

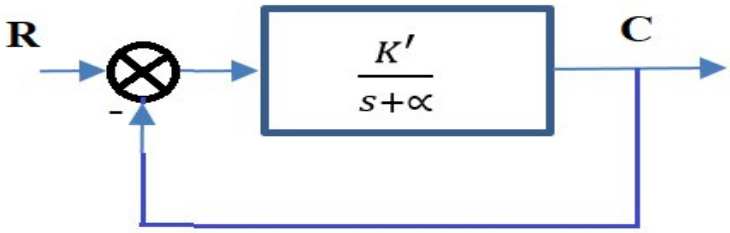
$$G(s) = \frac{K'}{s+\alpha} = \frac{K'/\alpha}{1+s/\alpha} = \frac{K}{1+sT}$$

Where K=K'/ α = DC gain of the system

T= 1/ α =Time constant of the system

Closed loop system:

Now provide a unity feedback for the above system to make the closed loop.



Transfer function of closed loop system T(s) = $\frac{K'}{s+\alpha+K'}$

The closed loop system has a pole at s= -(α+k')

$$T(s) = \frac{K'}{s+\alpha+K'} = \frac{K'/(\alpha+K')}{1+s/(\alpha+K')} = \frac{K_c}{1+sT_c}$$

Where Kc = $\frac{K'}{\alpha+K'}$ =DC gain of Closed loop system

Tc = $\frac{1}{\alpha+K'}$ = time constant of closed loop system.For open loop system K=K'/ α ,T= 1/ α

$$C(s) = \frac{K}{1+sT} R(s)$$

$C(t)=K\,e^{-t/T}$

Closed loop system

$$C(s) = \frac{K_c}{1+sT_c} R(s)$$

Impulse response of open loop & closed loop systems

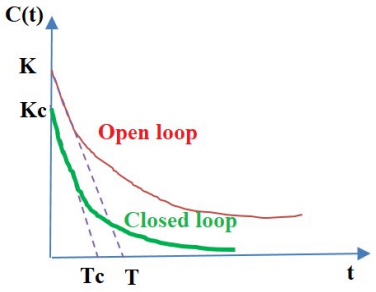
impulse input(R(s)=1)

$$= \frac{K_c}{1+sT_c}$$

Open loop system

$C(t)=Kc\,e^{-t/Tc}$

Both K' & α are positive numbers therefore feedback reduces the gain and time constant.Lesser the time constant faster is the response.

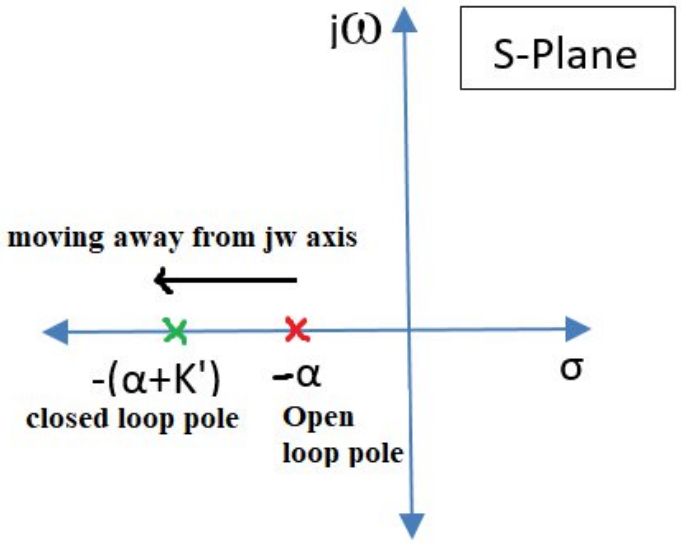


Lesser the time constant steeper the slope i.e. fast response.

Therefore response of feedback system is faster than open loop system.

Stability:

Stability depends on the location of poles of the system. Poles nearer the imaginary axis system is less stable and poles away from imaginary axis system is more stable.



Closed loop pole away from jμg axis. So closed loop system is relatively more stable than open

The effects of feedback on system dynamics are

- **It decreases the time constant of a system. Lesser the time constant faster is the time response. So response of closed loop system is faster than open loop response.**
- **Feedback controls the response by shifting the locations of poles away from imaginary axis. So system relative stability is increased.**

V. Effect of feedback on Bandwidth

A control system is a low pass filter. It responds to frequencies from 0 up to a certain value μg_b at which the gain drops to $1/\sqrt{2}$ of its DC value.

μg_b = band width

Open loop system:

$$G(s) = \frac{K}{1+sT}$$

$$G(j\ ) = \frac{K}{1+j\ T}$$

DC gain = K (DC means μg=0)

At $\mu g = \mu g_b$ gain = K/√2

$$\frac{K}{\sqrt{2}} = \frac{K}{1+j\ _bT} = \frac{K}{\sqrt{1+(\ _bT)^2}}$$

$$\sqrt{1+(\ _bT)^2} = \sqrt{2} \Rightarrow 1+(\mu g_b\ T)^2 = 2 \Rightarrow (\mu g_b\ T)^2 = 1$$

$\mu g_{bo} = 1/T$ = bandwidth of open loop system

Closed loop system:

$$T(s) = \frac{K_c}{1+sT_c}$$

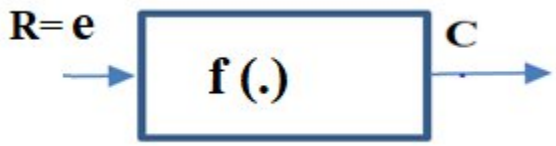
$\mu g_{bc} = 1/T_c$ = bandwidth of closed loop system

So bandwidth of closed loop is large since $T_c < T$
For large bandwidth system responds accurately to high frequencies i.e fast changing signals.
The bandwidth of closed loop system is larger than open loop system which implies faster response.

VI. Linearizing effect of feedback:

Yet another advantage of feedback is its linearizing effect which is illustrated by taking a simple single loop system shown in figure.

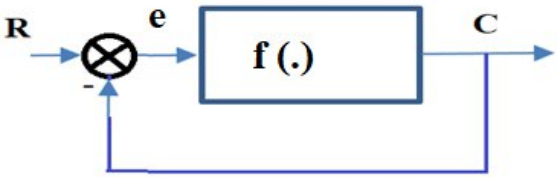
Open loop system



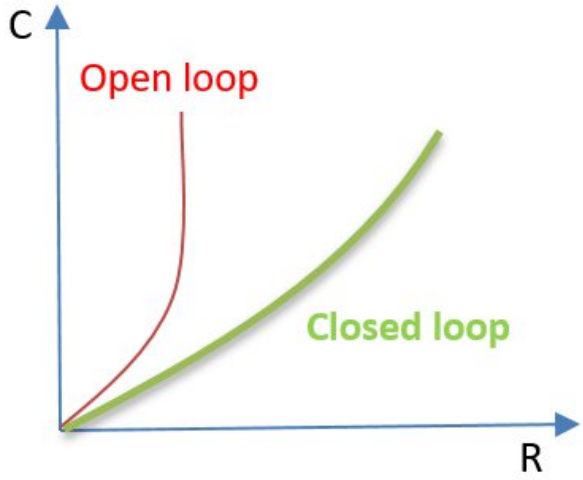
Assume f(.) is a nonlinear function
for example f(.) = e^2

R = e ⇒ $C = e^2 = R^2$ …(1)

Closed loop system



e = R - C ⇒ $C = e^2 = (R-C)^2$ …(2)

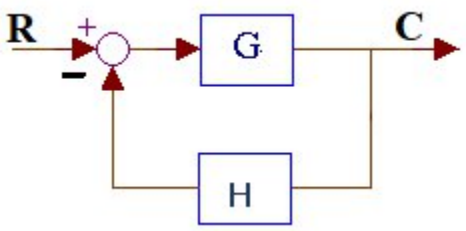


draw graph between C & R for open loop and closed loop systems using eqns 1 & 2. They are as shown in above figure. From the figure it is clear that the input-output relation is approximately linear over a much wider range for closed loop system compared to an open loop system.

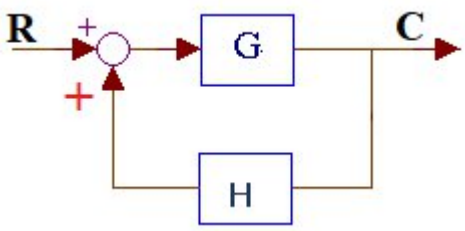
Types of feedback systems:

Feedback systems are two types based on the sign of feedback signal at summing point.

- Negative feedback systems (Degenerative feedback)
- Positive feedback systems (Regenerative feedback)

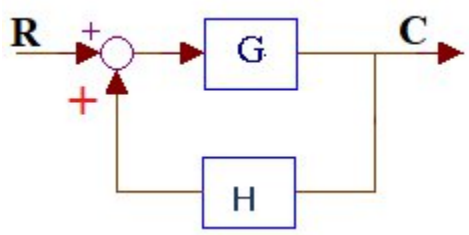


Negative feedback system



Positive feedback system

Regenerative or Positive feedback system:



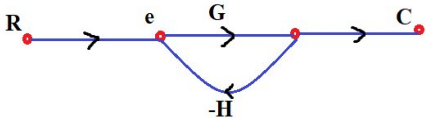
$$TF = gain = \frac{C(s)}{R(s)} = \frac{G}{1-GH}$$

Advantage : The gain of the +ve feedback is more compared to –ve feedback($\frac{G}{1+GH}$).

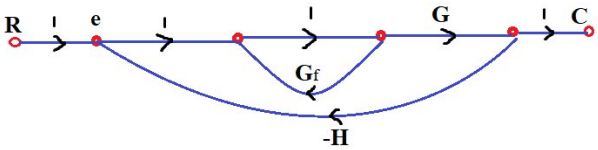
Disadvantage: there is a –ve sign in the sum in denominator. If for any value of s, GH =1 then denominator is zero.

It implies that for a finite input the output is infinity which is the condition of instability.

Applications: The +ve feedback is used to increase the gain of a –ve feedback system. Consider the below feedback system.



$$T = \frac{C}{R} = \frac{G}{1+GH}$$



$$T = \frac{C}{R} = \frac{G}{1+GH-G_f}$$

By selecting G_f near to unity

T= 1/H

Generally H is small. Therefore the system gain is very high.

Two advantages of providing positive feedback for above system are

1. Gain is increased.
2. Closed loop transfer function becomes insensitive to G.

Comparison of open loop and closed loop systems:

Open Loop System	Closed Loop System

ANNAMACHARYA UNIVERSITY, RAJAMPET

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RAJAMPET, Annamayya District, AP, INDIA

Department of Electrical & Electronics Engineering

III B.Tech I Semester

Linear Control Systems (24AEEE51T)
(Electrical and Electronics Engineering)

Unit-2 (Time Response Analysis)

Syllabus:

Standard test signals - Time response of first order systems _ Characteristic Equation of Feedback control systems, Transient response of second order systems, problems- Time domain specifications, problems _ Steady state response - Steady state errors and Static error co-efficients, problems, P, PI, PID Controllers.

Time Response

Time response of a system is the output of that system as a function of time when a time varying input is applied to it. It is represented by C(t).

TF = G(s) = C(s)/R(s)

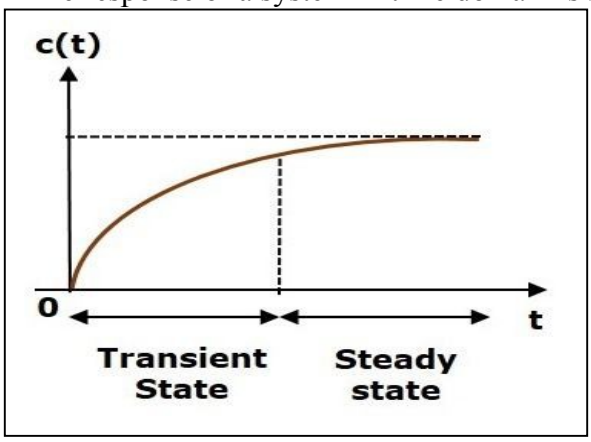
C(s) = G(s) * R(S)

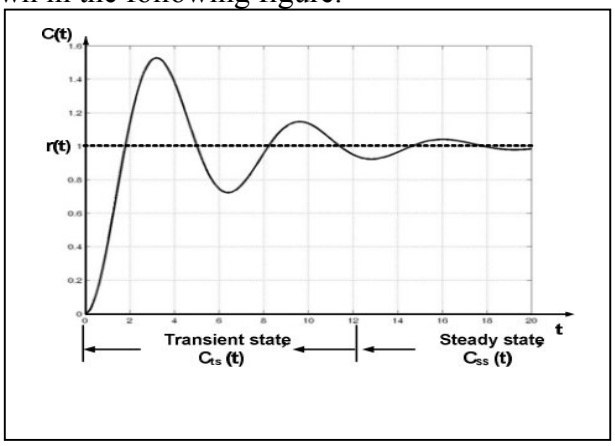
$C(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}[G(s)\,R(s)]$

The time response consists of two parts.

- Transient response
- Steady state response

The response of a system in time domain is shown in the following figure.





Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response C(t) as

$$C(t) = C_t(t) + C_{ss}(t)$$

Where,

- C_t (t) is the transient response
- C_{ss} (t) is the steady state res

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Def: It is defined as the part of the time response which decays to zero as time approaches ∞. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically, it can be written as

$$\mathscr{L}t_{\,t\to\infty} \quad C_t(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has become zero is known as **steady state response**. Or the response at t=∞ is called as steady state response.

$$C_{SS}(t) = Lt_{t\to\infty}\, C\,(\,t)$$

Example

Let us find the transient and steady state terms of the time response of the control system $c(t)=10+5e^{-t}$.

$$c(t)=10+5e^{-t}$$

Here, the second term $5e^{-t}$ will be zero as **t** approaches infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Transient response represents (i) Speed of the response (ii) Nature of response
Steady state response represents accuracy of the response.

Standard Input Signals

These standard input signals are

- an impulse, sudden shock
- a step, sudden change constant position
- a ramp and constant velocity
- a parabolic input. Constant acceleration

These are the time varying inputs. By using these standard test signals of control systems, analysis and design of control systems are carried out, defining certain performance measures for the system.

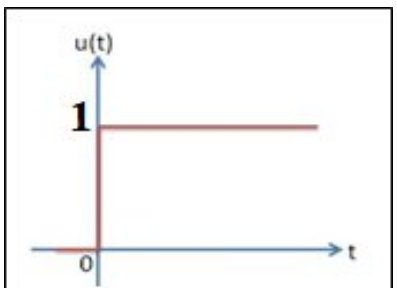
Unit Step Signal:

It is represented by u (t).

Mathematically it is expressed as

$$u(t) = 0 \;, t < 0$$
$$= 1,\; t>=0$$

Its graphical representation is as shown in fig.



So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values

of 't'.

The Laplace transformation of unit step function is 1/s.

$$\mathcal{L}[u(t)] = 1/s$$

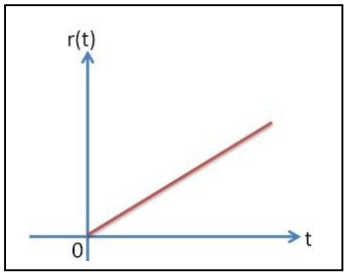
if the value is other than one that step function can be written as A u(t). the laplace transformation of this step function is A/s.

$\mathcal{L}[Au(t)] = A/s$

Unit Ramp Signal : It is represented by r (t). Mathematically it is expressed as

$$r(t) = 0, t < 0$$
$$= t, t >= 0$$

Its graphical representation is as shown in fig.



So, the unit ramp signal exists for all positive values of 't' including zero. And its value is' t' during this interval. The value of the unit ramp signal is zero for all negative values of 't'.

The Laplace transformation of unit step function is $1/s^2$.

$$\mathcal{L}[r(t)] = 1/s^2$$

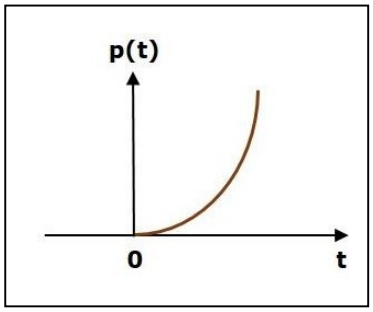
if the slope of the line is other than one the ramp signal is represented by A r(t) (or At). the laplace transform of this function is A/s^2.

Unit Parabola Signal :

Mathematically it is expressed as

$$f(t) = 0, t < 0$$
$$= At^2/2, t >= 0$$

Its graphical representation is as shown in fig.



So, the unit parabolic signal exists for all the positive values of **'t'** including zero. And its value increases non-linearly with respect to 't' during this interval. The value of the unit parabolic signal is zero for all the negative values of't'.

The Laplace transformation of unit step function is $1/s^3$.

$$\mathcal{L}[f(t)] = 1/s^3$$

$f(t) = 0, t < 0$

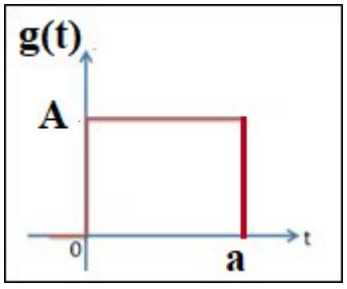
$= A\ t^2/2, t >= 0$

The Laplace transformation of unit step function is A/s^3.

Gate or pulse input: It is represented by g(t).

Mathematically it is expressed as

$$g(t) = 0, t <= 0$$
$$= A, 0 < t < a$$



$= 0, \quad t >= a$

$$\mathcal{L}[g(t)] = \frac{A}{s}(1 - e^{-as})$$

Unit Impulse Signal :

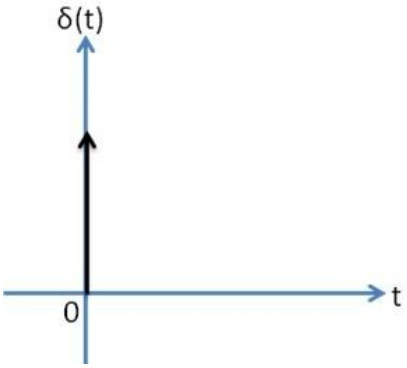
It is represented by δ(t).

Mathematically it is expressed as

$\delta(t) = 0 , t \neq 0$

$= \infty, t=0$

And $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



Its graphical representation is as shown in fig.

So, the unit impulse signal exists only at 't' is equal to zero. The value of unit impulse signal is zero for all other values of 't'. The area of this signal is equal to one.

The Laplace transformation of unit impulse function is 1.

$$\mathcal{L}[\delta(t)] = 1$$

Relation between various inputs:

1. Derivative of parabola is ramp.

2. Derivative of ramp is step.
3. Derivative of step is impulse.

$$\delta(t) = \frac{d\,u(t)}{dt}$$

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d\,u(t)}{dt}\right] = s\mathcal{L}[u(t)] = s * \frac{1}{s} = 1$$

Terminology Related To Transfer Function

Let us discuss terminology related to transfer function.

$$TF = G(s) = \frac{b_0 S^m + b_1 S^{m-1} + b_2 S^{m-2} + \ldots\ldots + b_{m-1} S + b_m}{S^n + a_1 S^{n-1} + a_2 S^{n-2} + \ldots\ldots + a_{n-1} S + a_n} = \frac{N(s)}{D(s)}$$

Characteristic equation:

The equation obtained by equating the denominator of the transfer function to zero is known as characteristic equation.

G(s) = N(s)/D(s)

D(s) =0 is the characteristic equation.

For closed loop system the transfer function is

$$T(s) = \frac{G}{1 + GH}$$

The characteristic equation of closed loop system is 1 + G(s) H(S)=0

Poles & Zeros of transfer function:

Zeros:

The roots of equation obtained by equating numerator of transfer function to zero are called as zeros of the transfer function.

$$T(s) = \frac{S(S+1)}{(S+2)(s+3)}$$

zeros of this system are roots of S (S+1) = 0

zeros are at S = 0, S = -1.

if you substitute this value of S in transfer function T(s) =

The values of "S" for which transfer function is zero are called as Zeros of that system.

The number of zeros = highest power of 's' in numerator.

Poles:

The roots of equation obtained by equating denominator of transfer function to zero are called as poles of the transfer function.

but that equation is called as characteristic equation. the roots of characteristic equation of the system are known as poles of that system.

$$T(s) = \frac{S(S+1)}{(S+2)(s+3)}$$

poles of this system are roots of (S+2) (S+3) = 0
poles are at S = -2, S = -3.

if you substitute this value of S in transfer function T(s) =

The values of "S" for which transfer function is '∞' are called as poles of that system.
The number of poles = highest power of 's' in denominator.

$$T(s) = \frac{S(S+1)}{(S+2)(s+3)}$$

Poles :
Zeros:

Poles and zeros may be
1. Real
2. Complex conjugate

$$G(s) = \frac{S+1}{S(S+2)(s^2+3s+16)}$$

Poles :
Zeros:

Complex poles always exist as a pair.

$$G(s) = \frac{S+2}{S(S+1)(s^2+16)}$$

Poles :
Zeros:

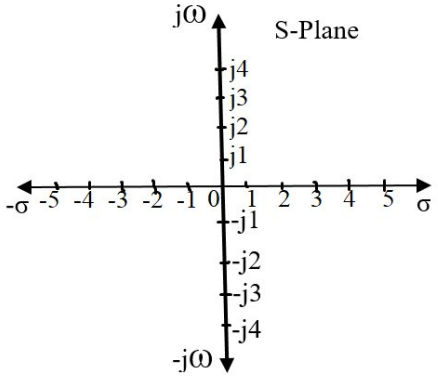
Simple poles:
If poles are non repeated poles they are known as simple poles.
If the poles are repeated or multiple they are called as multiple poles.

$$T(s) = \frac{S+1}{S(S+2)(s+3)^2}$$

Poles :
Simple Poles:
Multiple poles:

Pole at origin: If a pole exist at S=0 i.e. at origin it is called as pole at origin.

<u>pole-zero plot:</u>
Representation of the location of poles and zeros in s- plane is known as pole zero plot. poles are represented by "X" and zeros are represented by "O".



Order & Type of a system:

Order: Highest power of 'S' in the denominator of transfer function is known as order of the system.
Or
Highest power of 'S' in characteristic equation is known as order of the system.
No. of poles of a system = order of the system.

Type Number: The number of the poles at origin of S-plane is known as 'Type' or Type number of that system.

Examples:

$G(s) = \frac{1}{S+2}$ **Order : Type:**

$G(s) = \frac{1}{S(S+2)}$ **Order : Type:**

$G(s) = \frac{1}{S^2(S+2)}$ **Order : Type:**

$G(s) = \frac{S+2}{S(S^2+2S)}$ **Order : Type:**

Forms of Transfer Function:

Transfer function is represented in two forms.

1. Pole-Zero form
2. Time constant form

Pole zero form:

In this form it is easy to identify poles and zeros.
In time domain analysis (rootlocus) this form is more useful.

$$T(s) = \frac{K(S+Z_1)(S+Z_2)(S+Z_3)\ldots(S+Z_m)}{S^j(S+P_1)(S+P_2)(S+P_3)\ldots(S+P_n)}$$

$$T(s) = \frac{(S+1)}{S(S+2)(s+3)}$$

Time constant form:

In this form it is easy to calculate DC gain & time constant.
It is useful in frequency domain analysis.

$$T(s) = \frac{K(1+ST_{Z1})(1+ST_{Z2})(1+ST_{Z3})\ldots(1+ST_{Zm})}{S^j(1+ST_{P1})(1+ST_{P2})(1+ST_{P3})\ldots(1+ST_{Pn})}$$

We can convert one form into other.

$$T(s) = \frac{(S+1)}{S(S+2)(s+3)} = \frac{(1+S)}{S\ 2\ (S/2+1)\ 3\ (s/3+1)}$$

$$T(s) = \frac{\frac{1}{6}(1+S)}{S(1+S/2)(1+S/3)}$$

Time Response of First Order System

In this chapter, let us discuss the time response of the first order system.

A first order system is a system whose input - output relation is given by a first order differential equation.

$$a_0\frac{dy}{dt} + a_1 y = b_0 \mathbf{x}$$

Apply Laplace transformation to the above eqn.

$$a_0 SY(s) + a_1 Y(s) = b_0 X(s)$$

$$(a_0 S + a_1)Y(s) = b_0 X(s)$$

$$TF = \frac{Y(s)}{X(s)} = \frac{b_0}{a_0 S + a_1} = \frac{b_0}{a_1(S\frac{a_0}{a_1} + 1)} = \frac{b_0/a_1}{1 + S\frac{a_0}{a_1}} = \frac{K}{1 + ST}$$

Where K = DC Gain of the system

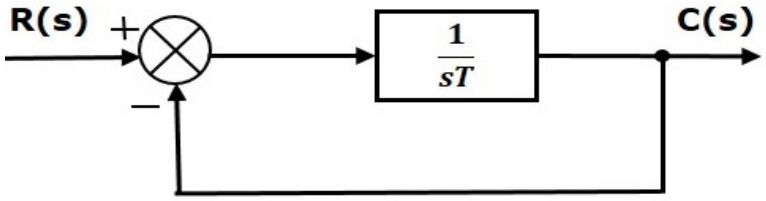
T= Time constant of the system

The power of 's' is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

Assume K=1.

$$TF = \frac{C(s)}{R(s)} = \frac{1}{1 + ST}$$

Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



$$\frac{C(s)}{R(s)} = \frac{1}{1+ST}$$

$$C(s) = \frac{1}{1+ST}R(s).$$

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, r(t) =u(t)

\R(s) = $\mathcal{L}$(u(t) = 1/s

$$\backslash C(s) = \frac{1}{1+ST} R(s) = \frac{1}{S(1+ST)} = \frac{1}{ST(1/T+S)} = \frac{1/T}{S(S+1/T)}.$$

Partial fraction expansion is

$$C(s) = \frac{A}{S} + \frac{B}{(S+\frac{1}{T})}$$

$$A = [C(s)\,S] \quad |s=0 \qquad B = \left\{C(s)\left(S+\frac{1}{T}\right)\right\} \quad |s=-1/T$$

$$A = \{\frac{1/T}{\left(S+\frac{1}{T}\right)}\} \quad |s=0 \qquad B = \frac{1/T}{S} \quad |s=-1/T$$

$$B = -\frac{1/T}{1/T} = -1$$

$$A = \frac{1/T}{1/T} = 1$$

$$C(s) = \frac{1}{S} - \frac{1}{(S+\frac{1}{T})}$$

$$C(t) = 1 - e^{-\frac{t}{T}}$$

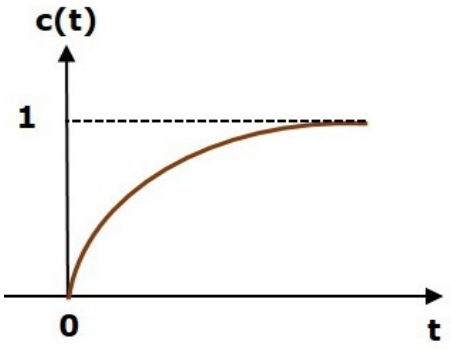
The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is –

$$C_t(t) = -e^{-\frac{t}{T}}$$

The steady state term in the unit step response is -

$$C_{SS}(t) = 1$$



The figure shows the unit step response.

At t=0 C(t) = 1-1 = 0; At t=∞ C(t) = 1

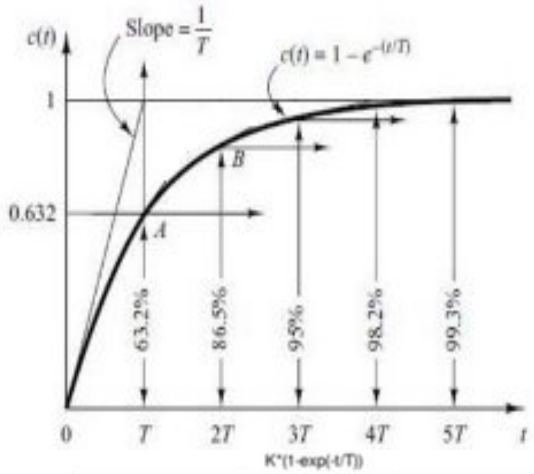
The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state.

Initial slope of C(t) curve

$$\frac{dC(t)}{dt} = (-\frac{1}{T})(-e^{-\frac{t}{T}})$$

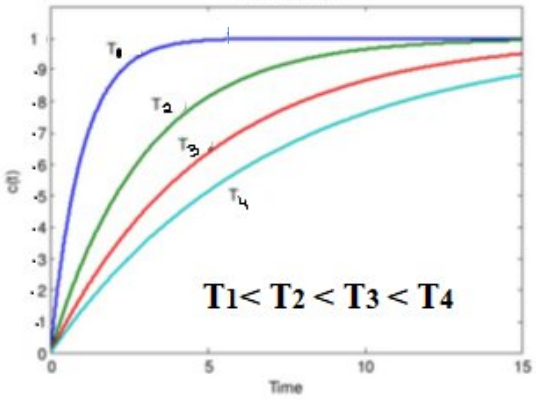
Initial slope means at t =0

Slope = 1/T



Time Constant: Time Constant is a measure of how quickly a 1st order system responds to an input.

Time required for the system to reach the final value if it maintains the initial slope of output.



At t = T $C(t) = 1 - e^{-\frac{T}{T}} = 0.632$

At t = 2T $C(t) = 1 - e^{-\frac{2T}{T}} = 0.86$

At t = 3T $C(t) = 1 - e^{-\frac{3T}{T}} = 0.95$

At t = 4T $C(t) = 1 - e^{-\frac{4T}{T}} = 0.982$

At t = 5T $C(t) = 1 - e^{-\frac{5T}{T}} = 0.993$

The time constant can be defined as the time it takes for the step response to rise up to 63% or 0.63 of its final value.

Time required to reach steady state = 4T

From this figure it is clear that shorter the time constant faster the response.

Steady state error

$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)] = \lim_{t\to\infty} [1 - 1 + e^{-\frac{t}{T}}] = 0$$

Or we can use $e_{SS} = \lim_{t\to\infty} e(t) = \lim_{S\to 0} SE(S)$

Steady state error of a first order system for step input is zero.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to first order system.

So, r(t) =r(t)

\R(s) = $\mathscr{L}$(r(t)) = 1/s^2

$$\backslash C(s) = \frac{1}{1+ST} R(s) = \frac{1}{S^2(1+ST)} = \frac{1}{S^2T(1/T+S)} = \frac{1/T}{S^2(S+1/T)}.$$

Partial fraction expansion is

$$C(s) = \frac{B}{S} + \frac{A}{S^2} + \frac{C}{(S + \frac{1}{T})}$$

$$A = \left[C(s)\, S^2\right] \quad |s = 0$$

$$A = \{\frac{1/T}{\left(S + \frac{1}{T}\right)}\} \quad |s = 0$$

$$A = \frac{1/T}{1/T} = 1$$

$$B = \frac{d}{dS}\left[C(s)\, S^2\right] \quad |s = 0$$

$$B = \frac{d}{dS}\{\frac{1/T}{\left(S + \frac{1}{T}\right)}\} \quad |s = 0$$

$$B = -\frac{1/T}{\left(S + \frac{1}{T}\right)^2} \quad |s = 0$$

$$B = -\frac{1/T}{1/T^2} = -T$$

$$C = \left\{C(s)\left(S + \frac{1}{T}\right)\right\} \quad |s = -1/T$$

$$C = \frac{1/T}{S^2} \quad |s = -1/T$$

$$C = \frac{1/T}{1/T^2} = T$$

$$C(s) = \frac{1}{S^2} - \frac{T}{S} + \frac{T}{(S + \frac{1}{T})}$$

$$C(t) = t - T + Te^{-\frac{t}{T}} = t - T(1 - e^{-\frac{t}{T}})$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

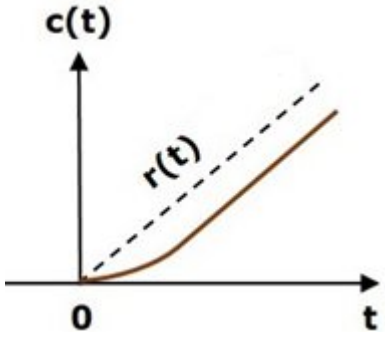
The transient term in the unit ramp response is -

$$C_t(t) = Te^{-\frac{t}{T}}$$

The steady state term in the unit ramp response is -

$$C_{ss}(t) = t - T$$

The following figure shows the unit ramp response.



Steady state error

$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)] = \lim_{t\to\infty} [t - t + T - Te^{-\frac{t}{T}}] = T$$

Or we can use $e_{SS} = \lim_{t\to\infty} e(t) = \lim_{S\to 0} SE(S)$

The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to first order system.

So, r(t) =δ(t)

\R(s) = $\mathscr{L}(\delta(t)) = 1$

$$\backslash C(s) = \frac{1}{1+ST}R(s) = \frac{1}{(1+ST)} = \frac{1}{T(1/T+S)} = \frac{1/T}{(S+1/T)}.$$

$$C(s) = \frac{1/T}{(S+1/T)}$$

$$C(t) = (1/T)e^{-\frac{t}{T}}$$

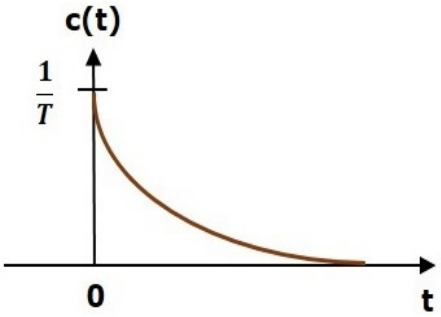
The **unit impulse response**, c(t) has only the transient term.

The transient term in the unit impulse response is –

$$C_t(t) = (\frac{1}{T})e^{-\frac{t}{T}}$$

The steady state term in the unit impulse response is -

$$C_{SS}(t) = 0$$



The figure shows the unit step response.

At t=0 C(t) = 1/T; At t=∞ C(t) = 0

Steady state error

$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)]$ (for impulse function the value is zero at all time except at t=o at which the value is '∞'

So we can use $e_{SS} = \lim_{t\to\infty} e(t) = \lim_{S\to 0} SE(S) = \lim_{S\to 0} S[R(S) - C(S)]$

$$e_{SS} = \lim_{S\to 0} S\left[1 - \frac{\frac{1}{T}}{\left(S+\frac{1}{T}\right)}\right] = \lim_{S\to 0} S\left[\frac{S+\frac{1}{T}-\frac{1}{T}}{\left(S+\frac{1}{T}\right)}\right] = \lim_{S\to 0}\left[\frac{S^2}{\left(S+\frac{1}{T}\right)}\right] = 0$$

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

So, $r(t) = f(t) = t^2/2 \quad t>0$

$\backslash R(s) = \mathcal{L}(r(t)) = 1/s^3$

$$\backslash C(s) = \frac{1}{1+ST} R(s) = \frac{1}{S^3(1+ST)} = \frac{1}{S^3 T(1/T+S)} = \frac{1/T}{S^3(S+1/T)}.$$

Partial fraction expansion is

$$C(s) = \frac{A}{S^3} + \frac{B}{S^2} + \frac{C}{S} + \frac{D}{(S+\frac{1}{T})}$$

$$A = \left[C(s)\, S^3\right] \quad |s=0$$

$$A = \left\{\frac{1/T}{\left(S+\frac{1}{T}\right)}\right\} \quad |s=0$$

$$A = \frac{1/T}{1/T} = 1$$

$$B = \frac{d}{dS}\left[C(s)\, S^2\right] \quad |s=0$$

$$B = \frac{d}{dS}\left\{\frac{1/T}{\left(S+\frac{1}{T}\right)}\right\} \quad |s=0$$

$$B = -\frac{1/T}{\left(S+\frac{1}{T}\right)^2} \quad |s=0$$

$$B = -\frac{1/T}{1/T^2} = -T$$

$$C = (1/2!)\frac{d^2}{dS^2}\left[C(s)\, S^2\right] \quad |s=0$$

$$= (1/2)\frac{d^2}{dS^2}\left\{\frac{1/T}{\left(S+\frac{1}{T}\right)}\right\}$$

$$= (1/2)\frac{d}{dS}\left\{-\frac{1/T}{\left(S+\frac{1}{T}\right)^2}\right\}$$

$$= (1/2)\frac{2*1/T}{\left(S+\frac{1}{T}\right)^3} \quad |s=0$$

$$C = \frac{1/T}{1/T^3} = T^2$$

$$D = \left\{C(s)\left(S+\frac{1}{T}\right)\right\} \quad |s=-1/T$$

$$D = \frac{1/T}{S^3} \quad |s=-1/T$$

$$D = -\frac{1/T}{1/T^3} = -T^2$$

$$C(s) = \frac{1}{S^3} - \frac{T}{S^2} + \frac{T^2}{S} - \frac{T^2}{(S+\frac{1}{T})}$$

$$C(t) = \frac{t^2}{2} - Tt + T^2 - T^2 e^{-\frac{t}{T}}$$

Summary:

For first order system:

No oscillations in response. The response exponentially reaches the final value from initial value.

The fastness of response is inversely proportional to time constant of the system.

For Step and impulse responses steady state error is zero. For ramp and parabola responses steady state error is ∞.

But for impulse response steady state value is zero.

So for first order system step input is preferred.

Problems:

1. A system has the following transfer function G(S)=20/(S+10). Determine its impulse response, Step response & ramp responses with zero initial conditions. Also find time constant of the system.

Sol:

$$G(S) = \frac{C(s)}{R(s)} = \frac{20}{S+10}.$$

$$C(s) = \frac{20}{S+10} R(s).$$

Impulse Response:
R(S) = 1

$$C(s) = \frac{20}{S+10}$$

$$C(t) = 20\, e^{-10t}$$

Step Response:
R(S) = 1/s

$$C(s) = \frac{20}{S(S+10)} = \frac{A}{S} + \frac{B}{S+10} = \frac{2}{S} - \frac{2}{S+10}$$

$$C(t) = 2(1 - e^{-10t})$$

Ramp Response:
R(S) = $1/s^2$

$$C(s) = \frac{20}{S^2(S+10)} = \frac{A}{S^2} + \frac{B}{S} + \frac{C}{S+10} = \frac{2}{S^2} - \frac{0.2}{S} + \frac{0.2}{S+10}$$

$$C(t) = 2t - 0.2(1 - e^{-10t})$$

Time constant:

$G(S) = \frac{20}{S+10}$ is in pole zero form. Convert into time constant form.

$$G(S) = \frac{20}{10\left(\frac{S}{10}+1\right)} = \frac{2}{(0.1S+1)} = \frac{K}{(TS+1)}$$

\Time constant of the system T=0.1 sec

DC gain of the system = 2

2. A system with unity feed back has the following open loop transfer function G(S)=5/S. Determine its impulse response, Step response & ramp responses with zero initial conditions. Also find time constant of the system.

Sol:

$$G(S) = \frac{5}{S} \quad \textbf{\& H(S) =1}$$

\The closed loop transfer function of the system = $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{5}{S+5}$

$$C(s) = \frac{5}{S+5} R(s).$$

Impulse Response:

R(S) = 1

$$C(s) = \frac{5}{S+5}$$

$$C(t) = 5\, e^{-5t}$$

Step Response:

R(S) = 1/s

$$C(s) = \frac{5}{S(S+5)} = \frac{A}{S} + \frac{B}{S+5} = \frac{1}{S} - \frac{1}{S+10}$$

$$C(t) = 1(1 - e^{-5t})$$

Ramp Response:

R(S) = $1/s^2$

$$C(s) = \frac{5}{S^2(S+5)} = \frac{A}{S^2} + \frac{B}{S} + \frac{C}{S+5} = \frac{1}{S^2} - \frac{5}{S} + \frac{5}{S+5}$$

$$C(t) = t - 5(1 - e^{-5t})$$

Time constant:

$G(S) = \frac{5}{S+5}$ is in pole zero form. Convert into time constant form.

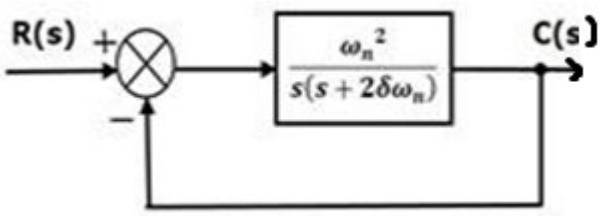
$$G(S) = \frac{5}{5\left(\frac{S}{5}+1\right)} = \frac{1}{(0.2S+1)} = \frac{K}{(TS+1)}$$

\Time constant of the system T=0.2 sec

DC gain of the system = 1

Time Response Of Second Order System

A second order system is a system whose input - output relation is given by a second order differential equation. Consider a standard second order unity feedback system shown in fig.



$$\frac{C(s)}{R(s)} = \frac{G(S)}{1+G(S)H(S)}$$

$$G(S) = \frac{\omega_n^2}{s(s+2\xi\omega_n)} \quad , \quad H(S) = 1$$

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Where ζ=damping ratio or factor

ω_n = Natural frequency of the system

Damping Ratio or Damping Factor :

Every dynamic system takes finite time to reach its steady state and during this period it oscillates or raises or decays exponentially. Every system has a tendency to oppose the oscillatory behavior of the system which is called damping. If the system has no damping, there is no opposition so it oscillates freely. If the system has low damping, oscillations exist but magnitude of these oscillations reduces as time increases. And if it has high damping then no oscillations are there in the response. This damping is measured by a factor which is known as Damping Ratio or Damping Factor.

Natural frequency: When there is no damping in the system it oscillates freely and the frequency of these oscillations is called as Natural frequency of the system.

The time response depends upon the poles of the system. The poles are dependent on the value of ζ. The characteristic equation of second order system is

$$\mathbf{S^2+2\,\zeta\,\omega_n\,S+\omega_n^2=0}$$

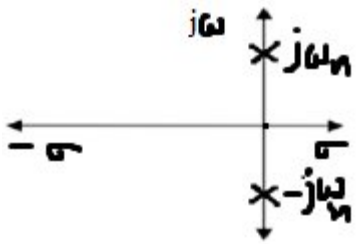
The two roots or poles of the system are

$\mathbf{S_1, S_2 = -\zeta\,\omega_n \pm \omega_n\sqrt{(\zeta^2-1)}}$

depending on ζ value different types of poles exist for the system.

ζ = 0 Undamped Case:

Roots are $S_1, S_2 = \pm j\,\omega_n$ The poles are pure imaginary.



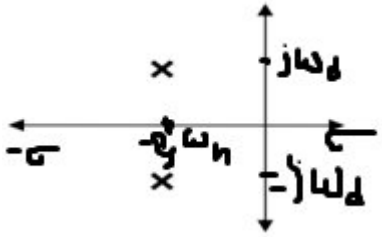
0 <ζ < 1 Underdamped Case:

Roots are $S_1, S_2 = -\zeta\,\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$

$S_1, S_2 = -\zeta\,\omega_n \pm j\omega_d$ Where ω_d is known as damped frequency of oscillations and

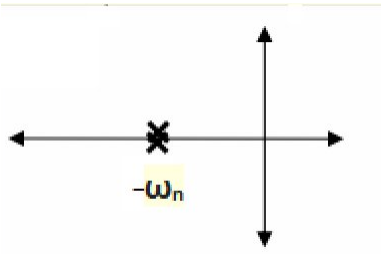
$\omega_d = \omega_n\sqrt{(1-\zeta^2)}$

The poles are complex conjugate.



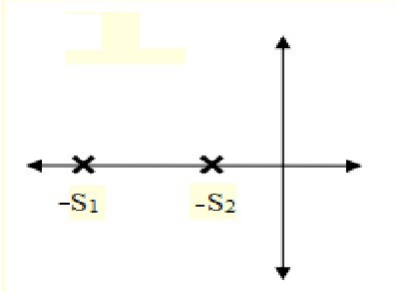
ζ = 1 Critically damped Case:

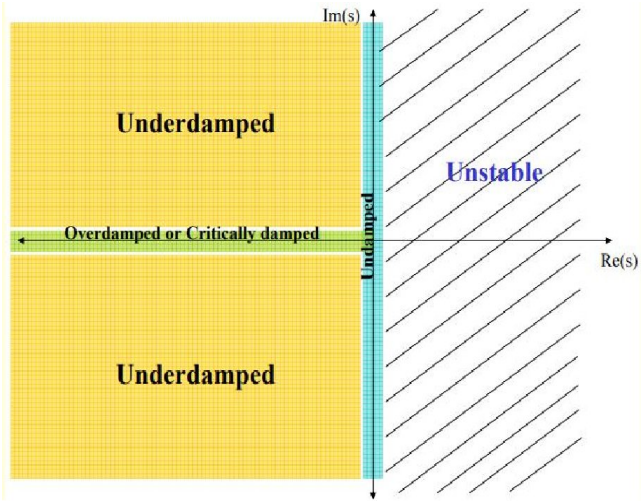
Roots are $S_1, S_2 = -\omega_n, -\omega_n$. The poles are real and equal.



ζ > 1 Over damped Case:

Roots are $S_1, S_2 = -\zeta\,\omega_n \pm \omega_n\sqrt{(\zeta^2-1)}$. The term under square root is positive so the poles are real and unequal.





Location of poles and the corresponding response are shown in above figure.

Step Response Of Second Order System

The standard transfer function of the 2nd order system is

$$\frac{C(S)}{R(S)} = G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

But R(s) = 1/s for step input

$$C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} R(S) = \frac{\omega_n^2}{S(S^2 + 2\zeta\omega_n S + \omega_n^2)}$$

Depending on ζ value different types of responses exist.

Undamped Response ($\zeta = 0$) :

Roots are $S_1, S_2 = \pm j\,\omega_n$ The poles are pure imaginary.

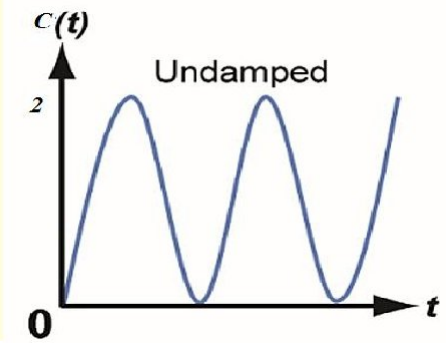
$$C(S) = \frac{\omega_n^2}{S(S^2 + 2\zeta\omega_n S + \omega_n^2)} = \frac{\omega_n^2}{S(S^2 + \omega_n^2)} = \frac{A}{S} + \frac{BS + C}{S^2 + \omega_n^2}$$

$$C(S) = \frac{1}{S} + \frac{-S + 0}{S^2 + \omega_n^2} = \frac{1}{S} - \frac{S}{S^2 + \omega_n^2}$$

A = 1, B = -1, C = 0

By applying inverse laplace

$C(t) = 1 - \cos(\omega_n t)$



The waveform of response as function of time is as shown below.

The undamped response oscillates with constant frequency(ω_n) and amplitude.

Under Damped Response($0 < \zeta < 1$) :

Poles are $S_1, S_2 = -\zeta\,\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$

$S_1, S_2 = -\zeta\,\omega_n \pm j\omega_d$ Where ω_d is known as damped frequency of oscillations and

$\omega_d = \omega_n\sqrt{(1-\zeta^2)}$

$$C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} R(S) = \frac{\omega_n^2}{S(S^2 + 2\zeta\omega_n S + \omega_n^2)} = \frac{A}{S} + \frac{BS + C}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

$$C(S) = \frac{1}{S} + \frac{-S - 2\zeta\omega_n}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

A = 1, B = -1, C = - 2ζ ω_n

$$C(S) = \frac{1}{S} - \frac{S + 2\zeta\omega_n}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

Rearrangement of terms is required to apply inverse laplace.

$$C(S) = \frac{1}{S} - \frac{S + 2\zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(S) = \frac{1}{S} - \frac{S + \zeta\omega_n + \zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2} = \frac{1}{S} - \frac{S + \zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(S) = \frac{1}{S} - \frac{S + \zeta\omega_n}{(S + \zeta\omega_n)^2 + \omega_d^2} - \frac{\left(\frac{\zeta\omega_n}{\omega_d}\right)\omega_d}{(S + \zeta\omega_n)^2 + \omega_d^2}$$

By applying inverse laplace

$$C(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \left(\frac{\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$C(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \left(\frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$C(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\{\sqrt{1-\zeta^2}\cos(\omega_d t) - \zeta\sin(\omega_d t)\}$$

Let Assume $\zeta = \cos\Theta$ *so* $\sqrt{1-\zeta^2} = \sin\Theta$

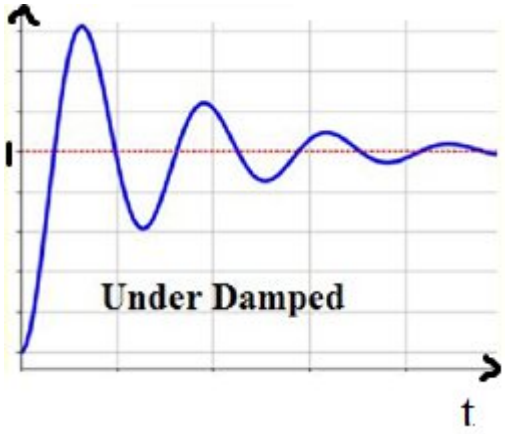
$Tan\,\Theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$ à $\Theta = Tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\{\sin\Theta\cos(\omega_d t) - \cos\Theta\sin(\omega_d t)\}$$

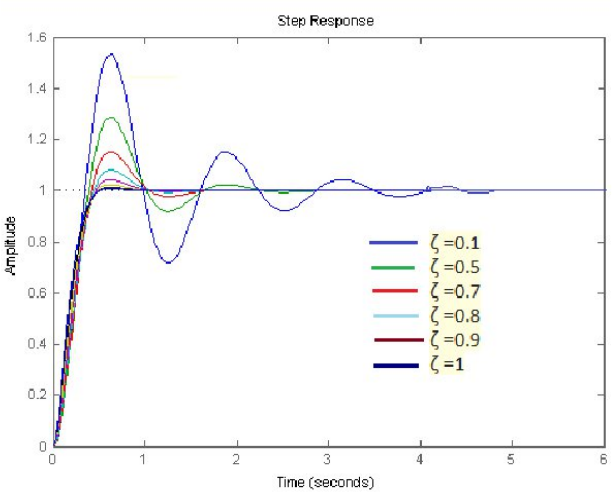
\The under damper response of second order system is

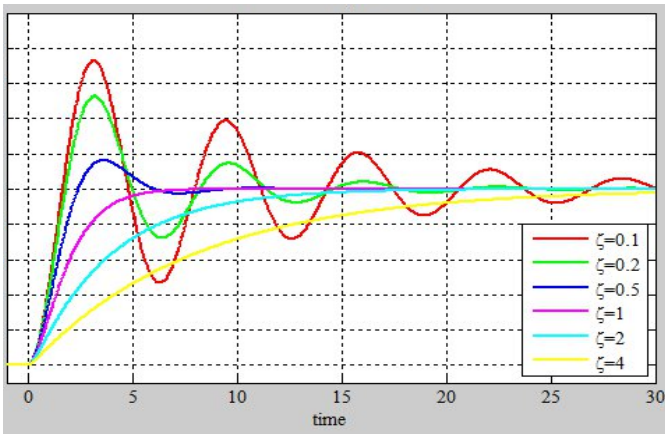
$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \Theta)$$

Where $\omega_d = \omega_n \sqrt{(1-\zeta^2)}$ and $\Theta = Tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$



As ζ increases from 0 amplitude of oscillations reduces as shown in below fig. And at ζ = 1 there are no oscillations at all.





Critically Damped Response(ζ = 1) :

Poles are real & equal. $S_1, S_2 = -\zeta\omega_n, -\zeta\omega_n$

$$C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n + \omega_n^2} R(S) = \frac{\omega_n^2}{S(S^2 + 2\omega_n + \omega_n^2)} = \frac{\omega_n^2}{S(S+\omega_n)^2}$$

$$C(S) = \frac{A}{S} + \frac{B}{S+\omega_n} + \frac{C}{(S+\omega_n)^2}$$

A= 1, B=-1, C= - ω_n

$$C(S) = \frac{1}{S} - \frac{1}{S+\omega_n} - \frac{\omega_n}{(S+\omega_n)^2}$$

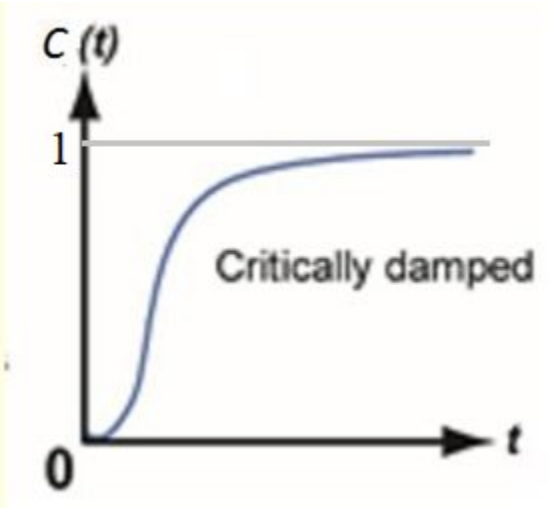
By applying inverse laplace

$$C(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$

So the critically damped response of 2nd order system is

$$C(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

The critically damped response is shown in below figure. As there are no sin or cos terms in expressions no oscillations exist in response.



Over Damped Response (ζ > 1) :

Poles are real & unequal. $S_1, S_2 = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$

$$C(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n + \omega_n^2} R(S) = \frac{\omega_n^2}{S(S-S_1)(S-S_2)}$$

$$\therefore C(S) = \frac{A}{S} + \frac{B}{(S-S_1)} + \frac{C}{(S-S_2)}$$

$$A = \frac{\omega_n^2}{S_1 S_2} \qquad B = \frac{\omega_n^2}{S_1(S_1 - S_2)} \qquad C = \frac{\omega_n^2}{S_2(S_2 - S_1)} = -\frac{\omega_n^2}{S_2(S_1 - S_2)}$$

But $S_1 S_2 = \omega_n^2$ & $S_1 - S_2 = 2\omega_n\sqrt{\zeta^2 - 1}$

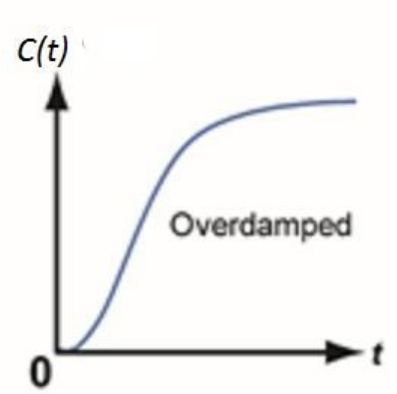
$$\therefore A = 1 \qquad B = \frac{\omega_n}{2S_1\sqrt{\zeta^2 - 1}} \qquad C = -\frac{\omega_n}{2S_2\sqrt{\zeta^2 - 1}}$$

By applying inverse laplace

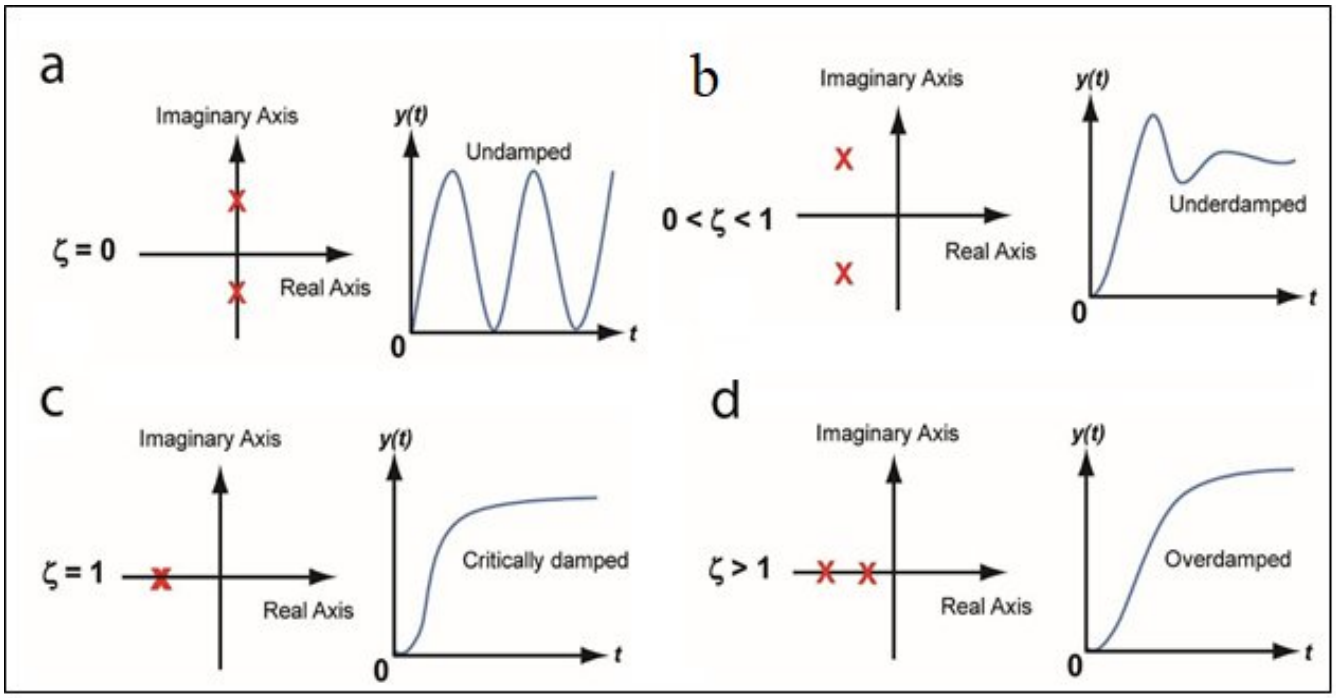
$$C(t) = 1 + \frac{\omega_n}{2S_1\sqrt{\zeta^2 - 1}} e^{S_1 t} - \frac{\omega_n}{2S_2\sqrt{\zeta^2 - 1}} e^{S_2 t}$$

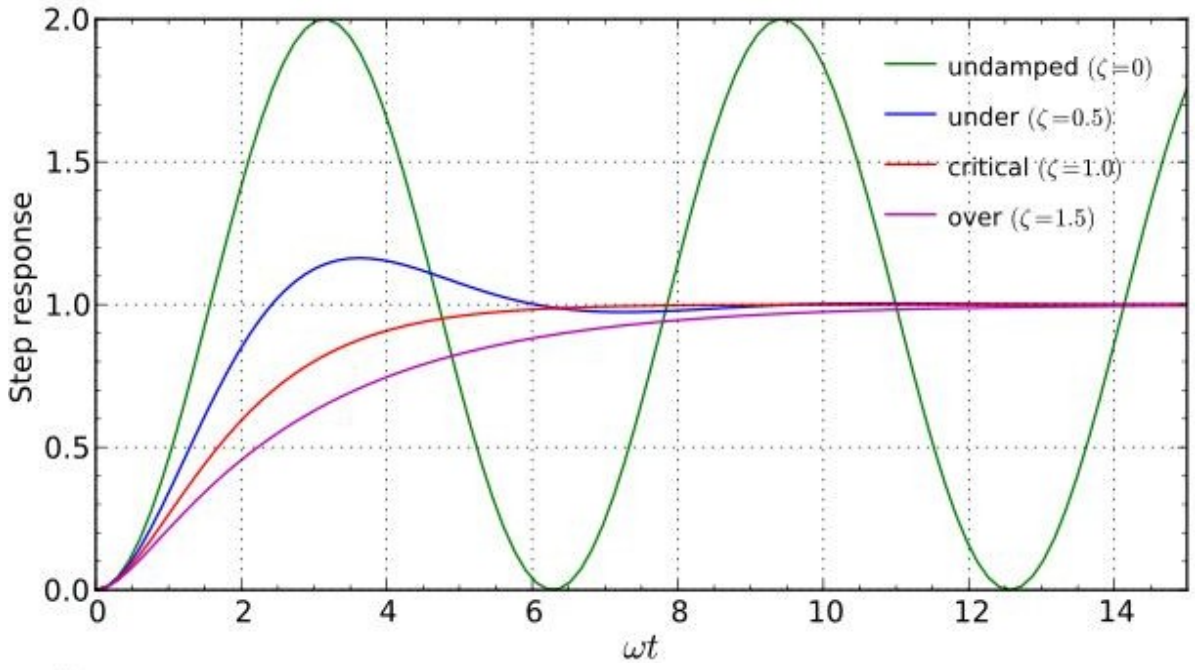
$$C(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[\frac{e^{S_1 t}}{S_1} - \frac{e^{S_2 t}}{S_2}\right]$$

The Over damped response of 2nd order system is as shown in the below figure.



Summary of Time response of second order system:





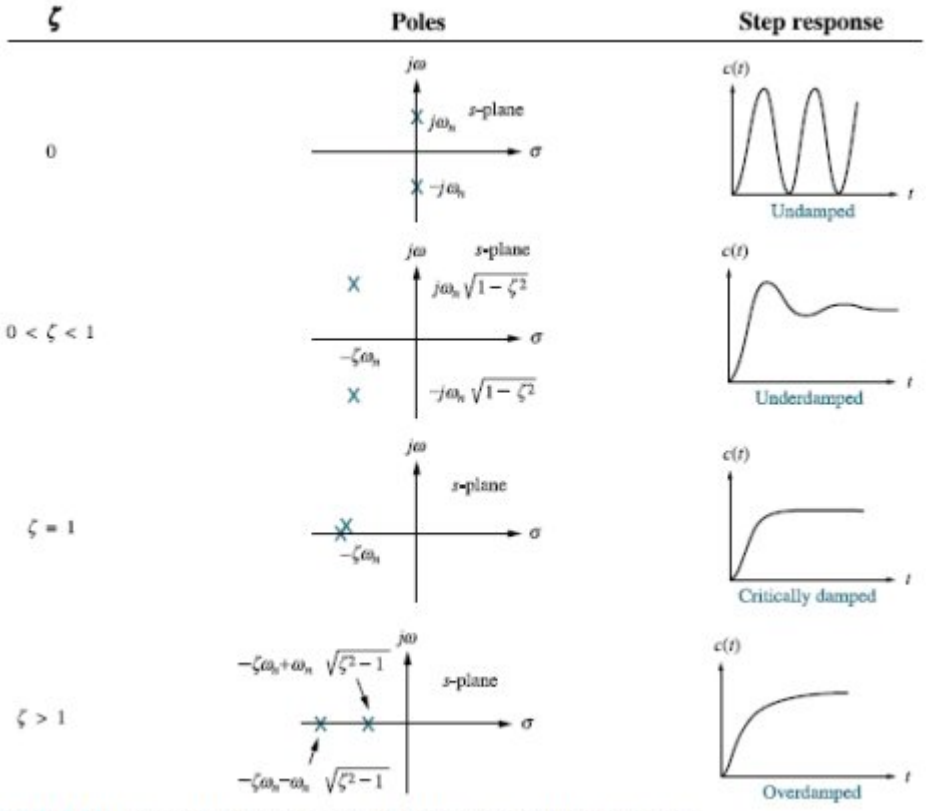
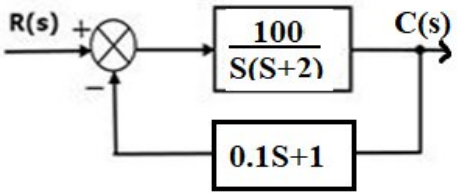


FIGURE 4.11 Second-order response as a function of damping ratio

Problems:

1. The positional control system with velocity feedback is shown in figure. Find the response of the system for unit step input.



Sol:

$$G(S) = \frac{100}{S(S+2)}$$

H(S) = 0.1 S + 1

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{100}{S^2+2S+100(0.1S+1)} = \frac{100}{S^2+2S+10S+100}$$

$$TF = \frac{C(S)}{R(S)} = \frac{100}{S^2+12S+100}$$

Characteristic equation of the system is

$S^2 + 12S + 100 = 0$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = 100 \rightarrow \quad \omega_n = 10\,\frac{rad}{sec}$$

$$2\zeta\omega_n = 12 \quad \rightarrow \quad \zeta = 0.6$$

Roots of the characteristic equation are S= -6 ± j8

Since ζ= 0.6 it is a under damped case.

So underdamped response of a second order system is

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\, sin\left(\omega_d t + \Theta\right)$$

Where $\Theta = Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 53.13^\circ$ and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 8$ rad/sec

$$C(t) = 1 - 1.25\,e^{-6t}\, sin(8t + 53.13)$$

Check the numerator of the transfer function. If it is $\omega_n{}^2$ above is the response. If numerator is other than $\omega_n{}^2$ multiply and divide the transfer function with $\omega_n{}^2$.

2. The open loop transfer function of a UFB system is 20/(S+1)(S+4). Find closed loop transfer function, damping factor , natural frequency and step response.

   Sol:

$$G(S) = \frac{20}{(S+1)(S+4)} = \frac{20}{S^2+5S+4}$$

H(S) = 1(since UFB)

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{20}{S^2+5S+4+20}$$

$$TF = \frac{C(S)}{R(S)} = \frac{20}{S^2 + 5S + 24} = \frac{20}{24}\left[\frac{24}{S^2 + 5S + 24}\right]$$

Characteristic equation of the system is

$S^2 + 5S + 24 = 0$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = 24 \rightarrow \quad \omega_n = 4.89 \frac{rad}{sec}$$

$$2\zeta\omega_n = 5 \quad \rightarrow \quad \zeta = 0.51$$

Roots of the characteristic equation are S= -2.5 ± j 4.21

Since ζ= 0.51 it is an under damped case.

So underdamped response of a second order system is

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \Theta\right)$$

Where $\Theta = Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 59.26^\circ$ and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.21$ rad/sec

$$C(t) = \frac{20}{24}\{1 - 1.11\, e^{-2.5t} \sin(4.21t + 59.26)$$

3. An unit step input is applied to a UFB system for which open loop transfer function is 16/S(S+8). Find transfer function of the system, ωn, ζ, ωd and response of the system.

Sol:

$$G(S) = \frac{16}{S(S+8)} = \frac{16}{S^2 + 8S}$$

H(S) = 1(since UFB)

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{16}{S^2 + 8S + 16}$$

$$TF = \frac{C(S)}{R(S)} = \frac{16}{S^2 + 8S + 16}$$

Characteristic equation of the system is

$S^2 + 8S + 16 = 0$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = 16 \rightarrow \quad \omega_n = 4 \frac{rad}{sec}$$

$$2\zeta\omega_n = 8 \quad \rightarrow \quad \zeta = 1$$

Since ζ= 1 it is critically damped case. And $\omega_d = 0\ rad/sec$

Roots of the characteristic equation are S= ± 4

$$C(S) = \frac{16}{S^2 + 8S + 16} R(S) = \frac{16}{S(S^2 + 8S + 16)} = \frac{16}{S(S+4)^2}$$

$$C(S) = \frac{A}{S} + \frac{B}{S+4} + \frac{C}{(S+4)^2} = \frac{1}{S} + \frac{-1}{S+4} + \frac{-4}{(S+4)^2}$$

$C(t) = 1- e^{-4t} - 4t e^{-4t} = 1- e^{-4t}(1+4t)$

if you remember ths critically damped response expression

$$C(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right) = 1 - e^{-4t}\left(1 + 4t\right)$$

4. A system is represented by a differential equation $\frac{d^2y}{dt^2} + 7\frac{dy}{dx} + 12y = 12x$ where y is output and x is input. Find transfer function and unit step response.

Sol:

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dx} + 12y = 12x$$

$S^2\,Y(S) + 7\,S\,Y(S) + 12\,Y(S) = 12\,X(S)$

$(S^2 + 7\,S + 12)\,Y(S) = 12\,X(S)$

$$TF = \frac{Y(S)}{X(S)} = \frac{12}{S^2 + 7S + 12}$$

Characteristic equation of the system is

$S^2 + 7S + 12 = 0$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = 12 \rightarrow \quad \omega_n = 3.46\frac{rad}{sec}$$

$$2\zeta\omega_n = 7 \quad \rightarrow \quad \zeta = 1.011$$

Since $\zeta = 1.011$ it is over damped case.

Roots of the characteristic equation are S= -3,-4

$$C(S) = \frac{12}{S^2 + 7S + 12}R(S) = \frac{12}{S(S^2 + 7S + 12)} = \frac{16}{S(S+3)(S+4)}$$

$$C(S) = \frac{A}{S} + \frac{B}{S+3} + \frac{C}{S+4} = \frac{1}{S} - \frac{4}{S+3} + \frac{3}{S+4}$$

$$C(t) = 1- 4\, e^{-3t} + 3\, e^{-4t}$$

5. A system is represented by a standard function $12/(s^2+16)$. Find ω_n, ζ, ω_d and step response of the system.

Sol:

$$TF = \frac{C(S)}{R(S)} = \frac{12}{S^2 + 16}$$

$$C(S) = \frac{12}{S^2 + 16}R(S) = \frac{12}{S(S^2 + 16)} = \frac{A}{S} + \frac{BS + C}{S^2 + 16} = \frac{0.75}{S} - \frac{0.75S}{S^2 + 16}$$

$$C(t) = 0.75- 0.75 \cos (4t)$$

Time Domain Specifications

The performance of a system in time domain is usually evaluated in terms of the following qualities.

- How fast it is able to respond to the input.
- How fast it is reaching the desired output,
- How fast it is reaching the steady state,
- What is the error between the desired output and the actual output, once the transients die down and steady state is achieved,
- Does it oscillate around the desired value, and
- Is the output continuously increasing with time or is it bounded.

To get answers of above questions time domain specifications are used.

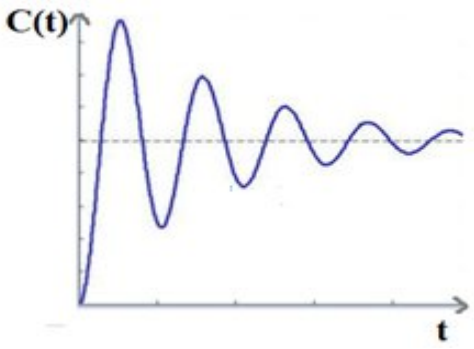
Time domain specifications are those specifications in terms of which the characteristics of the desired performance is specified.

From transient response 1.speed of response (t_d,t_r) 2. Oscillating Nature of response(t_p,μ_p) 3. Time to reach SS (t_s).

From steady state response 1. Accuracy of the response(e_{ss})

Definitions:

1. **Delay Time (t_d):** It is the time required for the response to reach 50% of the final value for the first time.
2. **Rise Time (t_r):** It is the time required for the response to reach 100% of the final value in first attempt for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value.
3. **Peak time (t_p):** It is the time required for the response to reach the maximum or Peak value of the response.
4. **Peak overshoot (μ_p)** : It is defined as the normalized difference between the peak value of the response and the final value.



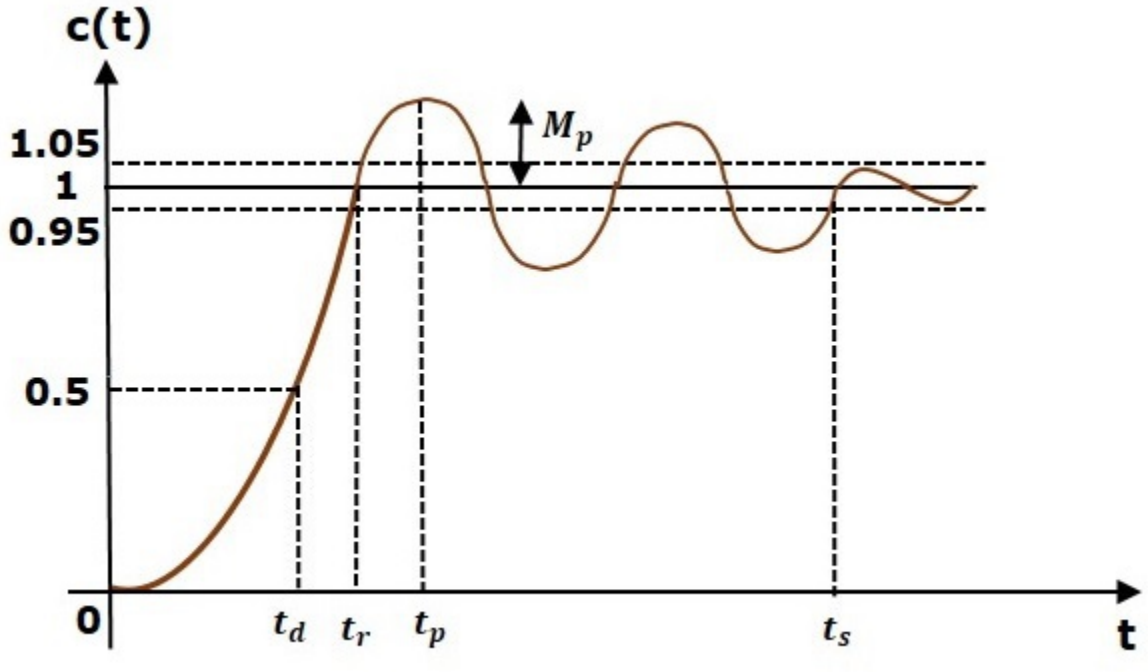
$$\mu_p = \frac{C(t_p) - C(\infty)}{C(\infty)}$$

$$\%\ \mu_p = \frac{C(t_p) - C(\infty)}{C(\infty)} * 100$$

5. **Settling time(t_s)** : It is the time required for the response to reach and remain within a specified tolerance limits (usually ± 2% to ± 5%) of it's final value.

6. **Steady State Error(e_{ss})**: It is the difference between the desired output and actual output as t -> ∞.

$$e_{SS} = \lim_{t \to \infty} e(t)$$



Expressions for the time domain specifications:

The underdamped step response of a second order system is given by

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \Theta)$$

Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\Theta = Tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})$

Delay Time: Empirical formula

$$t_d = \frac{1+0.7\zeta}{\omega_n}$$

Rise Time:

At $t = t_r$ $C(t) = 1$

$$1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \Theta)$$

$$0 = \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \Theta)$$

$$e^{-\zeta\omega_n t_r}\sin(\omega_d t_r + \Theta) = 0$$

$e^{-\zeta\omega_n t_r}$ is not zero for finite value of time. Therefore

$$\sin(\omega_d t_r + \Theta) = 0$$

$$\omega_d t_r + \Theta = 0,,2,3\ ....$$

But rise time means in first attempt. $\omega_d t_r + \Theta =$

$$\boldsymbol{t_r = \frac{-\Theta}{\omega_d}}$$ Θ should be in radians

Peak Time:

At $t = t_p$ $\frac{dC(t)}{dt} = 0$

$$\frac{dC(t)}{dt} = -\left(-\zeta\omega_n\right)\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \Theta) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \Theta)\,\omega_d = 0$$

$$(\zeta\omega_n)\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \Theta) - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\cos(\omega_d t + \Theta) = 0$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \Theta) - \cos(\omega_d t + \Theta) = 0$$

$$\zeta\sin(\omega_d t + \Theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \Theta) = 0$$

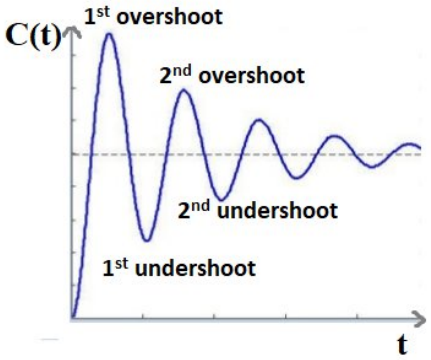
But ζ= cos Θ and $\sqrt{1-\zeta^2} = \sin\Theta$

$$\cos\Theta\sin(\omega_d t + \Theta) - \sin\Theta\cos(\omega_d t + \Theta) = 0$$

$$\sin(\omega_d t + \Theta - \Theta) = 0$$

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = 0,,2,3\ ....$$



But peak time means for first overshoot. $\omega_d t_p =$

$$\boldsymbol{t_p = \frac{}{\omega_d}}\ \text{Sec}$$

Peak Overshoot:

$$\mu_p = \frac{C(t_p) - C(\infty)}{C(\infty)}$$

$C(\infty) = C(t) \ |at\ t = \infty = 1$

$C(t_p) = C(t)\ (at\ t = t_p) \quad = P/\mu g_d$

$$C(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \Theta) = 1 - \frac{e^{-\zeta\omega_n \overline{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{}{\omega_d} + \Theta\right)$$

$$C(t_p) = 1 - \frac{e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(+\Theta)$$

$$C(t_p) = 1 + \frac{e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\Theta \quad (\text{but } \sqrt{1-\zeta^2} = \sin\Theta)$$

$$C(t_p) = 1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$\mu_p = \frac{1 + e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}} - 1}{1}$$

$$\mu_p = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$\%\mu_p = 100 e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

t_d, t_r, t_p depends on both ζ & μg_n. whereas μ_p depends only on ζ.

Settling Time:

As time increases Oscillations in the response will die out. That means no importance of sin or cos terms only dominant term is exponential term so sin terms can be neglected.

\At $t = t_s$ $\quad C(t) \approx 1 - e^{-\zeta\omega_n t_s}$

Tolerance ± 2 % ==> C(t) value should be between 1-0.02 to 1+ 0.02.

$0.98 \approx 1 - e^{-\zeta\omega_n t_s}$

$$e^{-\zeta\omega_n t_s} = 0.02 \quad (error)$$

$$-\zeta\omega_n t_s = \ln(0.02)$$

$$t_s = -\frac{\ln(0.02)}{\zeta\omega_n}$$

$$t_s = \frac{4}{\zeta\omega_n} = 4T$$

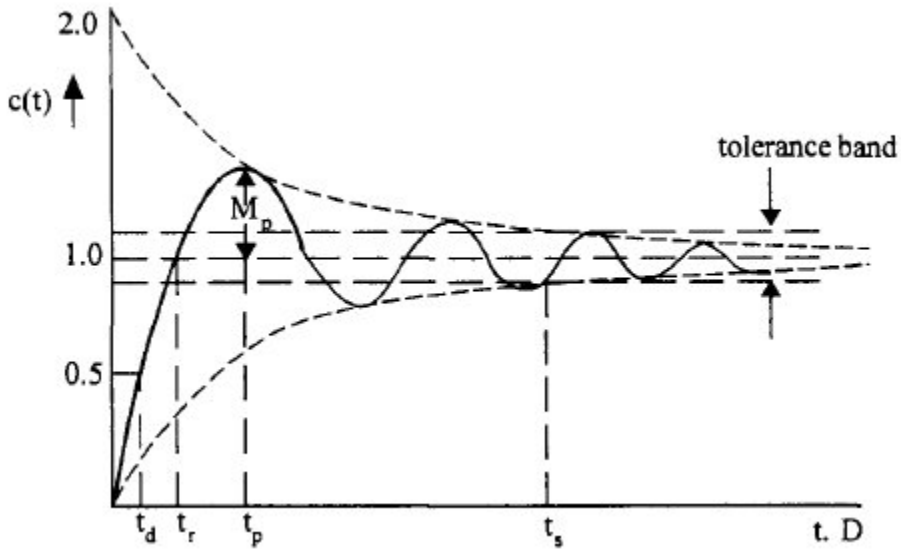
Time constant of second order system is inversely proportional to real part of the pole. T=1/(ζμgn)
For any other %error

$$t_s = -\frac{\ln(error)}{\zeta\omega_n}$$

- Both the settling time t_s and the time constant T are inversely proportional to the damping ratio ζ.
- Both the settling time t_s and the time constant T are independent of the system gain. That means even the system gain changes, the settling time t_s and time constant T will never change.

The following table shows the formulae of time domain specifications, substitution of necessary values and the final values.

Time domain specification	Formula
Delay time	$t_d=(1+0.7\zeta)/\omega n$
Rise time	$t_r = (\pi-\theta)/\omega_d$
Peak time	$t_p = \pi/\omega_d$
% Peak overshoot	$\%\mu_p = 100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$
Settling time	$t_s = \frac{4}{\zeta\omega_n}$ (for %2 error) $t_s = \frac{3}{\zeta\omega_n}$ (for %5 error)


2.0
c(t)
tolerance band
M_p
1.0
0.5
t_d
t_r
t_p
t_s
t. D

Problems:

1. A second order system is given by 25/(s2+6s+25). Find t_r, t_d, t_p, μ_p & t_s. Also calculate expression for its output if subjected to unit step input.

Sol:

$$\frac{C(S)}{R(S)} = \frac{25}{S^2 + 6S + 25}$$

Characteristic equation of the system is

$$S^2 + 6S + 25 = 0$$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = 25 \rightarrow \quad \omega_n = 5\frac{rad}{sec}$$

$$2\zeta\omega_n = 6 \quad \rightarrow \quad \zeta = 0.6$$

$\Theta = Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ = 0.927 rad and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ = 4 rad/sec

Delay Time $t_d = (1+0.7\zeta)/\omega n$ = 0.284 Sec

Rise Time $t_r = (\pi - \theta)/\omega_d$ = 0.553 Sec

Peak Time $t_p = \pi/\omega_d$ = 0.785 Sec

Peak Overshoot

$$\%\mu_p = 100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}} = 9.748\%$$

Settling Time $t_s = 4/\zeta\omega_n$ = 1.33 Sec

Roots of the characteristic equation are S= -3 ± j 4

Since ζ= 0.51 it is an under damped case.

So underdamped response of a second order system is

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \phi\right)$$

$$C(t) = 1 - 1.25\,e^{-3t}\sin(4t + 0.927rad)$$

2. A system is represented by a differential equation $2\frac{d^2y}{dt^2}+8\frac{dy}{dx}+16y=16x$ where y is output and x is input. Determine all time domain specifications for unit step response..

Sol:

$$2\frac{d^2y}{dt^2}+8\frac{dy}{dx}+16y=16x$$

$2S^2\,Y(S) + 8\,S\,Y(S) + 16\,Y(S) = 16\,X(S)$

$(2S^2 + 8\,S + 16)\,Y(S) = 16\,X(S)$

$$TF=\frac{Y(S)}{X(S)}=\frac{16}{2S^2+8S+16}$$

Characteristic equation of the system is

$S^2+4S+8=0$

Compare with standard second order system characteristic equation

$$S^2+2\zeta\omega_n S+\omega_n^2=0$$

$$\omega_n^2=8\ \rightarrow\ \omega_n=2.8\ \ {}^{rad}/_{sec}$$

$$2\zeta\omega_n=4\ \ \rightarrow\ \ \zeta=0.707$$

$\Theta = Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ = 0.785 rad and $\omega_d=\omega_n\sqrt{1-\zeta^2}$ = 1.994 rad/sec

Delay Time $t_d=(1+0.7\zeta)/\omega n$ = 0.53 Sec

Rise Time $t_r=(\pi-\theta)/\omega_d$ = 1.18 Sec

Peak Time $t_p=\pi/\omega_d$ = 1.57 Sec

Peak Overshoot

$$\%\mu_p=100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}=4.32\ \%$$

Settling Time $t_s=4/\zeta\omega_n$ = 1.806 Sec

3. The open loop transfer function of a UFB system is 24/(S+3)(S+4). Find t_r, t_d, t_p, μ_p & t_s. Also find the time at which 1st undershoot occurs, the time period of Oscillations and number of cycles completed before reaching steady state.

Sol:

$$G(S) = \frac{24}{(S+3)(S+4)} = \frac{24}{S^2+7S+12}$$

H(S) = 1(since UFB)

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{24}{S^2+7S+36}$$

$$TF = \frac{C(S)}{R(S)} = \frac{24}{S^2+7S+36}$$

Characteristic equation of the system is

$$S^2+7S+36=0$$

Compare with standard second order system characteristic equation

$$S^2+2\zeta\omega_n S+\omega_n^2=0$$

$$\omega_n^2 = 36 \rightarrow \quad \omega_n = 6 \quad {}^{rad}/_{sec}$$

$$2\zeta\omega_n = 7 \quad \rightarrow \quad \zeta = 0.58$$

$\Theta = Tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ = 0.94 rad=54.549◦ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ = 4.88 rad/sec

Delay Time $t_d=(1+0.7\zeta)/\omega n$ = 0.2343 Sec

Rise Time $t_r = (\pi-\theta)/\omega_d$ = 0.45 Sec

Peak Time $t_p = \pi/\omega_d$ = 0.64 Sec

Peak Overshoot

$$\%\mu_p = 100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}} = 10.68\ \%$$

Settling Time $t_s = 4/\zeta\omega_n = 1.149$ Sec

The time at which 1st undershoot occurs is given by $t = 2 * \pi/\omega_d = 1.28$ sec

Time period of oscillations:

Frequency of oscillations $= \omega_d = 2Pf = 2P/T$

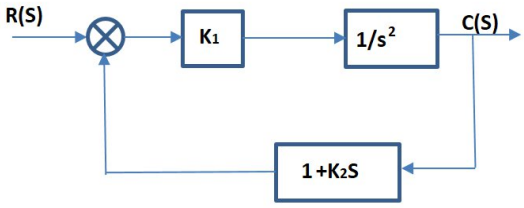
\Time period of oscillation $T = 2P/\omega_d = 1.28$ sec

number of cycles completed before reaching steady state:

Time period of oscillation is 1.28 sec. so time taken to complete one cycle is 1.28 sec.

Number of cycles completed before reaching SS(1.149sec) = (1/1.28)* 1.149 = 0.897 cycles

4. For a control system shown in figure find the values of K1 & K2 so that μp 25% & tp = 4 sec. Assume unit step input.



Sol: $G(S) = \frac{K_1}{S^2}$ *and* $H(S) = 1 + K_2S$

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1 + G(S)H(S)} = \frac{K_1}{S^2 + K_1 + K_1K_2S}$$

$$TF = \frac{K_1}{S^2 + K_1K_2S + K_1}$$

Characteristic equation of the system is

$$S^2 + K_1K_2S + K_1 = 0$$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_nS + \omega_n^2 = 0$$

$$\omega_n^2 = K_1 \ldots.(1)$$

$$2\zeta\omega_n = K_1K_2 \text{ .....(2)}$$

Given Peak Overshoot

$$\%\mu_p = 25\ \% = 100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$0.25 = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$\frac{-\zeta}{\sqrt{1-\zeta^2}} = \ln(0.25) = -1.386$$

$$\zeta = 1.386\sqrt{1-\zeta^2}$$

$$(\zeta)^2 = 1.92099(1-\zeta^2)$$

\ζ= 0.403

Given Peak Time $t_p = \pi/\omega_d = 4$ Sec

$\omega_d = 0.78$ è $\omega_n = 0.85$ rad/sec

from eqn 1

$$K_1 = \omega_n^2 = 0.7225$$

from eqn 2

$$2\zeta\omega_n = K_1K_2 \text{ è } K_2 = 2\zeta\omega_n/K_1 = 0.948$$

5. The open loop transfer function of a UFB system is K/S(1+ST) where K & T are constants. Determine the factor by which gain K should be multiplied so that overshoot of the unit step response is reduced from 75 % to 25%.

Sol: $G(S) = \frac{K}{S(1+ST)}$ *and* $H(S) = 1$ (UFB)

\closed loop transfer function

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{K}{TS^2+S+K}$$

$$TF = \frac{K/T}{S^2+\frac{S}{T}+\frac{K}{T}}$$

Characteristic equation of the system is

$$S^2+\frac{S}{T}+\frac{K}{T} = 0$$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = K/T \;\rightarrow\; \omega_n = \sqrt{K/T}$$

$$2\zeta\omega_n = \frac{1}{T} \;\rightarrow\; \zeta = \frac{1}{2\sqrt{KT}} \quad ...(1)$$

Let assume when $\%\mu_p = 75\%$ $\zeta = \zeta_1$ & $K = K_1$ and

when $\%\mu_p = 25\%$ $\zeta = \zeta_2$ & $K = K_2$

$0.75 = e^{\frac{-\zeta_1}{\sqrt{1-\zeta_1^2}}}$ à $\zeta_1 = 0.0911$

$0.25 = e^{\frac{-\zeta_2}{\sqrt{1-\zeta_2^2}}}$ à $\zeta_2 = 0.403$

From eqn 1 $\zeta_1 = \frac{1}{2\sqrt{K_1 T}}$ and $\zeta_2 = \frac{1}{2\sqrt{K_2 T}}$

$$\frac{\zeta_1}{\zeta_2} = \frac{2\sqrt{K_2 T}}{2\sqrt{K_1 T}} = \sqrt{\frac{K_2}{K_1}} = \frac{0.0911}{0.403} = 0.226$$

$K_2/K_1 = 0.051$ -> $K_2 = 0.051\, K_1$

To reduce the peak overshoot from 75% to 25%, K should be multiplied by a factor 0.051.

6. A second order system is represented by a transfer function $1/(Js_2 + BS + K)$. A step input of 10 NM is applied to the system and the results are maximum overshoot 6% , time for peak overshoot is 1 sec and steady state value of output is 0.5 rad. Determine the values of J,B &K.

Sol:

$$TF = \frac{C(S)}{R(S)} = \frac{1}{JS^2 + BS + K}$$

Characteristic equation of the system is

$$S^2 + \frac{B}{J}S + \frac{K}{J} = 0$$

Compare with standard second order system characteristic equation

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

$$\omega_n^2 = K/J \;\rightarrow\; \omega_n = \sqrt{K/J} ...(1)$$

$$2\zeta\omega_n = \frac{B}{J} \;\rightarrow\; \zeta = \frac{B}{2\sqrt{KJ}} \quad ...(2)$$

Given Peak Overshoot

$$\%\mu_p = 6\ \% = 100e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$0.06 = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}}}$$

$$\frac{-\zeta}{\sqrt{1-\zeta^2}} = \ln(0.06) = -1.386$$

$$\zeta = 0.66$$

Given Peak Time $t_p = \pi/\omega_d = 1$ Sec

$\omega_d = 3.14$ è $\omega_n = 4.18$ rad/sec

given steady state value of output is $\mathscr{L}t_{t\to\infty} C(t) = \mathscr{L}t_{S\to 0} SC(S) = 0.5$

$$C(S) = \frac{R(S)}{JS^2 + BS + K} \quad and\ R(S) = 10/s$$

$$C(S) = \frac{10}{S(JS^2 + BS + K)}$$

$$\mathscr{L}t_{S\to 0} SC(S) = \mathscr{L}t_{S\to 0} S\frac{10}{S(JS^2 + BS + K)} = \frac{10}{K} = 0.5$$

\K = 20

from eqn 1

$\omega_n^2 = \frac{K}{J}$ -> J=K/$\omega_n^2 = 1.14$

from eqn 2

$2\zeta\omega_n = \frac{B}{J}$ è B = 2ζμg_nJ = 6.3

7. For a control system G(S) = K/S_2 & H(s) = As+B.
   (i) For K=0 determine the value of A & B for which μgn =20 & ζ=0.5.
   (ii) For the above values of A & B if K=40 determine μgn & ζ.

Sol:by varying the value of gain K we can vary the type of response.

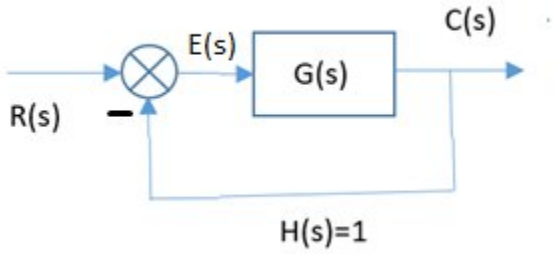
Steady State Error

All previously discussed Time domain specifications are dependent on only system. But Steady state error depends on both system & input.

Steady State Error(e_{ss}): It is the difference between the desired output and actual output as t -> ∞.

$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} SE(S)$$

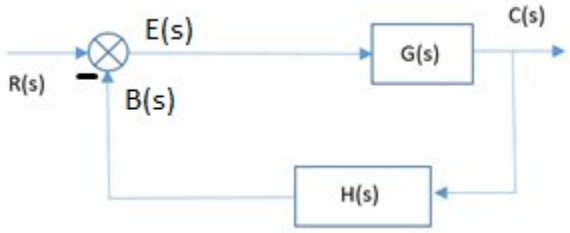
For unity feedback systems



$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} [r(t) - c(t)]$$

$$e_{SS} = \lim_{s\to 0} SE(S) = \lim_{s\to 0} [R(s) - C(s)]$$

For non-unity feedback systems



E(s)=R(s)-B(s)= R(s) – H(s) C(s)

= R(s) – H(s) G(s) E(s)

E(s) + H(s) G(s) E(s) = R(s)

(1 + H(s) G(s)) E(s) = R(s)

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{S\to 0} SE(S) = \lim_{S\to 0} \frac{SR(s)}{1 + G(s)H(s)}$$

Steady state error depends on both input (R(s)) and system(G(s)H(s)).

Effect of input on steady state Error (or) Static Error Coefficients

Static error coefficients are those constants in terms of which steady state error is specified when standard inputs(step, ramp, parabola) are applied to the system.

Step Input & Positional Error Constant:

R(t) = A u(t) =A t>0

=0 t<= 0

R(s) = A/s

$$e_{ss} = \lim_{S\to 0} \frac{SR(s)}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{S*A/S}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{A}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{A}{\lim_{S\to 0}[1 + G(s)H(s)]} = \frac{A}{1 + \lim_{S\to 0}[G(s)H(s)]}$$

For a given system $\lim_{S\to 0}[G(s)H(s)$ *is a constant and is known as Positional Error Constant.*

It is represented by Kp.

$$K_P = \lim_{S\to 0} G(s)H(s)$$
$$e_{ss} = \frac{A}{1+K_P} \text{ for step input}$$

Ramp Input & Velocity Error Constant:

R(t) = A r(t) =At t>0
=0 t<= 0

$R(s) = A/s^2$

$$e_{ss} = \lim_{S\to 0} \frac{SR(s)}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{S*A/S^2}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{A}{S(1 + G(s)H(s))}$$

$$e_{ss} = \frac{A}{\lim_{S\to 0}[S + SG(s)H(s)]} = \frac{A}{\lim_{S\to 0} SG(s)H(s)}$$

For a given system $\lim_{S\to 0}[sG(s)H(s)]$ *is a constant and is known as Velocity Error Constant.*

It is represented by Kv.

$$K_V = \lim_{S\to 0} sG(s)H(s)$$
$$e_{ss} = \frac{A}{K_V} \text{ for ramp input}$$

Parabola Input & Acceleration Error Constant:

$R(t) = At^2/2$ t>0
=0 t<= 0

$R(s) = A/s^3$

$$e_{ss} = \lim_{S\to 0} \frac{SR(s)}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{S*A/S^3}{1 + G(s)H(s)} = \lim_{S\to 0} \frac{A}{S^2[1 + G(s)H(s)]}$$

$$e_{ss} = \frac{A}{\lim_{S\to 0}[S^2 + S^2G(s)H(s)]} = \frac{A}{0 + \lim_{S\to 0}[S^2G(s)H(s)]}$$

For a given system $\lim_{S\to 0}[S^2G(s)H(s)]$ *is a constant and is known as Acceleration Error Constant.*

It is represented by K_a.

$$K_a = \lim_{S\to 0} S^2G(s)H(s)$$
$$e_{ss} = \frac{A}{K_a} \text{ for parabola input}$$

Effect of type number on steady state error:

General loop transfer function

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{S^j(1+sT_1)(1+sT_2)\ldots}$$

Where j – type number of the system.

Let us see the effect of type number on steady state error.

TYPE -0: j=0

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots}$$

Step Input

$$K_P = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = K$$

$$e_{ss} = \frac{A}{1+K_P} = \frac{A}{1+K}$$

Where A magnitude of input & K gain of loop transfer function.

Steady state error for type-0 system for step input is finite.

Ramp Input

$$K_v = \lim_{S\to 0} SG(s)H(s) = \lim_{S\to 0} S\frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = 0$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{0} = \infty$$

Steady state error for type-0 system for ramp input is ∞.

Parabola Input

$$K_a = \lim_{S\to 0} S^2G(s)H(s) = \lim_{S\to 0} S^2\frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = 0$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$

Steady state error for type-0 system for parabola input is ∞.

TYPE -1: j=1

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{s(1+sT_1)(1+sT_2)\ldots}$$

Step Input

$$K_P = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{s(1+sT_1)(1+sT_2)\ldots} = \infty$$

$$e_{ss} = \frac{A}{1+K_P} = \frac{A}{1+\infty} = 0$$

Steady state error for type-1 system for step input is zero.

Ramp Input

$$K_v = \lim_{S\to 0} SG(s)H(s) = \lim_{S\to 0} S\frac{K(1+sT_a)(1+sT_b)\ldots}{S(1+sT_1)(1+sT_2)\ldots} = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = K$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{K}$$

Steady state error for type-1 system for ramp input is *finite*.

Parabola Input

$$K_a = \lim_{S\to 0} S^2 G(s)H(s) = \lim_{S\to 0} S^2 \frac{K(1+sT_a)(1+sT_b)\ldots}{S(1+sT_1)(1+sT_2)\ldots} = \lim_{S\to 0} S\frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = 0$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty$$

Steady state error for type-1 system for parabola input is ∞.

TYPE -2: j=2

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\ldots}{s^2(1+sT_1)(1+sT_2)\ldots}$$

Step Input

$$K_P = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{s^2(1+sT_1)(1+sT_2)\ldots} = \infty$$

$$e_{ss} = \frac{A}{1+K_P} = \frac{A}{1+\infty} = 0$$

Steady state error for type-2 system for step input is zero.

Ramp Input

$$K_v = \lim_{S\to 0} SG(s)H(s) = \lim_{S\to 0} S\frac{K(1+sT_a)(1+sT_b)\ldots}{s^2(1+sT_1)(1+sT_2)\ldots} = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{s(1+sT_1)(1+sT_2)\ldots} = \infty$$

$$e_{ss} = \frac{A}{K_v} = \frac{A}{\infty} = 0$$

Steady state error for type-2 system for ramp input is *zero*.

Parabola Input

$$K_a = \lim_{S\to 0} S^2 G(s)H(s) = \lim_{S\to 0} S^2 \frac{K(1+sT_a)(1+sT_b)\ldots}{s^2(1+sT_1)(1+sT_2)\ldots} = \lim_{S\to 0} \frac{K(1+sT_a)(1+sT_b)\ldots}{(1+sT_1)(1+sT_2)\ldots} = K$$

$$e_{ss} = \frac{A}{K_a} = \frac{A}{K}$$

Steady state error for type-2 system for parabola input is finite.

Input ->	**Step Input**		**Ramp Input**		**Parabola Input**	
Type	$K_P = \lim_{S\to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1+K_P}$	$K_v = \lim_{S\to 0} SG(s)H(s)$	$e_{ss} = \frac{A}{K_v}$	$K_a = \lim_{S\to 0} S^2G(s)H(s)$	$e_{ss} = \frac{A}{K_a}$
0	**K**	**A/(1+K)**	**0**	∞	**0**	∞
1	∞	**0**	**K**	**A/K**	**0**	∞
2	∞	**0**	∞	**0**	**K**	**A/K**
3	∞	**0**	∞	**0**	∞	**0**

For same input if type number is increased steady state error is decreased. But higher than type 2 is not preferred due to stability reasons.

Limitations of static error constants:

1. Only useful when inputs are step, Ramp & parabola.
2. No information of e(t). some times e_{ss} is low. But initially error is very high. But that information is unknown in this method of calculation.

To overcome these limitations dynamic error coefficients or generalized error coefficients are used.

Dynamic error constants (or) generalized error constants:

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

We can represent this ratio as

$$\frac{E(s)}{R(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \dots$$

Where $C_0, C_1, C_2 \dots$ are known as dynamic error coefficients.

$$E(s) = [C_0 + C_1 s + \frac{C_2}{2!}s^2 + \dots]R(s)$$

$$E(s) = C_0 R(s) + C_1 s R(s) + \frac{C_2}{2!}s^2 R(s) \dots \quad (1)$$

By applying inverse laplace

$$e(t) = C_0\, r(t) + C_1\, r'(t) + C_2\, r''(t) + \dots \quad (2)$$

$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} SE(S)$$

Expression for dynamic error constants

$$C_n = \lim_{S\to 0} \frac{d^n}{dS^n}\left(\frac{1}{1+G(s)H(s)}\right) \quad for\ n = 0,1,2\dots$$

$$C_0 = \lim_{S\to 0}\left(\frac{1}{1+G(s)H(s)}\right)$$

$$C_1 = \lim_{S\to 0}\frac{d}{dS}\left(\frac{1}{1+G(s)H(s)}\right)$$

$$C_2 = \lim_{S\to 0}\frac{d^2}{dS^2}\left(\frac{1}{1+G(s)H(s)}\right)$$

Evaluation of e_{ss} for a given system:

Method 1: (For any input)

$$e_{SS} = \lim_{s\to 0} SE(S) = \lim_{s\to 0}\frac{SR(S)}{1+G(s)H(s)}$$

Method 2- Using Dynamic error constants: (For any input)

Find C_0, C_1, C_2 ….
Then find either E(s) or e(t) using these values.

$$E(s) = C_0 R(s) + C_1 sR(s) + \frac{C_2}{2!} s^2 R(s) \ldots \quad (1)$$

(Or)

$$e(t) = C_0\, r(t) + C_1\, r'(t) + C_2\, r''(t) + \ldots \quad (2)$$

$$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} SE(S)$$

2 methods to calculate C_0, C_1, C_2 ….

M1:

$$C_n = \lim_{S\to 0} \frac{d^n}{dS^n}\left(\frac{1}{1+G(s)H(s)}\right) \quad for\ n = 0,1,2\ldots$$

M2: Express $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$ as $\frac{E(s)}{R(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \ldots$

Write numerator and denominator of the above ratio from lowest power to highest power and divide numerator with denominator.

Method 3- Using Static error constants: (For Step, Ramp, Parabola inputs)

If input is step (r(t) = A) *then* $K_P = \lim_{S\to 0} G(s)H(s)$ & $e_{ss} = \frac{A}{1+K_P}$

If input is ramp (r(t) = At) *then* $K_v = \lim_{S\to 0} SG(s)H(s)$ & $e_{ss} = \frac{A}{K_v}$

If input is parabola ($r(t) = At^2/2$) *then* $K_a = \lim_{S\to 0} S^2 G(s)H(s)$ & $e_{ss} = \frac{A}{K_a}$

If input is sum of the standard inputs $r(t)=A_1 + A_2 t + A_3 t^2/2$ (ex: $r(t)=2+3t+5t^2$)

Consider $r_1(t) = A_1$ and calculate $e_{ss1} = \frac{A_1}{1+K_P}$

$r_2(t) = A_2 t$ and calculate $e_{ss2} = \frac{A_2}{K_v}$

$r_3(t) = A_3 t^2/2$ and calculate $e_{ss3} = \frac{A_3}{K_a}$

\Total $e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$

Problems:

1. An unity feedback system has $G(s) = \frac{K(2S+1)}{S(4S+1)(S+1)^2}$.determine the value of K if the steady state error to be less than 0.1 when an input 1+5t is applied.

Sol:

$$G(s) = \frac{K(2S+1)}{S(4S+1)(S+1)^2} \quad \& H(s) = 1$$

$R(t) = 1 + 5t$ \R(s) = $(1/S) + (5/S^2)$

Method 1:

$$e_{SS} = \lim_{s\to 0} \frac{SR(S)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{S[\frac{1}{S}+\frac{5}{S^2}]}{1+\frac{K(2S+1)}{S(4S+1)(S+1)^2}}$$

$$= \lim_{s\to 0} \frac{S[\frac{S+5}{S^2}]}{\frac{[S(4S+1)(S+1)^2]+K(2S+1)}{S(4S+1)(S+1)^2}} = \lim_{s\to 0} \frac{(S+5)S(4S+1)(S+1)^2}{S[S(4S+1)(S+1)^2]+K(2S+1)}$$

$$e_{SS} = \lim_{s\to 0} \frac{(S+5)(4S+1)(S+1)^2}{[S(4S+1)(S+1)^2]+K(2S+1)} = \frac{5}{K}$$

Method 2: R(t) = 1 + 5t sum of standard inputs. So static error constants can be used to find e_{ss}.

Let $r_1(t) = 1$ & $r_2(t) = 5t$

For $r_1(t) = 1$ $K_P = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{K(2S+1)}{S(4S+1)(S+1)^2} = \infty$

$$e_{ss1} = \frac{A}{1+K_P} = \frac{1}{1+\infty} = 0$$

For $r_2(t) = 5t$ $K_v = \lim_{S\to 0} SG(s)H(s) = \lim_{S\to 0} S\frac{K(2S+1)}{S(4S+1)(S+1)^2} = \lim_{S\to 0} \frac{K(2S+1)}{(4S+1)(S+1)^2} = K$

$$e_{ss2} = \frac{A}{K_v} = \frac{5}{K}$$

\Total $e_{ss} = e_{ss1} + e_{ss2} = 0 + \frac{5}{K} = 5/K$

$e_{ss} = 0.1 = 5/K$

\K = 50 for $e_{ss} = 0.1$

\K should be greater than 50 so that error is less than 0.1.

2. For a unity feedback system open loop transfer function is $G(s) = \frac{K}{S^2(S+2)(S+3)}$.determine the value of K to limit steady state error to 10 when an input $1+10t+20t^2$ is applied.

Sol:

$$G(s) = \frac{K}{S^2(S+2)(S+3)} \quad \& H(s) = 1$$

R(t) = $1+10t+20t^2$

3. For a unity feedback system open loop transfer function is $G(s) = \frac{10(S+2)}{S^2(S+1)}$.determine static error coefficients and steady state error when an input $R(s) = \frac{3}{S} - \frac{2}{S^2} + \frac{1}{3S^3}$.

Sol:

$$G(s) = \frac{10(S+2)}{S^2(S+1)} \quad \& H(s) = 1$$

$R(s) = \frac{3}{S} - \frac{2}{S^2} + \frac{1}{3S^3}$ → r(t) = $3-2t+(1/6)t^2$

r(t) = = $3-2t+(1/2)t^2$ sum of standard inputs. So static error constants can be used to find e_{ss}.

Let $r_1(t) = 3$, $r_2(t) = -2t$ & $r_3(t) = (1/6)t^2$

For $r_1(t) = 3$ $K_P = \lim_{S\to 0} G(s)H(s) = \lim_{S\to 0} \frac{10(S+2)}{S^2(S+1)} = \infty$

$$e_{ss1} = \frac{A}{1+K_P} = \frac{3}{1+\infty} = 0$$

For $r_2(t) = -2t$ $\quad K_v = \lim_{S\to 0} SG(s)H(s) = \lim_{S\to 0} S\frac{10(S+2)}{S^2(S+1)} = \lim_{S\to 0}\frac{10(S+2)}{S(S+1)} = \infty$

$$e_{ss2} = \frac{-2}{K_v} = 0$$

For $r_3(t) = (1/6)t^2$ $\quad K_a = \lim_{S\to 0} S^2G(s)H(s) = \lim_{S\to 0} S^2\frac{10(S+2)}{S^2(S+1)} = \lim_{S\to 0}\frac{10(S+2)}{(S+1)} = \frac{10(0+2)}{(0+1)} = 20$

$$e_{ss3} = \frac{A}{K_a} = \frac{1/3}{20} = 1/60$$

\Total $e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \frac{1}{60}$

4. For the above problem find dynamic error constants and then find steady state error.

Sol:

$$G(s) = \frac{10(S+2)}{S^2(S+1)} \quad \& \; H(s) = 1$$

$R(s) = \frac{3}{S} - \frac{2}{S^2} + \frac{1}{3S^3} \quad \rightarrow \quad r(t) = 3\text{-}2t\text{+}(1/6)t^2$

Dynamic error constants:

First decide the number of dynamic error constants we need to calculate. For that input is useful.

$r(t) = 3\text{-}2t\text{+}(1/6)t^2$

so $r'(t) = -2\text{+}(1/3)t$

$r''(t) = 1/3$

$r'''(t) = 0$

$e(t) = C_0\, r(t) + C_1\, r'(t) + C_2\, r''(t) + C_3\, r'''(t) + \ldots.$

Since $r'''(t) = 0$ and also higher derivatives are zero we need to calculate $C_0, C_1 \& C_2$

Method1:

$$C_n = \lim_{S\to 0}\frac{d^n}{dS^n}\left(\frac{1}{1+G(s)H(s)}\right) \quad for\ n = 0,1,2...$$

$$C_0 = \lim_{S\to 0}\left(\frac{1}{1+G(s)H(s)}\right) = \lim_{S\to 0}\frac{1}{1+\frac{10(S+2)}{S^2(S+1)}} = \lim_{S\to 0}\frac{S^2(S+1)}{S^2(S+1)+10(S+2)} = 0$$

$$C_1 = \lim_{S\to 0}\frac{d}{dS}\left(\frac{1}{1+G(s)H(s)}\right) = \lim_{S\to 0}\frac{d}{dS}\left(\frac{S^2(S+1)}{S^2(S+1)+10(S+2)}\right) = 0$$

$$C_2 = \lim_{S\to 0}\frac{d^2}{dS^2}\left(\frac{1}{1+G(s)H(s)}\right) = 1/10$$

Method 2: Express $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$ as $\frac{E(s)}{R(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \ldots$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{10(S+2)}{S^2(S+1)}} = \frac{S^2(S+1)}{S^2(S+1)+10(S+2)} = \frac{S^3+S^2}{S^3+S^2+10S+20}$$

$\frac{E(s)}{R(s)} = \frac{S^2+S^3}{20+10S+S^2+S^3} = \frac{S^2}{20} + \dots$

Compare with $\frac{E(s)}{R(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \dots$

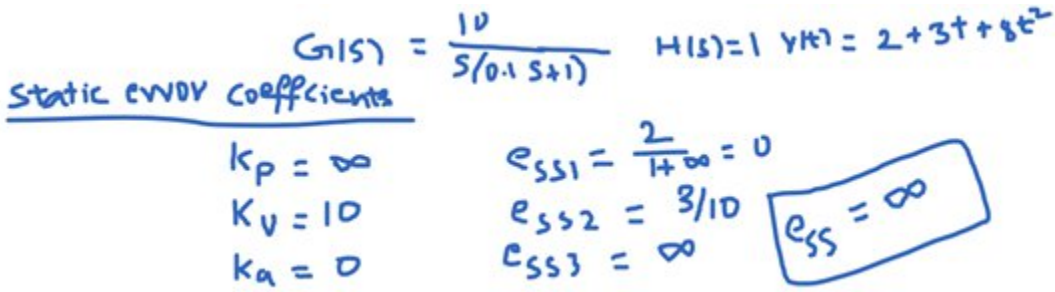
\C_0=0, C_1=0 & (C_2/2!) =1/20 -> C_2 =1/10

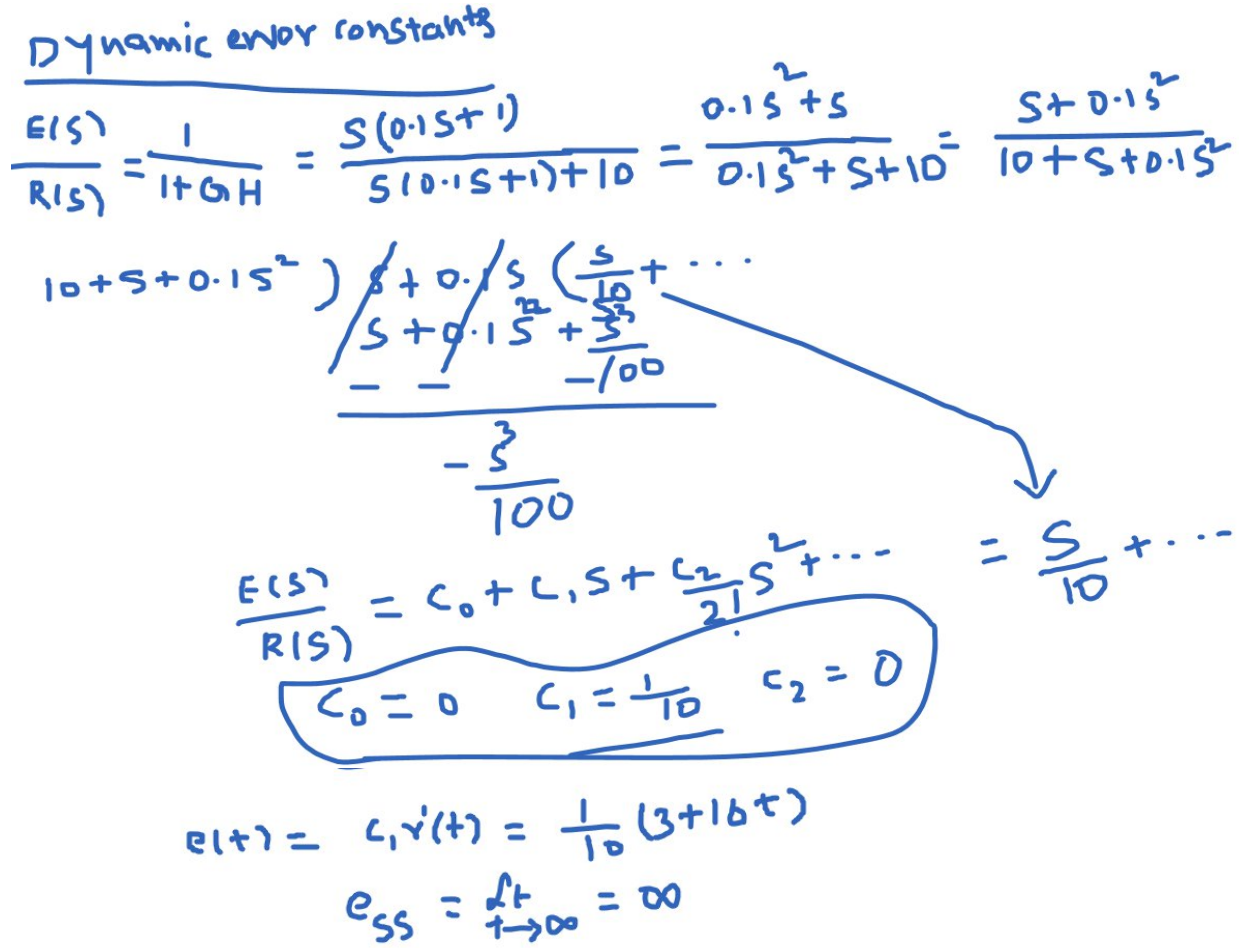
\e(t) = C_0 r(t) + C_1 r'(t) + (C_2/2!) r'' (t) + C_3 r''' (t)+ ….

e(t) = (1/20) r'' (t) = 1/60

$e_{SS} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} \frac{1}{60} = \frac{1}{60}$

5. Evaluate static error coefficients, dynamic error coefficients and steady state error for a given unity feedback system with open loop transfer function $\frac{10}{S(0.1S+1)}$ when subjected to an input of $2+3t+8t^2$.



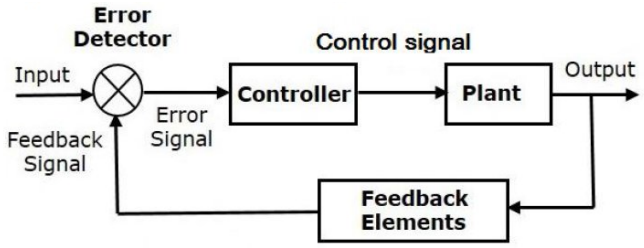

Dynamic error constants
E(s)/R(s) = 1/(1+GH) = s(0.1s+1)/s(0.1s+1)+10 = (0.1s^2+s)/(0.1s^2+s+10) = (s+0.1s^2)/(10+s+0.1s^2)
10+s+0.1s^2) s+0.1s (s/10 + ...
s+0.1s^2+s^3/100
-s^3/100
= s/10 + ...
E(s)/R(s) = c0 + c1 s + (c2/2!) s^2 + ...
c0 = 0 c1 = 1/10 c2 = 0
e(t) = c1 r'(t) = 1/10 (3+10t)
e_ss = lim t→∞ = ∞

P,PI,PID controllers

By varying Gain K value response of a system can be controlled.

Controller is an external device which modify error signal to get desired performance from a control system.

Desired response :
Good transient response: Less t_d,t_r, t_p,μ_p,t_s
Good steady state response: Neglisible or zero e_{ss}.



Input to controller is error signal e(t) and output is the control signal which is input to plant or system.

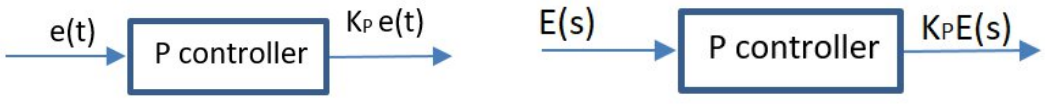
Controller modify the error signal by using 3 actions:

(i) Proportional action - P control
Output of controller is proportional to input i.e error signal e(t).
(ii) Integral action - I control
Output of controller is proportional to integral of input i.e error signal e(t).
(iii) derivative action - D control
Output of controller is proportional to derivative of input i.e error signal e(t).

P Controller:

If the output of controller is proportional to error signal then it is known as P controller.

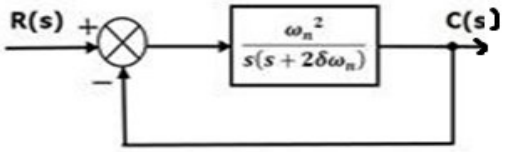
O/P α e(t) -> O/P = K_P e(t) where K_P is proportional constant or proportional gain.



Therefore the transfer function of P controller is K_P.



Consider a 2nd order system shown in fig.



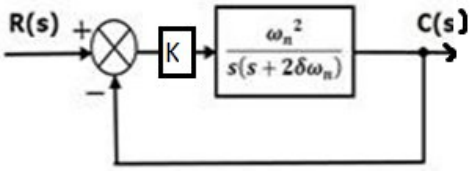
$$G(s) = \frac{\omega_n^2}{S(S+2\zeta\omega_n)}$$

$$for\ step\ input \quad K_P = \lim_{S\to 0} G(s)H(s) = \infty \qquad e_{ss} = \frac{A}{1+K_P} = 0$$

$$\text{for ramp input } K_V = \lim_{S\to 0} sG(s)H(s) = \lim_{S\to 0} s\frac{\omega_n^2}{S(S+2\zeta\omega_n)} = \frac{\omega_n}{2\zeta} \qquad e_{ss} = \frac{A}{K_V}$$

$$\text{Transfer function} = \frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$$

with P controller:



$$G(s) = \frac{K\omega_n^2}{S(S+2\zeta\omega_n)}$$

$$for\ step\ input \quad K_P = \lim_{S\to 0} G(s)H(s) = \infty \qquad e_{ss} = \frac{A}{1+K_P} = 0$$

$$\text{for ramp input } K_V = \lim_{S\to 0} sG(s)H(s) = \lim_{S\to 0} s\frac{K\omega_n^2}{S(S+2\zeta\omega_n)} = K\frac{\omega_n}{2\zeta} \qquad e_{ss} = \frac{A}{K_V}$$

since k_v is increased e_{ss} is reduced,

Therefore for same input if you apply P controller steady state error will be reduced.

$$\text{Transfer function} = \frac{K\omega_n^2}{S^2+2\zeta\omega_n S+K\omega_n^2}$$

Let ζ' & μg_n' are damping factor and natural frequency of the system with P controller.

$$S^2 + 2\zeta'\omega_n' S + (\omega_n')^2 = S^2 + 2\zeta\omega_n S + K\omega_n^2$$

$$(\omega_n')^2 = K\omega_n^2 \rightarrow \omega_n' = \sqrt{K}\omega_n$$

$$2\zeta'\omega_n' = 2\zeta\omega_n \rightarrow \zeta' = \frac{\zeta\omega_n}{\omega_n'} = \frac{\zeta\omega_n}{\sqrt{K}\omega_n} = \frac{\zeta}{\sqrt{K}}$$

Therefore ζ' is less than ζ. So transient response becomes worse with more oscillations & high peaks.

P controller makes steady state response better and transient response worse.

Effects:

1. Increases μg_n of the system.
2. decreases ζ of the system.
3. So transient response becomes worse.
4. e_{ss} reduces for same type of input.

So in general PI controller improves steady state response but worsens transient part.

PI Controller:

It is a controller whose output consists two components, one proportional to error signal and the other proportional to integral of error signal.

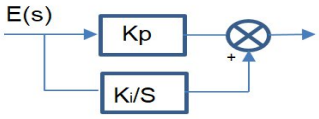
O/P1 α e(t) -> O/P1 = Kp e(t) where K_P is proportional constant or proportional gain.

O/P2 α $\int e(t)\,dt$ -> O/P2 = $K_i \int e(t)\,dt$ where K_i is known as integral constant.

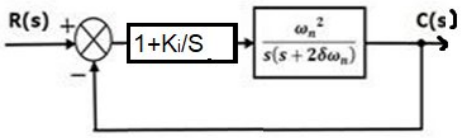




Therefore the transfer function of PI controller is $K_P + K_i/S$.



Consider the 2nd order system with PI controller.



Assume K_P=1

$$G(s) = \frac{(1+\frac{K_i}{S})\omega_n^2}{S(S+2\zeta\omega_n)} = \frac{(S+K_i)\omega_n^2}{S^2(S+2\zeta\omega_n)}$$

$$for\ step\ input \quad K_P = \lim_{S\to 0} G(s)H(s) = \infty \qquad e_{ss} = \frac{A}{1+K_P} = 0$$

$$\text{for ramp input } K_V = \lim_{S\to 0} sG(s)H(s) = \lim_{S\to 0} s\frac{(S+K_i)\omega_n^2}{S^2(S+2\zeta\omega_n)} = \infty \qquad e_{ss} = \frac{A}{K_V} = 0$$

e_{ss} is reduced to zero.

Therefore, for same input if you apply PI controller steady state error will be greatly reduced.

$$\text{Transfer function} = \frac{(S + K_i)\omega_n^2}{S^3 + 2\zeta\omega_n S^2 + (S + K_i)\omega_n^2}$$

If you add PI controller, order of the system is increased by 1. As order increases system becomes relatively less stable. So K_i should be designed such that system will remain stable.

Effects:

5. Increases order of the system.
6. Increases type number of OLTF.
7. Design of K_i must be proper to maintain stability. Otherwise system will become unstable.
8. e_{ss} reduces greatly for same type of input.

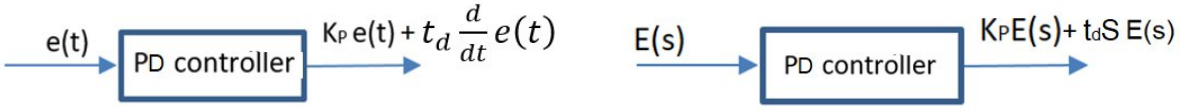
So in general PI controller improves steady state response but effects transient part.

PD Controller:

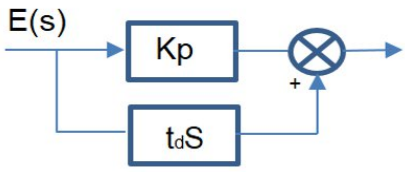
It is a controller whose output consists two components, one proportional to error signal and the other proportional to derivative of error signal.

O/P1 α e(t) -> O/P1 = Kp e(t) where K_P is proportional constant or proportional gain.

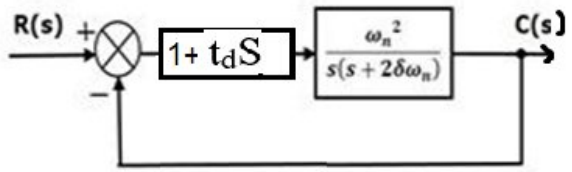
O/P2 α $\frac{d}{dt}e(t)$ -> O/P2 = $t_d\frac{d}{dt}e(t)$ where t_d is time delay constant



Therefore, the transfer function of PD controller is $K_P + t_dS$.



Consider the 2nd order system with PD controller.



Assume $K_P=1$.

$$G(s) = \frac{(1 + t_d S)\omega_n^2}{S(S + 2\zeta\omega_n)}$$

$$for\ step\ input \quad K_P = \lim_{S\to 0} G(s)H(s) = \infty \qquad e_{ss} = \frac{A}{1 + K_P} = 0$$

$$\text{for ramp input } K_V = \lim_{S\to 0} sG(s)H(s) = \lim_{S\to 0} s\frac{(1+t_d S)\omega_n^2}{S(S+2\zeta\omega_n)} = \lim_{S\to 0}\frac{(1+t_d S)\omega_n^2}{(S+2\zeta\omega_n)} = \frac{\omega_n}{2\zeta} \qquad e_{ss} = \frac{A}{K_V}$$

There is no change in e_{ss} with PD controller.

$$\text{Transfer function} = \frac{(1+t_d S)\omega_n^2}{S^2+2\zeta\omega_n S+(1+t_d S)\omega_n^2} = \frac{(1+t_d S)\omega_n^2}{S^2+2\zeta\omega_n S+t_d\omega_n^2 S+\omega_n^2} = \frac{(1+t_d S)\omega_n^2}{S^2+(2\zeta\omega_n+t_d\omega_n^2)S+\omega_n^2}$$

Let ζ' & $\mu g_n'$ are damping factor and natural frequency of the system with PD controller.

$$S^2 + 2\zeta'\omega_n' S + (\omega_n')^2 = S^2 + (2\zeta\omega_n + t_d\omega_n^2)S + \omega_n^2$$

$$(\omega_n')^2 = \omega_n^2 \rightarrow \omega_n' = \omega_n$$

$$2\zeta'\omega_n' = 2\zeta\omega_n + t_d\omega_n^2 \quad \rightarrow \zeta' = \zeta + \frac{t_d\omega_n}{2}$$

Therefore, ζ' is greater than ζ. So transient response improves. i.e. it has less oscillations & low peaks.

Effects:

1. No change in order & type of OLTF the system.
2. ζ increased.
3. No change in μg_n.
4. Reduces μ_p, t_p,t_s.
5. e_{ss} does not effected.

So in general PD controller improves transient response without effecting steady state error.

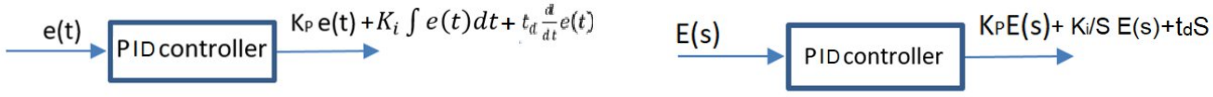
PID Controller:

It is a controller whose output consists three components, one proportional to error signal ,second proportional to integral of error signal and the third proportional to derivative of error signal.

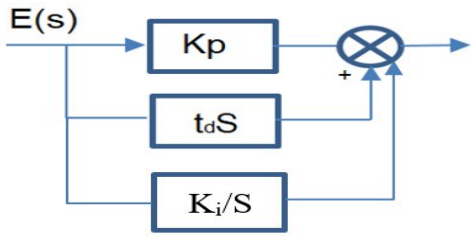
O/P1 α e(t) -> O/P1 = Kp e(t) where K_P is proportional constant or proportional gain.

O/P2 α $\int e(t)\,dt$ -> O/P2 = $K_i \int e(t)\,dt$ where K_i is integral constant

O/P3 α $\frac{d}{dt}e(t)$ -> O/P3 = $t_d \frac{d}{dt}e(t)$ where t_d is time delay constant



Therefore the transfer function of PID controller is $K_P + K_i/S + t_dS$.



Effects:

1. Type number & order of OLTF are increased by 1.
2. e_{ss} reduces greatly for same input.
3. ζ increased.
4. Reduces μ_p, t_p,t_s.

So in general PID controller improves both transient & steady state responses.

ANNAMACHARYA UNIVERSITY, RAJAMPET

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016
RAJAMPET, Annamayya District, AP, INDIA

Department of Electrical & Electronics Engineering

III B.Tech I Semester

Linear Control Systems (24AEEE51T)
(Electrical and Electronics Engineering)

Unit-3 (Stability Analysis in Time Domain)

Syllabus:

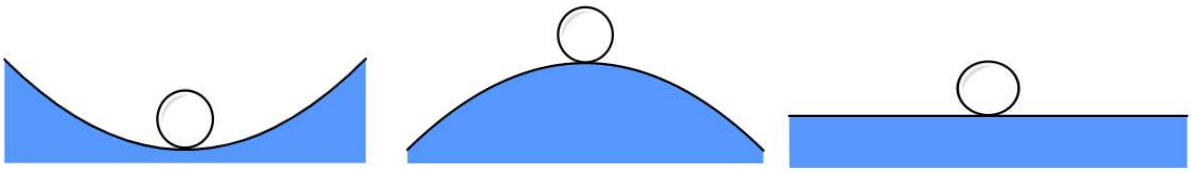
The concept of stability – Routh's stability criterion, problems-absolute stability and conditional stability – limitations of Routh's stability criterion. The Root locus concept, problems –effects of adding poles and zeros to G(S)H(S) on the root loci.

Introduction to Stability

In this unit we will discuss

- Stability concept- different notions of stability
- BIBO stability
- Effect of location of poles on stability
- RH criteria
- Root locus method

Stability: what do you mean by stability.

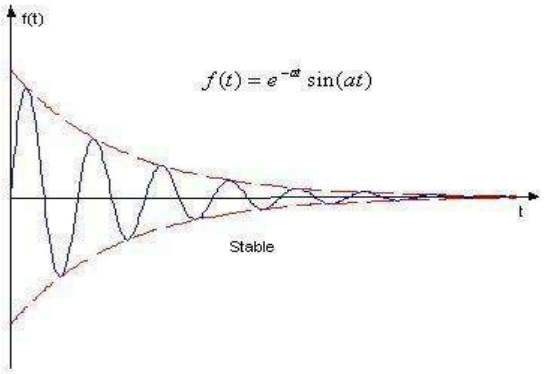


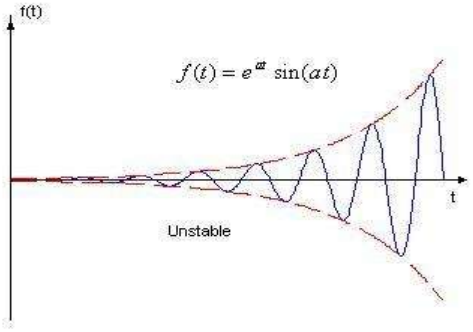
(1) (2) (3)

(1) In this system the ball will return to its original position if it is displaced by a small amount.
(2) Here the ball will move further and further away from the initial point or rest point if it is displaced by even a small amount.
(3) In this case if the ball is moved it will stay fixed at its new location. It will not move closer to or further from the original equilibrium.

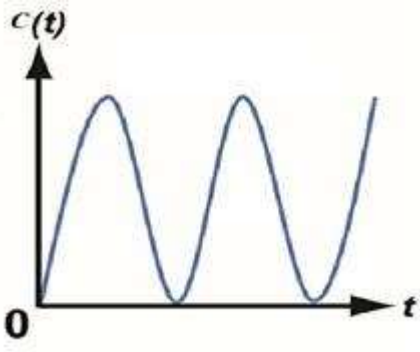
Stable: A system is Stable if the natural response approaches zero as time approaches infinity .

Unstable: A system is Unstable if the natural response approaches infinity as time approaches infinity





Marginally stable or limitedly Stable: A system is Marginally Stable if the natural response neither decays nor grows but remains constant or oscillates within a bound. A marginally stable system may become unstable under certain circumstances, and may be perfectly stable under other circumstances. It is impossible to tell by inspection whether a marginally stable system will become unstable or not.

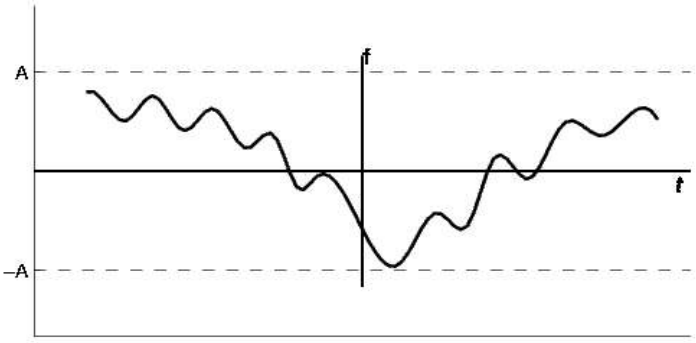


Two notions of Stability:

(1) Bounded Input Bounded Output (BIBO) Stability
(2) Asymptotic Stability

BIBO Stability: A system is BIBO stable if every bounded input signal results in a bounded output.

Bounded input: it is an input whose value always lie between a higher and lower values.



A bounded signal is a signal for which there exists a constant A such that

|f(t)|<A for all values of 't'.(A<∞)

Asymptotic stability:

In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

$$\lim_{t\to\infty} C(t) = 0$$

For nonlinear systems both are different. An asymptotic stable system is BIBO stable also but not vice versa. For LTI systems both are same. So simply stable.

Condition for BIBO Stability

A SISO system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0 S^m + b_1 S^{m-1} + b_2 S^{m-2} + \cdots \ldots + b_{m-1} S + b_m}{S^n + a_1 S^{n-1} + a_2 S^{n-2} + \cdots \ldots + a_{n-1} S + a_n}$$

C(s) = G(s) * R(S)

C(t) = $\mathcal{L}^{-1}$ [C(s)] = $\mathcal{L}^{-1}$ [G(s) R(s)]

If the input is impulse input

r(t) = δ(t) -> R(s) = 1

C(t) = $\mathcal{L}^{-1}$ [G(s) R(s)] = $\mathcal{L}^{-1}$ [G(s)] = g(t) = impulse response

Impulse response of a system is equal to inverse laplace of the transfer function.

For any other inputs

C(t) = $\mathcal{L}^{-1}$ [G(s) R(s)] = g(t)*r(t) (By Convolution theorem)

The time response of any system is equal to time convolution of impulse response and the input.

$$C(t) = \int_0^\infty g(\tau)\, r(t-\tau) dt$$

Taking absolute value on both sides,

$$|C(t)| = \left|\int_0^\infty g(\tau)\, r(t-\tau) dt\right|$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand.

$$|C(t)| \le \int_0^\infty |g(\tau)\, r(t-\tau)|\, dt$$

$$|C(t)| \le \int_0^\infty |g(\tau)||\, r(t-\tau)|\, dt$$

But input is bounded.

$|r(t)| \le A_1 < \infty$

For the system to be stable output should be bounded. So

$|C(t)| \le A_2 < \infty$

$$|C(t)| \le \int_0^\infty |g(\tau)|\, A_1 dt \le A_2 < \infty$$

$$A_1 \int_0^\infty |g(\tau)|\, dt \le A_2 < \infty$$

$$\int_0^\infty |g(\tau)|\, dt \le A$$

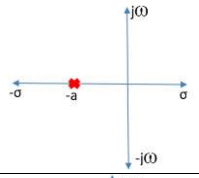
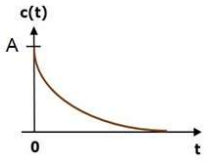
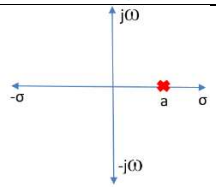
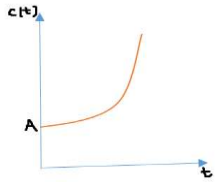
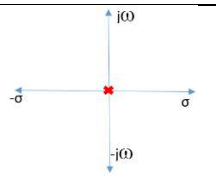
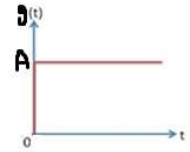
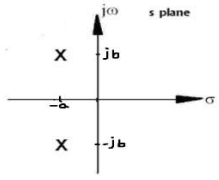
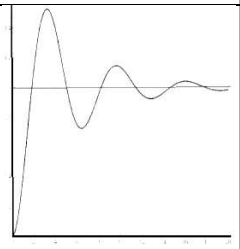
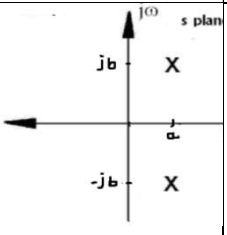
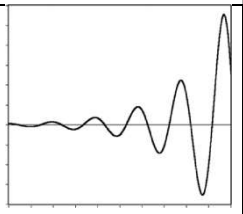
So $\int_0^\infty |g(\tau)|\, dt$ should be finite (less than infinity) for the system to be stable.

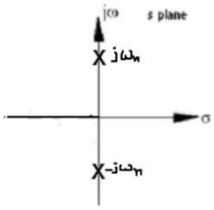
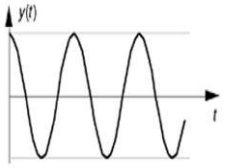
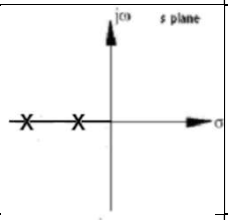
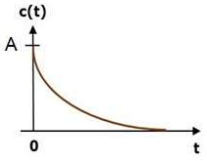
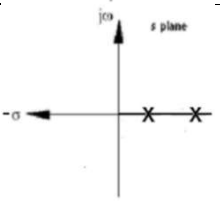
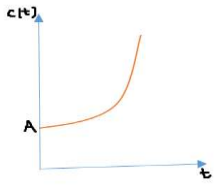
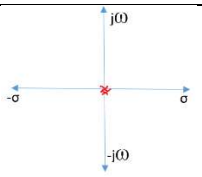
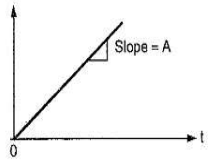
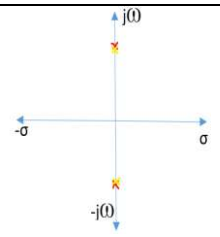
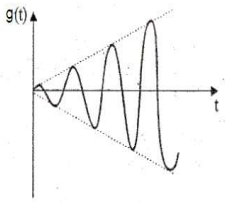
The impulse response of the system i.e. g(t) should be absolutely integrable for the system to be stable.

The area under the impulse response curve should be finite for the system to be stable.

Effect of location of poles on stability:

Let us see the impulse responses of various transfer functions and check the stability of those systems.

Transfer Function	Pole zero plot	Location of the poles	Impulse response	Impulse response	Comment on stability
$G(s)=\frac{A}{S+a}$ S=-a is –ve real pole		Pole on left half real axis	g(t) =A e^{-at}		The area under the curve is finite. So Stable
$G(s)=\frac{A}{S-a}$ S=a is +ve real pole		Pole on right side real axis	g(t) =A e^{at}		The area under the curve is infinite. So unstable
$G(s)=\frac{A}{S}$ S=0 is a pole at origin		Pole on imaginary axis	g(t) =A		The area under the curve is infinite but it is bounded. So marginally Stable
$G(S)=\frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$ Poles are S = -a±jb complex poles with –ve real part		Poles on left half of S-plane	g(t) =A e^{-at} sin (bt+Θ)		The area under the curve is finite. So Stable
$G(S)=\frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2}$ Poles are S = a±jb complex poles with +ve real part		Poles on right half of S-plane	g(t) =A e^{at} sin (bt+Θ)		The area under the curve is infinite. So unstable

$G(S) = \frac{\omega_n^2}{S^2 + \omega_n^2}$ Poles are S = ±jωn pure imaginary		Poles on imaginary axis	g(t) = cos (ω_nt)		The area under the curve is infinite but it is bounded. So marginally Stable
$G(S) = \frac{A}{(S+a)(S+b)}$ Poles are S = -a,-b –ve real poles		Poles on left half of s-plane	g(t)=K_1e^{-at} + K_2 e^{-bt}		The area under the curve is finite. So Stable
$G(S) = \frac{A}{(S-a)(S-b)}$ Poles are S = a,b +ve real poles		Poles on right half of s-plane	g(t)=K_1e^{at} + K_2 e^{bt}		The area under the curve is infinite. So unstable
G(s) = A/S^2 Poles are S=0,0 Double pole at origin		Multiple poles on imaginary axis	g(t)=At		The area under the curve is infinite. So unstable
$G(S) = \frac{A}{(S^2+b^2)^2}$ Poles are $S = \pm jb, \pm jb$ Multiple pure imaginary		Multiple poles on imaginary axis	g(t)= At sin(bt)		The area under the curve is infinite. So unstable

Conclusions:

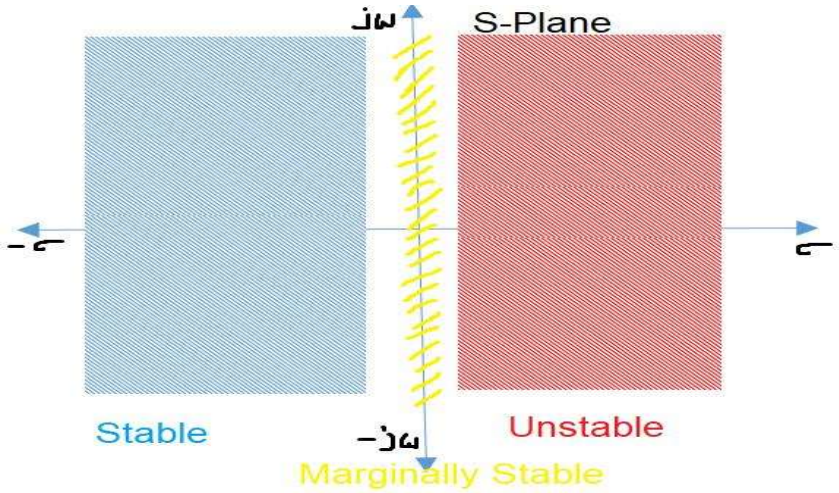
- A system is stable if all poles lie in the left half of S-plane.

  (Or) all the poles have –ve real part.

- A system is unstable if at least one pole lie in the right half of S-plane and/or multiple poles on the imaginary axis.

  (Or) at least one pole has –ve real part.

- A system is marginally stable if some simple poles lie on imaginary axis and other poles in the left half of S-plane.



Two types of stability:

If the question is "Is the system stable?". The aswer is either Yes or no. that type of stability is known as absolute stability. Once it is known that the system is stable the next question is how much near it is to instaility condiction. Is it nearer or away from instability condition. This type of stability is relative stability.

Absolute stability:

- A linear system is absolutely stable w.r.t a parameter of the system if it is stable for all values of that parameter.
- A linear system is conditionally stable w.r.t a parameter of the system if it is stable for certain range of values of that parameter only.

Relative stability:

It is a measure of how near a system is to instability condition. It is a quantitative measure of how fast the transients die out in the system. So it can be measured from settling times of each poles or pair of poles. The settling time is inversely proportional to real part of the pole.

If the real part is high the pole is away from the imaginary axis and if it is low then the pole is nearer to imaginary axis.

Therefore if the poles of the system are nearer to imaginary axis that system is less stable and a system with poles far away from imaginary axis is more stable.

RH CRITERION

Dr. M. PADMA LALITHA

STABILITY

A process to find out if a system is stable or not.

1. A system is stable if every bounded input yields a bounded output.
2. A system is unstable if any bounded input yields an unbounded output.

STABILITY

- Most important issue to design a control system.
- An unstable feedback system is of no practical value.
- A stable system should exhibit a bounded output if the corresponding input is bounded.

EXAMPLE OF UNSTABLE SYSTEM

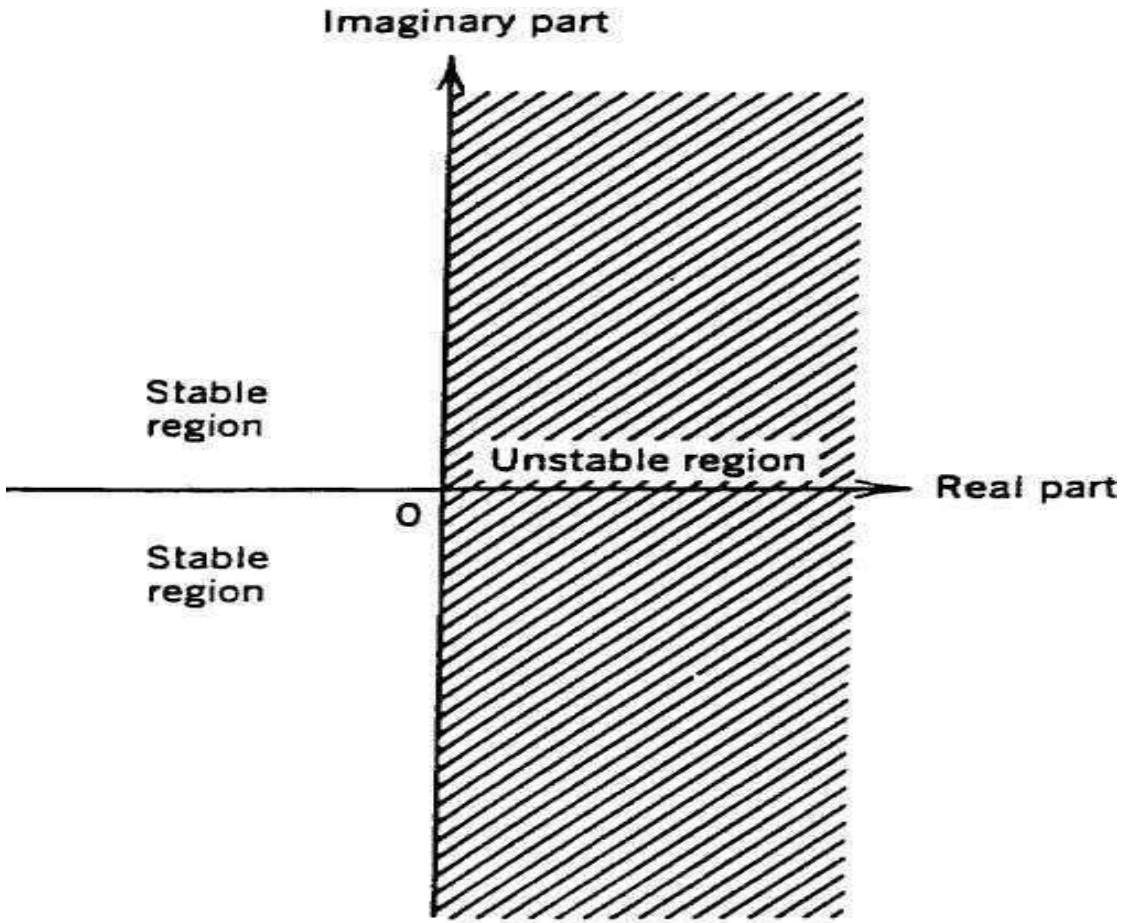
Dramatic photographs showing the collapse of the Tacoma Narrows suspension bridge on November 7, 1940.

(a) Photograph showing the twisting motion of the bridge's center span just before failure.

(b) A few minutes after the first piece of concrete fell, this second photograph shows a 600-ft section of the bridge breaking out of the suspension span and turning upside down as it crashed in Puget Sound, Washington.



LOCATION OF POLES & STABILITY



STABILITY

- Using these concepts:

Stable System: If all the roots of the characteristic equation lie on the left half of the 'S' plane then the system is said to be a stable system.

Marginally Stable System: If all the roots of the system lie on left half of the 'S' plane except some non repeated roots on the imaginary axis of the 'S' plane then the system is said to be marginally stable.

Unstable System: If one root of the system lie on the right half of the 'S' plane or multiple roots on imaginary axis then the system is said to be an unstable system.



ROUTH-HURWITZ STABILITY CRITERION

- The characteristic equation of the nth order system

$$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-1} \ldots\ldots\ldots + a_{n-1}S + a_n = 0$$

Necessary but not sufficient conditions for Stability

1. **All the coefficients of the equation should have the same sign.**
2. **There should be no missing term.**

If the above conditions are not satisfied, then the system is said to be unstable.

If all the coefficients have the same sign and there are no missing terms, we have no guarantee that the system will be stable. For this, **Routh Hurwitz Criterion** is used to check the stability of the system.

EXAMPLES

- $3s^4+4s^3+5s+1 = 0$

  s^2 ⟶ missing.

The system is unstable

- $5s^3-3s^2+5s+1 = 0$

  -ve sign ⟶ for coefficient s^2

The system is unstable

CONSTRUCTION OF ROUTH ARRAY

- The characteristic equation of the nth order system

$$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-1} \ldots\ldots\ldots + a_{n-1}S + a_n = 0$$

Step 1: Arrange all the coefficients of the above equation in two rows:

Row 1	a_0	a_2	a_4	.....
Row 2	a_1	a_3	a_5	.....

First row elements- all even coefficents i.e a_0, a_2, a_4.....
Second row elements- all odd coefficents i.e a_1, a_3, a_5..

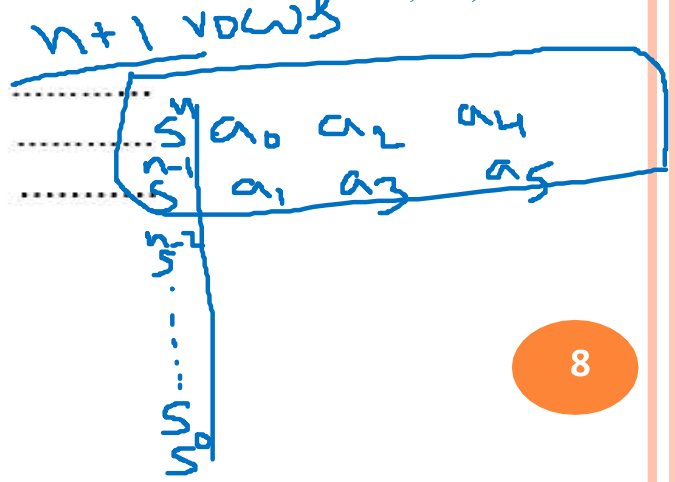
Step 2: The elements of third row can be calculated as:

Row 1	a_0	a_2	a_4
Row 2	a_1	a_3	a_5
Row 3	b_1	b_3	b_5

Where,

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{a_0a_3 - a_1a_2}{a_1}$$

$$b_3 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{a_0a_5 - a_1a_5}{a_1}$$



CONSTRUCTION OF ROUTH ARRAY (CONTD…)

Step 3: Now fourth row elements are calculated by using second and third row:

Row 1	a_0	a_2	a_4	.....
Row 2	a_1	a_3	a_5	.....
Row 3	b_1	b_3	b_5	.....
Row 4	c_1	c_3	c_5	.....

Where,

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -\frac{a_1 b_3 - b_1 a_3}{b_1}$$

$$c_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix} = -\frac{a_1 b_5 - b_1 a_5}{b_1}$$

Step 4: This procedure of forming new rows is continued till the last row (n+1 the row).

ROUTH-HURWITZ STABILITY CRITERION

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

If no sign changes in first column of Routh array then the system is stable

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

ROUTH-HURWITZ STABILITY CRITERION

The necessary & sufficient condition for stability

The system is stable if and only if there are no sign changes in first column of Routh array. If any sign changes exist in first column then the system is unstable and number of poles in the right half of S-plane is equal to number of sign changes.

EXAMPLE-1

Check the stability of the system whose characteristic equation is given by

$$s^4 + 2s^3+6s^2+4s+1 = 0$$

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix},$$

$$b_1 = \frac{-(1 \times 4 - 2 \times 6)}{2} = \frac{-(-8)}{2} = 4$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1$$

$$c_1 = -\frac{1}{4}\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 3.5$$

$$d_1 = -\frac{1}{3.5}\begin{vmatrix} 4 & 1 \\ 3.5 & 0 \end{vmatrix} = 1$$

There are no sign changes in first column of Routh array so the system is stable.

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

s^4	1	6	1
s^3	2	4	10
s^2	4	1	
s	3.5		
s^0			

$$b_1 = \frac{12-4}{2} = 4$$

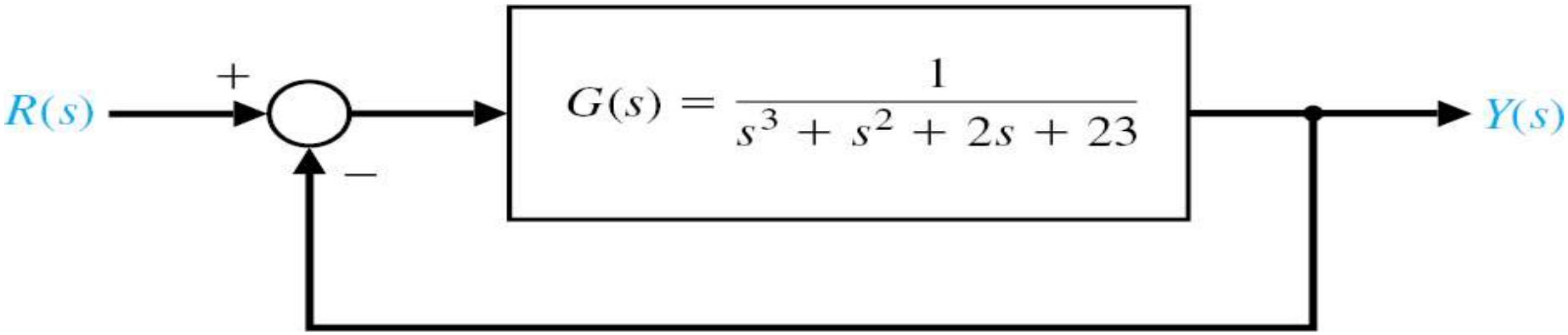
$$b_2 = \frac{(2\times1)-(1\times0)}{2} = 1$$

$$c_1 = \frac{(16-2)}{4} = 3.5$$

$$c_2 =$$

EXAMPLE-2

Find the stability of the given system.



Closed-loop control system TF is

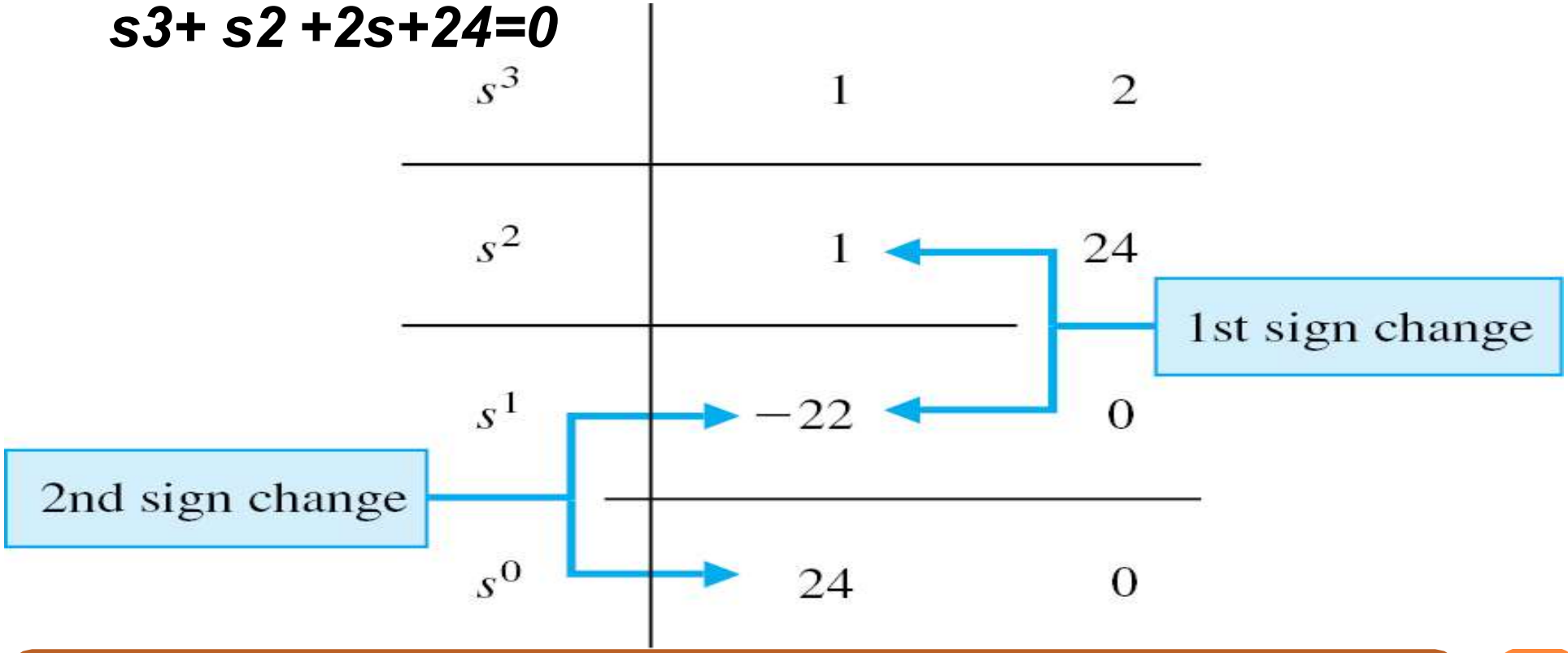
$T(s) = Y(s)/R(s) = 1/(s^3 + s^2 + 2s + 24)$.

So Characteristic Equation is $s^3 + s^2 + 2s + 24 = 0$

EXAMPLE-2 (CONTD…)

Routh array for the system with *Characteristic Equation*

s3+ s2 +2s+24=0



There are two sign changes in first column so there are two poles in right half of S-plane and hence the system is unstable.

ROUTH-STABILITY CRITERION: SPECIAL CASES

Two special cases may occur:

- **When all the elements of any row of the Routh array are zero.**
- **Only the first element of a row is zero**

  – all zero row

s^5	1	2	4
s^4	1	2	4
s^3	0	0	0

  – First element of a row is zero

s^3	1	2	6
s^2	2	4	8
s^1	0	2	0

CASE1: ALL ZERO ROW

Find the stability of the system whose characteristic equation is

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0$$

s^5	1	6	8
s^4	7	42	56
s^3	28 ~~0~~	84 ~~0~~	0
s^2	21	56	
s^1	9.33		
s^0	56		

- **Form Even polynomial A(s) with the elements of the row just above all zero row. Here that row corresponds to s^4**
- **A(s) = $7s^4 + 42s^2 + 56$**
- Differentiate A(s)w.r.t S.
- dA(s)/ds = $28s^3 + 84s$
- Replace all zero with the elements of dA(s)/ds i.e 28 & 84.

No sign changes in first column so the system is marginally stable due to presence of all zero row.

CASE2: FIRST ELEMENT OF A ROW IS ZERO

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

s^5	1	2	11
s^4	2	4	10
s^3	~~0~~ ε	6	
s^2	$\frac{4\varepsilon-12}{\varepsilon}$ -ve	10	
s^1	6 +ve		
s^0	10		

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, Q has some roots in RHP.

Two sign changes in the first column ➡ Two roots in RHP

$$\varepsilon \to \underbrace{\frac{4\varepsilon - 12}{\varepsilon}}_{<0} \to 6$$

So the system is unstable.

EXAMPLE

Find the values of controller gain K that make the feedback control system of following equation stable.

$$10\,s^3 + 17s^2 + 8s + 1 + K = 0$$

S^3	10	8
S^2	17	$1 + K$
S	$7.41 - 0.588K$	0
S^0	$1 + K$	

All elements of first column are positive provided that

$1 + K > 0$ or $K < -1$. and

$7.41 - 0.588K > 0$ *or*

$K < 7.41/0.588$

$K < 12.6$

So the system is stable if 0 < K < 12.6

ADVANTAGES & LIMITATIONS

Advantages:

- We can find the stability of the system without solving the characteristic equation for any order system.
- We can easily determine the absolute and conditional stability of the system.
- By this method, we can determine the range of K for stability.
- By this method, we can also determine the point of intersection for root locus with an imaginary axis.

Limitations:

- This criterion is applicable only for a linear system.
- It does not provide the exact location of poles on the right and left half of the S plane so relative stability can not be determined. And also it cannot distinguish between real & complex roots.
- It is valid only for real coefficients of the characteristic equation.

Difficulties in forming Routh array:

Two difficulties may occur while forming Routh array.

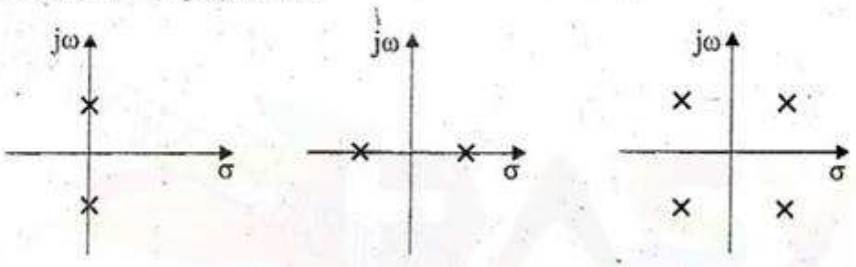
1. All zero row.
2. First element of a row is zero.

When these situations occur the next row elements become infinity so we can not construct Routh array further.

Case 1: All Zero Row

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is also called ***auxiliary polynomial.*** The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The fig shows the roots of an even polynomial.



METHOD

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner and the array is interpreted as follows.
   a. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.
   b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

Case 2: First element of a Row is zero

When first element of a row is zero, next row elements becomes infinity. To overcome this difficulty there are two methods.

Method 1: Replace '0' with a small positive number 'ε' and proceed with construction of remaining routh array. Check the sign of the lelements of first column by making $\varepsilon \to 0$. The interpretation is as follows.

1. If no sign changes in the first column system and no all zero row after by letting $\varepsilon \to 0$ system is stable.
2. If sign changes are there in the first column system is unstable.
3. If no sign changes in the first column system and all zero row present after making $\varepsilon \to 0$ solve as discussed in previous case.

Method 2: Replace 'S' with '1/Z' in characteristic equation. We will get another characteristic equation interms of 'Z'. Construct routh array for this and examine the stability.

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

$$s = 1/z$$

$$\frac{1}{z^5} + \frac{1}{z^4} + \frac{2}{z^3} + \frac{2}{z^2} + \frac{3}{z} + 5 = 0$$

$$1 + z + 2z^2 + 2z^3 + 3z^4 + 5z^5 = 0$$

$$5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0$$

z^5	5	2	1
z^4	3	2	1
z^3	−1.33	−0.66	
z^2	0.511	1	
z	1.942	0	
z^0	1		

There are 2 Sign changes.

∴ there are two poles in RHS.

& hence the system is Unstable

$s^6+4s^5+3s^4-16s^2-64s-48=0$. Check the stability of given characteristic equation using routh method. Also find the location of poles of the system.

s^3 term missing & some of the terms are −ve.

So the system is unstable. $s^6+4s^5+3s^4+0s^3-16s^2-64s-48=0$

s^6	1	3	−16	−48
s^5	4	0	−64	0
s^4	3	0	−48	
s^3	0	0	→ All zero row	
s^3	12	0	0	
s^2	∈	−48		
s^1	$\frac{576}{\epsilon}$	0		
s^0	−48			

$A(s) = 3s^4+0s^2-48$

$\frac{d}{ds}A(s) = 12s^3$

$\epsilon \to 0$

1 4 3 0 12 ∈ $\frac{576}{\epsilon}$ −48

1 4 3 0 12 0 ∞ −48

1 sign change in 1st column there is one pole in Right half of S-plane. So the system is unstable.

1 Pole — Right half of S-plane. 6 Poles.

2 poles — on imaginary axis $A(s) = 3s^4-48=0$

3 pole — on Left half of S-plane

$3s^4 = 48$

$s^4 = 16$ $s^2 = 4$

$s^2 = \pm 4$ $s = \pm 2$

$s^2 = -4 \Rightarrow s = \pm j2$

s^6	1	3	−16	−48
s^5	4	0	−64	0
s^4	3	0	−48	0
s^3	0	0	0	

$A(s) = 3s^4 - 48 = 0$

$\frac{dA}{ds} = 12s^3$

s^6	1	3	− 16	− 48
s^5	4	0	− 64	0
s^4	3	0	− 48	0
s^3	12	0	0	0
s^2	[ε] 0	− 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	− 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

$s^6+3s^5+4s^4+6s^3+5s^2+3s+2=0$. Check the stability of given characteristic equation using routh method. Also find the location of poles of the system.

A unity feedback system is characterised by the open loop transfer function $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$

Using the routh criterion calculate the range of K for the system to be stable.

Sol:

Charcteristic equation of the system is

1+G(s)H(s) =

S^6	
S^5	
S^4	
S^3	
S^2	
S	
S^0	

A unity feedback system is characterised by the open loop transfer function $G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$.By applying the routh criterion calculate the range of K for the system to be stable, marginal value of K and frequency of sustained oscillations.

Sol:

Charcteristic equation of the system is

1+G(s)H(s) =

S^6	
S^5	
S^4	
S^3	
S^2	
S	
S^0	

The open loop transfer function of a unity feedback control system is givenby $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$

Using the routh criterion calculate the value of K interms of T1 & T2 for the system to be stable.

Sol:

Charcteristic equation of the system is

1+G(s)H(s) =

S^6	
S^5	
S^4	
S^3	
S^2	
S	
S^0	

Determine the values of K & b so that the system whose open loop transfer function is givenby $G(s) = \frac{K(s+1)}{s^3+bs^2+3s+1}$oscillates at a frequency of 2 rad/sec. assume unity feedback.

Sol:

Charcteristic equation of the system is

1+G(s)H(s) =

S^6	
S^5	
S^4	
S^3	
S^2	
S	
S^0	

$s^6+2s^5+8\,s^4+12s^3+20\,s^2+16\,s+16=0$. Check the stability of given characteristic equation using routh method. Also find comment on the location of poles of the system.

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0

Row of zeros

$$A(s) = 2s^4 + 12s^2 + 16 = 0$$

$$\frac{dA}{ds} = 8s^3 + 24s = 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	
s^1	2.67	0		
s^0	16			

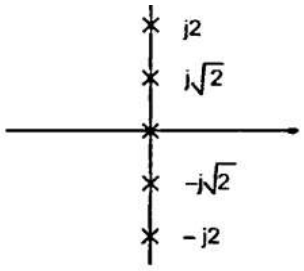
$$2s^4 + 12s^2 + 16 = 0$$

$$s^4 + 6s^2 + 8 = 0$$

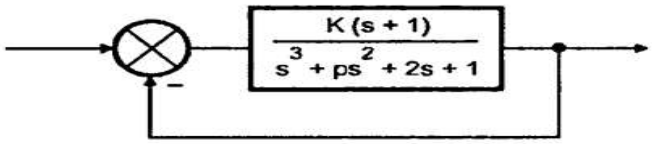
Put $s^2 = y$

$\therefore$ $y^2 + 6y + 8 = 0$

$$y = -6 \pm \frac{\sqrt{36-32}}{2}$$



Example 8.10 : *A given system oscillates with frequency 2 rad/sec. Find values of `$K_{mar}$' and `p'. No poles are in R.H.S.* **(M.U. : May-1996)**



Advantages & Limitations Of RH Criteria:

Advantages:

- We can find the stability of the system without solving the characteristic equation for any order system.
- We can easily determine the absolute and conditional stability of the system.
- By this method, we can determine the range of K for stability.
- By this method, we can also determine the point of intersection for root locus with an imaginary axis.

Limitations:

- This criterion is applicable only for a linear system.
- It does not provide the exact location of poles on the right and left half of the S plane so relative stability can not be determined. And also it cannot distinguish between real & complex roots.
- It is valid only for real coefficients of the characteristic equation

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	18	5	.... Row-1
s^3	:	8	16		.... Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.

s^4	:	1	18	5	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	16	5		.... Row-3
s^1	:	1.7			.... Row-4
s^0	:	5			.... Row-5

Column-1

$s^2: \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$

$s^2: 16 \quad 5$

$s^1: \frac{16\times2-5\times1}{16}$

$s^1: 1.6875 \approx 1.7$

$s^0: \frac{1.7\times5-0\times16}{17}$

$s^0: 5$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

s^6 :	1	8	20	16	.... Row-1
s^5 :	1	6	8		....Row-2
s^4 :	1	6	8		....Row-4
s^3 :	0	0			....Row-4
s^3 :	1	3			....Row-4
s^2 :	3	8			....Row-5
s^1 :	0.33				....Row-6
s^0 :	8				....Row-7

Column-1

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4+6s^2+8=0$$

Let, $s^2=x$

$$\therefore x^2+6x+8=0$$

The roots of quadratic are, $x=\dfrac{-6\pm\sqrt{6^2-4\times8}}{2}$

$$=-3\pm1=-2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s=\pm\sqrt{x}=\pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$=+j\sqrt{2},-j\sqrt{2},+j2 \text{ and } -j2$$

s^4 : $\dfrac{1\times8-6\times1}{1}$ $\dfrac{1\times20-8\times1}{1}$ $\dfrac{1\times16-0\times1}{1}$

s^4 : 2 12 16

divide by 2

s^4 : 1 6 8

s^3: $\dfrac{1\times6-6\times1}{1}$ $\dfrac{1\times8-8\times1}{1}$

s^3: 0 0

The auxiliary equation is, $A = s^4+6s^2+8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds}=4s^3+12s$$

The coefficients of $\dfrac{dA}{ds}$ are used to form s^3 row.

s^3: 4 12

divide by 4

s^3: 1 3

s^2: $\dfrac{1\times6-3\times1}{1}$ $\dfrac{1\times8-0\times1}{1}$

s^2: 3 8

s^1: $\dfrac{3\times3-8\times1}{3}$

s^1: 0.33

s^0: $\dfrac{0.33\times8-0\times3}{0.33}$

s^0: 8

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

Construct Routh array and determine the stability of the system represented by the characteristic equation, $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

s^3	:	ϵ	-2	.... Row-3
s^2	:	$\frac{2\epsilon+2}{\epsilon}$	5	.... Row-4
s^1	:	$\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$		.... Row-5
s^0	:	5		.... Row-6

On letting $\epsilon \to 0$, we get

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	0	-2		.... Row-3
s^2	:	∞	5		.... Row-4
s^1	:	-2			.... Row-5
s^0	:	5			.... Row-6

− to +

+ to −

Column-1

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

s^3: $\frac{1\times 2-2\times 1}{1}$ $\frac{1\times 3-5\times 1}{1}$

s^3: 0 -2

Replace 0 by ϵ

s^3: ϵ -2

s^2: $\frac{\epsilon\times 2-(-2\times 1)}{\epsilon}$ $\frac{\epsilon\times 5-0\times 1}{\epsilon}$

s^2: $\frac{2\epsilon+2}{\epsilon}$ 5

s^1: $\frac{\frac{2\epsilon+2}{\epsilon}\times(-2)-(5\times\epsilon)}{\frac{2\epsilon+2}{\epsilon}}$

s^1: $\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$

s^0: $\frac{\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}\times 5-0\times\frac{2\epsilon+2}{\epsilon}}{\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}}$

s^0: 5

RESULT

(a). The system is unstable.

(b). Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s) = \frac{K}{s(s+1)(s+2)}$.

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$

The characteristic equation is, $s(s+1)(s+2)+K=0$

$\therefore s(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

Column-1

$s^1 : \frac{3\times 2 - K\times 1}{3}$
$s^1 : \frac{6-K}{3}$
$s^0 : \frac{\frac{6-K}{3}\times K - 0\times 3}{(6-K)/3}$
$s^0 : K$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

ROOT LOCUS

Objectives

Upon completion of this topic you will be able to:

- Plot the Root Locus for a given Transfer Function by varying gain of the system.
- Analyze the stability of the system from the root locus plot.
- Determine K parameter for any given point on rootlocus

Def: **Root Locus** is the locus of the roots of characteristic equation *of the closed-loop system* when the gain K changes from 0 to ∞.

Root Locus Concept:

Consider open loop transfer function of a system $G(s) = \frac{K}{s(s+4)}$

Open loop poles of the system are S=0, -4

Characteristic eqn, of the system is

$$S^2 + 4S + K = 0$$

The roots of the eqn. (closed loop poles) are

$$S_1, S_2 = -2 \pm \sqrt{(4-K)}$$

Let $S_1 = -2 + \sqrt{(4-K)}$ & $S_2 = -2 - \sqrt{(4-K)}$

At K=0 $S_1 = 0$ & $S_2 = -4$

Therefore at K=0 closed loop poles are same as open loop poles.

Root Locus starts from open loop poles.

If we vary K value poles moves from these values.

For 0<K<4

S1 starts decreasing from 0 and S2 starts increasing from -4.

For example let K= 1.

$S_1=$ & $S_2=$

For example let K= 2.

$S_1=$ & $S_2=$

S1 moves left side and S2 moves right sides from their initial positions.

At K=4 $S_1 = S_2 = -2$

Double pole exist at K=4.

The two poles meets at this point.

At K>4 S_1 & S_2 becomes complex numbers so locus moves to 2 & 3rd quadrants.

If we increase K value no change in real part but imaginary value increases with K. So locus forms vertical line.

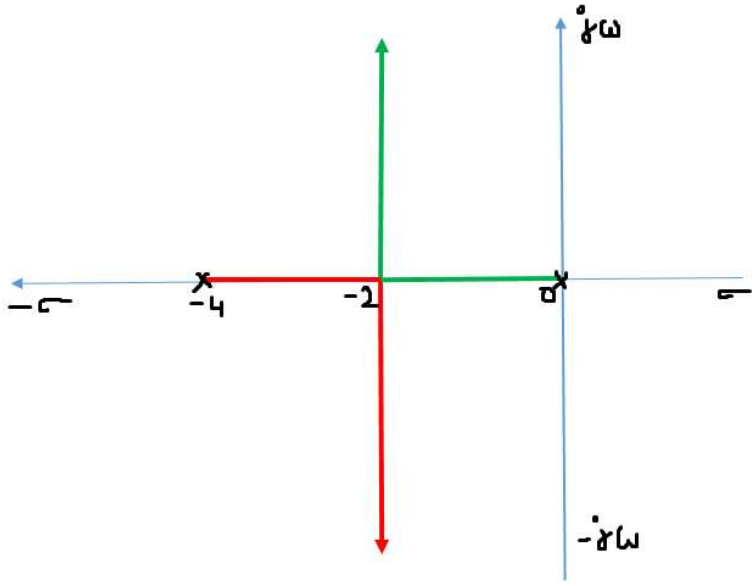
For example let K= 10.

S_1= & S_2=

For example let K= 50.

S_1= & S_2=

At K=α imaginary part of S_1& S_2 becomes infinity. The locus ends at infinity. So the complete rootlocus is as shown in figure.



No. of Rootlocus branches = No. of poles

Starting point of RL branches are Open loop poles.

End points of RL branches are infinity or Zeros of the system.

Break points are the points on root locus where root locus branches meets. At these points multiple poles exist for characteristic eqn.

Angle and Magnitude Criteria

The characteristic equation of a closed loop system is

$$1 + G(s)\,H(s) = 0 \quad .....(1)$$

The values of '**s**' that satisfy this equation are part of the root locus. The above eq. can be written as

$$G(s)\,H(s) = -1$$

The values of '**s**' that satisfy this equation lies on the root locus. We can split this equation in to two equations

$$|G(s)H(s)| = 1 \quad .....(2)$$

$$\angle G(s)H(s) = \pm 180^\circ (2q+1)\ ,\ q = 0, 1, 2. .....(3)$$

Equation (2) is known as **Magnitude condition** which states that any point S= S_a

will be a point on rootlocus if magnitude of G(s)H(s) is equal to 1 for that value of S.

Equation (3) is known as **Angle condition** which states that any point S= S_a will be a point on rootlocus if angle of G(s)H(s) is odd multiple of 180^0 for that value of S.

Generally Angle condition is used to check whether a point lies on rootlocus or not and magnitude condition is used to determined the corresponding K value for a given point on rootlocus.

Let $$\mathbf{G(s)\,H(s) = \frac{K(S+Z1)(S+Z2)(S+Z3)(S+Z4).....}{(S+P1)(S+P2)(S+P3)(S+P4).....}}$$

Magnitude condition:

$$\mathbf{G(s)\,H(s) = -1}$$

Therefore

$$\mathbf{\frac{K|S+Z1||S+Z2||S+Z3||S+Z4|\,.....}{|S+P1||S+P2||S+P3||S+P4|\,.....} = 1}$$

$$\mathbf{\frac{K\prod_{i=1}^{m}|S+Z_i|}{\prod_{i=1}^{n}|S+P_i|} = 1}$$

Therefore

$$\mathbf{K = \frac{\prod_{i=1}^{n}|S+P_i|}{\prod_{i=1}^{m}|S+Z_i|}}$$

Where $|S+P_i|$ represents the length of vector from point $S=S_a$ to all open loop poles and $|S+Z_i|$ represents the length of vector from point $S=S_a$ to all open loop zeros. So

$$\mathbf{K = \frac{\text{Product of length of vectors from point } S = S_a \text{ to all open loop poles}}{\text{Product of length of vectors from point } S = S_a \text{ to all open loop zeros}}}$$

The above expression can be used to determine K value for any given point on rootlocus.

Angle Condition:

$$\angle G(s)H(s) = \pm 180^\circ\,(2q+1) \qquad for\, q = 0, 1, 2 \ldots$$

$$(\angle(s+Z_1) + \angle(s+Z_2) + \angle(s+Z_3) \ldots.) - (\angle(s+P_1) + \angle(s+P_2) + \angle(s+P_3) + \cdots.) = \pm 180^\circ\,(2q+1)$$

$$\sum \angle(s+Z_i) - \sum \angle(s+P_i) = \pm 180^\circ\,(2q+1)$$

Where $\angle(s+P_i)$ represents the angle made by vectors from point S=Sa to all open loop poles with +ve real axis and $\angle(s+Z_i)$ represents the angle made by vectors from point S=Sa to all open loop zeros with +ve real axis. This condition is used to determine whether a point lies on rootlocus or not.

Ex: Check whether s=-1+j $\sqrt{3}$ **lies on the root locus** of a system or not whose open loop transfer function $G(s)H(s) = \frac{K}{(s+1)(s+2)(s+4)}$. If it does so then find the gain K at this point.

Sol :- To find whether s=-1+j $\sqrt{3}$ lies on the root locus or not we use angle criterion.

$\angle$ G(s)H(s) at s=-1+j $\sqrt{3}$ is

$$= \frac{\angle K}{\angle(j\sqrt{3})\angle(1+j\sqrt{3})\angle(3+j\sqrt{3})}$$

$$= \tan^{-1}\left[\frac{0}{K}\right] - \tan^{-1}\left[\frac{\sqrt{3}}{0}\right] - \tan^{-1}\left[\frac{\sqrt{3}}{1}\right] - \tan^{-1}\left[\frac{\sqrt{3}}{3}\right]$$

$$= 0 - 90^0 - 60^0 - 30^0$$

$$= -180^0$$

i.e $\angle G(s)H(s)$ an odd multiple of $\pm 180^0$. Therefore, point lies on the root locus.

Lets find the gain at this point using magnitude criterion.

$|G(s)H(s)| = 1$

$$\left| \frac{K}{(j\sqrt{3})(1+j\sqrt{3})(3+j\sqrt{3})} \right| = 1$$

$$\frac{K}{\sqrt{3}\times 2\times\sqrt{12}} = 1 \qquad \text{or} \qquad K=12$$

Rules for Construction of Root Locus

Rule 1: Symmetry of Root Locus

The root locus is symmetric with respect to the real axis.

Rule 2: Starting and End points of Rootlocus

The number of root locus branches = n = The number of finite open loop poles
The root locus originates at the open loop poles (for K =0) and terminates at the Open loop zeros or infinity (as $K = \infty$).
Let m is the number of open loop zeros.
Then m rootlocus branches terminates at open loop zeros and remaining (n-m) branches terminates at infinity.

Rule 3: Rootlocus on Real axis

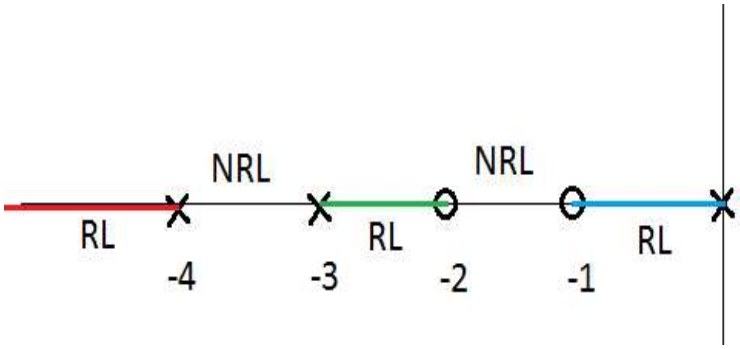
The open loop poles and zeros divided the rootlocus in to number of segments. If **odd number of open loop poles and zeros exist to the right side of a segment** on the real axis, then that **segment is part of root locus**.

Ex:

$$G(S) = \frac{K(S+1)(S+2)}{S(S+3)(S+4)}$$

No. of Open loop poles= n S=0,-3,-4
No. of Open loop zeros= m S=-1,-2
No. of rootlocus branches = n=3.
The three rootlocus branches starts from three poles and two branches terminates at zeros s= -1 & -2 and remaining one branch terminates at infinity.
These open loop poles and zeros divides the real axis into number of segments.

The colored segments are part of rootlocus(RL) because they have odd number of poles and zeros to their side.



Rule 4: Centroid and Angle of Asymptotes

We know that (n-m) of the root locus branches terminates at infinity. They do so in parallel with Asymptotes. The intersection point of asymptotes with the real axis is known as **centroid**.

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n - m)}$$

The **Angle of asymptotes Φ_a** is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n\text{-}m)$$

Example:

$$G(S) = \frac{K(S+3)}{S(S+1)(S+2)(S+4)}$$

Clearly, this open loop transfer function has three poles at s= 0, −1, −2,-4. It has one zero at s=-3.

centroid σ_a = [0 + (−1) + (−2) + (−4)] − [(−3)] /(4 – 1) = −4/3

Angle of asymptotes Φ_a=±(2q+1) 180^0/(4-1)

=±(2q+1) 180^0/3

=± 60^0 for q=0

=± 180^0 for q=1

Rule 5: Break-away and Break-in points

The intersection point of two or more rootlocus branches is known as **Break point**.

The breakaway or break-in points are the points at which multiple roots exist for the characteristic equation 1 + G(s)H(s) = 0.

The value of K is maximum at the points where the branches of root loci break away. The value of K is minimum at the points where the branches of root loci break in. Break points may be real, imaginary or complex.

Write K in terms of s from the characteristic equation 1+G(s)H(s)=0

Break points are roots of the equation

$$\frac{dK}{dS} = 0$$ **For which K is positive real.**

If part of the root locus resides between two adjacent poles on the real axis, then there exists a break-away point between these two poles.

If part of the root locus resides between two adjacent zeros(one can be at infinity) on the real axis, then there exists a break-in point between these two zeros.

Rule 6: Angle of Departure and the Angle of Arrival

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

Angle of Departure: The initial angle made by rootlocus branch with real axis when leaving a complex pole is known as angle of departure.

The formula for the **angle of departure** θ_d is

$$\theta_d = 180^0 + \phi$$

Angle of Arrival: The angle made by rootlocus branch with real axis when arriving at a complex zero is known as angle of arrival.

The formula for the **angle of arrival** θ_a is

$$\theta_a = 180^0 - \phi$$

Where ϕ is net angle contribution by a complex pole or zero which given by sum of angles made by the vectors from the given complex pole or zero to all other poles and zeros.

$$\phi = \sum\theta_Z - \sum\theta_P$$

Rule 7: Intersection point with imaginary axis

Substitute $S=j\omega$ in the characteristic equation

$$1+G(S)H(S)=0$$

Equate the real terms to zero and imaginary terms to zero. Solve these two equations for K & **ω.** (Routh Hurwitz also can be used.)

Rule 8: Determination of K for a point on rootlocus

For any given point $S=S_a$ on rootlocus corresponding K value can be determined using the below expression.

$$\mathbf{K} = \frac{\text{Product of length of vectors from point } S = S_a \text{ to all open loop poles}}{\text{Product of length of vectors from point } S = S_a \text{ to all open loop zeros}}$$

Example1: Draw rootlocus for the system whose OLTF is

$$G(S)H(S) = \frac{K}{S(S+3)}$$

Solution:

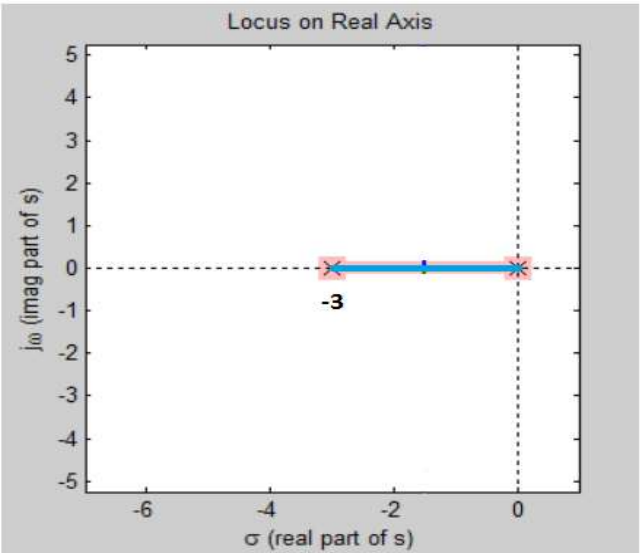
$$G(S)H(S) = \frac{K}{S(S+3)}$$

number of open loop poles =n=2 & open loop pole are S=0,-3
number of open loop zeros =m=0

Starting and End points of Rootlocus

The number of root locus branches = n = 2
The root locus originates at the open loop poles i.e at S=0,-3.
2 rootlocus branches terminates at infinity(since no open loop zeros).

Rootlocus on Real axis



Centroid and Angle of Asymptotes

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n - m)}$$

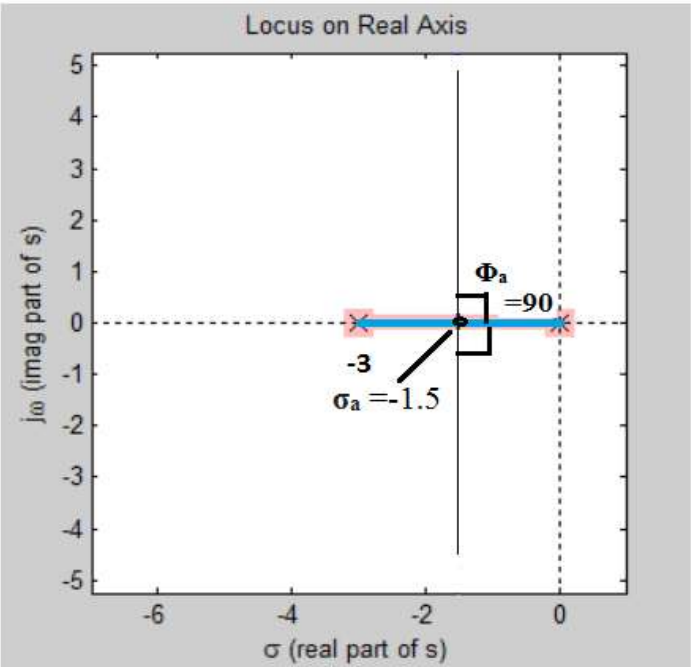
σ_a = (-3-0)/(2-0) = -1.5

The **Angle of asymptotes** Φ_a is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n\text{-}m)$$

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{2}$$

$= \pm 90^0$ for q=0



Break-away and Break-in points

$$G(S)H(S) = \frac{K}{S(S+3)}$$

characteristic equation 1+G(s)H(s)=0

$$1 + \frac{K}{S(S+3)} = 0$$

K= -S(S+3) ….(1)

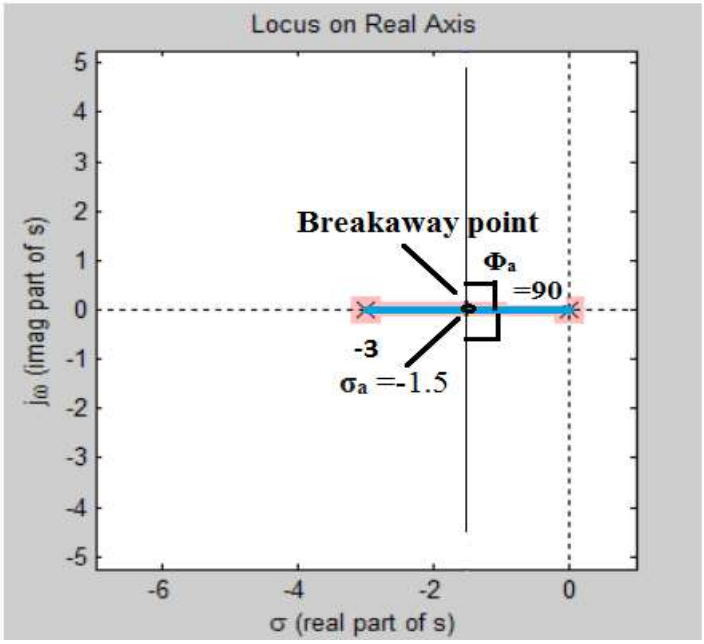
$$\frac{dK}{dS} = 0$$

-2S-3=0

Root of this equation is **S=-1.5**

Substitute **S=-1.5** in eq(1) k=2.25 which is +ve real.

Therefore **S=-1.5** is a breakaway point.



Angle of Departure and the Angle of Arrival

No complex conjugate open loop poles and complex conjugate open loop zeros for the given Transfer function.

Intersection point with imaginary axis

The characteristic equation is

$$1+G(S)H(S)=0$$

$$1+\frac{K}{S(S+3)}=0$$

$$S^2+3S+K=0$$

Substitute S=jω in the characteristic equation.

$$(j\omega)^2+3(j\omega)+K=0$$

$$-\omega^2+j3\omega+K=0$$

Equating the real terms to zero

$$-\omega^2 + K = 0$$
$$K = \omega^2$$

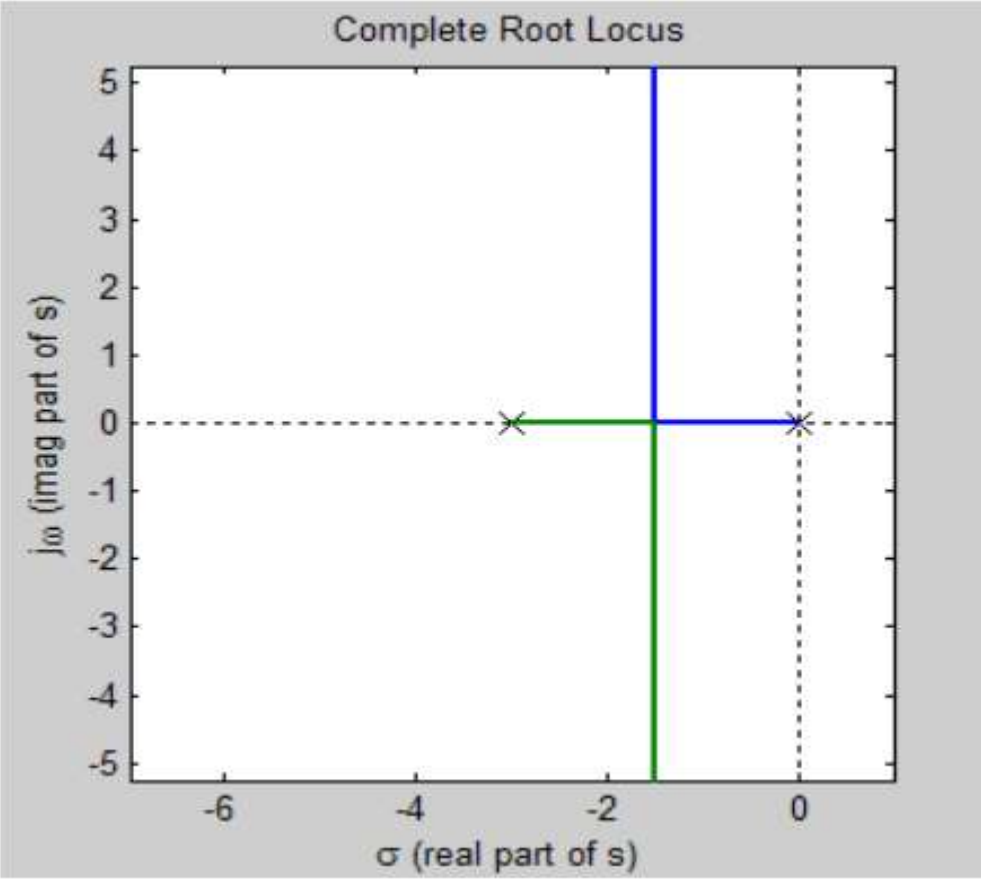
Equating imaginary terms to zero.

$$3\omega = 0$$

Therefore $\omega = 0$ and $K = 0$.

Therefore no intersection point with imaginary axis(origin is the intersection point).

Complete Rootlocus:



Example2: Draw rootlocus for the system whose OLTF is

$$G(S) = \frac{K}{S(S^2 + 5S + 6)}$$

Solution:

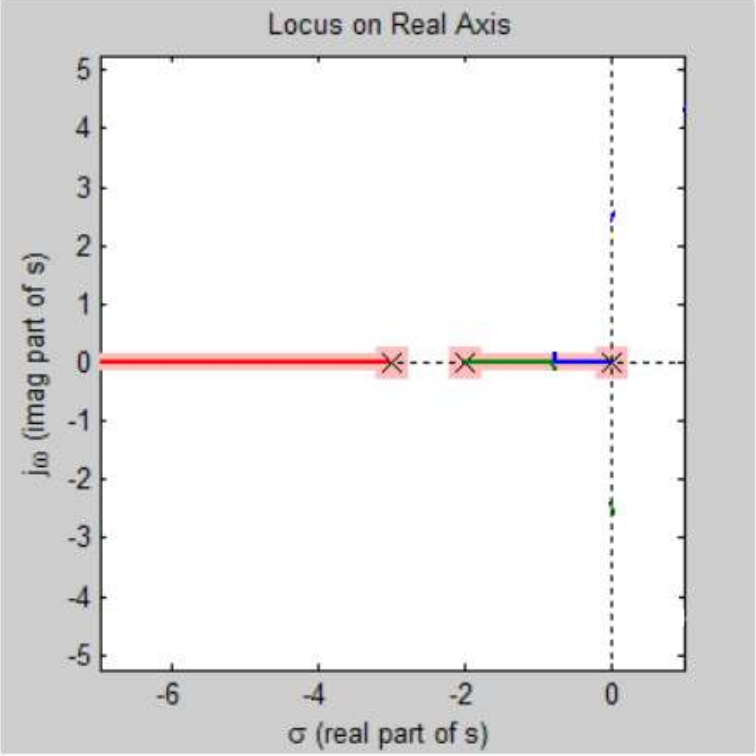
$$G(S) = \frac{K}{S(S^2 + 5S + 6)} = \frac{K}{S(S+2)(S+3)}$$

number of open loop poles =n=3 & open loop pole are S=0,-2,-3
number of open loop zeros =m=0

Starting and End points of Rootlocus

The number of root locus branches = n = 3
The root locus originates at the open loop poles i.e at S=0,-2,-3.
All 3 rootlocus branches terminates at infinity(since no open loop zeros).

Rootlocus on Real axis



Centroid and Angle of Asymptotes

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n - m)}$$

σ_a = (-2-3-0)/(3-0) = -1.667

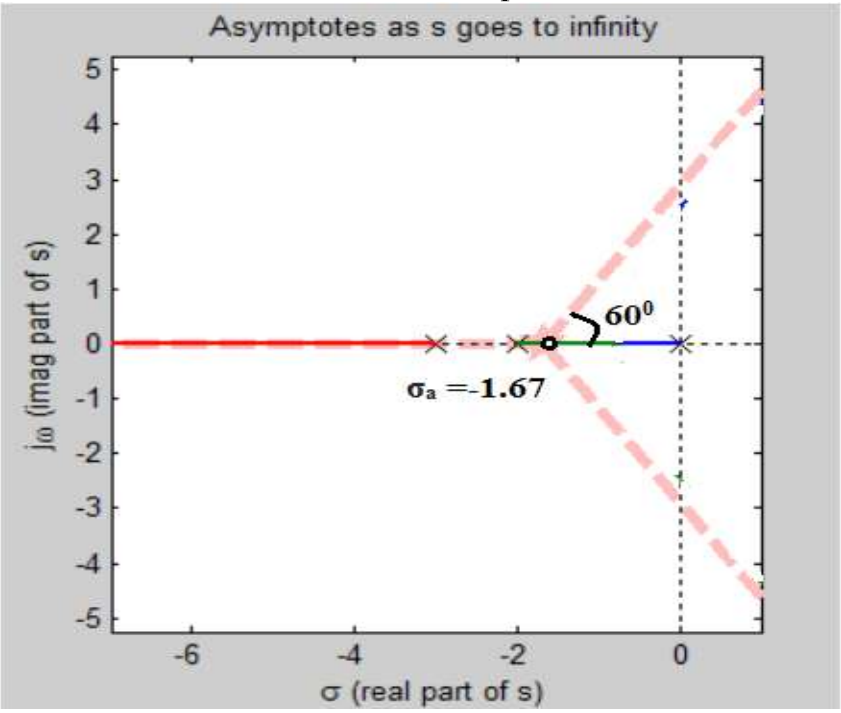
The **Angle of asymptotes Φ_a** is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n\text{-}m)$$

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{3}$$

$= \pm 60^0$ for q=0

$= \pm 180^0$ for q=1



Break-away and Break-in points

$$G(S) = \frac{K}{S(S^2 + 5S + 6)}$$

characteristic equation 1+G(s)H(s)=0

$$1 + \frac{K}{S(S^2 + 5S + 6)} = 0$$

$$K = -S(S^2 + 5S + 6) \quad \ldots.(1)$$

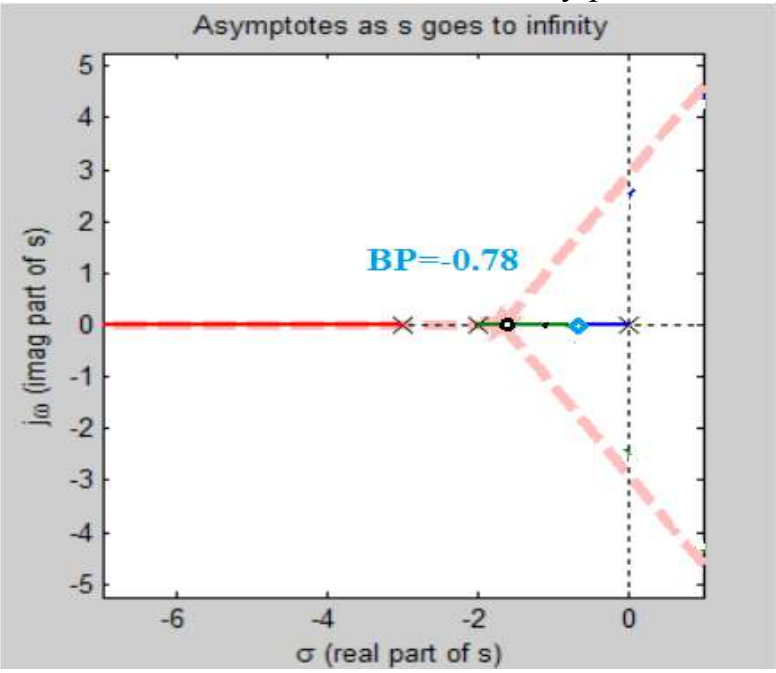
$$\frac{dK}{dS} = 0$$

$$-2S^2-10S-6=0$$

Roots of this equation are s = -2.5, -0.78.

Substitute **S=-0.78 in** eq(1) results in +ve real K.

Therefore **S=-0.78** is a breakaway point.



Angle of Departure and the Angle of Arrival

No complex conjugate open loop poles and complex conjugate open loop zeros for the given Transfer function.

Intersection point with imaginary axis

The characteristic equation is

$$1+G(S)H(S)=0$$

$$1+\frac{K}{S(S+2)(S+3)}=0$$

$$S^3+5S^2+6S+K=0$$

Substitute S=jω in the characteristic equation.

$$(j\omega)^3+5(j\omega)^2+6(j\omega)+K=0$$

$$-j\omega^3-5\omega^2+j6\omega+K=0$$

Equating the real terms to zero

$$-5\omega^2+K=0$$

$$K=5\omega^2$$

Equating imaginary terms to zero.

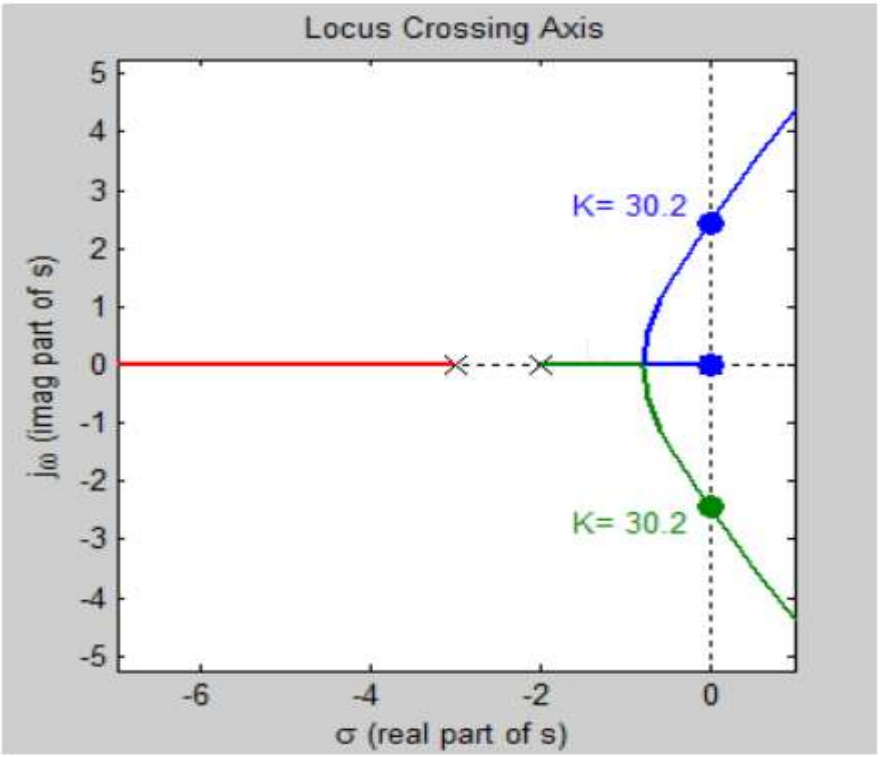
$$-\omega^3+6\omega=0$$

$$\omega^2=6 \ \&$$

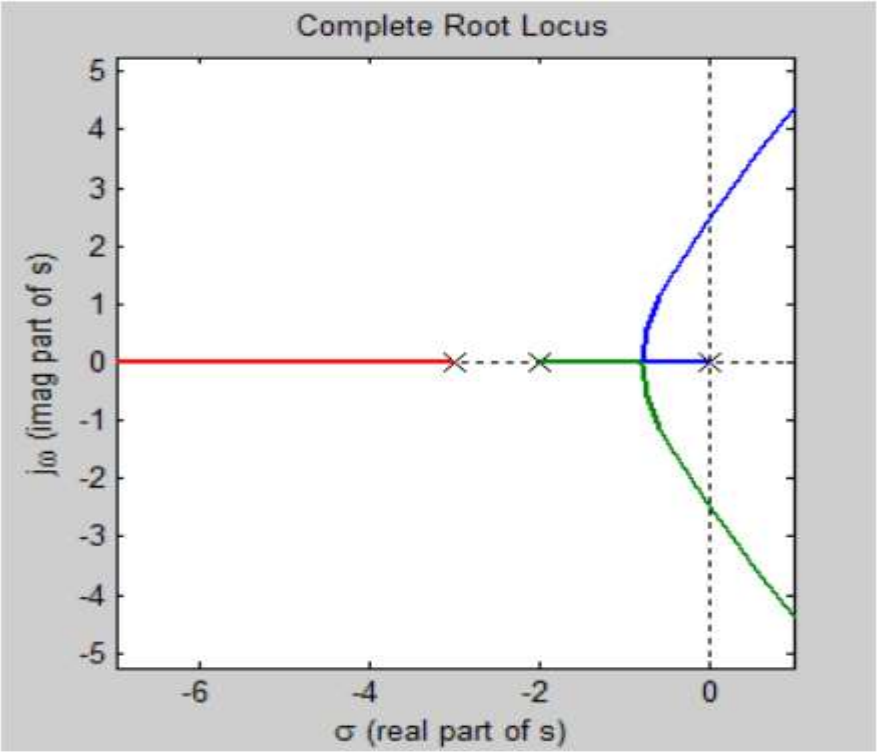
$$\omega=2.45 \text{ rad/sec}$$

Therefore K=30.

Therefore intersection points with imaginary axis are s= ±j 2.45.



Complete Rootlocus:



Example3: Draw rootlocus for the system whose OLTF is

$$G(S) = \frac{K(S+1)}{S^3+4S^2+6S+4}$$

Solution:

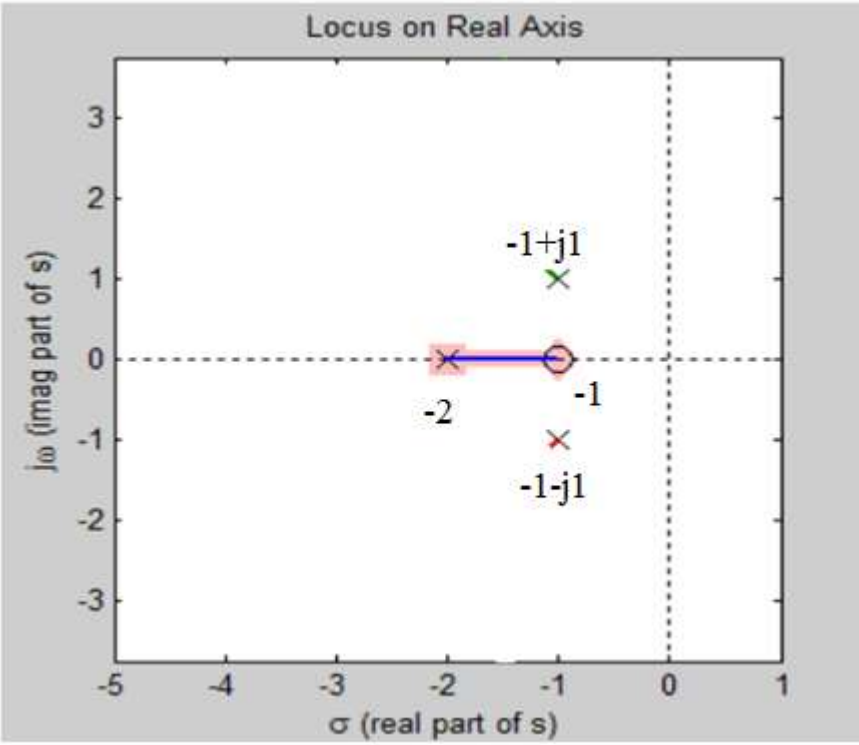
$$G(S) = \frac{K(S+1)}{S^3+4S^2+6S+4}$$

number of open loop poles =n=3 & open loop pole are S = -2, -1 ± j1.
number of open loop zeros =m=1& open loop zero is S = -1.

Starting and End points of Rootlocus

The number of root locus branches = n = 3
The root locus originates at the open loop poles i.e at S = -2, -1 ± j1.
1 rootlocus branch terminates at open loop zero i.e at S = -1 and remaining 2 rootlocus branches terminates at infinity.

Rootlocus on Real axis



Centroid and Angle of Asymptotes

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n - m)}$$

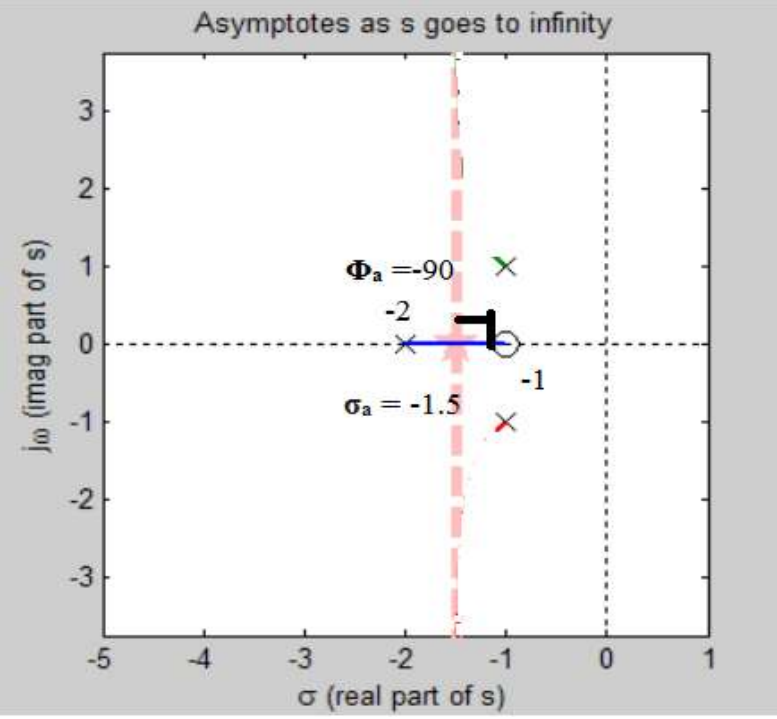
σ_a = (-2-1-1-(-1))/(3-1) = -1.5

The **Angle of asymptotes** Φ_a is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n\text{-}m)$$

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{2}$$

$= \pm 90^0$ for q=0



Break-away and Break-in points

$$G(S) = \frac{K(S+1)}{S^3+4S^2+6S+4}$$

characteristic equation 1+G(s)H(s)=0

$$1 + \frac{K(S+1)}{S^3+4S^2+6S+4} = 0$$

$$K = -\frac{S^3+4S^2+6S+4}{(S+1)} \ldots\ldots\ldots (1)$$

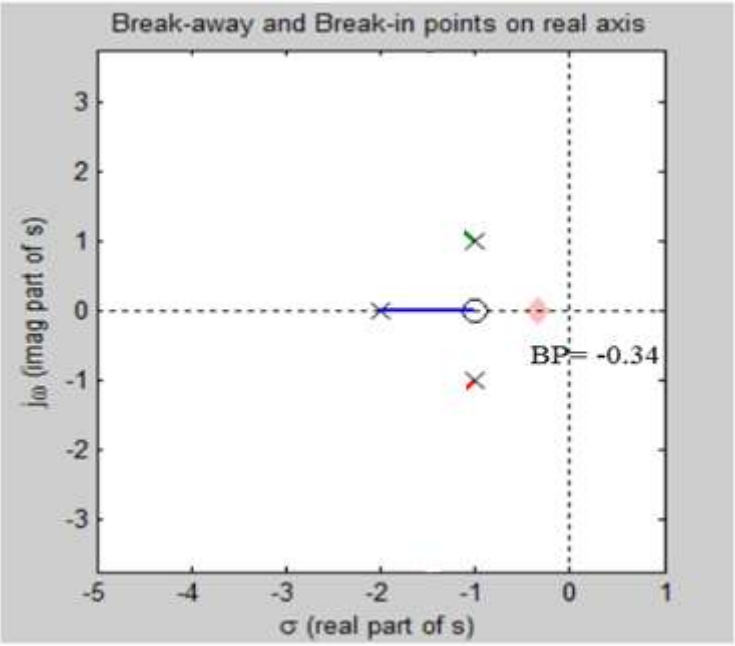
$$\frac{dK}{dS} = 0$$

$$2s^3 + 7s^2 + 8s + 2 = 0.$$

Roots of this equation are s = s = -1.6 ±0.65j, -0.34.

Substitute **S= -0.34 in** eq(1) results in +ve real K.

Therefore **S= -0.34** is a breakaway point.



Angle of Departure and the Angle of Arrival

G(S) has complex conjugate open loop poles so Angle of Departure need to be calculated.

The formula for the **angle of departure** θ_d is

$$\theta_d = 180^0 + \phi$$

Where ϕ is net angle contribution by a complex pole or zero which given by sum of angles made by the vectors from the given complex pole or zero to all other poles and zeros.

$$\phi=\sum\theta_Z-\sum\theta_P$$

consider pole A= -1+j1. Draw vectors from this pole to remaining poles and zeros.

$$\phi=\theta_{Z1}-\theta_{P1}-\theta_{P3}$$

where $\theta_{Z1} = 90^0$

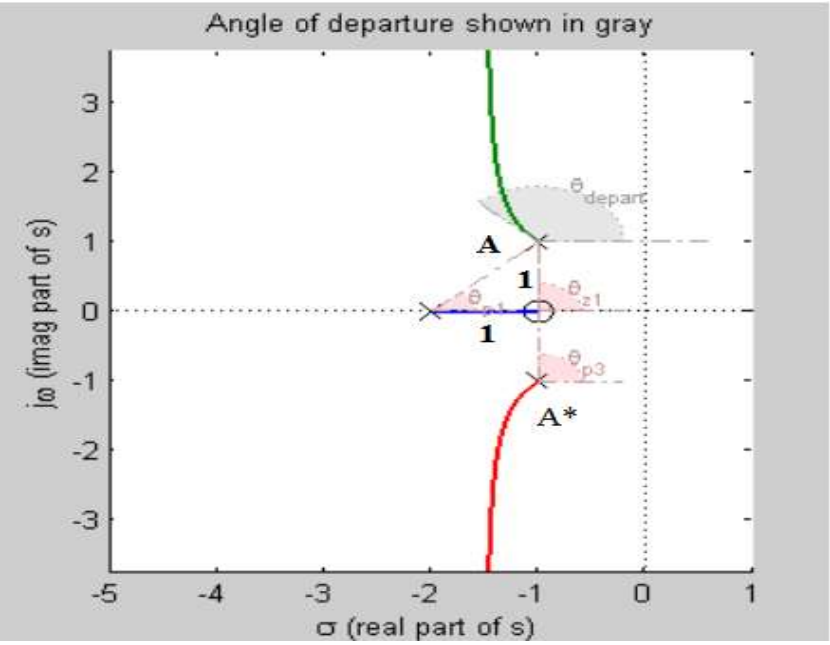
$\theta_{P1} = \tan^{-1}(1/1)= 45^0$

$\theta_{P3} = 90^0$

Therefore $\phi= -45^0$

and $\theta_d = 180^0 + \phi = \mathbf{135^0}$ (This angle is shown in gray.)

Therefore Angle of departure at pole A*= $\mathbf{-135^0}$



Intersection point with imaginary axis

The characteristic equation is

$$\mathbf{1+G(S)H(S)=0}$$

$$\mathbf{1+\frac{K(S+1)}{S^3+4S^2+6S+4}=0}$$

$$\mathbf{s^3 + 4\,s^2 + 6\,s + 4 + K(\,s + 1\,) = 0}$$

Substitute S=jω in the characteristic equation.

$$\mathbf{(j\omega)^3+4(j\omega)^2+(6+K)(j\omega)+4+K=0}$$

$$-j\omega^3-4\omega^2+j(6+K)\omega+4+K=0$$

Equating the real terms to zero

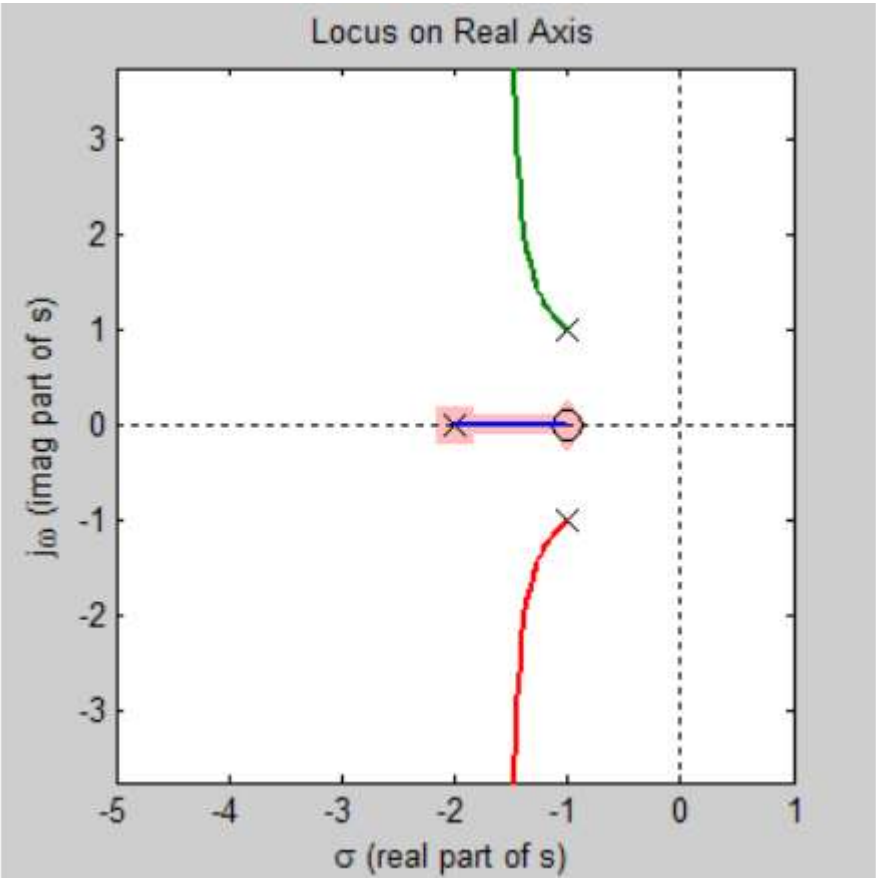
$$-4\omega^2+4+K=0$$
$$4\omega^2=(4+K)$$

Equating imaginary terms to zero.

$$-\omega^3+(6+K)\omega=0$$
$$\omega^2=6+K$$
$$4(6+K)=(4+K)$$
$$24+4K=4+K$$
$$3K=-20$$
$$K=-20/3$$

K is not a positive real number so no intersection point with imaginary axis.

Complete Rootlocus:



Example4: Draw rootlocus for the system whose OLTF is

$$G(S) = \frac{K(S+3)}{S^2 - S - 2}$$

Solution:

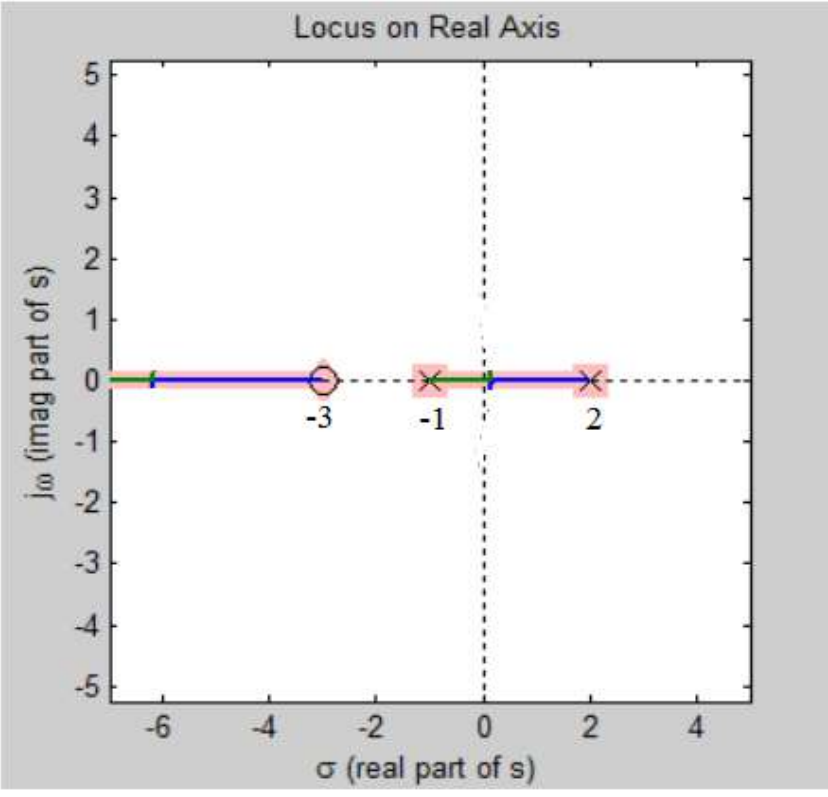
$$G(S) = \frac{K(S+3)}{S^2 - S - 2} = \frac{K(S+3)}{(S-2)(S+1)}$$

number of open loop poles =n=2 & open loop pole are S=2,-1.
number of open loop zeros =m=1& open loop zero is S=-3.

Starting and End points of Rootlocus

The number of root locus branches = n = 2
The root locus originates at the open loop poles i.e at S=2,-1.
1 rootlocus branch terminates at open loop zero i.e at S= -3 and remaining 1 rootlocus branch terminates at infinity.

Rootlocus on Real axis



Centroid and Angle of Asymptotes

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n - m)}$$

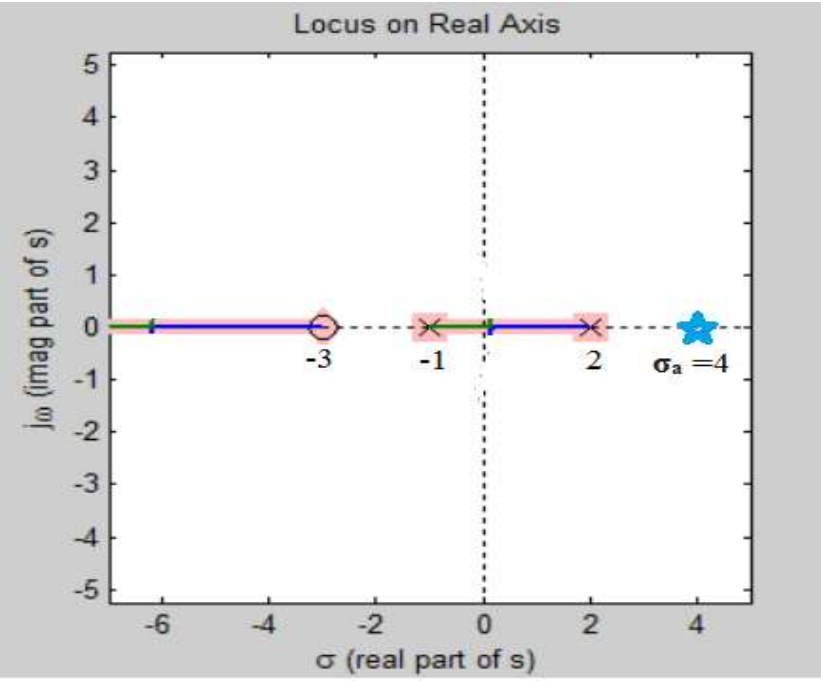
$\sigma_a = (2\text{-}1+3)/(2\text{-}1) = 4$

The **Angle of asymptotes Φ_a** is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n\text{-}m)$$

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{1}$$

$= \pm 180^0$ for q=0



Break-away and Break-in points

$$G(S) = \frac{K(S+3)}{S^2 - S - 2}$$

characteristic equation 1+G(s)H(s)=0

$$1 + \frac{K(S+3)}{S^2 - S - 2} = 0$$

$$K = -\frac{S^2 - S - 2}{S+3} \quad \text{.....(1)}$$

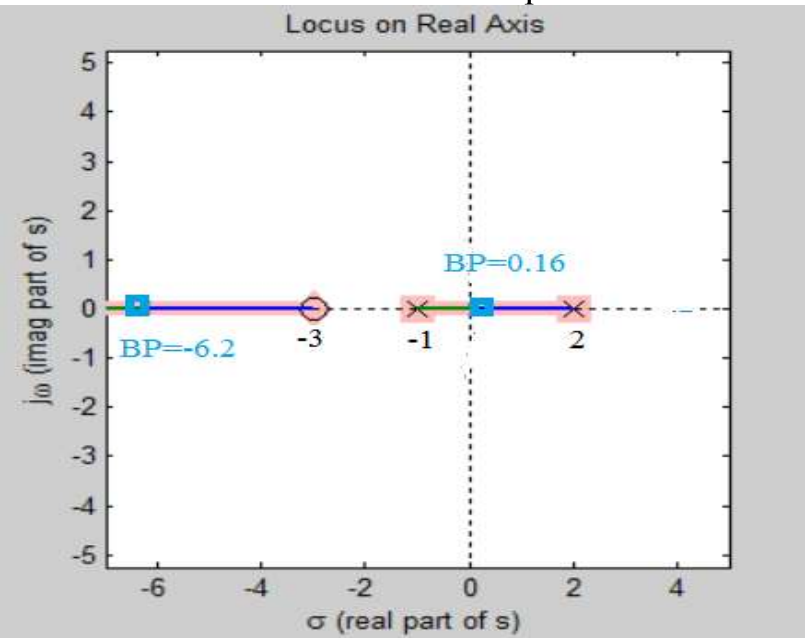
$$\frac{dK}{dS} = 0$$

$$s^2 + 6\,s - 1 = 0.$$

Roots of this equation are s = -6.2, 0.16.

Both results in +ve real K.

Therefore **both are** break points.



Angle of Departure and the Angle of Arrival

No complex conjugate open loop poles and complex conjugate open loop zeros for the given Transfer function.

Intersection point with imaginary axis

The characteristic equation is

1+G(S)H(S)=0

$$1+\frac{K(S+3)}{S^2-S-2}=0$$

$$S^2+(K-1)S-2+3K=0$$

Substitute S=jω in the characteristic equation.

$$(j\omega)^2+(K-1)(j\omega)+3K-2=0$$

$$-\omega^2+j(K-1)\omega+3K-2=0$$

Equating the real terms to zero

$$-\omega^2+3K-2=0$$

$$3K-2=\omega^2$$

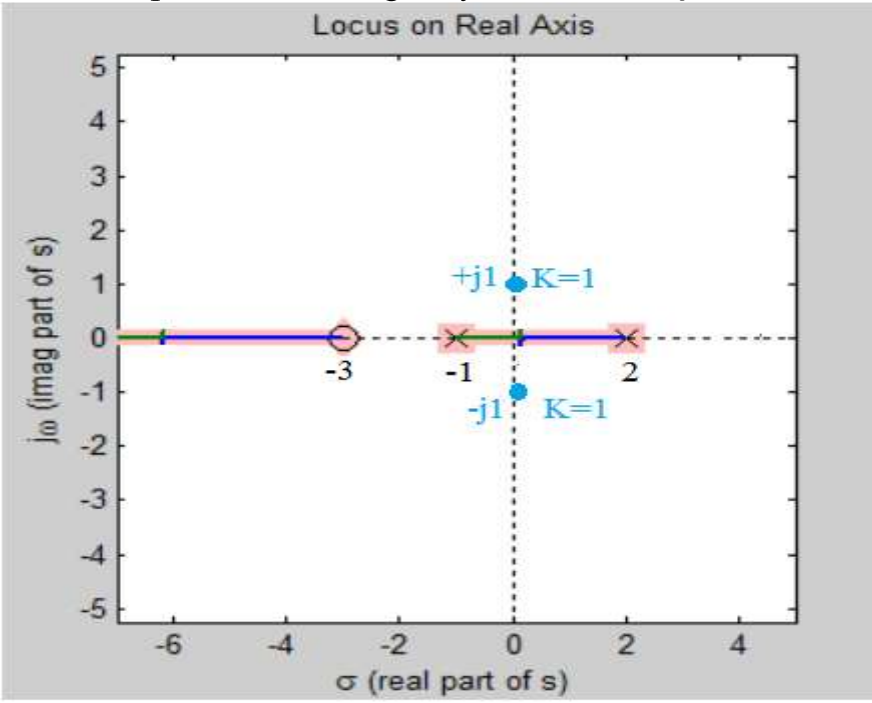
Equating imaginary terms to zero.

$$(K-1)\omega=0$$

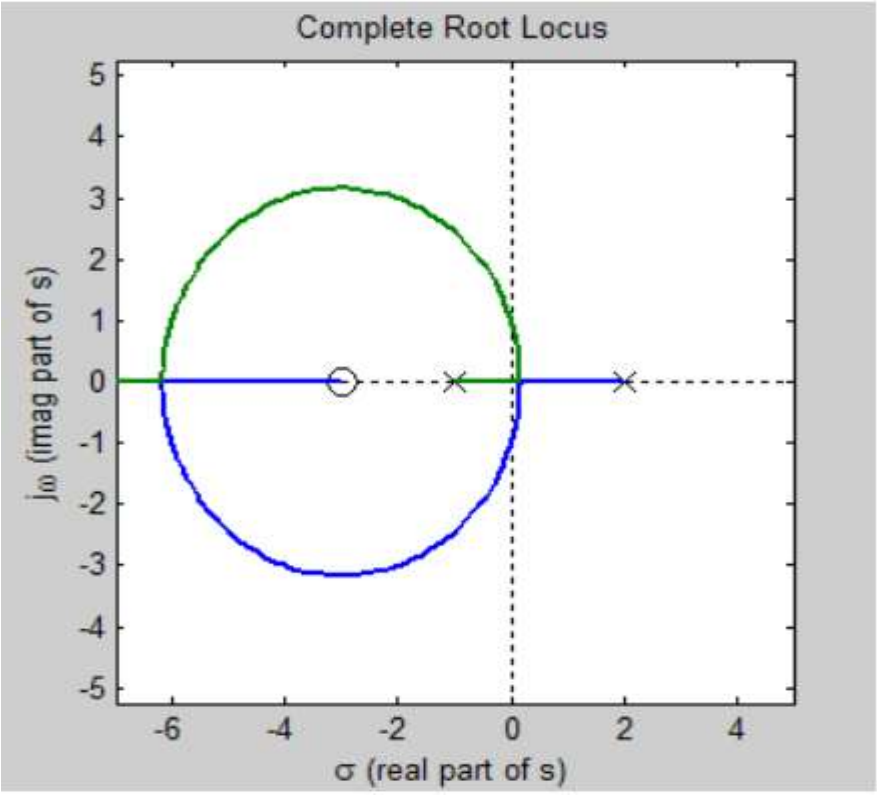
$$K-1=0$$

Therefore K=1 and ω = 1 rad/sec

Therefore intersection points with imaginary axis are s= ±j 1.



Complete Rootlocus:



Example 5: Draw rootlocus for the system whose OLTF is

$$G(S) = \frac{K}{S(S+4)(S^2+4S+20)}$$

Solution:

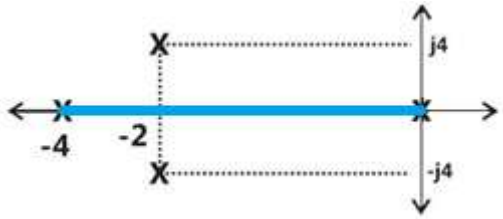
$$G(S) = \frac{K}{S(S+4)(S^2+4S+20)}$$

number of open loop poles =n=4 & open loop pole are S=0,-4,-2±j4.
number of open loop zeros =m=0.

Starting and End points of Rootlocus

The number of root locus branches = n = 4
The root locus originates at the open loop poles i.e at S=0,-4,-2±j4.
All 4 rootlocus branches terminates at infinity(Since no open loop zeros).

Rootlocus on Real axis



Centroid and Angle of Asymptotes

centroid σ_a is given by

$$\sigma_a = \frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{(n-m)}$$

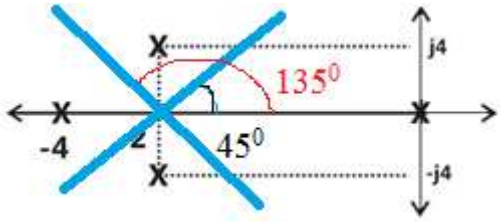
σ_a = (0-4-2-2)/(4-0) = -2

The **Angle of asymptotes** Φ_a is given by

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{(n-m)} \quad q = 0,1,2 \ldots (n-m)$$

$$\Phi_a = \frac{\pm(2q+1)180^\circ}{4}$$

$= \pm 45^0$ for q=0
$= \pm 135^0$ for q=1



Break-away and Break-in points

$$G(S) = \frac{K}{S(S+4)(S^2+4S+20)}$$

characteristic equation 1+G(s)H(s)=0

$$1 + \frac{K}{S(S+4)(S^2+4S+20)} = 0$$

$$K = -s^4 - 8s^3 - 36s^2 - 80s \qquad \text{.....(1)}$$

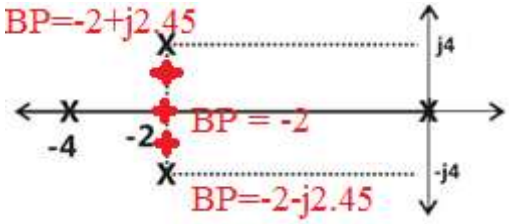
$$\frac{dK}{dS} = 0$$

$$4s^3 + 24s^2 + 72s + 80 = 0$$

Roots of this equation are s = -2,– 2± j2.45

All results in +ve real K.

Therefore **all are** break points.



Angle of Departure and the Angle of Arrival

G(S) has complex conjugate open loop poles so Angle of Departure need to be calculated.

The formula for the **angle of departure** θ_d is

$$\theta_d = 180^0 + \phi$$

Where ϕ is net angle contribution by a complex pole or zero which given by sum of angles made by the vectors from the given complex pole or zero to all other poles and zeros.

$$\phi=\sum\theta_Z-\sum\theta_P$$

consider pole A= -2+j4. Draw vectors from this pole to remaining poles and zeros.

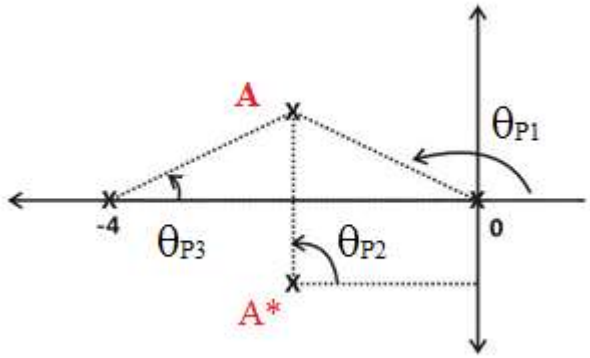
$$\phi=-\theta_{P1}-\theta_{P2}-\theta_{P3}$$

where $\theta_{P2} = 90^0$

$$\theta_{P1} = 180^0-\tan^{-1}(4/2)= 116.56^0$$

$$\theta_{P3} = \tan^{-1}(4/2)= 63.4^0$$

Therefore $\phi= -270^0$



and $\theta_d = 180^0 + \phi = \mathbf{-90^0}$ (This angle is shown in gray.)

Therefore Angle of departure at pole A*= $+90^0$

Intersection point with imaginary axis

The characteristic equation is

$$\mathbf{1+G(S)H(S)=0}$$

$$\mathbf{1+\frac{K}{S(S+4)(S^2+4S+20)}=0}$$

$$\mathbf{s^4+8s^3+36s^2+80s+K=0}$$

Substitute S=jω in the characteristic equation.

$$\mathbf{(j\omega)^2+(K-1)(j\omega)+3K-2=0}$$

$$\mathbf{(j\omega)^4+8(j\omega)^3+36(j\omega)^2+80(j\omega)+K=0}$$

$$\mathbf{\omega^4-j8\omega^3-36\omega^2+j80\omega+K=0}$$

Equating the real terms to zero

$$\omega^4 - 36\,\omega^2 + K = 0$$

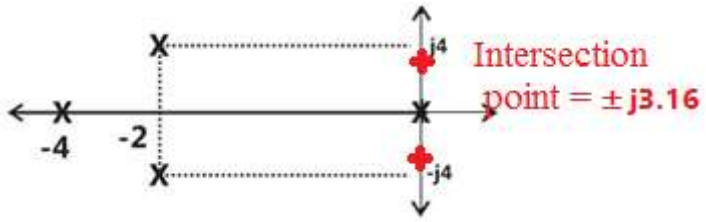
Equating imaginary terms to zero.

$$-8\omega^3 + 80\omega = 0$$

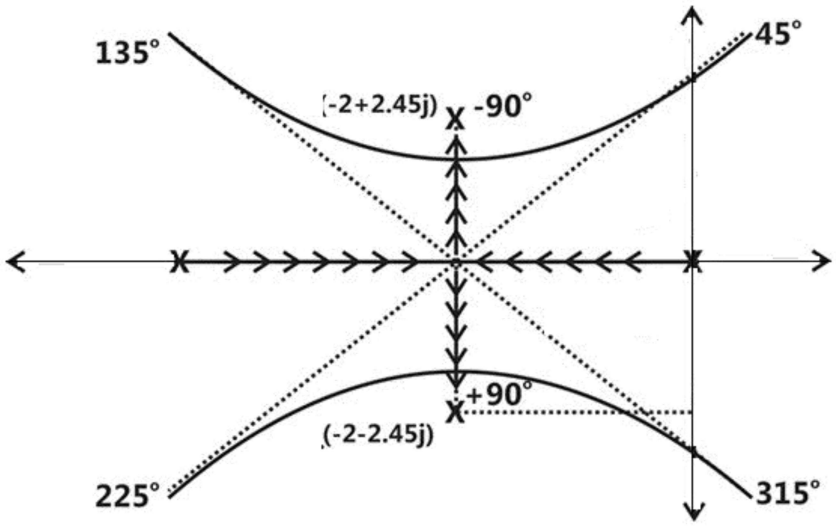
$$\omega^2 = 10 \text{ and } \omega = 3.16 \text{ rad/sec}$$

Therefore $K = 260$

Therefore intersection points with imaginary axis are $s = \pm j\,3.16$.



Complete Rootlocus:



ANNAMACHARYA UNIVERSITY, RAJAMPET

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016
RAJAMPET, Annamayya District, AP, INDIA

Department of Electrical & Electronics Engineering

III B.Tech I Semester

Linear Control Systems (24AEEE51T)
(Electrical and Electronics Engineering)

Unit-4 (Frequency Response Analysis)

Syllabus:

Frequency domain specifications-Definitions, Bode Diagrams-Stability Analysis from Bode Plots, Problems. Polar Plots-Stability Analysis from polar plots, problems-Nyquist stability criterion-Nyquist Plots, problems. Compensation techniques – Lag, Lead, and Lag-Lead Compensator design using Bode plots.

Frequency Response Analysis

Learning Outcomes:

After the completion of this unit, students are able to

1. Define frequency domain specifications
2. Draw bode plot for given transfer function
3. Draw polar plot for given transfer function
4. Determine stability from bode plot & polar plot
5. Determine stability of a system using nyquist stability criterion

We have already discussed time response analysis of the control systems and the time domain specifications of the second order control systems. In this unit, let us discuss the frequency response analysis of the control systems

Frequency Response:

The steady state response of a system for a sinusoidal input is known as the **frequency response**.

The steady state output of a system as a function of frequency for a sinusoidal input with constant amplitude and varying frequency.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

Let the input signal be −

$$r(\omega) = A \sin(\omega t)$$

then the output $c(\omega) = M \sin(\omega t + \theta) = M\angle\theta$

At each frequency, output has a magnitude and phase.

At $\omega=0$ $c = M_o\angle\theta_o$

At $\omega= \omega_1$ $c = M_1\angle\theta_1$

At $\omega= \omega_2$ $c = M_2\angle\theta_2$

At $\omega= \infty$ $c = M_\alpha \angle\theta_\alpha$

If we plot ‘c’ at different frequencies that is known as frequency response. There are many frequency responses viz- bode plot , polar plot, M&N circles etc.

Advantages of Frequency Domain Analysis

Some of the advantages of frequency domain analysis are:

1. Transfer functions which are complicated to determine the behavior of the experimentally can be determined using the frequency response analysis.
2. Design of the system and adjusting the parameters of the system can be easily carried out.
3. Corrective measurement for noise disturbance generated in the system and parameters variation can be easily determined using frequency analysis.
4. Absolute and Relative stability of the closed-loop system can be estimated from the knowledge of the open loop frequency system.
5. Frequency domain analysis can also be carried out for the nonlinear control systems.
6. Time response can be found from frequency response using fourier integral but limited to second order systems.
7. For complex systems probably Nyquist is the only method to determine stability.
8. Time response analysis is limited to second order system where as frequency response can be obtained for any order of the system.

Disadvantages:

1. For systems with large time constants, obtaining frequency response is cumbersome as the time required for the response to reach steady state for each frequency is large.
2. Frequency response test can not be performed on non-interruptible systems.

Time response is determined using transfer function. frequency response can be determined using **sinusoidal transfer function**.
Sinusoidal transfer function is transfer function of the system when S= jω
Transfer function -G(s)
Sinusoidal transfer function –G(jω)

Frequency Domain Specifications

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of theses specifications.

Resonant Peak(Mr):

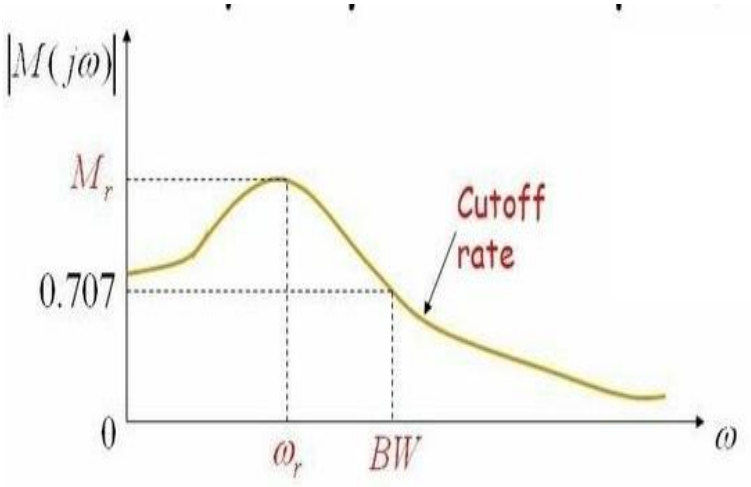
The peak value of the magnitude of a transfer function is called resonant peak.

It is similar to peak overshoot in time domain specifications. Large Mr corresponds to large overshoot.

Resonant Frequency(ωr):

The frequency at which resonant peak occurs is called as resonant frequency.

This is related to frequency of oscillations in time response and thus indicated the speed of transient response.



Bandwidth:

The bandwidth is the range of frequencies for which magnitude of the system is more than or equal to 1/√2 of its DC gain (-3db).

The frequency at which magnitude is equal to 1/√2 is called as cutoff frequency.

The band width is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and also represents speed of the response.

It is related to rise time in time domain. Large bandwidth corresponds to small rise time and hence fast response.

Cut-off Rate:

The slope of the log magnitude curve near the cutoff frequency is known as cutoff rate.it indicates the ability of the system to distinguish the input from noise.

Gain margin: gain margin is the reciprocal of the magnitude of open loop transfer function at phase cross over frequency.

Gain margin = $\frac{1}{|G(j\omega_{PC})|}$

ω_{PC} is the phase cross over frequency.

The frequency at which the phase of open loop transfer function is 180° is known as **Phase Cross Over frequency**.

The gain margin the maximum amount by which the gain of the system can be increased without loosing stability.

Phase Margin:

The phase margin is that amount of additional phase lag at gain cross over frequency required to bring the system to the verge of instability.

$\gamma = 180° + \varphi_{gc}$

φ_{gc} is the phase of open loop transfer function at gain cross over frequency.

The gain cross over frequency is the frequency at which gain of open loop transfer function is equal to one(or 0db).

$\varphi_{gc} = \angle G(j\omega_{gc})$

Expressions for frequency domain specifications for second order system:

The standard transfer function of the 2nd order system is

$$\frac{C(S)}{R(S)} = T(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

The sinusoidal transfer function of the 2nd order system is

$$T(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2(-\frac{\omega^2}{\omega_n^2} + \frac{j2\zeta\omega}{\omega_n} + 1)}$$

$$T(j\omega) = \frac{1}{(1 - \frac{\omega^2}{\omega_n^2}) + \frac{j2\zeta\omega}{\omega_n}}$$

Let u= ω/ωn = normalized frequency

Therefore

$$T(j\omega) = \frac{1}{(1-u^2)+j2\zeta u} \ldots (1)$$

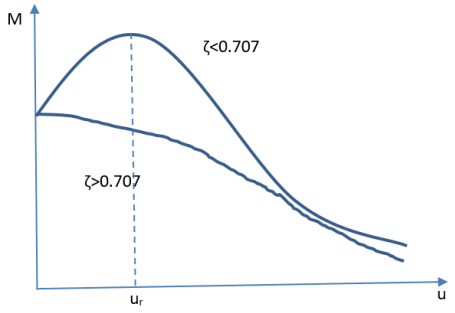
Magnitude of transfer function M= $|T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}$...(2)

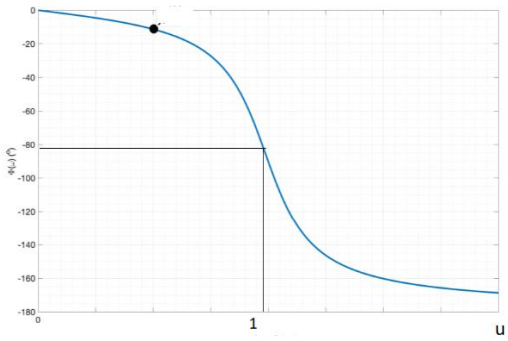
Phase of the transfer function φ = ∠T(jω) = - Tan^{-1} (2ζu/(1-u^2)) ...(3)

Both magnitude and phase are dependent on ζ.

At ω=0	u=0	M=1	&	φ =0
At ω=ωn	u=1	M=1/2ζ	&	φ =-90°
At ω=α	u=α	M=0	&	φ =-180°

The magnitude plot is as shown in figure.





The frequency at which M has a peak is known as resonant frequency. At this frequency slope of magnitude curve is zero.

Let u_r= ωr/ωn = normalized resonant frequency

$$\frac{dM}{du} = 0 \; at \; u = u_r$$

$4u_r^2+8\zeta^2-4=0 \rightarrow u_r^2 = 1-2\zeta^2$

$$u_r = \sqrt{1-2\zeta^2}$$

Therefore

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \ldots\ldots(4)$$

for $0<\zeta<1/\sqrt{2}$ ωr has +ve real value so it exist that means resonant peak occurs for frequency less than ωn.

For $\zeta >= 1/\sqrt{2}$ ωr is complex . That means resonant peak does not exist and maximum value of magnitude is its DC gain(ω=0).

Resonant peak is M at u=ur. So from eqn 2

$$M_r = \frac{1}{\sqrt{(1-u_r{}^2)^2 + (2\zeta u_r)^2}} = \frac{1}{\sqrt{(1-1+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}}$$

$$M_r = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4)}} == \frac{1}{\sqrt{-4\zeta^4 + 4\zeta^2}}$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2)}} \ldots..(5)$$

Control systems are low pass filters. So bandwidth is equal to cutoff frequency. That is frequency at which magnitude is equal to 1/√2 of its DC value.



Let $u_b = \omega b/\omega n$ = normalized bandwidth

When u=ub the magnitude M = 1/√2

$$M = \frac{1}{\sqrt{(1-u_b{}^2)^2 + (2\zeta u_b)^2}} = 1/\sqrt{2}$$

$$(1-u_b{}^2)^2 + (2\zeta u_b)^2 = 2$$

$$1 - u_b{}^4 - 2u_b{}^2 + 4\zeta^2 u_b{}^2 = 0$$

$$u_b{}^4 - 2u_b{}^2(1-2\zeta^2) - 1 = 0$$

Let $x = u_b{}^2$

$x^2-2(1-2\zeta^2)x-1=0$ → $x = 1-2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}$

therefore $u_b = [1-2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}]^{1/2}$

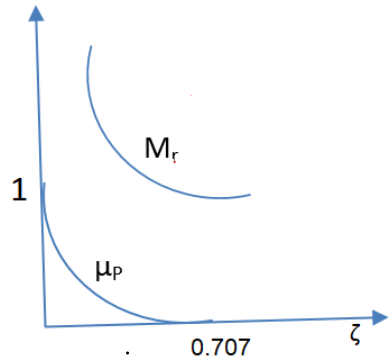
therefore $\omega_b = \omega_n [1-2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\ ]^{1/2}$

Correlation between time domain and frequency domain specifications:

correlation exist for a second order system.

Time domain	frequency domain
$\omega_d = \omega_n\sqrt{1-\zeta^2}$	$\omega_r = \omega_n\sqrt{1-2\zeta^2}$
$\mu_p = e^{-\frac{\zeta\Pi}{\sqrt{1-\zeta^2}}}$	$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

The sketch of Mr and μp various values of ζ are shown in figure.

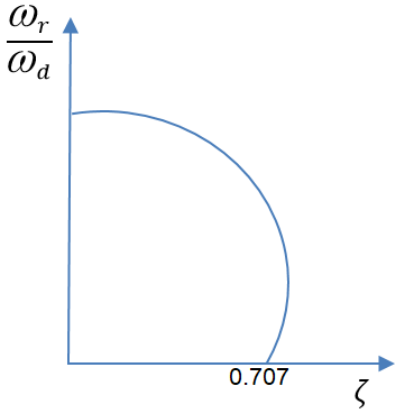


There is a one to one correspondence between Mr & μ_P. If you know one value, you can calculate the other. But this correspondence exists only upto ζ = .707.

for ζ > .707 Mr does not exist. And correlation breakdown. This is not a serious problem as for this range, the oscillation of the response are well damped and μ_P is negligible.

Similarly, there is a correlation between ω_r& ω_d.

$$\frac{\omega_r}{\omega_d} = \sqrt{\frac{1-2\zeta^2}{1-\zeta^2}}$$



Problems:

1. A system of 3rd order shows resonance peak of 2 and resonant frequency 3 rad/sec. Determine transfer function of equivalent 2nd order system and hence find tr,tp,ts and μp. draw a sketch of step response & frequency response.

Sol:

$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 2 \Rightarrow \zeta = 0.2588 \text{ or } 0.966$

Mr=2 → ωr= 3 rad/sec

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2)}} = 2 \rightarrow \quad \zeta = 0.2588 \ or \ 0.966$$

For ζ>0.707 Mr does not exist. $\omega_r = \omega_n\sqrt{1-2\zeta^2} = 3.2$

So ζ= 0.2588

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} = 3 \;\longrightarrow\; \omega_n = 3.223\ \frac{rad}{sec}$$

The equivalent second order system transfer function

$$G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{10.387}{S^2 + 1.668S + 10.387}$$

tr= (Π-Θ)/ωd =0.592

tp=Π/ωd = 1.02 sects= 4/(ζωn) = 4.795 sec $\mu_p = e^{-\frac{\zeta\Pi}{\sqrt{1-\zeta^2)}}} = 0.43$

2. System exhibits 50% overshoot in its step response and the time to peak is 3s. determine transfer function of system, resonant peak and resonant frequency.

Sol:

$$\mu_p = e^{-\frac{\zeta\Pi}{\sqrt{1-\zeta^2)}}} = 0.5 \rightarrow \quad \zeta = 0.2154$$

tp=Π/ωd = 3 sec -→ ωd =1.047 rad/sec → ωn =1.095 rad/sec

$\omega_r = \omega_n\sqrt{1-2\zeta^2}$ =1.025 rad/sec

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2)}} = 2.377$$

3. A unity feedback system has $G(s) = \frac{K}{S(1+TS)}$.when subject to a unit step input it shows 25% μp while it shows resonant frequency of 8 rad/sec in frequency domain. Calculate K & T. also calculate M_r & bandwidth.

Sol: ζ=0.4036 ωn=9.744

K=0.071 T=0.1271

Mr=1.354

Bode Plot

Bode plot is a graphical representation of frequency response of a linear time invariant system.

The Bode plot consists of two plots −

- Magnitude plot
- Phase plot

It is drawn on semi-log graph sheet. In both the plots, x-axis(logarithmic scale) represents angular frequency(ω), Whereas, y-axis(linear scale) represents magnitude of open loop transfer function in 'db' in the magnitude plot and phase angle of open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB is -

$$M_{db}=20\log|G(j\omega)H(j\omega)|$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi=\angle G(j\omega)H(j\omega)$$

Basic factors of Transfer Function:

The basic factors of a transfer function are

1. Constant K
2. Derivative Term KS or KS^n (Zeros at origin)
3. Integral term K/S or K/S^n (Poles at origin)
4. First order term in denominator (Real poles) $1/(1+ST)$ or $1/(1+ST)^n$
5. First order term in Numerator (Real Zeros) $(1+ST)$ or $(1+ST)^n$
6. Second order term in denominator (complex poles) $1/(S^2+2\zeta\omega_nS+\omega_n^2)$
7. Second order term in numerator (complex zeros) $S^2+2\zeta\omega_nS+\omega_n^2$

Bode plots of each of these factors are explained below.

1. Constant K

Consider the open loop transfer function

$$\mathbf{G(s) = K}$$

Sinusoidal transfer function $\mathbf{G(j\omega) = K}$ (substitute $S=j\omega$)

$$\mathbf{|G(j\omega)| = K}$$

$$\mathbf{\angle G(j\omega) = 0^0}$$

Magnitude Plot:

Magnitude $M = |G(j\omega)| = K$

Magnitude **M in db = 20log|G(jω)| = 20 logK**
Magnitude is constant with respect to ω.
If K=1, then magnitude is 0 dB.
If K>1, then magnitude will be positive dB.
If K<1, then magnitude will be negative dB.

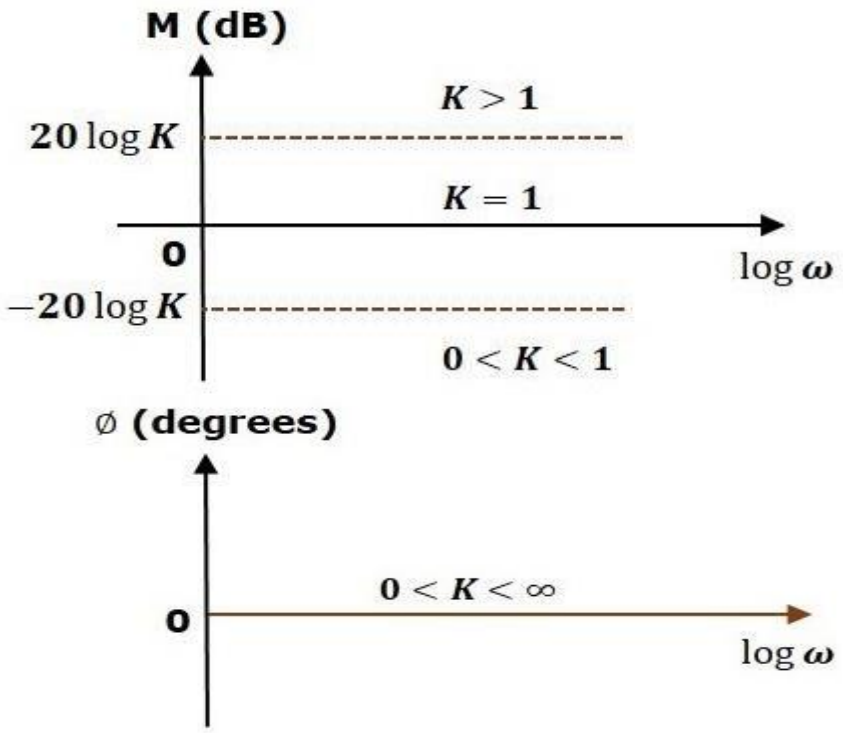
Phase Plot:

Phase angle ϕ = ∠G(jω) = 0^0

Phase is constant with respect to ω. So phase plot is horizontal line at 0°.

Bode Plot for Constant K:

The following figure shows the corresponding Bode plot.



The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. **For the positive values of K**, the horizontal line will shift **20 log K dB above the 0 dB line**. For the **negative values of K**, the horizontal line will shift **20 log K dB below the 0 dB line.**

Key Concept: Bode Plot of Gain Term

- **For a constant term, the magnitude plot is a horizontal line.**
- **The phase plot is also a horizontal line at 0° (for a positive constant).**

2.Derivative Term (Zeros at origin) KS or KS^n

A. Consider the open loop transfer function

$$\mathbf{G(s) = KS}$$

Sinusoidal transfer function $\mathbf{G(j\omega) = jK\omega}$ (substitute S=jω)

$$\mathbf{|G(j\omega)| = K\omega}$$

$$\mathbf{\angle G(j\omega) = 90°}$$

Magnitude Plot:

Magnitude M = |G(jω)| = Kω

Magnitude M in db = 20log|G(jω)| = 20 log(Kω)

At ω=0.1/K rad/sec, the magnitude is -20 dB.
At ω=1/K rad/sec, the magnitude is 0 dB.
At ω=10/K rad/sec, the magnitude is 20 dB.

So the magnitude plot is a line having a slope of 20 dB/dec and touching 0 dB line at ω=1/K rad/sec.

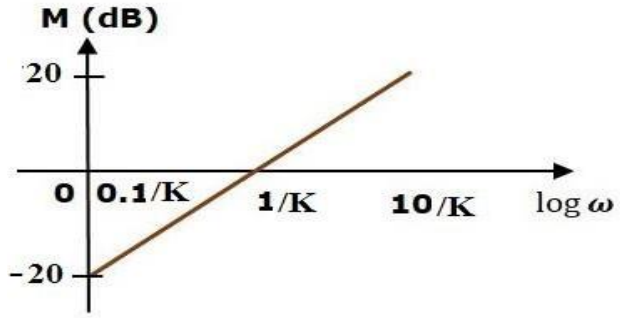
Phase Plot:

Phase angle $\phi = \angle G(j\omega) = 90^0$

Phase is constant with respect to ω. So phase plot is horizontal line at 90°.

Bode Plot for derivative term KS or jKω:

The following figure shows the corresponding Bode plot.





In this case, the phase plot is a horizontal line at 90^0.

B. Consider the open loop transfer function

$$\mathbf{G(s) = KS^n}$$

Sinusoidal transfer function $\mathbf{G(j\omega) = K(j\omega)^n}$ (substitute S=jω)

$$|G(j\omega)| = K\omega^n$$

$$\angle G(j\omega) = n*90^0$$

Magnitude Plot:

Magnitude $\mathbf{M = |G(j\omega)| = K\omega^n}$

Magnitude M in db = 20 log|G(jω)| $= 20\log(K\omega^n) = 20\log(K^{1/n}\omega)^n$

$= n*20\log(K^{1/n}\omega)$

At $\omega=0.1/K^{1/n}$ rad/sec, the magnitude is - n*20 dB.

At $\omega=1/K^{1/n}$ rad/sec, the magnitude is 0 dB.

At $\omega=10/K^{1/n}$ rad/sec, the magnitude is n*20 dB.

So the magnitude plot is a line having a slope of n*20 dB/dec and touching 0 dB line at $\omega=1/K^{1/n}$ rad/sec.

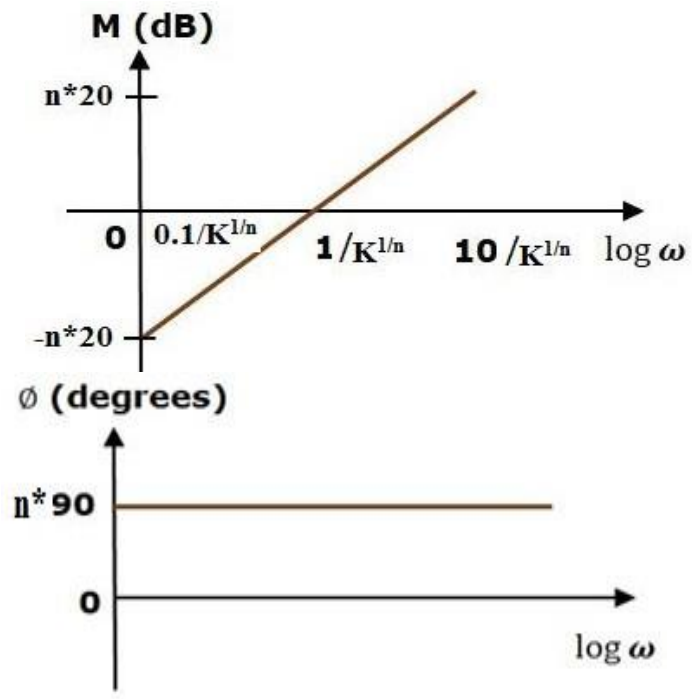
Phase Plot:

Phase angle $\phi = \angle G(j\omega) = n*90^0$

Phase is constant with respect to ω. So phase plot is horizontal line at n*90°.

Bode Plot for derivative term KS^n or $jK\omega^n$:

The following figure shows the corresponding Bode plot.



Key Concept: <u>Bode Plot of Derivative Term</u>

- **For a derivative term, the magnitude plot is a straight line with slope +n*20 db/dec.**
- **The phase plot is a horizontal line at $+n*90^0$. Where n is the power of 'S'.**

In this case, the phase plot is a horizontal line at $n*90^0$.

3.Integral Term(Poles at origin) K/S or K/S^n

A. Consider the open loop transfer function

$$\mathbf{G(s) = K/S}$$

Sinusoidal transfer function $\mathbf{G(j\omega) = K/j\omega}$ (substitute S=jω)

$$\mathbf{|G(j\omega)| = K/\omega}$$

$$\mathbf{\angle G(j\omega) = -90°}$$

Magnitude Plot:

Magnitude M = |G(jω)| = K/ω

Magnitude M in db = 20log|G(jω)| = 20 log(K/ω)

At ω=0.1K rad/sec, the magnitude is 20 dB.
At ω=K rad/sec, the magnitude is 0 dB.
At ω=10K rad/sec, the magnitude is -20 dB.

So the magnitude plot is a line having a slope of -20 dB/dec and touching 0 dB line at ω=K rad/sec.

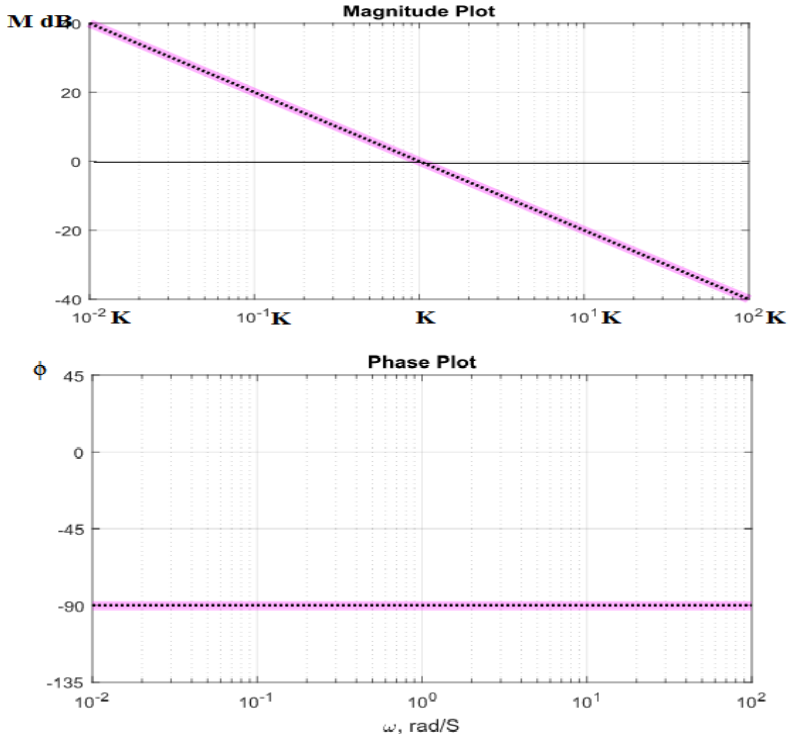
Phase Plot:

Phase angle $\phi = \angle G(j\omega) = -90^0$

Phase is constant with respect to ω. So phase plot is horizontal line at -90°.

Bode Plot for integral term K/S or K/jω:

The following figure shows the corresponding Bode plot.



In this case, the phase plot is a horizontal line at -90^0.

B. Consider the open loop transfer function

$$\mathbf{G(s) = K/S^n}$$

Sinusoidal transfer function $G(j\omega) = \mathbf{K/(j\omega)^n}$ (substitute $S=j\omega$)

$$|G(j\omega)| = K/\omega^n$$

$$\angle G(j\omega) = -n*90^0$$

Magnitude Plot:

Magnitude $\mathbf{M = |G(j\omega)| = K\omega^n}$

Magnitude M in db = $\mathbf{20\ log|G(j\omega)| = 20\ log(K/\omega^n)} = 20\ log(K^{1/n}/\omega)^n$

$= n*20\ log(K^{1/n}/\omega)$

At $\omega=0.1K^{1/n}$ rad/sec, the magnitude is n*20 dB.

At $\omega=K^{1/n}$ rad/sec, the magnitude is 0 dB.

At $\omega=10K^{1/n}$ rad/sec, the magnitude is -n*20 dB.

So the magnitude plot is a line having a slope of -n*20 dB/dec and touching 0 dB line at $\omega=K^{1/n}$ rad/sec.

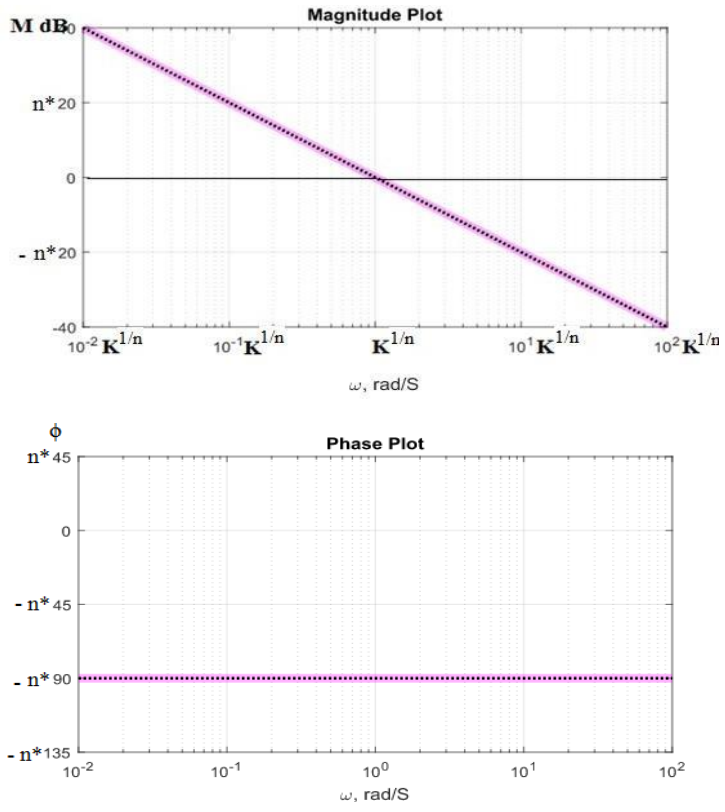
Phase Plot:

$$\textbf{Phase angle } \phi = \angle G(j\omega) = -n*90^0$$

Phase is constant with respect to ω. So phase plot is horizontal line at -n*90°.

Bode Plot for derivative term $\mathbf{K/S^n}$ or $\mathbf{K/(j\omega)^n}$:

The following figure shows the corresponding Bode plot.



Key Concept: Bode Plot of Integral Term

- **For an integral term, the magnitude plot is a straight line with slope -n*20 db/dec.**
- **The phase plot is a horizontal line at $-n*90^0$.**

Where n is the power of 'S'

In this case, the phase plot is a horizontal line at $-n*90^0$.

4.First order term in denominator (Real Poles)

A. Consider the open loop transfer function

$$\mathbf{G(s) = 1/(1+ST)}$$

Sinusoidal transfer function $\mathbf{G(j\omega) = 1/(1+j\omega T)}$ (substitute S=jω)

$$\mathbf{|G(j\omega)| = 1/\sqrt{(1+\omega^2T^2)}}$$
$$\mathbf{\angle G(j\omega) = -\tan^{-1}(\omega T)}$$

Magnitude Plot:

$$\text{Magnitude } M = |G(j\omega)| = 1/\sqrt{(1+\omega^2T^2)}$$

Magnitude M in db = $\mathbf{20\log|G(j\omega)| = 20\log(1/\sqrt{(1+\omega^2T^2)})}$
$= -20\log(\sqrt{(1+\omega^2T^2)})$

For **low frequencies ω«1** So ω^2 term can be neglected.
Therefore **M= 0 dB.**
For **high frequencies ω»1** So '1' can be neglected.
Therefore

$$M= -20\log(\sqrt{(\omega^2T^2)}) \text{ dB} = -20\log(\omega T)$$

At ω=1/ T rad/sec, the magnitude is 0 dB.
At ω=10/ T rad/sec, the magnitude is -20dB.
At ω=100/ T rad/sec, the magnitude is -40 dB.
So the magnitude plot can be approximated by two lines one is a horizontal line at 0 dB upto ω=1/T rad/sec and for ω>1/T rad/sec, it is a straight line having a slope of -20 dB/dec.

Corner frequency or Break frequency: The frequency at which slop of the magnitude plot changes is known as **Corner frequency**.

For first order term corner frequency is ω_c=1/ T

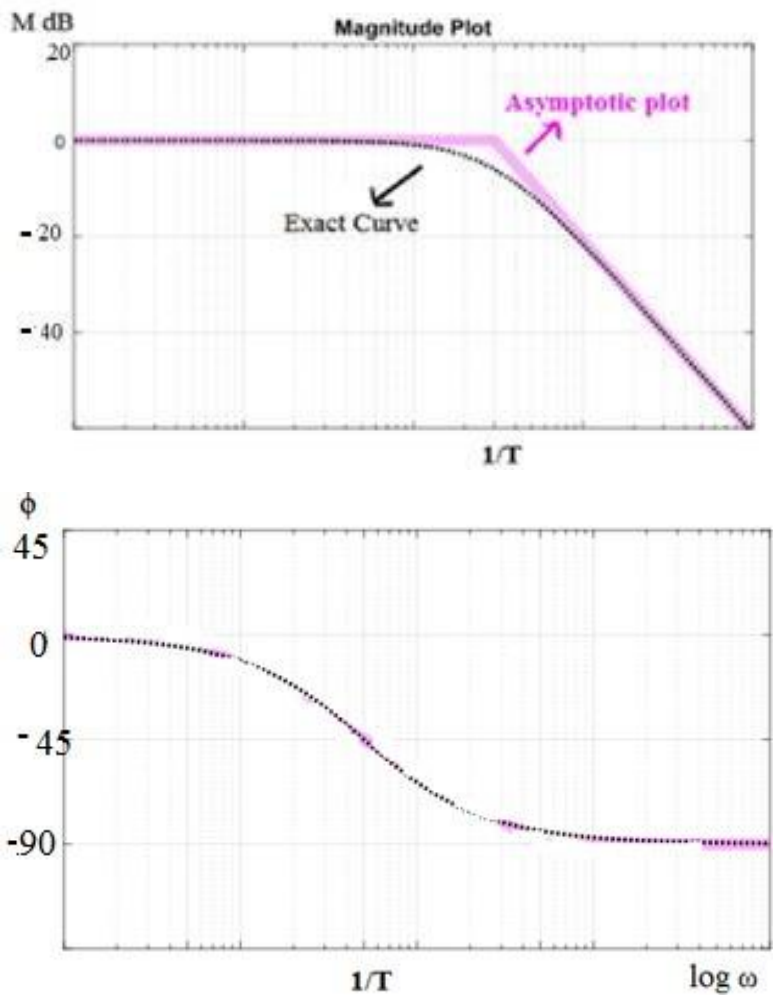
Phase Plot:

$$\text{Phase angle } \phi = \angle G(j\omega) = -\tan^{-1}(\omega T)$$

As $\omega \rightarrow 0$ $\phi \rightarrow 0^0$
At $\omega = 1/T$ $\phi = -45^0$
As $\omega \rightarrow \infty$ $\phi \rightarrow -90^0$

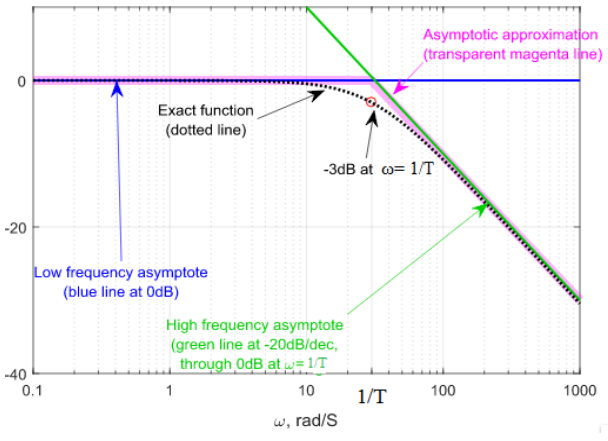
Bode Plot for first order term 1/(1+ST):

The following figure shows the corresponding Bode plot.



More on Magnitude plot:

In the above figure exact bode plot is represented by black dotted line.



At low frequencies i.e. for ω< 1/T the magnitude is approximately zero represented by blue line asymptote.

At high frequencies i.e. for ω> 1/T the magnitude plot is approximately by an asymptote line represented by green line asymptote.

The error between asymptote lines and exact plot is maximum at corner frequency i.e. at ω= 1/T.

At ω= 1/T the exact magnitude is

$$M \text{ in db} = 20 \log(1/\sqrt{(1+\omega^2T^2)})$$
$$= -20 \log(\sqrt{(1+1)}) = -3 \text{ dB}$$

We are assuming magnitude is zero at ω= 1/T. So the error is -3 dB.

B. Consider the open loop transfer function

$$\mathbf{G(s) = 1/(1+ST)^n}$$

Sinusoidal transfer function $G(j\omega) = 1/(1+j\omega T)^n$ (substitute $S=j\omega$)

$$|G(j\omega)| = 1/\sqrt{(1+\omega^2T^2)^n}$$

$$\angle G(j\omega) = -n*\tan^{-1}(\omega T)$$

Magnitude Plot:

$$\text{Magnitude } M = |G(j\omega)| = 1/\sqrt{(1+\omega^2T^2)^n}$$

$$\text{Magnitude M in db} = 20\log|G(j\omega)| = 20\log(1/\sqrt{(1+\omega^2T^2)})^n$$

$$= -20n\log(\sqrt{(1+\omega^2T^2)})$$

For low frequencies $\omega \ll 1$ So can be neglected Therefore M= 0 dB

For high frequencies $\omega \gg 1$ So '1' can be neglected.

Therefore

$M= -20n\log(\sqrt{(\omega^2T^2)})$ dB $=-20n\log(\omega T)$

At $\omega=1/T$ rad/sec, the magnitude is 20n dB.

At $\omega=10/T$ rad/sec, the magnitude is 0 dB.

At $\omega=100/T$ rad/sec, the magnitude is -20n dB.

So the magnitude plot can be approximated by two lines one is a horizontal line at 0 dB upto $\omega=1/T$ rad/sec and for $\omega>1/T$ rad/sec, it is a line having a slope of -20n dB/dec.

Phase Plot:

$$\text{Phase angle } \phi = \angle G(j\omega) = -n\tan^{-1}(\omega T)$$

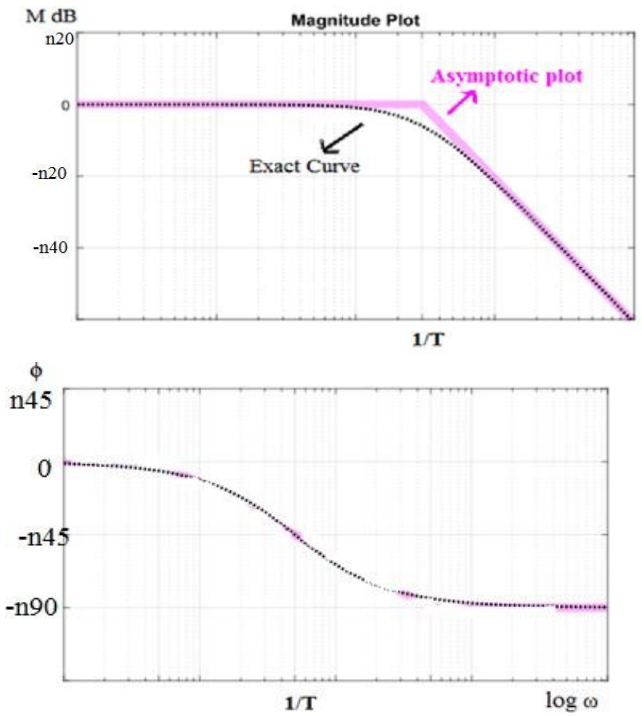
$$\text{As } \omega \rightarrow 0 \ \phi \rightarrow 0^0$$

$$\text{At } \omega = 1/T \ \phi = -n45^0$$

$$\text{As } \omega \rightarrow \infty \ \phi \rightarrow -n90^0$$

Bode Plot for first order term $1/(1+ST)^n$:

The following figure shows the corresponding Bode plot.



Key Concept: Bode Plot of 1st order Term in denominator

- **For first order term corner frequency is ω=1/T**
- **For this term, the magnitude plot is a horizontal line at 0 db for ω≤1/T and a straight line with slope -n*20 db/dec for ω>1/T.**
- **The phase minimum value is zero and maximum value is – n*90^0.**
  Where n is the power of 'S'.

5. First order term in Numerator (Real Zeros)

A. Consider the open loop transfer function

$$G(s) = (1+ST)$$

Sinusoidal transfer function $G(j\omega) = (1+j\omega T)$ (substitute $S=j\omega$)

$$|G(j\omega)| = \sqrt{(1+\omega^2T^2)}$$

$$\angle G(j\omega) = \tan^{-1}(\omega T)$$

Magnitude Plot:

$$\text{Magnitude } M = |G(j\omega)| = \sqrt{(1+\omega^2T^2)}$$

$$\text{Magnitude } M \text{ in db} = 20\log|G(j\omega)| = 20\log(\sqrt{(1+\omega^2T^2)})$$
$$= 20\log(\sqrt{(1+\omega^2T^2)})$$

For low frequencies $\omega \ll 1$ So ω^2 term can be neglected.
Therefore M= 0 dB.
For high frequencies $\omega \gg 1$ So '1' can be neglected.
Therefore
$M = 20\log(\sqrt{(\omega^2T^2)})$ dB $= 20\log(\omega T)$
At $\omega = 1/T$ rad/sec, the magnitude is -20 dB.
At $\omega = 10/T$ rad/sec, the magnitude is 0 dB.
At $\omega = 100/T$ rad/sec, the magnitude is +20 dB.

So the magnitude plot can be approximated by two lines one is a horizontal line at 0 dB upto ω=1/T rad/sec and from ω=1/T rad/sec, it is a line having a slope of +20 dB/dec.

For first order term corner frequency is ω=1/ T

Phase Plot:

Phase angle $\phi = \angle G(j\omega) = \tan^{-1}(\omega T)$

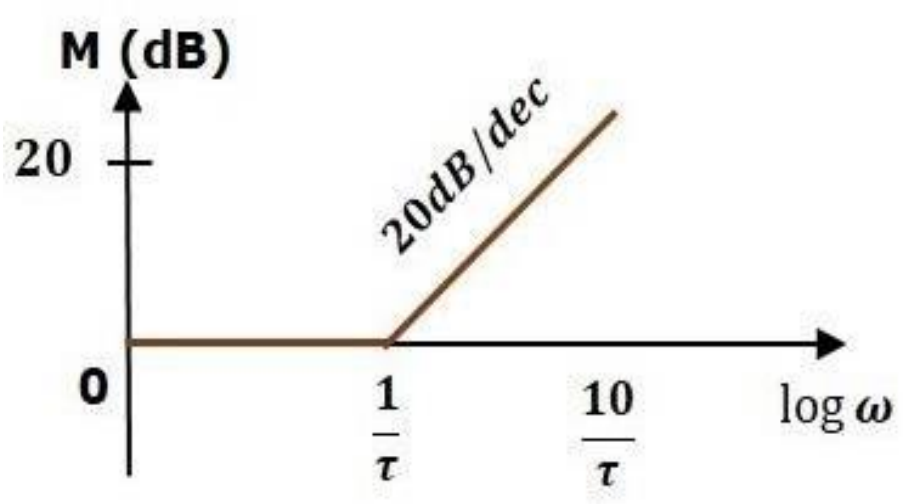
As $\omega \rightarrow 0$ $\phi \rightarrow 0^0$

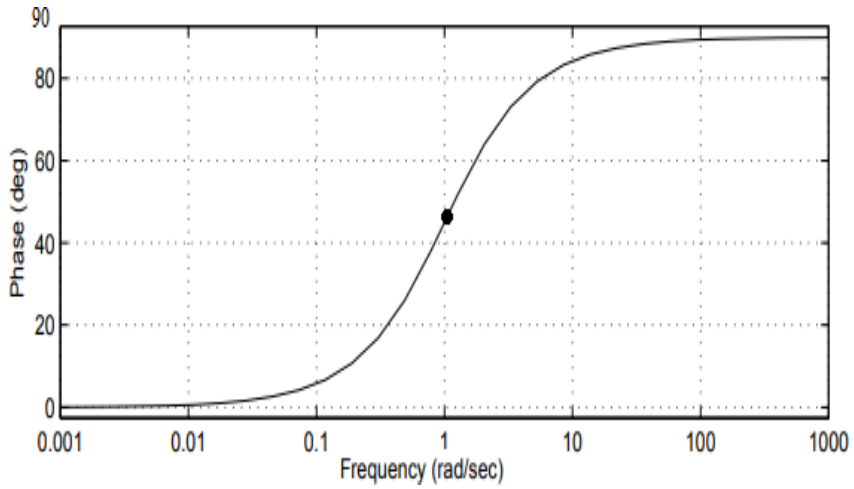
At $\omega = 1/T$ $\phi = +45^0$

As $\omega \rightarrow \infty$ $\phi \rightarrow +90^0$

Bode Plot for first order term (1+ST):

The following figure shows the corresponding Bode plot.





B. Consider the open loop transfer function

$$\mathbf{G(s) = (1+ST)^n}$$

Sinusoidal transfer function $G(j\omega) = (1+j\omega T)^n$ (substitute $S=j\omega$)

$$|G(j\omega)| = \sqrt{(1+\omega^2T^2)^n}$$
$$\angle G(j\omega) = n*\tan^{-1}(\omega T)$$

Magnitude Plot:

$$\text{Magnitude } M = |G(j\omega)| = \sqrt{(1+\omega^2T^2)^n}$$

$$\text{Magnitude M in db} = 20\log|G(j\omega)| = 20\log(\sqrt{(1+\omega^2T^2)})^n$$
$$= +20n\log(\sqrt{(1+\omega^2T^2)})$$

For low frequencies $\omega \ll 1$ So ω^2 term can be neglected Therefore M= 0 dB
For high frequencies $\omega \gg 1$ So '1' can be neglected.
Therefore
$M_{dB} = +20n\log(\sqrt{(\omega^2T^2)})$ dB $= +20n\log(\omega T)$

At $\omega = 1/T$ rad/sec, the magnitude is- 20n dB.
At $\omega = 10/T$ rad/sec, the magnitude is 0 dB.
At $\omega = 100/T$ rad/sec, the magnitude is +20n dB.

So the magnitude plot can be approximated by two lines one is a horizontal line at 0 dB upto $\omega = 1/T$ rad/sec and from $\omega = 1/T$ rad/sec, it is a line having a slope of
-20n dB/dec.

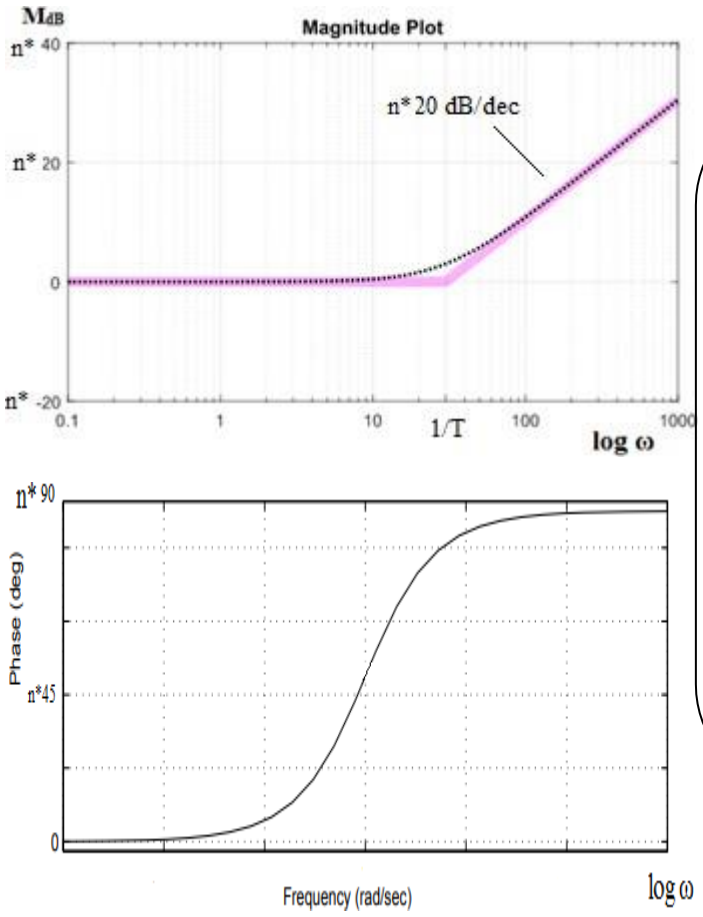
Phase Plot:

$$\text{Phase angle } \phi = \angle G(j\omega) = +n\tan^{-1}(\omega T)$$

As $\omega \rightarrow 0$ $\phi \rightarrow 0^0$
At $\omega = 1/T$ $\phi = +n45^0$
As $\omega \rightarrow \infty$ $\phi \rightarrow +n90^0$

Bode Plot for first order term $(1+ST)^n$:

The following figure shows the corresponding Bode plot.



Key Concept: Bode Plot of 1st order Term in numerator

- **For first order term corner frequency is ω=1/T**
- **For this term, the magnitude plot is a horizontal line at 0 db for ω≤1/T and a straight line with slope +n*20 db/dec for ω>1/T.**
- **The phase minimum value is zero and maximum value is +n*90^0.**
  Where n is the power of 'S'.

6. Second order term in denominator (Complex Poles)

The magnitude and phase plots of a second order term (complex conjugate pair of poles-under damped) is more complicated than those for a simple pole. Consider the transfer function (with $0<\zeta<1$):

$$G(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

Where ω_n is the natural frequency of the system.
The sinusoidal transfer function

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + \frac{j2\zeta\omega}{\omega_n} + 1\right)}$$

$$G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \frac{j2\zeta\omega}{\omega_n}}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + (\frac{2\zeta\omega}{\omega_n})^2}}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 1 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}}$$

$$\angle G(j\omega) = -tan^{-1}\frac{(\frac{2\zeta\omega}{\omega_n})}{(1-\frac{\omega^2}{\omega_n^2})}$$

Magnitude Plot:

Magnitude M = $|G(j\omega)| = \frac{1}{\sqrt{\frac{\omega^4}{\omega_n^4}-2\frac{\omega^2}{\omega_n^2}+1+4\zeta^2\frac{\omega^2}{\omega_n^2}}}$

Magnitude M in db = $20\log|G(j\omega)| = 20\log\left(\frac{1}{\sqrt{\frac{\omega^4}{\omega_n^4}-2\frac{\omega^2}{\omega_n^2}+1+4\zeta^2\frac{\omega^2}{\omega_n^2}}}\right)$

$$=-20\log\left(\sqrt{\frac{\omega^4}{\omega_n^4} - 2\frac{\omega^2}{\omega_n^2} + 1 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}\right)$$

For low frequencies $\omega \ll \omega_n$ So all powers of (ω / ω_n) can be neglected.
Therefore M= $-20\log(\sqrt{1}) = 0$ dB

For high frequencies $\omega \gg \omega_n$ So all terms except (ω^4/ ω^4_n) can be neglected.
Therefore
M= $-20n\log(\sqrt{(\omega 4 / \omega 4)})$ dB $=-40\log(\omega/ \omega_n)$

At $\omega= \omega_n$ rad/sec, the magnitude is 0 dB.
At $\omega=10\ \omega_n$ rad/sec, the magnitude is -40 dB.
At $\omega=100\ \omega_n$ rad/sec, the magnitude is -80 dB.

So the magnitude plot can be approximated by two asymptote lines one is a horizontal line at 0 dB upto $\omega = \omega_n$ rad/sec and from $\omega = \omega_n$ rad/sec, it is a line having a slope of -40 dB/dec.

For second order term corner frequency is $\omega = \omega_n$

Phase Plot:

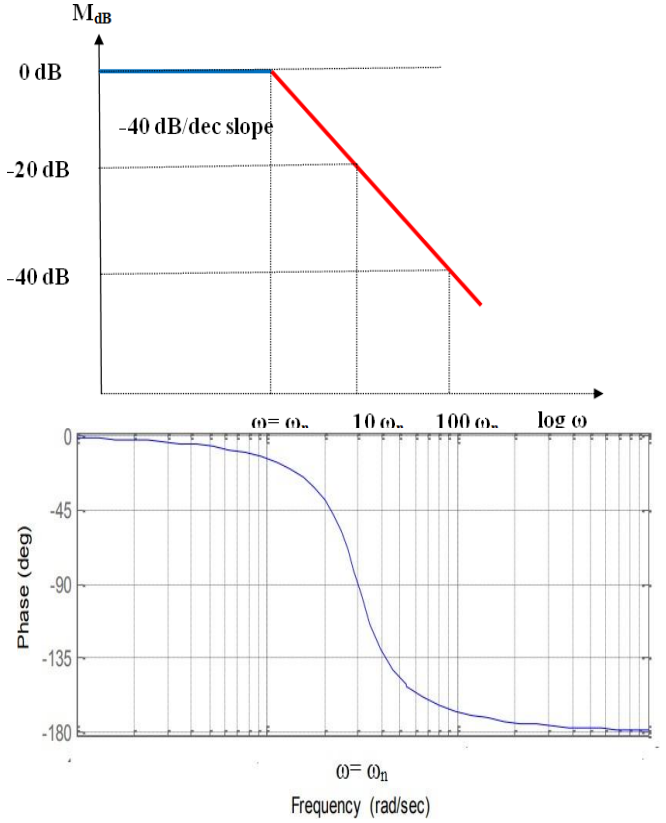
$$\text{Phase angle } \phi = \angle G(j\omega) = -tan^{-1}\frac{(\frac{2\zeta\omega}{\omega_n})}{(1-\frac{\omega^2}{\omega_n^2})}$$

As $\omega \rightarrow 0$ $\phi \rightarrow 0^0$
At $\omega = \omega_n$ $\phi = -90^0$
As $\omega \rightarrow \infty$ $\phi \rightarrow -180^0$

At other frequencies ϕ is dependent on ζ value.

Bode Plot for second order term in denominator:

The following figure shows the corresponding Bode plot.



Key Concept: Bode Plot of 2nd order Term in denominator

- **For second order term corner frequency is $\omega = \omega_n$.**
- **For this term, the magnitude plot is a horizontal line at 0 db for $\omega \leq \omega_n$ and a straight line with slope -40 db/dec for $\omega > \omega_n$.**
- **The phase minimum value is 0^0 and maximum value is -180^0.**

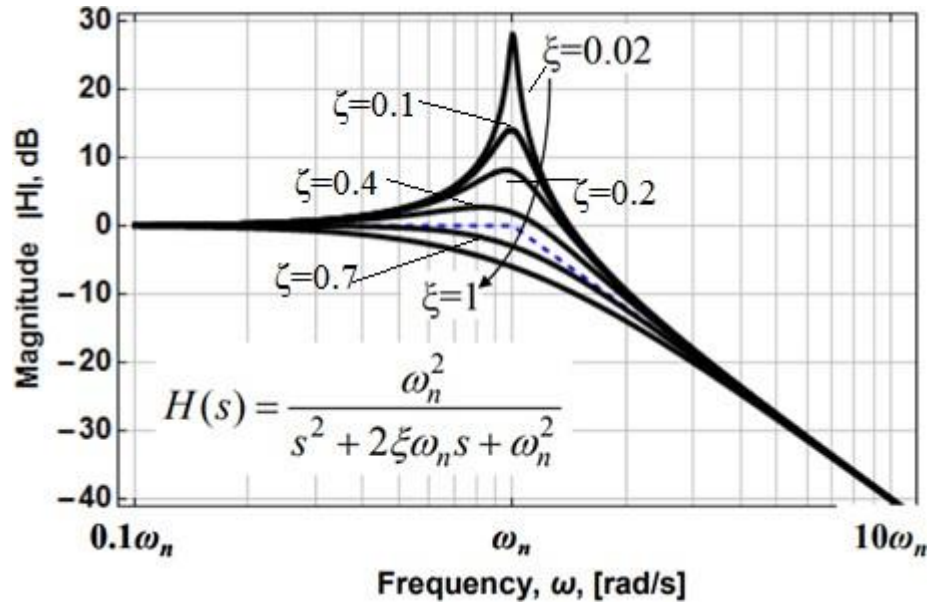
The exact curves are dependent on ζ value. The below figure shows the exact bode plot for different ζ values.

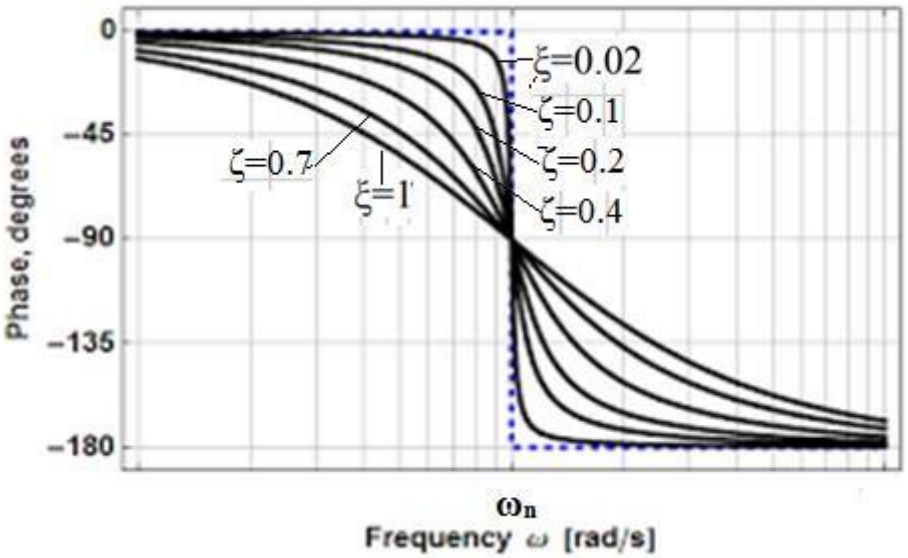
At $\omega = \omega_n$

Magnitude M in db = $20\log|G(j\omega)| = 20\log\left(\frac{1}{2\zeta}\right)$

$= -20\log(2\zeta)$

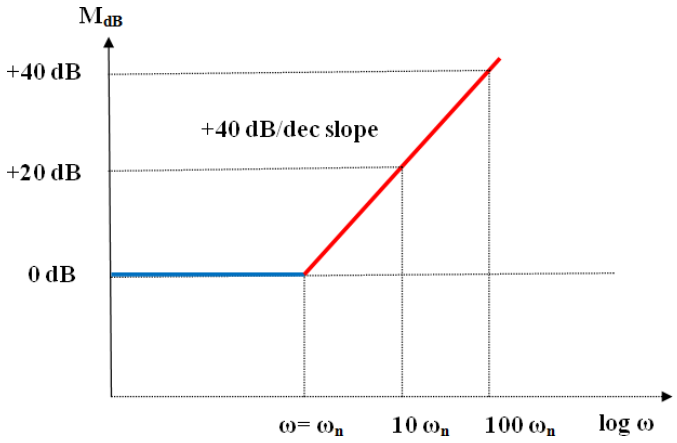
So lower the ζ value higher the peak of magnitude plot.

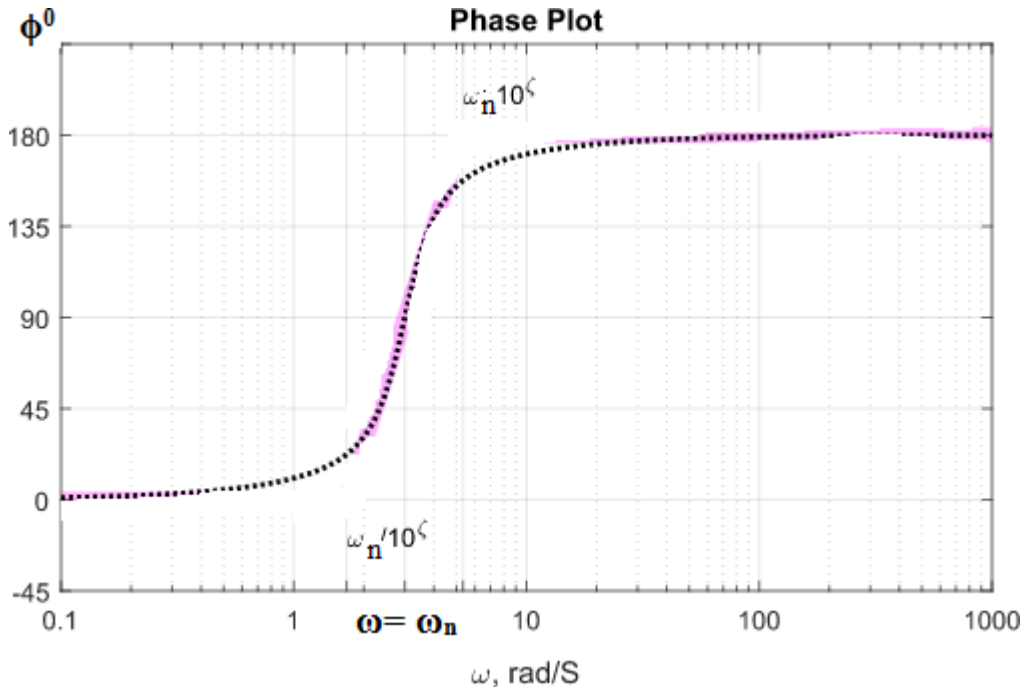




7.Second order term in numerator(Complex zeros)

In the similar way bode plot for second order term n numerator can be drawn. The magnitude plot and phase plot are as shown below.





Key Concept: Bode Plot of 2nd order Term in numerator

- **For second order term corner frequency is ω= ωn.**
- **For this term, the magnitude plot is a horizontal line at 0 db for ω≤ ωn and a straight line with slope +40 db/dec for ω> ωn.**
- **The phase minimum value is 0^0 and maximum value is $+180^0$.**

Summary:

Type of term	G(jω)H(jω)	Slope (dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	20logK	0
Zero at origin	jω	20	20logω	90
'n' zeros at origin	$(j\omega)^n$	20*n	20nlogω	90*n
Pole at origin	1/jω	−20	−20logω	−90
'n' poles at origin	$1/(j\omega)^n$	−20*n	−20*n logω	−90*n
Simple zero	1+jωT	20	0 for ω<1/T 20logωT for ω>1/T	0 to 45 for ω<1/T 45 at ω=1/T 45 to 90 for ω>1/T
Simple pole	1/(1+jωT)	−20	0 for ω<1/T -20logωT for ω>1/T	0 to -45 for ω<1/T -45 at ω=1/T -45 to -90 for ω>1/T
Second order derivative term	$S^2 + 2\zeta\omega_n S + \omega_n^2$	40	0 for ω< ω_n 40log(ω/ ω_n) for ω> ω_n	0 to 90 for ω<ω_n 90 at ω= ω_n 90 to 180 for ω>ω_n
Second order integral term	$\frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$	−40	0 for ω< ω_n -40log(ω/ ω_n) for ω> ω_n	−0 to -90 for ω<ω_n −90 at ω= ω_n -90 to −180 for ω>ω_n

Procedure to draw Bode Plot

In this section, let us understand in detail how to construct Bode plot.

Procedure to draw magnitude plot of Bode Plot:

Step 1: Represent the given open loop transfer function in the **time constant form.**

Step 2: Substitute $s=j\omega$ in the transfer function to obtain **sinusoidal transfer function.**

Step 3: Find the corner frequencies and arrange them in ascending order. Prepare a Table as shown below.

Sl. No.	Term	Corner Frequency (rad/sec)	Slope of the term (db/dec)	Net Slope (db/dec)
1	K (or)	------	0	Same as previous column
	KS^n (or)		+n*20	
	K/S^n		-n*20	Ex: a
2		ω_{c1}	Ex: b	Above slope + slope of this term Ex: a+b
3		ω_{c2}		
4		ω_{c3}		
. . .	Terms in ascending order of corner frequencies	. . .	. . .	. . .

The first term in the table is always K (or) KS^n (or) K/S^n. For all these terms there is no corner frequency.

The slope of this term is

For K it is 0 db/dec.

For KS^n it is +n*20 db/dec.

For K/S^n it is -n*20 db/dec.

The last column of the first row is same as previous cell i.e slope of the term.

After completing the first row arrange the terms in ascending order of their corner frequencies.

The corner frequency of the first order terms (1+ST)n or 1/(1+ST)n is 1/T.

The corner frequency of the second order terms is ω_n. The slope of any other term is ±n*20 dB/dec.

The slope is '+ve' if the term is in numerator and The slope is '-ve' if the term is in denominator.

And 'n' is the power of 'S' in the term.

Net slop at any row is sum of the previous net slope and slope of that row term.

Step 4: Choose a frequency (ω_l) lower than the lowest corner frequency. And determine the magnitude at ω_l and ω_{c1} using the below expressions.

$M_{dB} = 20 \log(K)$ if first term is K.

$= 20 \log(K\omega^n)$ if first term is KS^n.

$= 20 \log(K/\omega^n)$ if first term is K/S^n.

Substitute $\omega = \omega_l$ and $\omega = \omega_{c1}$ and determine M.

Step 5: Find magnitude at other corner frequencies using below expression

Gain at ω_y = Gain at ω_x + net slope at ω_x *$(\log(\omega_y/\omega_x)$ …(1)

Step 6: Choose a frequency (ω_h) higher than the highest corner frequency and determine the magnitude at this frequency using the same above expression (1).

Step 7: Mark all these points on semi log sheet and join them by straight lines.

Phase plot:

Write the expression for Φ.

$\Phi = \angle G(j\omega)$

Find angle of each of the terms in transfer function.

For K no angle.

For the term KS^n angle is $+n*90^0$.

For the term K/S^n angle is $-n*90^0$.

For other terms (a+jb) angle is $\pm\tan^{-1}(b/a)$. Angle is +ve if the term is in numerator and Angle is -ve if the term is denominator.

Φ= sum of angles of all terms in the transfer function.

It is a function of ω. find Φ at all frequencies at which magnitude is determined.

Note:

- Choose ω_l and ω_h such that magnitude is crossing 0dB.
- Find Φ for more frequencies around the frequency which is crossing -180^0.

Stability analysis using Bode Plot

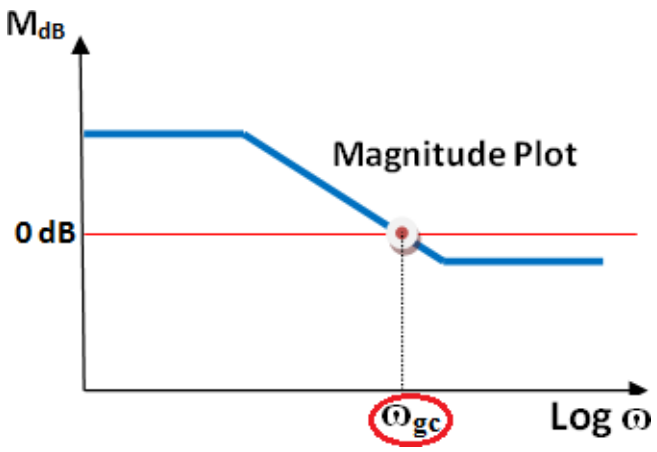
In this section, let us understand in detail how to determine stability from Bode plot.

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of the following parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

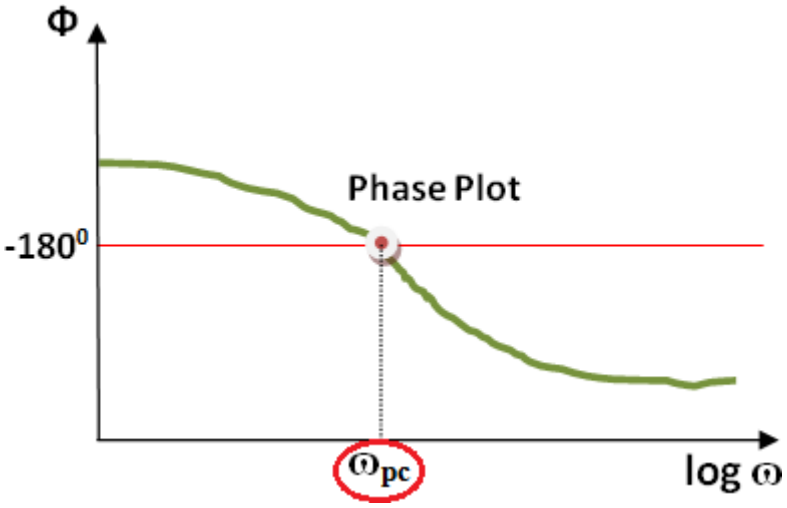
Gain Cross over Frequency(ω_{gc}):

- The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by ω_{gc}.



Phase Cross over Frequency(ω_{pc}):

- The frequency at which the phase plot is having the phase of -180^0 is known as **phase cross over frequency**. It is denoted by ω_{pc}.

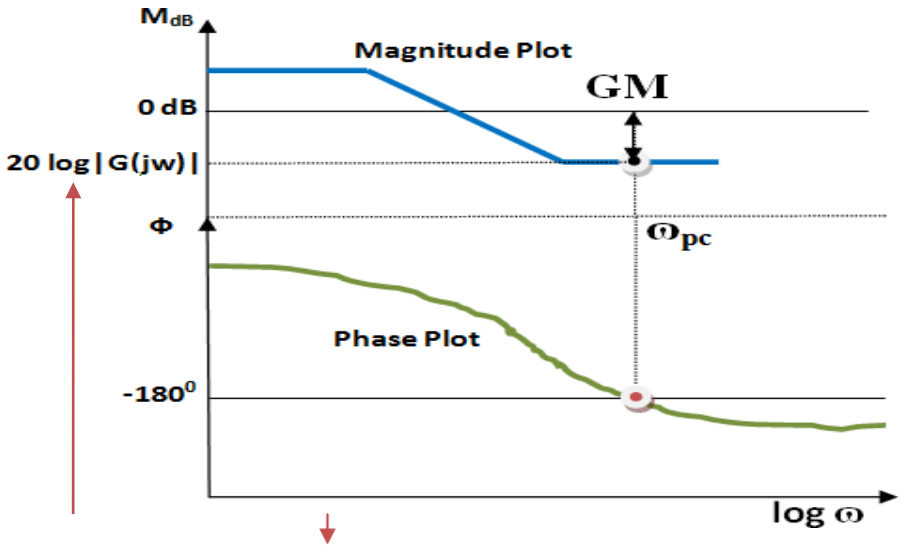


Gain Margin

Gain margin GM is equal to negative of the magnitude in dB at phase cross over frequency.

$$GM = -20\log(|G(j\omega_{pc}|) = -ve(\text{magnitude at } \omega_{pc} \text{ from bode plot})$$

Where, $|G(j\omega_{pc}|$ is the magnitude at phase cross over frequency.



From above figure let $x = 20\log(|G(j\omega)|)$ Then
GM = -x dB

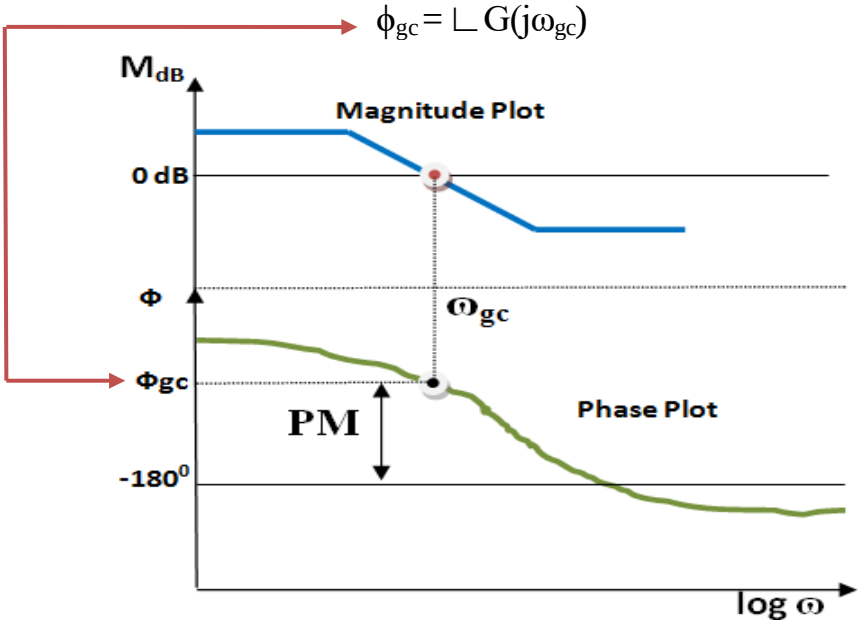
Phase Margin

The formula for phase margin γ is

$$\gamma = 180^0 + \phi_{gc}$$

Where, ϕ_{gc} is the phase angle at gain cross over frequency. i.e.

$$\phi_{gc} = \angle G(j\omega_{gc})$$

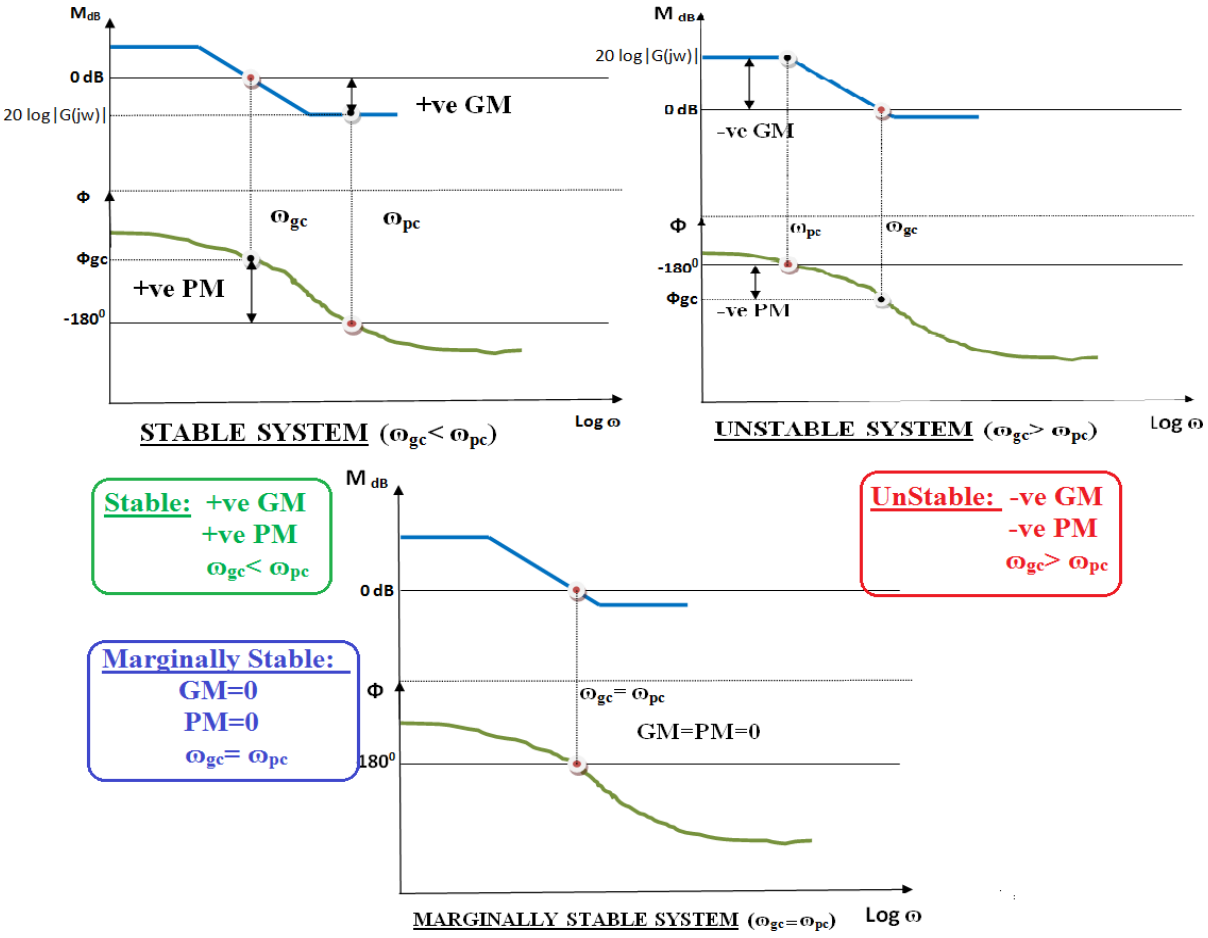


Stability from Bode plot:

The stability of the control system is based on the gain margin and phase margin and is listed below.

- If both gain margin and phase margin are positive, then the control system is **stable**.
- If both gain margin and phase margin are equal to zero, then the control system is **marginally stable**.
- If gain margin and phase margin are negative, then the control system is **unstable**.

The following figures are example bode plots of stable, unstable and marginally stable systems.



From the above figures we can draw conclusions as listed below.

- If the gain cross over frequency ω_{gc} is less than the phase cross over frequency ω_{pc}, **($\omega_{gc} < \omega_{pc}$)** then the control system is **stable**.
- If the gain cross over frequency ω_{gc} is equal to the phase cross over frequency ω_{pc}, **($\omega_{gc} = \omega_{pc}$)** then the control system is **marginally stable**.
- If the gain cross over frequency ω_{gc} is greater than the phase cross over frequency ω_{pc}, **($\omega_{gc} > \omega_{pc}$)** then the control system is **unstable**.

Bode Plot solved examples

Solved Examples:

Example 1:

Draw Bode plot for the open loop transfer function of a closed loop control system whose

$$G(s)H(s) = \frac{100}{(s+2)(s+5)}$$

Solution:

- Convert the open loop transfer function into time constant form.

$$G(s)H(s) = \frac{100}{(s+2)(s+5)}$$

$$\Rightarrow G(s)H(s) = \frac{10}{(0.5s+1)(0.2s+1)}$$

- The transfer function has the following three basic factors 10, 1/(0.5s+1), 1/(0.2s+1). The corner frequencies of these terms are given below.

No Corner frequency for the term '10'.

Corner frequency of the term '1/(0.5s+1)' is $\omega_c = 1/T = 1/0.5 = 2$ rad/sec.

Corner frequency of the term '1/(0.2s+1)' is $\omega_c = 1/T = 1/0.2 = 5$ rad/sec.

Sl. No.	Term	Corner Frequency (rad/sec)	Slope of the term (db/dec)	Net Slope (db/dec)
1	10	------	0	0
2	1/(0.5s+1)	$\omega_{c1}=2$	-20 (the power of S is '1' and the term is in denominator)	Above slope + slope of this term -20+0= -20
3	1/(0.2s+1)	$\omega_{c2} = 5$ Ascending order of corner frequencies	-20 (the power of S is '1' and the term is in denominator)	-20-20= -40

- Choose $\omega_l = 0.2$(less than $\omega_{c1}=2$).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is

  $M_{dB}=-20 \log(10)$ **(if 's' exist in first term replace with ω)**

  At $\omega = \omega_l = 0.2$ $M_{dB}=-20 \log(10) = -20$ dB
  At $\omega = \omega_{C1} = 2$ $M_{dB}=-20 \log(10) = -20$ dB
- AT $\omega = \omega_{C2} = 5$ gain is calculated using the eqn
  Gain at ω_y = Gain at ω_x + net slope at ω_x *(log(ω_y/ω_x)
  Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *(log(ω_{c2}/ω_{c1})

  M_{dB} = -20 +(-20)*log(5/2) = -27.96 dB.

- Choose ω_h = 10(gre ter than ω_{c2}=5).
  The expression to calculate M_{dB} at ω_h
  Gain at ω_y = Gain at ω_x + net slope at ω_x *(log(ω_y/ω_x)
  Gain at ω_h = Gain at ω_{c2} + net slope at ω_{c2} *(log(ω_h/ω_{c2})
  = -27.96 + (-40)* log(10/5) = -40 adB.

Phase plot:

$$G(s)H(s) = \frac{10}{s\,(0.5s + 1)(0.2s + 1)}$$

The sinusoidal transfer function is

$$G(j\omega)H(j\omega) = \frac{10}{j\omega\,(0.5j\omega + 1)(0.2j\omega + 1)}$$

$$G(j\omega)H(j\omega) = \frac{10}{j\omega\,(1 + j0.5\omega)(1 + j0.2\omega)}$$

- **Angle of the term '10' is zero.**
- **Angle of the term '1/jω' is -90^0** (since the term is in denominator angle is –ve and the power of 'j' is 1 so 1*90).
- **Angle of the term '1/(1+j0.5ω)' is –tan^{-1}(0.5 ω)** (since the term is in denominator angle is –ve and the term is (a+jb) so tan^{-1}(b/a)).
- **Angle of the term '1/(1+j0.2ω)' is –tan^{-1}(0.2 ω)** (since the term is in denominator angle is –ve and the term is (a+jb) so tan^{-1}(b/a)).

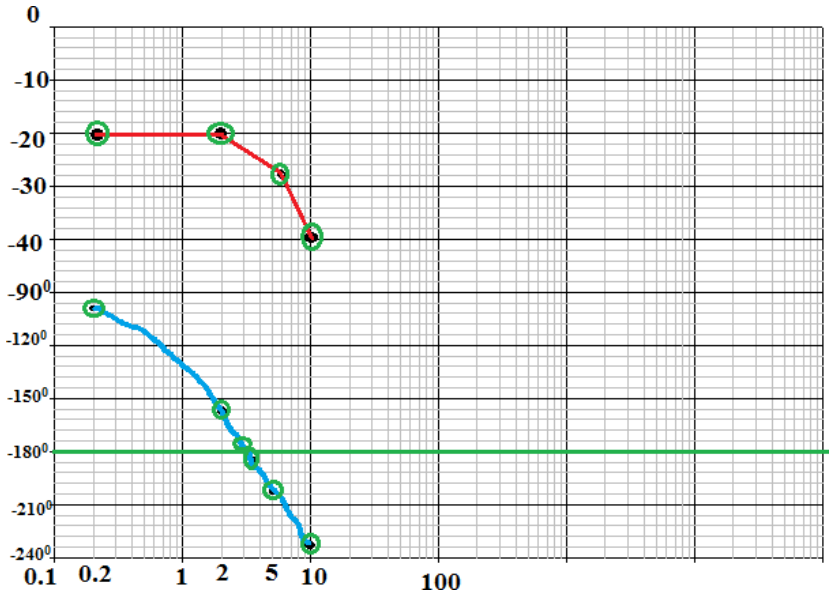
Therefore

$\Phi = \angle G(j\omega)\,H(j\omega)$ = **-90^0 –tan^{-1}(0.5 ω) –tan^{-1}(0.2 ω)**

ω(rad/sec)	Φ(deg)
0.2	-98
2	-156.8
5	-203.2
10	-232.125
3	-177.27
3.5	-185.25

Crossing -180^0. So need to calculate more point s between these

The bode plot is as shown below.


0
-10
-20
-30
-40
-90⁰
-120⁰
-150⁰
-180⁰
-210⁰
-240⁰
0.1
0.2
1
2
5
10
100

Example 1:

Draw Bode plot for the open loop transfer function of a closed loop control system whose

$$\boldsymbol{G(s) = \frac{16(1+0.2s)^2}{s^2(1+0.02s)(1+0.01s)}}$$

Solution:

- Convert the open loop transfer function into time constant form. It is already in time constant form.

$$\boldsymbol{G(s) = \frac{16(1+0.2s)^2}{s^2(1+0.01s)(1+0.02s)}}$$

Sinusoidal transfer function

$$\boldsymbol{G(j\omega) = \frac{16(1+j0.2\omega)^2}{(j\omega)^2(1+j0.01\omega)(1+j0.02\omega)}}$$

-
- The transfer function has the following basic factors. The corner frequencies of these terms are given below.

$16/S^2$	no corner frequency
$(1+0.2s)^2$	$\omega_c = 1/T = 1/0.2 = 5$ rad/sec
$1/(1+0.01s)$	$\omega_c = 1/0.01 = 100$ rad/sec
$1/(1+0.02s)$	$\omega_c = 1/0.02 = 50$ rad/sec

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
$16/S^2$	-----	-40	-40
$(1+0.2s)^2$	5	40	0
$1/(1+0.02s)$	50	-20	-20
$1/(1+0.01s)$	100	-20	-40

- Choose ω_l = 2(less than ω_{c1}=5).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is

  M_{dB}=-20 log(16/ω^2) **(if 's' exist in first term replace with ω)**

  At ω= ω_l = 2 MdB=-20 log(16/4) = 12.04 db dB
  At ω= ω_{C1} = 5 MdB=-20 log(16/25) = -3.876 dB
- AT ω= ω_{C2} = 50 gain is calculated using the eqn
  Gain at ω_y = Gain at ω_x + net slope at ω_x *log(ω_y/ω_x)

  Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *log(ω_{c2}/ω_{c1})
  = -3.876 + (0)*log(50/5) = -3.876 db
  Gain at ω_{c3} = Gain at ω_{c2} + net slope at ω_{c2} *log(ω_{c3}/ω_{c2})
  = -3.876 + (-20)*log(100/50) = -9.8966 db

- Choose ω_h = 400(larger than ω_{c3}=100).
  Gain at ω_h = Gain at ω_{c3} + net slope at ω_{c3} *log(ω_h/ω_{c3})
  = -9.8966 + (-40)*log(400/100) = -33.979 db

Phase plot:

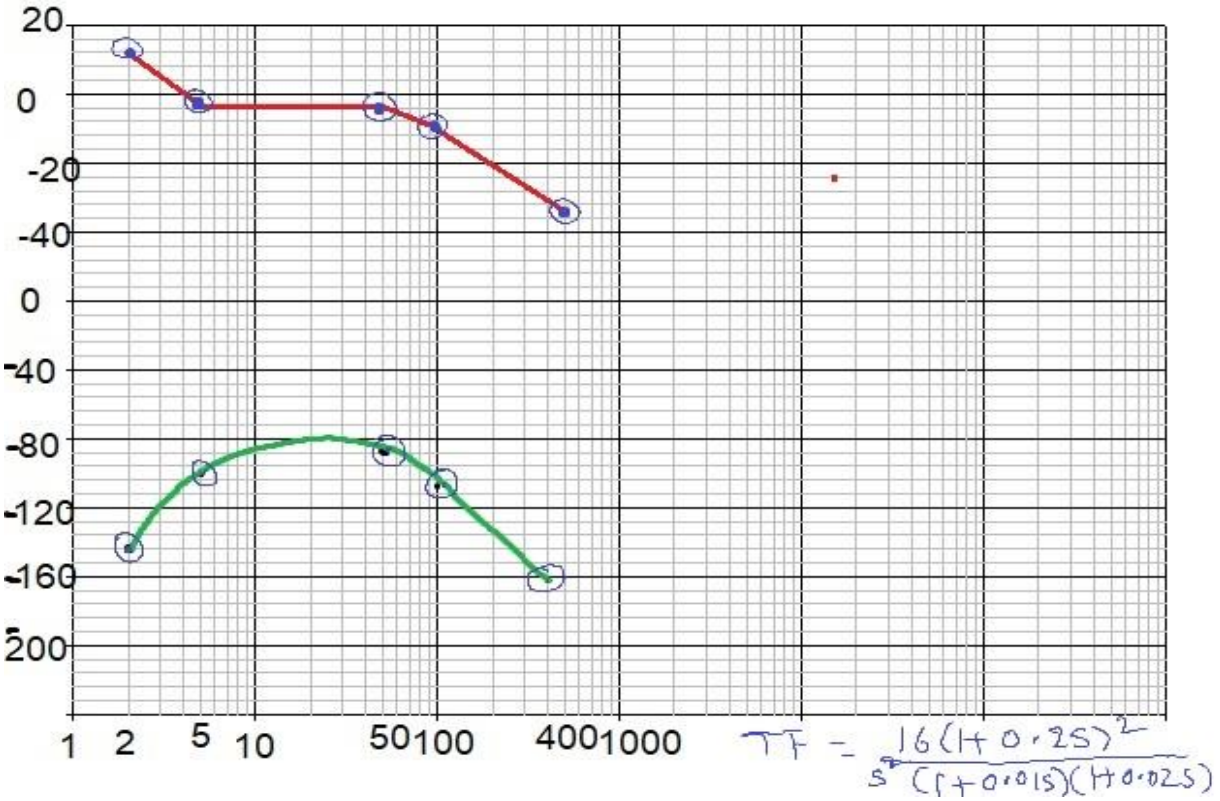
$$G(j\omega) = \frac{16(1 + j0.2\omega)^2}{(j\omega)^2(1 + j0.01\omega)(1 + j0.02\omega)}$$

- **Angle of the term '16' is zero.**
- **Angle of the term '$(1+j0.2\omega)^2$' is $2*\tan^{-1}(0.2\ \omega)$** (since the term is in numarator angle is +ve and the term is (a+jb) so $\tan^{-1}$(b/a)).
- **Angle of the term '$1/(j\omega)^2$' is $-2*90^0$** (since the term is in denominator angle is –ve and the power of 'j' is 2 so 2*90).
- **Angle of the term '$1/(1+j0.01\omega)$' is $-\tan^{-1}(0.01\ \omega)$** (since the term is in denominator angle is –ve and the term is (a+jb) so $\tan^{-1}$(b/a)).
- **Angle of the term '$1/(1+j0.02\omega)$' is $-\tan^{-1}(0.02\ \omega)$** (since the term is in denominator angle is –ve and the term is (a+jb) so $\tan^{-1}$(b/a)).

$\Phi=\angle G(j\omega) = \tan^{-1}(0.2\omega) - (2*90) - \tan^{-1}(0.01\omega) - \tan^{-1}(0.02\omega)$ …(1)

ω(rad/sec)	Φ(degrees)
ω=2	-139.835
ω=5	-98.572
ω=50	-82.99
ω=100	-114.16
ω=400	-160.271

Bode Plot:


20
0
-20
-40
0
-40
-80
-120
-160
-200
1
2
5
10
50
100
400
1000
TF = 16(1+0.2S)^2 / S^2(1+0.01S)(1+0.02S)

Example 2:

Draw Bode plot for the open loop transfer function of a closed loop control system whose $\boldsymbol{G(s) = \frac{80}{s(s+2)(s+20)}}$. Determine gain margin & phase margin. Also determine stability of the system.

Solution:

- Convert the open loop transfer function into time constant form.

$$\boldsymbol{G(s) = \frac{80}{s(s+2)(s+20)} = \frac{80}{s2\left(\frac{s}{2}+1\right)20\left(\frac{s}{20}+1\right)} = \frac{2}{s(1+0.5s)(1+0.05s)}}$$

Sinusoidal transfer function

$$\boldsymbol{G(j\omega) = \frac{2}{j\omega(1+j0.5\omega)(1+j0.05\omega)}}$$

- The transfer function has the following basic factors. The corner frequencies, slope and net slope of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
2/S	-----	-20	-20
1/(1+0.5s)	ω_{c1}=2	-20	-40
1/(1+0.05s)	ω_{c2}=20	-20	-60

- Choose ω_l = 0.2(less than ω_{c1}=2).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is

  M_{dB}=-20 log(2/ω) **(if 's' exist in first term replace with ω)**

  At ω= ω_l = 0.2 M_{dB}=20 log(2/0.2) = 20 db
  At ω= ω_{C1} = 2 M_{dB}=20 log(2/2) = 0 dB
- AT ω= ω_{C2} = 20 gain is calculated using the eqn
  Gain at ω_y = Gain at ω_x + net slope at ω_x *log(ω_y/ω_x)

  Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *log(ω_{c2}/ω_{c1})

  = 0 + (-40)*log(20/2) = -40 db

- Choose ω_h = 40(larger than ω_{c3}=20).
  Gain at ω_h = Gain at ω_{c3} + net slope at ω_{c3} *log(ω_h/ω_{c3})
  = -40 + (-60)*log(40/20) = -58.06 db

Phase plot:

$$G(j\omega) = \frac{2}{j\omega(1 + j0.5\omega)(1 + j0.05\omega)}$$

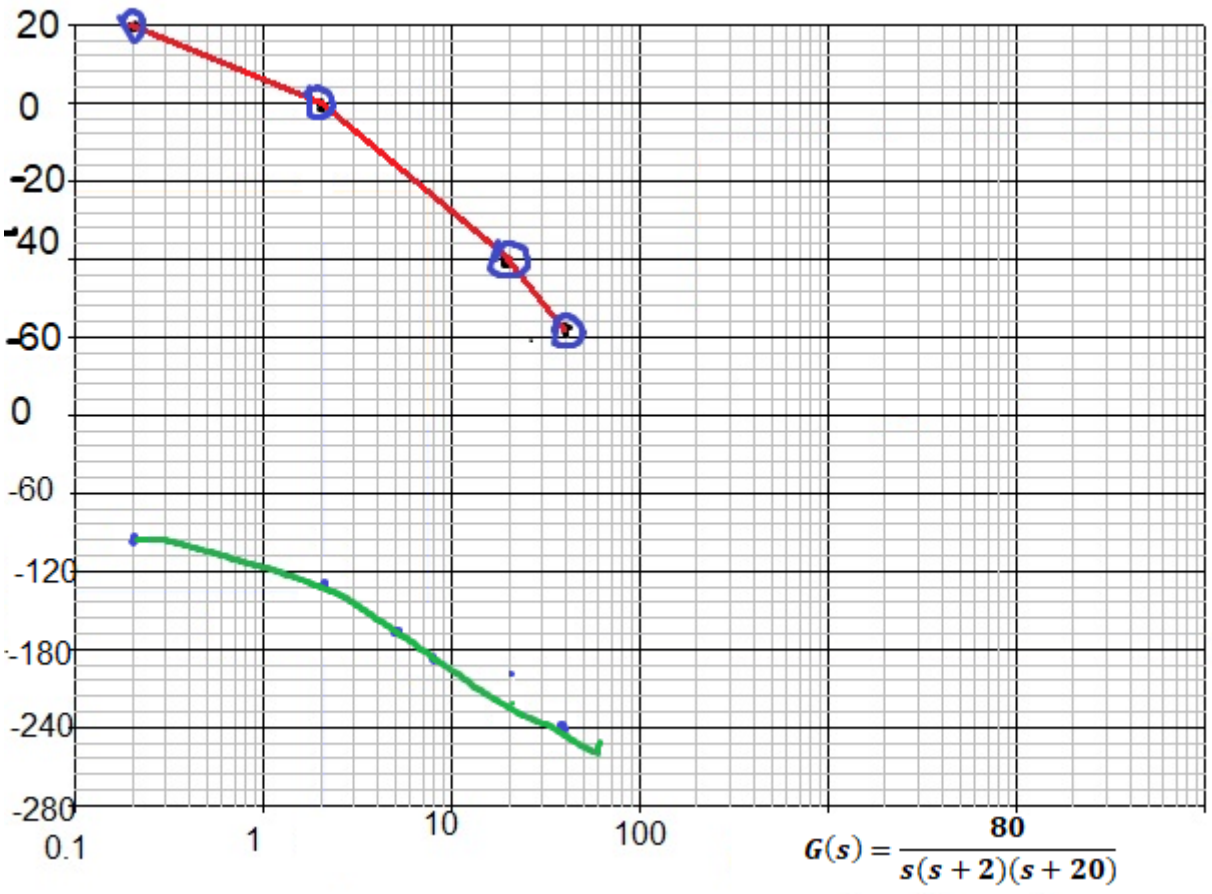
$$\Phi = \angle G(j\omega) = -90° - \tan^{-1}(0.5\omega) - \tan^{-1}(0.05\omega) \ldots(1)\ \omega_{gc}$$

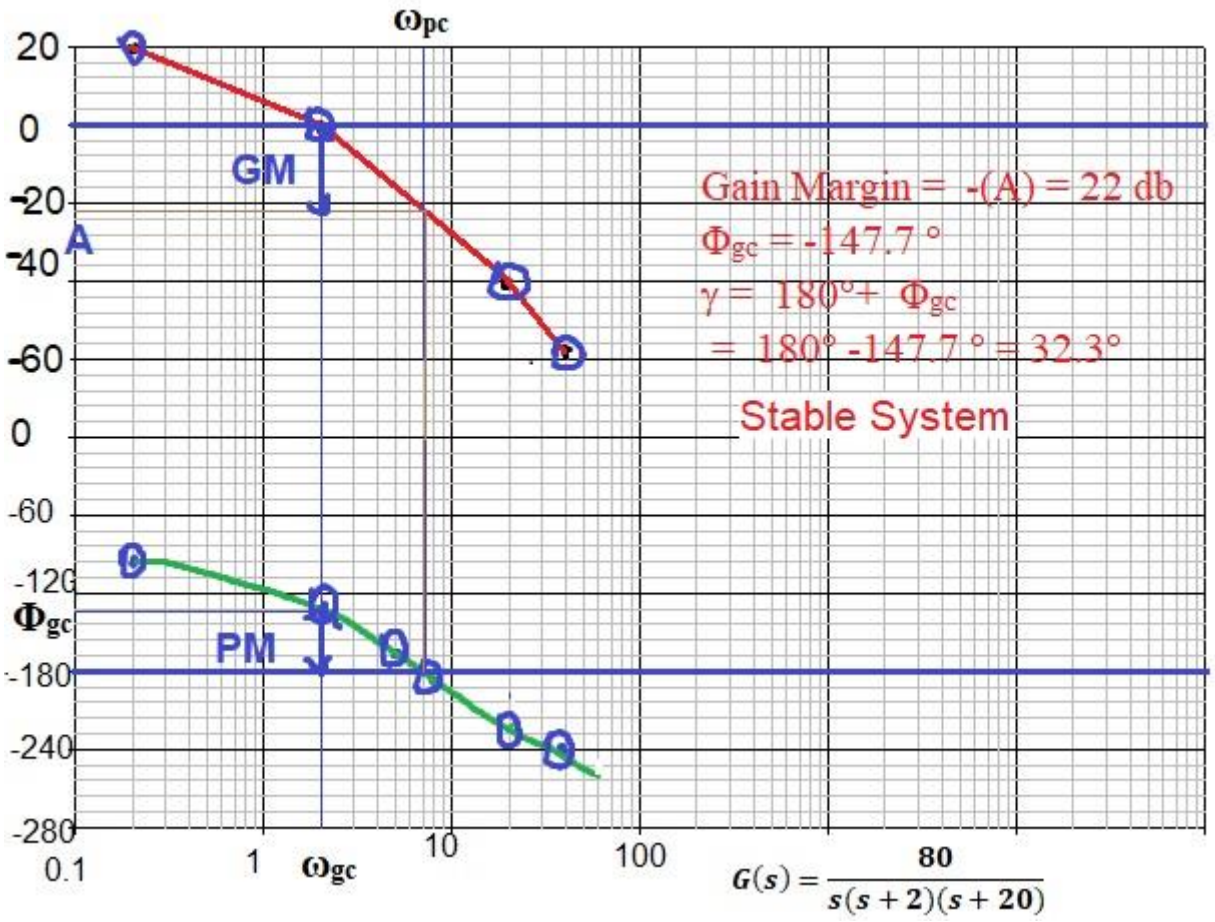
ω(rad/sec)	Φ(degrees)
ω=0.2	-96.27
ω=2	-140.7
ω=5	-172.22
ω=8	-187.76
ω=20	-219.28
ω=40	-240.58
∞	-270

Crossing -180^0. So need to calculate more points between these two.

Bode Plot:

The bode plot is as shown below.





ω_{gc} = 2 rad/sec
ω_{pc} = 7 rad/ sec
Therefore system is stable since $\omega_{gc} < \omega_{gc}$

Gain Margin = -20 log (|G(jωpc)|) =-(-22) = 22 db

Φ_{gc} = -147.7 °

γ = 180°+ Φ_{gc} = 180° -147.7 ° = 32.3°

Example 3:

Draw Bode plot for the closed loop control system whose open loop transfer function is $\boldsymbol{G(s) = \frac{KS^2}{(1+0.2s)(1+0.02s)}}$. Determine the gain K for the gain cross over frequency to be 5 rad/sec.

Solution:

- Convert the open loop transfer function into time constant form.

$$\boldsymbol{G(s) = \frac{KS^2}{(1+0.2s)(1+0.02s)}}$$

Sinusoidal transfer function

$$\boldsymbol{G(j\omega) = \frac{K(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}}$$

Draw bode plot for K=1.

- The transfer function has the following basic factors. The corner frequencies, slope and net slope of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
S^2	-----	40	40
1/(1+0.2s)	ω_{c1}=5	-20	20
1/(1+0.02s)	ω_{c2}=50	-20	0

- Choose ω_l = 1(less than ω_{c1}=5).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is
  $M_{dB} = 20 \log(\omega^2)$ **(if 's' exist in first term replace with ω)**

At $\omega = \omega_l = 1$ $M_{dB} = 20 \log(1) = 0$ db
At $\omega = \omega_{C1} = 5$ $M_{dB} = 20 \log(25) = 27.95$ dB

- AT $\omega = \omega_{C2} = 50$ gain is calculated using the eqn

Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *log(ω_{c2}/ω_{c1})
= 27.95 + (20)*log(50/5) = 47.95 db

- Choose ω_h = 100(larger than ω_{c2}=50).
  Gain at ω_h = Gain at ω_{c2} + net slope at ω_{c2} *log(ω_h/ω_{c2})
  = 47.95 + 0)*log(100/50) = 47.95 db

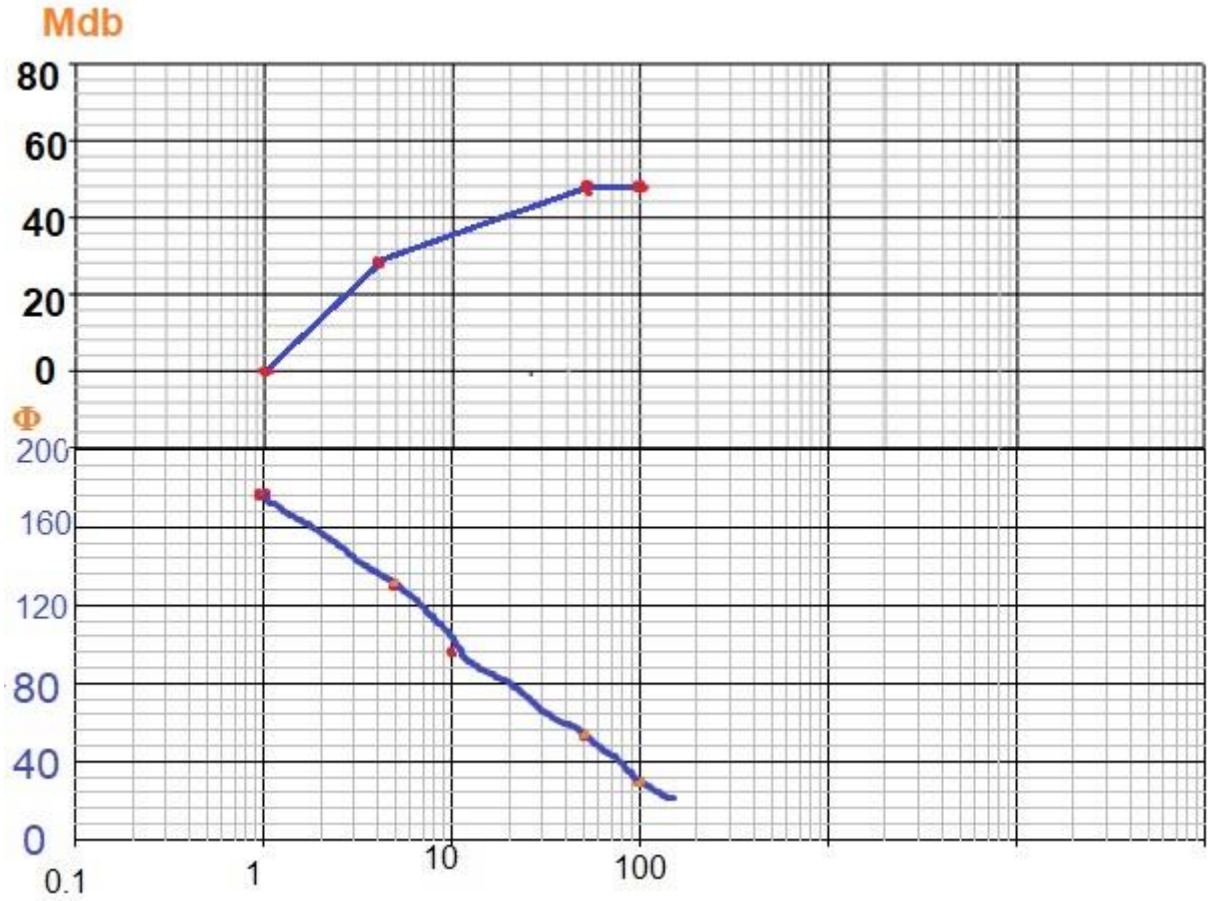
Phase plot:

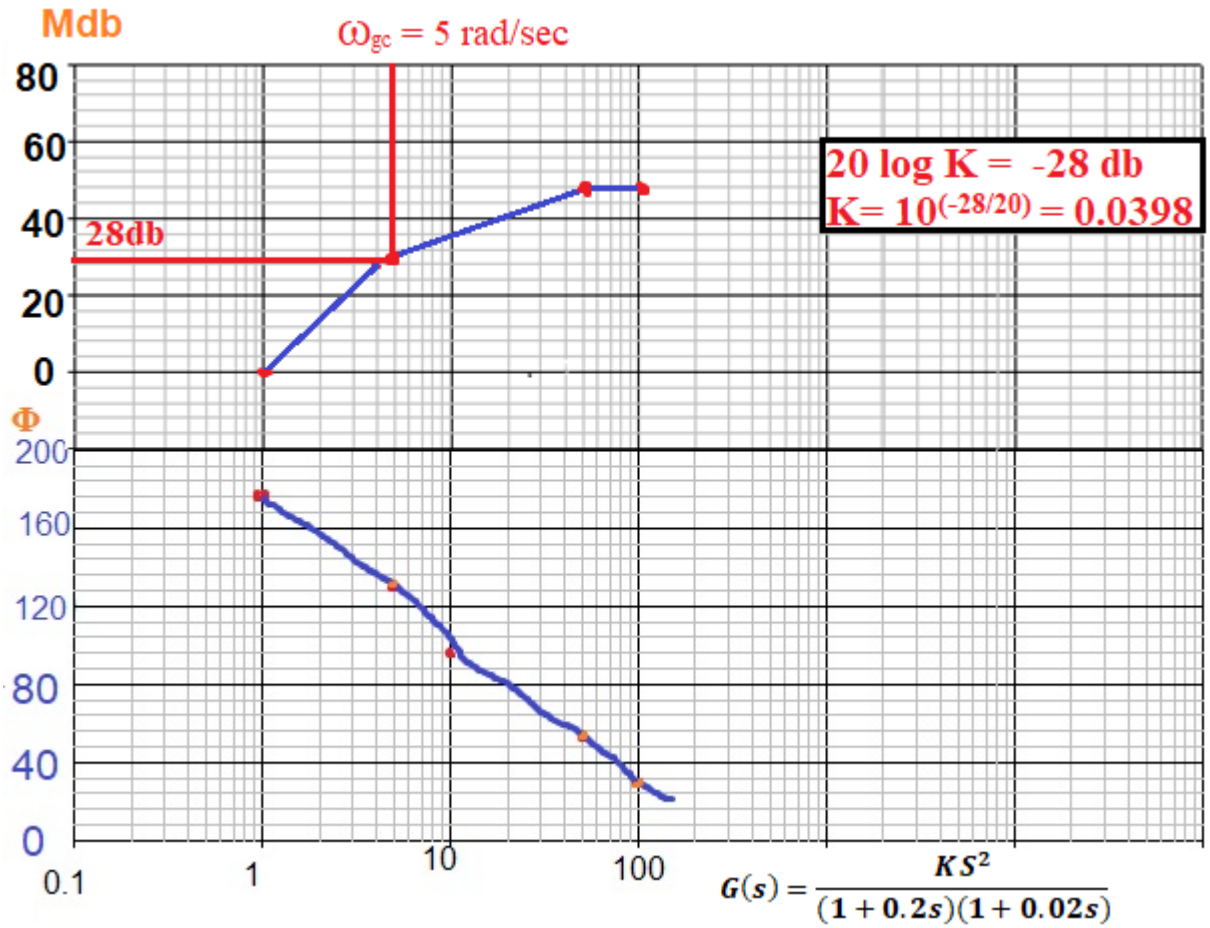
$$\boldsymbol{G(j\omega) = \frac{K(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}}$$

$$\Phi=\angle G(j\omega) = 2^* 90° - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega) \ldots(1)$$

ω(rad/sec)	Φ(degrees)
ω=1	167.6
ω=5	129.3
ω=10	105.3
ω=50	50.7
ω=100	29.5
ω=∞	0

Bode Plot: The bode plot is as shown below.





Given ω_{gc} = 5 rad/sec that means at this frequency the magnitude should be 0 db.
But from graph magnitude at 5 rad/sec = 28 db.
To make it zero we need to add -28 db.
Therefore **20 log K = -28 db**
K= $10^{(-28/20)}$ = 0.0398

Example 4:

Draw Bode plot for the open loop transfer function of a closed loop control system whose $\boldsymbol{G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}}$. Determine gain margin & phase margin. Also determine stability of the system.

Solution:

- Convert the open loop transfer function into time constant form.

$$\boldsymbol{G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{75(1+0.2s)}{s*100\left(\frac{s^2}{100}+.16s+1\right)}}$$

$$\boldsymbol{= \frac{0.75(1+0.2s)}{s(0.01s^2+0.16s+1)}}$$

Sinusoidal transfer function

$$\boldsymbol{G(j\omega) = \frac{0.75(1+j0.2\omega)}{j\omega(-0.01\omega^2+j0.16\omega+1)}}$$

- The transfer function has the following basic factors. The corner frequencies, slope and net slope of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
0.75/S	-----	-20	-20
(1+0.2s)	ω_{c1}=5	20	0
$1/(\boldsymbol{s^2+16s+100})$	$\omega_{c2}=\omega_n$=10	-40	-40

- Choose ω_l = 1(less than ω_{c1}=5).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is

  M_{dB} = 20 log(0.75/ω) **(if 's' exist in first term replace with ω)**

At ω= ω_l = 1 M_{dB}=20 log(0.75) = -2.49 db
At ω= ω_{C1} = 5 M_{dB}=20 log(0.75/5) = -16.5 dB
Further slope is negative only so choose a frequency less than 1 so that 0 db line will be crossed.
Choose ω_l = 0.5 M_{dB} = 20 log(0.75/0.5)=3.5db

- AT ω= ω_{C2} = 10 gain is calculated using the eqn

  Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *log(ω_{c2}/ω_{c1})
  = -16.5 + (0)*log(50/5) = -16.5 db

- Choose ω_h = 20(larger than ω_{c3}=10).
  Gain at ω_h = Gain at ω_{c2} + net slope at ω_{c2} *log(ω_h/ω_{c2})
  = -16.5 + (-40)*log(20/10) = -28.5 db

Phase plot:

$$G(j\omega) = \frac{0.75(1 + j0.2\omega)}{j\omega(-0.01\omega^2 + j0.16\omega + 1)}$$

$$\Phi = \angle G(j\omega) = \tan^{-1}(0.2\omega) - 90° - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) \ldots(1)$$

For the second order term the angle varies from 0 to 180. But calculator calculates tan-1 only between 0 to 90. Hence a correction of 180 should be required.

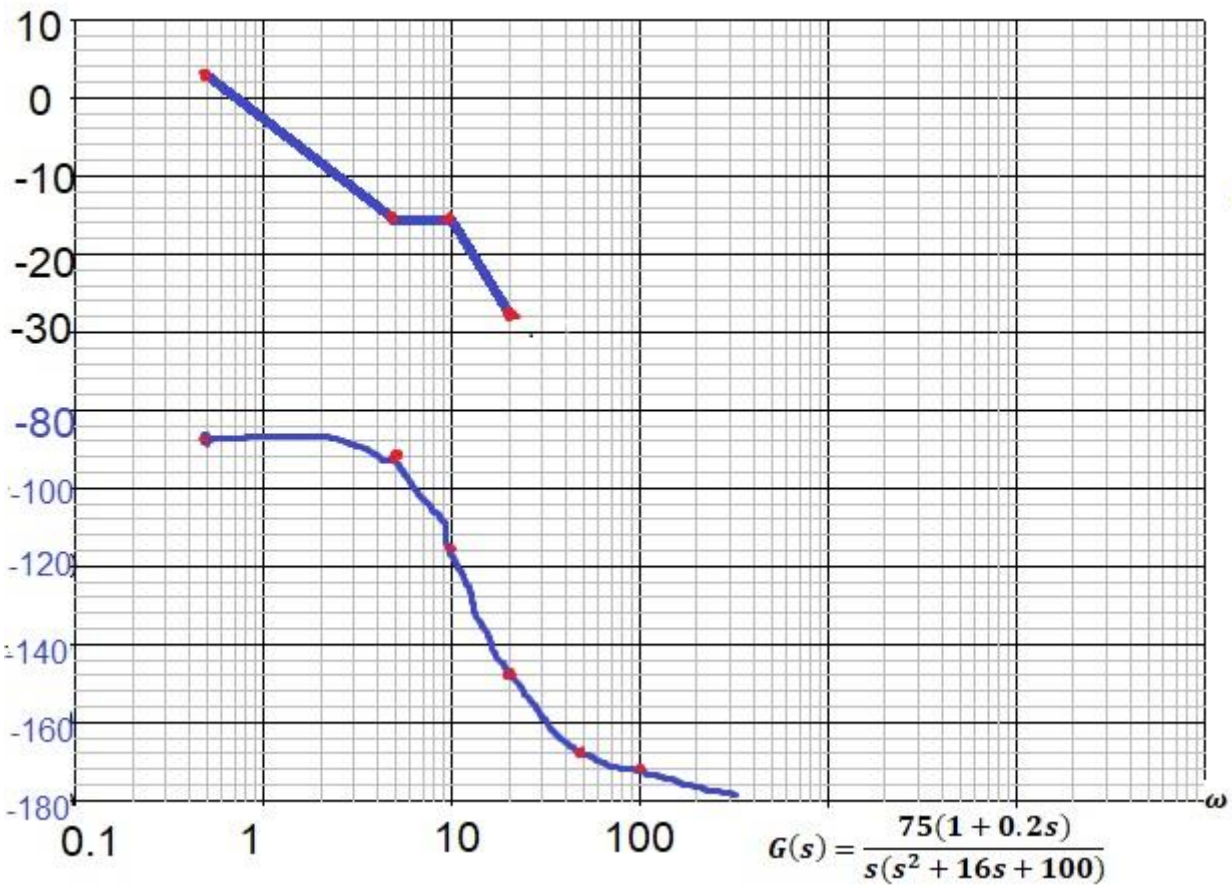
For $\omega <= \omega n$ $\quad \Phi = \tan^{-1}(0.2\omega) - 90° - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right)$

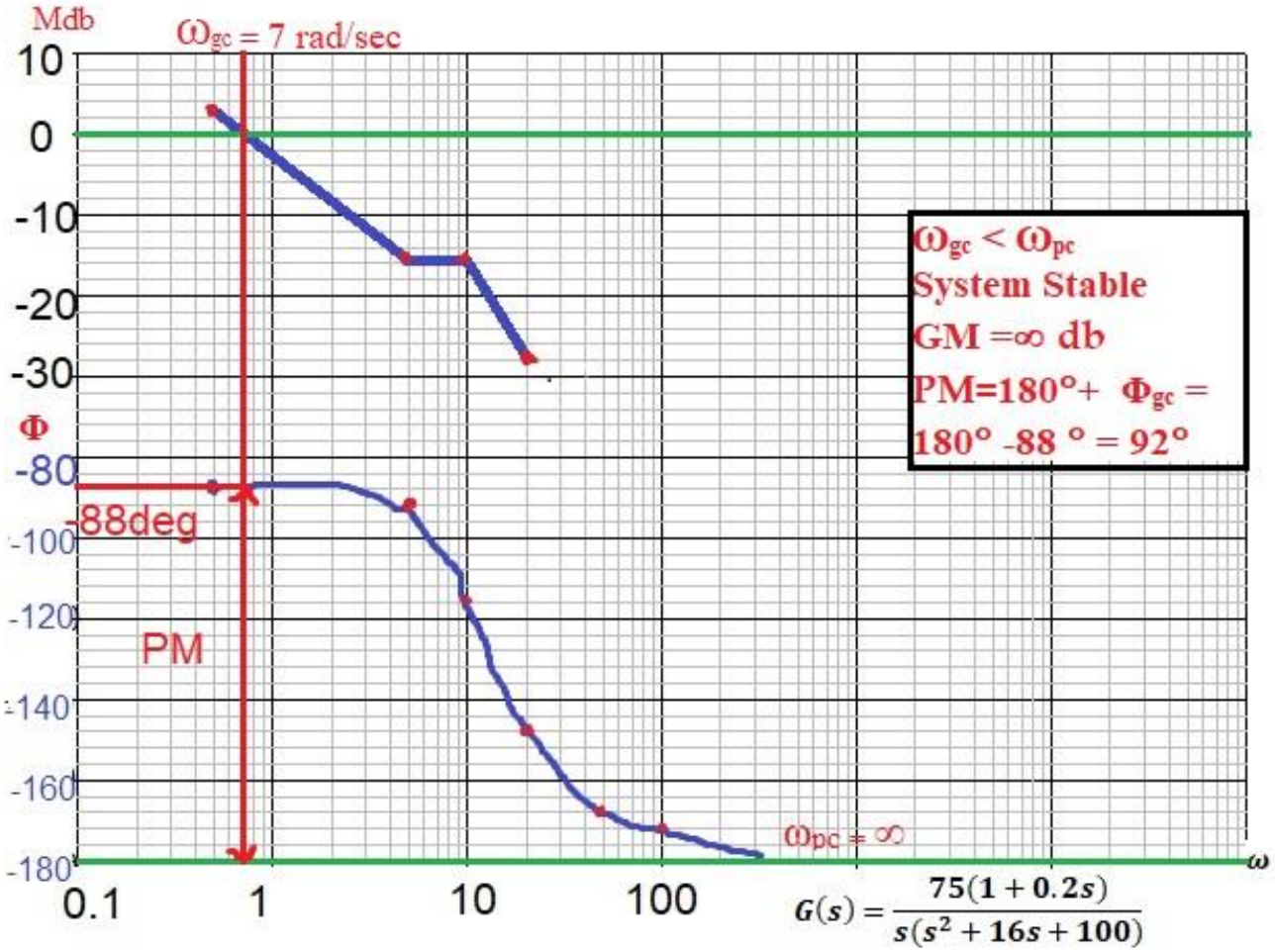
For $\omega > \omega n$ $\quad \Phi = \tan^{-1}(0.2\omega) - 90° - [\tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) + 180]$

$$\Phi = 90° + \tan^{-1}(0.2\omega) - \tan^{-1}\left(\frac{0.16\omega}{1-0.01\omega^2}\right) + 180]$$

ω(rad/sec)	Φ(degrees)
ω=1	-87.9
ω=5	-91.8
ω=10	-116.6
ω=20	-147.3
ω=50	-167.3
ω=100	-173.7
∞	-180

Bode Plot:The bode plot is as shown below.


10
0
-10
-20
-30
-80
-100
-120
-140
-160
-180
0.1
1
10
100
ω
$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$



ω_{gc} = 7 rad/sec
ω_{pc} = ∞ rad/ sec
Therefore system is stable since $\omega_{gc} < \omega_{pc}$

Gain Margin = -20 log (|G(jωpc)|) =-(-∞) = ∞ db

Φ_{gc} = -88 °

γ = 180°+ Φ_{gc} = 180° -88 ° = 92°

Example 5:

Draw Bode plot for the open loop transfer function of a closed loop control system whose $\boldsymbol{G(s) = \frac{Ke^{-0.2s}}{s(s+2)(s+8)}}$. Determine K so that system is stable with (i) gain margin 2 db (ii) phase margin 45°.

Solution: this type of systems are known as systems with transportation lag.

- Convert the open loop transfer function into time constant form.

$$\boldsymbol{G(s) = \frac{Ke^{-0.2s}}{s(s+2)(s+8)} = \frac{Ke^{-0.2s}}{s*2(0.5s+1)*8(s0.125+1)}}$$

$$\boldsymbol{= \frac{0.0625Ke^{-0.2s}}{s(1+0.5s)(1+0.125s)}}$$

Sinusoidal transfer function

$$\boldsymbol{G(j\omega) = \frac{0.0625Ke^{-0.2j\omega}}{s(1+j0.5\omega)(1+j0.125\omega)}}$$

$$\boldsymbol{|e^{-0.2j\omega}| = sin(0.2\omega) - jcos(0.2\omega) = 1\angle(-0.2\omega)}$$

So there is no change in magnitude due to this term but for phase plot we need to add (-0.2ω)rad in the phase eqn.

Bode plot for K=1

- The transfer function has the following basic factors. The corner frequencies, slope and net slope of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
0.0625/S	-----	-20	-20
1/(1+0.5s)	ω_{c1}=2	-20	-40
1/(1+0.125s)	ω_{c2}=8	-20	-60

- Choose ω_l = 1(less than ω_{c1}=2).
  The expression to calculate M_{dB} at ω_l & ω_{c1} is

  M_{dB}=20 log(0.0625/ω) **(if 's' exist in first term replace with ω)**

  At ω= ω_l = 1 M_{dB}=20 log(0.0625/1) = -24.08 db
  At ω= ω_{C1} = 2 M_{dB}=20 log(0.0625/2) = -30 dB
- At ω= ω_{C2} = 8 gain is calculated using the eqn

  Gain at ω_{c2} = Gain at ω_{c1} + net slope at ω_{c1} *log(ω_{c2}/ω_{c1})
  = -30 + (-40)*log(8/2) = -54 db

- Choose ω_h = 20(larger than ω_{c3}=8).

Gain at ω_h = Gain at ω_{c2} + net slope at ω_{c2} *log(ω_h/ω_{c2})
= -54 + (-60)*log(20/8) = -77.87 db

Phase plot:

$$G(j\omega) = \frac{0.0625Ke^{-0.2j\omega}}{s(1+j0.5\omega)(1+j0.125\omega)}$$

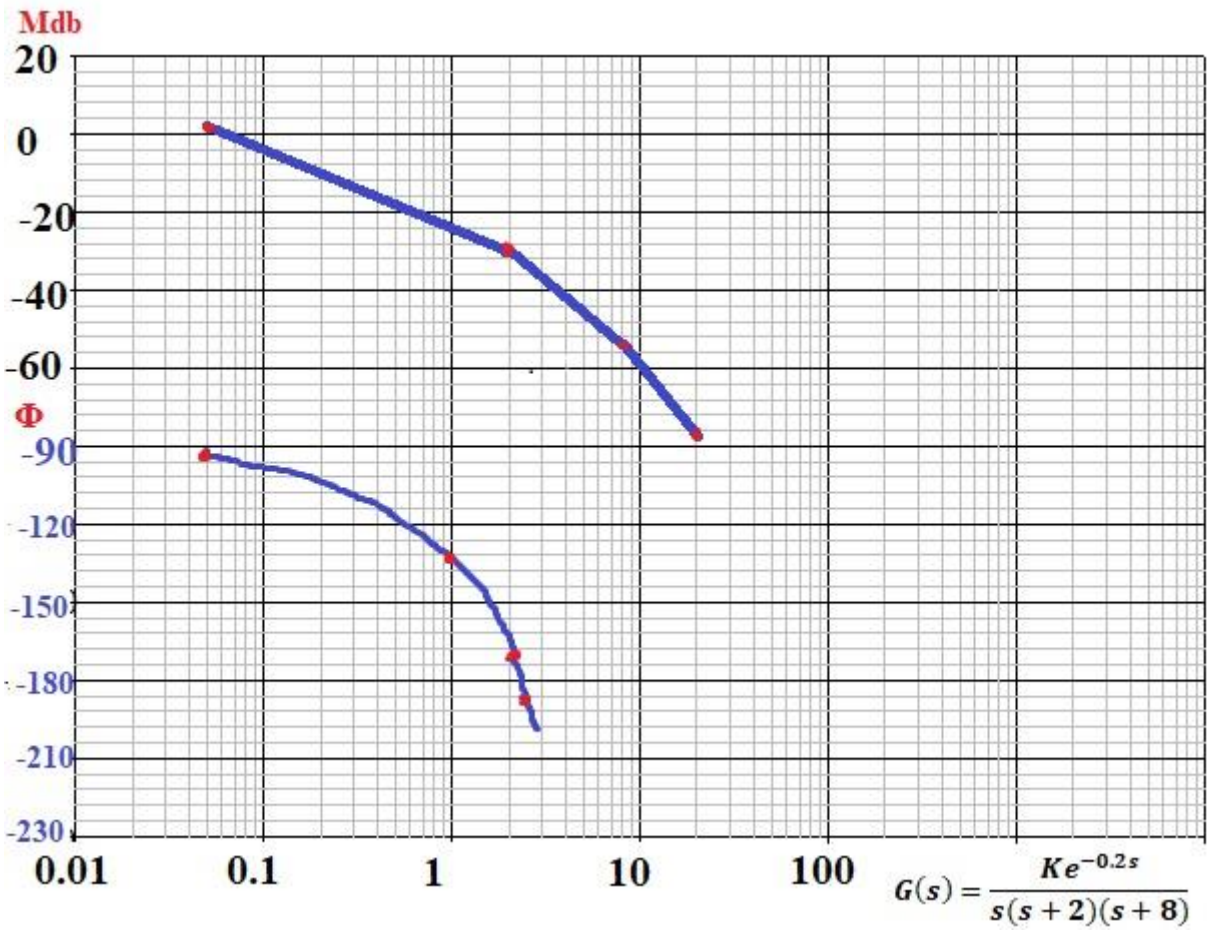
$$|e^{-0.2j\omega}| = sin(0.2\omega) - jcos(0.2\omega) = 1\angle(-0.2\omega)$$

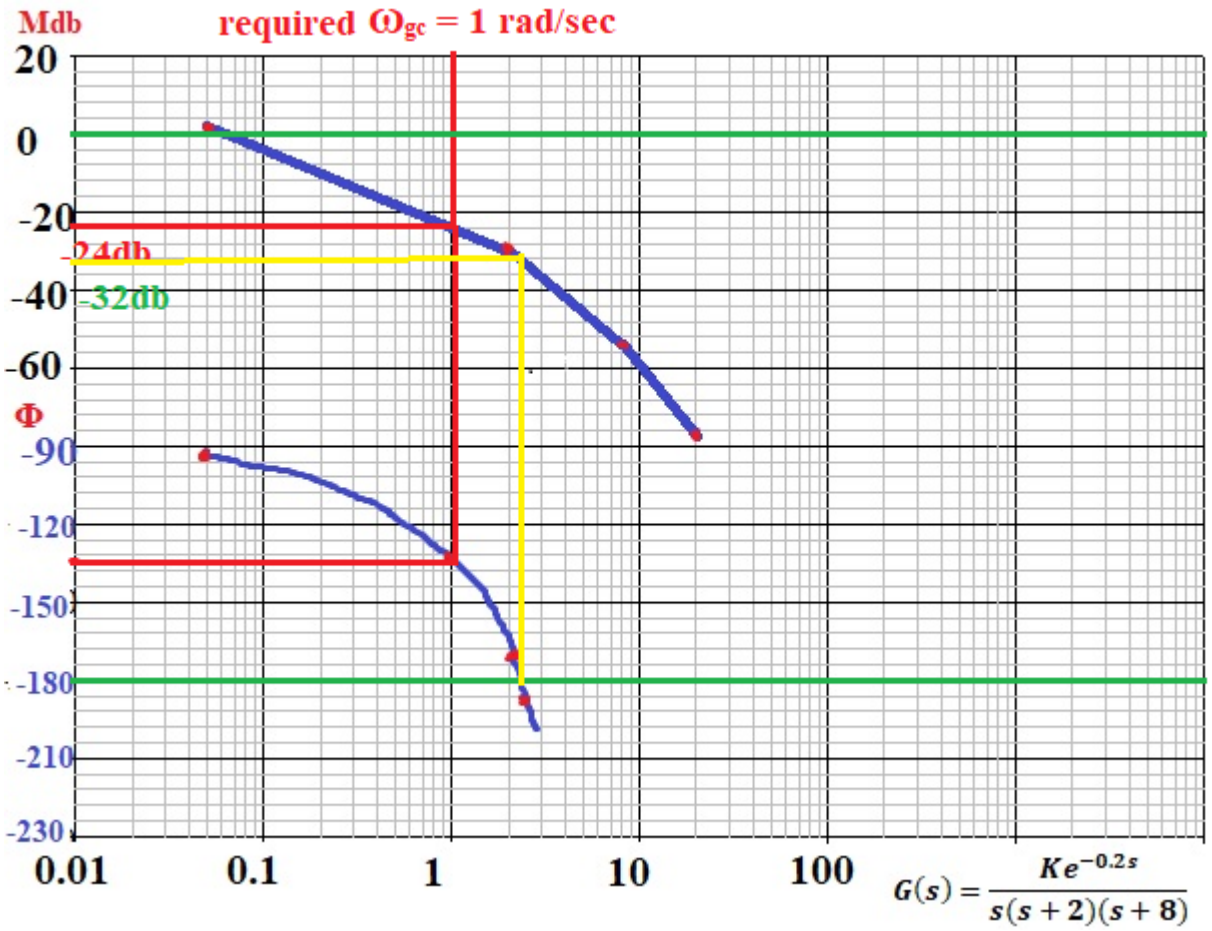
$$\Phi = \angle G(j\omega) = \left(-0.2\omega * \frac{180^\circ}{\Pi}\right) - 90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(0.125\omega) \ldots(1)$$

ω(rad/sec)	Φ(degrees)
ω=1	-134.4
ω=2	-171.9
ω=2.3	
ω=2.5	
ω=3	-201.2
ω=8	

Crossing -180^0. So need to calculate more points between these two.

Bode Plot:The bode plot is as shown below.


Mdb
20
0
-20
-40
-60
Φ
-90
-120
-150
-180
-210
-230
0.01
0.1
1
10
100
$G(s) = \frac{Ke^{-0.2s}}{s(s+2)(s+8)}$



(i) Given Gain Margin = 6 db

For K=1 bode plot GM = -(-32) = 32 db

To get GM = 6 db we need to add -26 db to the magnitude.

Therefore 20 log K = -26 db

$K = 10^{(-26/20)} = 0.05$

(ii) Given phase margin = 45° = 180°+ Φ_{gc}

Therefore required Φ_{gc} = -180° +45° = -135°

From Bode plot frequency at this angle is **ω_{gc}** = -24 db

To make this frequency **ω_{gc} the magnitude at this frequency should be 0db.**

So we need to add 24 db to the magnitude.

Therefore 20 log K = 24 db

$K = 10^{(24/20)} = 15.84$

Practice problems:

Draw Bode Diagram for the transfer functions:

1. $G(s) = \frac{100(s+1)}{(s^2+110s+1000)}$
2. $G(s) = \frac{100s}{(s+1)^2(s+10)}$
3. $G(s) = \frac{30(s+10)}{s^2+3s+50}$
4. $G(s) = \frac{4(s^2+s+25)}{s^3+100s^2}$

COMPENSATION

The purpose of control system is to get desired response. The requirements of response are usually specified as performance specifications. They are generally related to accuracy, relative stability and speed of response.

In time domain specifications are given in terms of rise time, peak overshoot, settling time etc.

In frequency domain they are given in terms of phase margin, Gain margin, resonant peak & band width.

Both these things are specifications of transient state performance. The steady state performance is given by steady state error and error constants.

When time domain specifications are given root locus is used to design the system.

When frequency domain specifications are given frequency response plots like bode plots are used to design the system.

The first step to get desired response is varying K value. But practically adjustment of gain alone will not be sufficient to meet the given specifications. In many cases increasing the gain may improve steady state performance but results in poor transient state performance. Sometimes the system may become unstable.

So it is necessary to introduce additional devices in the system to alter its behavior and to meet the desired requirements. This is known as compensation.

The redesigning or alteration of system using an additional device to achieve given specifications is called compensation. A device introduced in the system to alter its behavior is known as compensator.

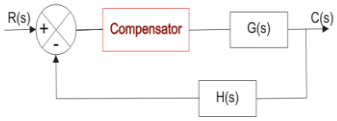
The compensators basically introduces poles and zeros to the system to alter the behavior.

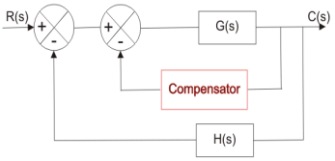
The design problem: can be stated as follows.

When a set of specifications are given for a system, we need to design a suitable compensator so that overall system will meet required specifications.

Types of Compensation

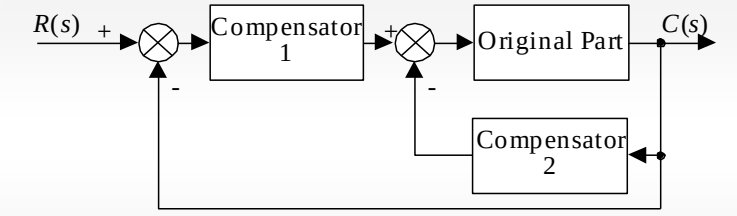
- Connecting compensating circuit between error detector and plants known as **series compensation.**





- When a compensator used in a feedback manner called **Feedback/Parallel compensation.**

- Combination of series and feedback compensator is called series-parallel compensation or load compensation.



Choice of Compensation

The choice between various types is depends on following factors.

- Nature of Signals
- Power levels at various points
- Components available
- Designers experience
- Economic consideration

Types of Compensators

- **Electrical*or Electronic**
- Mechanical
- Hydraulic
- pneumatic etc.

Types of Electrical or Electronic Compensators

The types of series compensators are

- Lag Compensator
- Lead Compensator
- Lag-lead Compensator

In control systems compensation is required in the following situations.

1. When the system is unstable the compensation is required to make the system stable.
2. When the system is stable compensation is provided to obtain desired performance.(to meet the given specifications)

System with **Type number 2** and above are absolutely unstable systems, so **Lead Compensation** is required because they increases the margin of stability.

Type 1 or 0 stability is achieved by adjusting the gain, so **any** compensator can be used to obtain desired performance.

Lead Compensator:

- The Lead Compensator is an electrical network which produces a sinusoidal output having the phase lead when a sinusoidal input is applied.
- Used When Steady state response is OK and Transient response NOT OK.

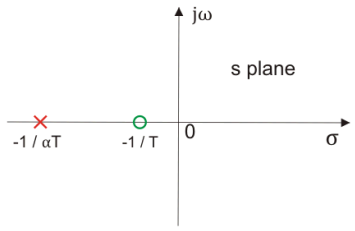
Transfer Function of lead compensator:

$$G_c(s) = \frac{s+Z_c}{s+P_c} = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \quad \alpha < 1, T > 0$$

One pole at s= -1/αT

One zero at s= -1/T

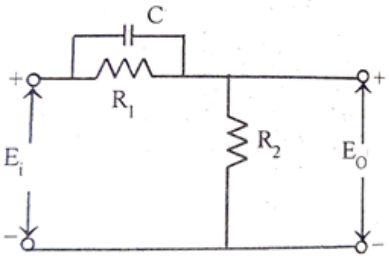
Pole-Zero plot:

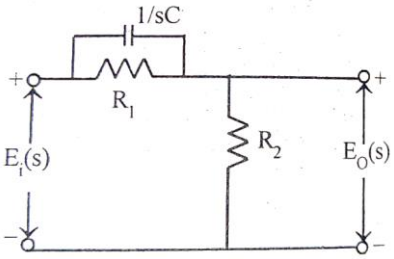


1/T < 1/αT

Zero is nearer to jω axis. So dominating. Lead compensator add one pole and one dominant zero to the system. So provides lead angle(zero means numerator term)

Electrical network for Lead compensator:





$$Z_1 = \frac{R_1 * 1/sC}{R_1 + 1/sC} = \frac{R_1}{R_1Cs + 1}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{\frac{R_1}{R_1Cs+1} + R_2} = \frac{R_2R_1Cs + R_2}{R_1 + R_1R_2Cs + R_2}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_1R_2C(s + \frac{1}{R_1C})}{R_1R_2C(s + \frac{R_1+R_2}{R_1R_2C})}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{s+\frac{1}{R_1C}}{s+\frac{R_1+R_2}{R_1R_2C}} = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

T = $R_1C > 0$ & $\alpha = \frac{R_2}{R_1+R_2} < 1$

Bode plot & Frequency response:

$$G_c(s) = \frac{s+Z_c}{s+P_c} = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = \frac{\frac{1}{T}(1+sT)}{\frac{1}{\alpha T}(1+s\alpha T)} = \frac{\alpha(1+sT)}{(1+s\alpha T)}$$

Sinusoidal transfer function

$$G(j\omega) = \frac{\alpha(1+j\omega T)}{(1+j\omega\alpha T)}$$

- The transfer function has the following basic factors. The corner frequencies of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
α	-----	0	0
(1+sT)	1/T	20	20
1/(1+s αT)	1/αT	-20	0

At ω=0 M_{db} =20 log α = -ve (since α<1)

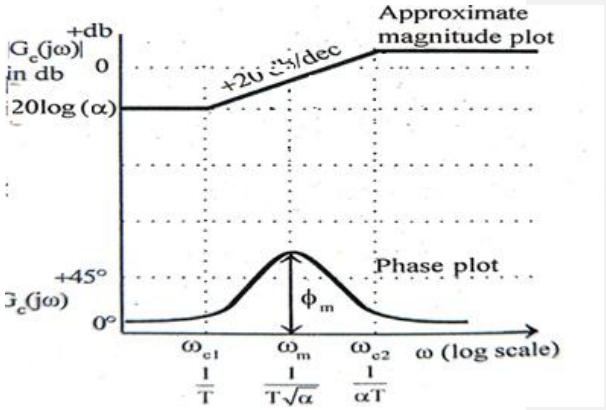
Φ=∠G(jω) = $\tan^{-1}(\omega T) - \tan^{-1}(\omega\alpha T)$

At ω=0 Φ=0

At ω=∞ φ=0

As ω increases from 0 rad/sec φ increases from 0 and reaches a maximum value and then decreases to reach 0 again.

This maximum lead angle is represented by φ_m and the frequency at which this angle occurs is represented by ω_m.



Maximum Lead angle φ_m:

$$G(j\omega) = \frac{\alpha(1+j\omega T)}{(1+j\omega\alpha T)}$$

$$M = |G(j\omega)| = \frac{\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2\alpha^2T^2}}$$

Φ=∠G(jω) = $\tan^{-1}(\omega T) - \tan^{-1}(\omega\alpha T)$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

$$\frac{d\varphi}{d\omega} = \frac{T}{1+\omega^2T^2} - \frac{\alpha T}{1+\omega^2\alpha^2T^2} = 0$$

$$1 + \omega^2\alpha^2T^2 - \alpha(1+\omega^2T^2) = 0$$

$$1 + \omega^2\alpha^2T^2 - \alpha - \alpha\omega^2T^2 = 0$$

$$\omega^2\alpha^2T^2 - \alpha\omega^2T^2 = \alpha - 1$$

$$\alpha\omega^2T^2(\alpha - 1) = \alpha - 1$$

$$\omega^2 = \frac{1}{\alpha T^2}$$

Therefore

$$\boldsymbol{\omega_m = \frac{1}{T\sqrt{\alpha}}} = \sqrt{\frac{1}{T}\frac{1}{\alpha T}} = \sqrt{\omega_{c1}\omega_{c2}}$$

ω_m is geometric mean of two corner frequencies.

Therefore $\boldsymbol{\Phi_m = \tan^{-1}(\omega_m T) - \tan^{-1}(\omega_m \alpha T)}$

$$tan\ \varphi_m = \frac{\omega_m T - \omega_m \alpha T}{1 + \omega_m T \omega_m \alpha T} = \frac{1-\alpha}{2\sqrt{\alpha}}$$

Therefore $\boldsymbol{sin\ \varphi_m = \frac{1-\alpha}{1+\alpha}}$

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Magnitude at ω_m is

$$M = \frac{\alpha\sqrt{1+\frac{1}{\alpha T^2}T^2}}{\sqrt{1+\frac{1}{\alpha T^2}\alpha^2T^2}} = \frac{\alpha\sqrt{\frac{\alpha+1}{\alpha}}}{\sqrt{1+\alpha}} = \sqrt{\alpha}$$

Magnitude in db = 20 log √α

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Design of lead compensator:

Specifications: System error constant and Phase margin

STEP 1: At zero frequency the gain of lead compensator is negative that means it provides attenuation. To cancel this the practical lead compensator is connected in series with an amplifier having gain Kc.

Transfer Function of practical lead compensator is

$$G_c(s) = K_c\alpha\frac{1+sT}{1+s\alpha T} = \frac{K(1+sT)}{1+s\alpha T}$$

The open loop transfer function of compensated system is

$$G_c(s)G(s) = \frac{K(1+sT)}{1+s\alpha T}G(s)$$

Let $G_1(s) = K\,G(s)$

Determine K value which satisfies the given error constant.

STEP 2: Using this value of K draw the Bode plot for $G_1(s)$. Determine the phase margin and gain crossover frequency. If phase margin is not satisfactory design lead compensator.

STEP3: let γ_s = specified phase margin

γ = actual phase margin

Determine the amount of phase lead ϕ_m to be contributed by lead network using

$$\boldsymbol{\phi_m = \gamma_s - \gamma + \epsilon}$$

Where Є-margin of safty as gain cross over frequency may shift due to compensation=5 to 15°. Initially Choose $\epsilon = 5^0$

STEP 4: calculate α using $\alpha = \frac{1-sin\,\emptyset_m}{1+sin\,\emptyset_m}$

From the bode plot determine the frequency ω_m at which the magnitude is **20 log √α** db.

Calculate T using the relation $\omega_m = 1/(T\sqrt{\alpha})$ → **$T = 1/(\omega_m\sqrt{\alpha})$**

The transfer function of the compensator $G_c(s) = \frac{1+sT}{1+s\alpha T}$

STEP 5: Determine the open loop transfer function of the compensated system after connecting $G_c(S)$ in cascade with G1(S) as given below

$$\mathbf{G_o(s)} = G_1(s)*G_c(s) = \mathbf{KG(s)*G_c(s)}$$

Step 6: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε value till the specifications met.

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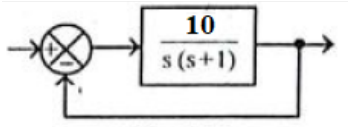
Pb: for the system shown in figure design a lead compensator such that the closed loop system will satisfy the following specifications. (i) static velocity error constant=20sec (ii) PM =50°(iii) GM>= 10db



STEP 1: At zero frequency the gain of lead compensator is negative that means it provides attenuation. To cancel this the practical lead compensator is connected in series with an amplifier having gain Kc.

Transfer Function of practical lead compensator is

$$G_c(s) = K_c\alpha\frac{1+sT}{1+s\alpha T} = \frac{K(1+sT)}{1+s\alpha T}$$

Let $G_1(s) = K\,G(s) = \frac{10K}{s(s+1)}$

K_v=20 sec = $\lim_{s\to 0} sG_1(s) = 10K$

Therefore K = 2

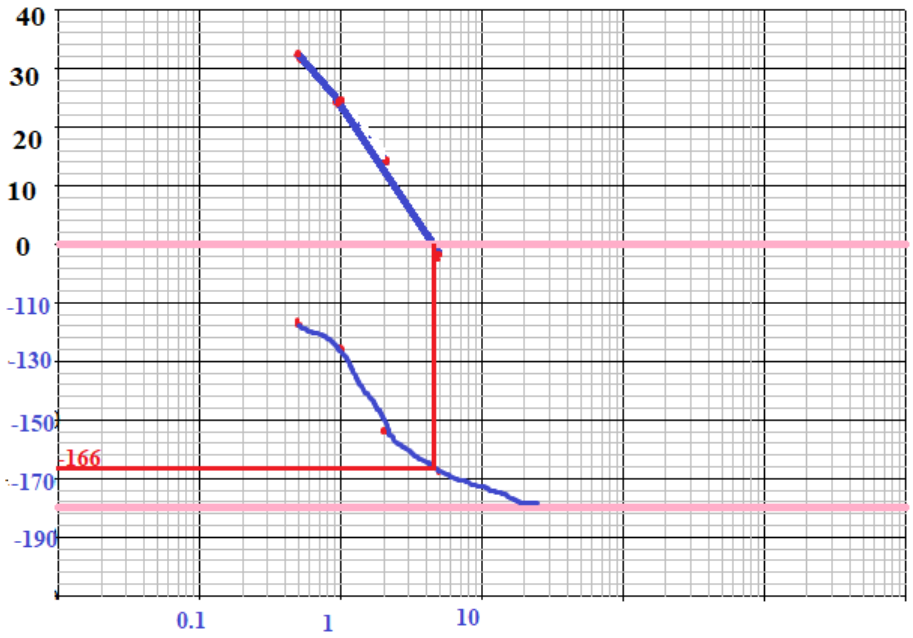
STEP 2: Using this value of K draw the Bode plot for $G_1(s) = \frac{20}{s(s+1)}$.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
20/S	-----	-20	-20
(1+s)	1	-20	-40

ω(rad/sec)	M db
ω_l=0.5	32.04
ω=1	26.02
ω_h=5	-1.938

Φ=∠G(jω) = 90°- $\tan^{-1}(\omega)$

ω(rad/sec)	Φ(degrees)
ω_l=0.5	-116.56
ω=1	-135
ω_h=5	-168.69
ω=∞	-180



ω_{gc}=4 rad/sec, ϕ_{gc}=-166°, PM=180 +, ϕ_{gc} = 14°

ω_{pc}=∞ rad/sec & GM = +∞db

STEP3: γ_s = specified phase margin=50°

γ = actual phase margin = 14°(from bode plot)

Choose ϵ=5^0

$\phi_m=\gamma_s - \gamma+\epsilon$ = 50 – 14 + 5 = 41°

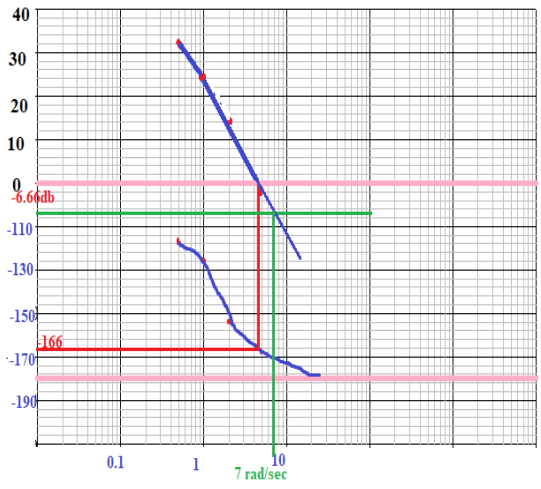
STEP 4:

calculate α using $\alpha = \frac{1-sin\,\emptyset_m}{1+sin\,\emptyset_m} = 0.207$

20 log √α = -6.82 db

From the bode plot determine the frequency ω_m at which the magnitude is **20 log √α** db.

ω_m = 7 rad/sec (from bode plot)



Calculate T using the relation $\omega_{m=} 1/(T\sqrt{\alpha})$ → **$T=1/(\omega_m \sqrt{\alpha})=0.314$ sec**

The transfer function of the compensator $G_c(s) = \frac{1+sT}{1+s\alpha T} = \frac{1+0.314s}{1+0.065s}$

STEP 5: Determine the open loop transfer function of the compensated system after connecting $G_c(S)$ in cascade with $G_1(S)$ as given below

$$\mathbf{G_o(s)} = G_1(s)*G_c(s) = \frac{20(1+0.314s)}{s(s+1)(1+0.065s)}$$

Step 6: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε value till the specifications met.

Effects of Phase Lead Compensation

- Lead compensator add a dominant zero to the system. This increases damping ratio of the system.
- Increased damping ratio results in less rise time , less overshoot ,less settling time etc.
- So it improves the transient performance of the system.
- The velocity constant K_v increases.
- The slope of the magnitude plot reduces at the gain crossover frequency so that phase margin & Gain margin improve. So relative stability improves.
- Increases bandwidth of closed loop system.
- Hence Response become faster.
- Addition of phase lead near gain crossover frequency.
- Increase of gain at higher frequencies.
- The steady state error does not get effected.

Disadvantages of Phase Lead Compensation

- Lead compensator requires amplifiers at low frequency to increase the gain. This results in more number of elements, larger space, greater weight and higher cost.
- Large BW is not desirable sometimes. It makes system more susceptible to the noise.
- The compensated system may have a large undershoot than overshoot. So tendency to over compensate system may lead to conditionally stable system.
- Steady state error is not improved.
- The maximum lead angle provided by a single lead compensator is about 60°. If more than required need to design multi stage compensator.

Lag Compensator:

- The Lag Compensator is an electrical network which produces a sinusoidal output having the phase lag when a sinusoidal input is applied.
- It acts a low pass filter and so high frequency noise signals are attenuated .
- It increases the order of the system by one
- Large improvement in Steady state performance but slow response due to reduced bandwidth

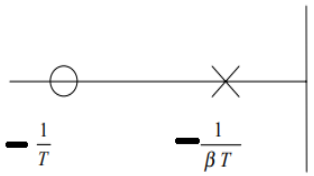
Transfer Function of lag compensator:

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad \beta>1,\ T>0$$

One pole at s= -1/βT

One zero at s= -1/T

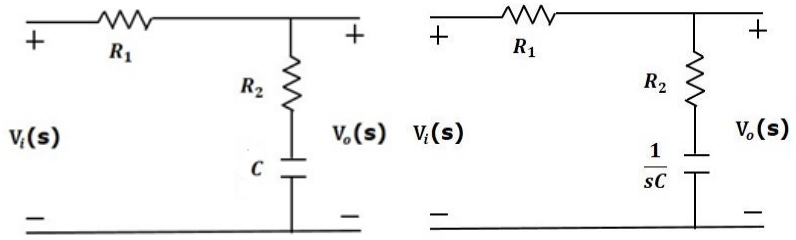
Pole-Zero plot:



1/T > 1/βT

Pole is nearer to jω axis. So dominating. Lag compensator add one zero and one dominant pole to the system.

Electrical network for Lag compensator:



$$Z_2 = R_2 + 1/sC$$

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1+Z_2} = \frac{R_2+1/sC}{R_2+1/sC+R_1} = \frac{R_2Cs+1}{(R_1+R_2)Cs+1}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2C(s+\frac{1}{R_2C})}{(R_1+R_2)C(s+\frac{1}{(R_1+R_2)C})}$$

$$\frac{E_0(s)}{E_i(s)} = (\frac{R_2}{R_1+R_2})\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}} = (\frac{R_2}{R_1+R_2})\frac{s+\frac{1}{R_2C}}{s+\frac{R_2}{(R_1+R_2)R_2C}} = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

T = $R_2C > 0$ & $\beta = \frac{R_1+R_2}{R_2} > 1$

Bode plot & Frequency response:

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \frac{1}{\beta}\frac{\frac{1}{T}(1+sT)}{\frac{1}{\beta T}(1+s\beta T)} = \frac{(1+sT)}{(1+s\beta T)}$$

Sinusoidal transfer function

$$G(j\omega) = \frac{(1+j\omega T)}{(1+j\omega\beta T)}$$

- The transfer function has the following basic factors. The corner frequencies of these terms are given below.

Term	Corner frequency (rad/sec)	Slope of the term (db/dec)	Net slope (db/dec)
1	-----	0	0
1/(1+s βT)	1/βT	-20	-20
(1+sT)	1/T	20	0

At $\omega=0$ M_{db} =20 log 1 = 0 db

$\Phi = \angle G(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\omega\beta T)$

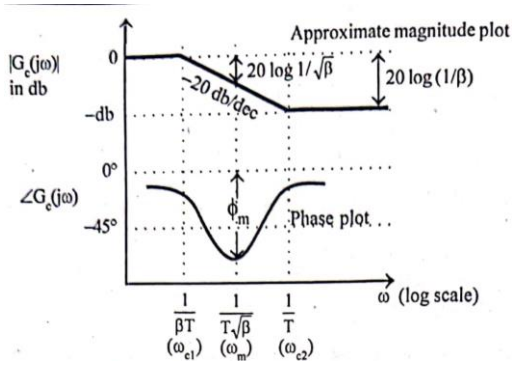
At $\omega=0$ $\Phi=0$

At $\omega=\infty$ $\varphi=0$

As ω increases from 0 rad/sec φ decreases from 0 and reaches a minimum value and then increases to reach 0 again.

This maximum lag angle is represented by φ_m and the frequency at which this angle occurs is represented by ω_m.

$\omega_m = 1/(T\sqrt{\beta})$ = geometric mean of two corner frequencies.



Design of Lag compensator:

Specifications: System error constant and Phase margin

STEP 1: Transfer Function of practical lag compensator is

$$G_c(s) = \frac{1 + sT}{1 + s\beta T}$$

Determine K value which satisfies the given error constant for G(s).

STEP 2: Using this value of K draw the Bode plot for G(s). Determine the phase margin and gain crossover frequency. If phase margin is not satisfactory design lag compensator.

STEP3: let γ_s = specified phase margin

Determine the new phase margin using

$$\gamma_n = \gamma_s + \epsilon$$

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Where γ_s = specified phase margin

Є-margin of safety as gain cross over frequency may shift due to compensation=5 to 15°.

Initially Choose $\epsilon = 5^0$

STEP 4: Determine new phase required at gain cross over frequency ω_{gcn} using

$$\Phi_{gcn} = \gamma_n - 180 \qquad (\gamma_n = 180 + \Phi_{gcn})$$

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Determine new gain cross over frequency ω_{gcn}. ω_{gcn} is the frequency at which the phase of G(jω) is Φ_{gcn} from bode plot.

STEP 5: Determine the parameter β of the compensator. The value of β is given by the magnitude of G(jω) at new gain cross over frequency ω_{gcn}. Find the db gain of (A_{gcn}) at new gain cross over frequency ω_{gcn} from bode plot.

$$A_{gcn} = 20\log \beta \quad \text{or} \quad \beta = 10^{A_{gcn}/20}$$

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Choose $\omega_{c2} = \omega_{gcn}/10$ (or) $\omega_{gcn}/2$

Therefore $T = 10/\omega_{gcn}$

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Step 6: The transfer function of the lag compensator $G_c(s) = \frac{1+sT}{1+s\beta T}$

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Step 7: Determine the open loop transfer function of the compensated system after connecting $G_c(S)$ in cascade with G(S) as given below $G_o(s) = G(s) * G_c(s) = G(s) * \frac{1+sT}{1+s\beta T}$

Step 8: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε or ωn value till the specifications met.

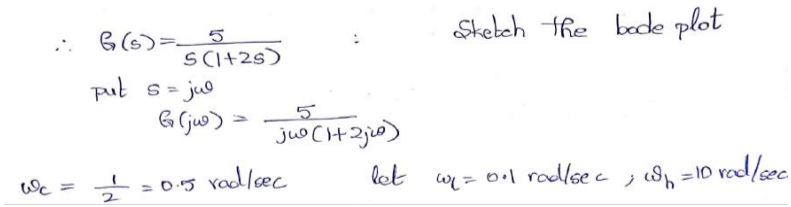
Pb: For a unity feedback system with open loop transfer function is $G(s) = \frac{K}{s(1+2s)}$. Design a lag compensator such that the closed loop system will satisfy the following specifications. (i) steady state error for ramp input should be less than or equal to 0.2 (ii) PM =40°

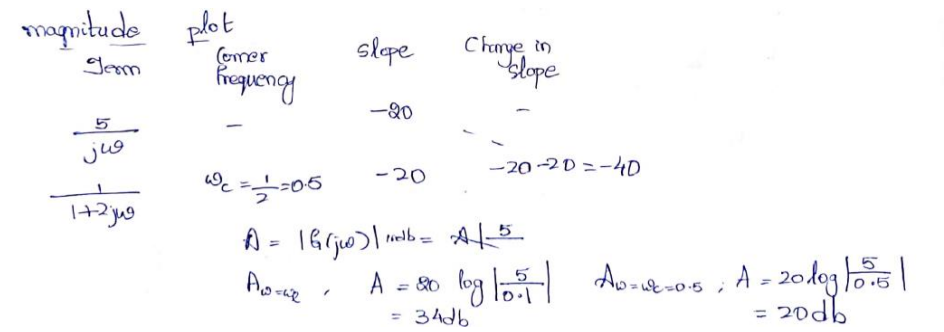
STEP 1: Determine K value of G(s) which satisfies the given error constant.

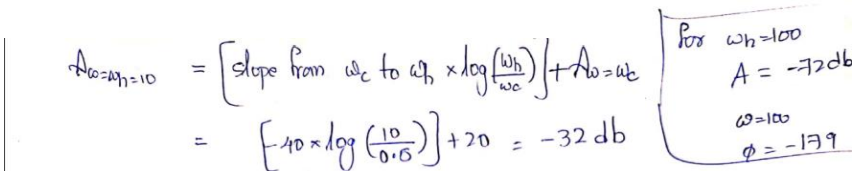
e_{ss} for ramp input = A/ K_v = 1/ K_v = 0.2 → K_v = 5 $= \lim_{s\to 0} sG(s) = K$

Therefore $G(s) = \frac{5}{s(1+2s)}$

STEP 2: Draw the Bode plot for G(s). Determine the phase margin and gain crossover frequency. If phase margin is not satisfactory design lag compensator.







Phase plot

$\phi = -90° - \tan^{-1}(2\omega)$

ω	ϕ
0.1	-101
0.3	-121
0.5	-135
6	-175
8	-176
10	-177

A = -12(

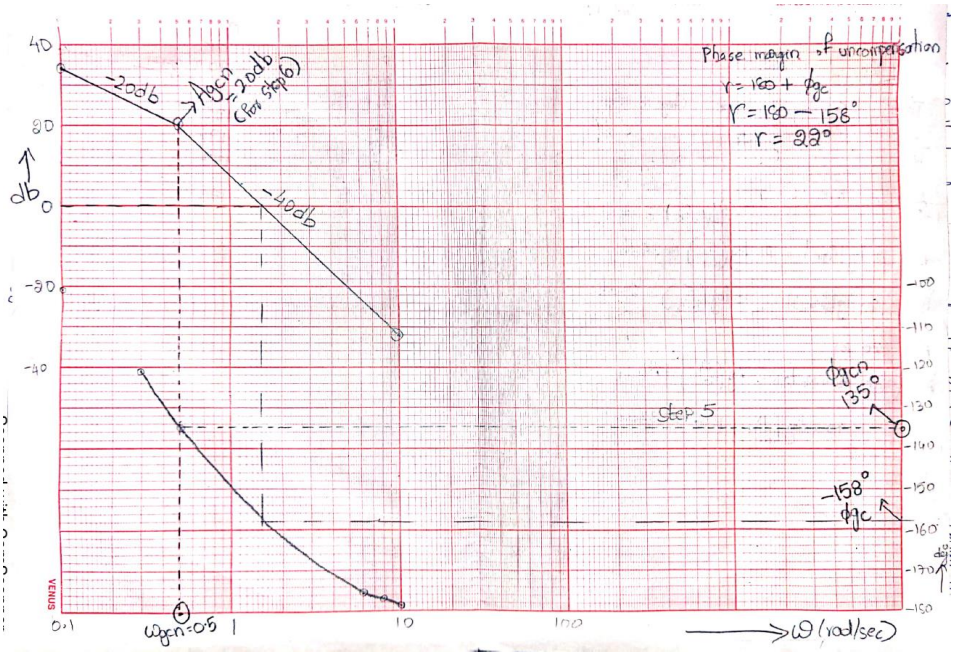
ω=100

ϕ = -17°

Step_3 Determine phase margin of uncompensation, if phase margin does not satisfy the requirement then lag compensation is required

$\gamma = 22°$ from plot

Given $\gamma_d = 40°$ ∴ compensation is required to improve phase margin



PM = 22°. Compensation is required.

STEP3: let γ_s = specified phase margin=40°

Є-margin of safety =5^0

$\gamma_n=\gamma_s+ Є =45°$

STEP 4: Determine $\Phi_{gcn} =\gamma_n-180 =-135°$

Determine ω_{gcn} from bode plot the frequency at which the phase is Φ_{gcn}.

ω_{gcn} = 0.5 rad/sec

STEP 5: Find the db gain of (A_{gcn}) at ω_{gcn} from bode plot.

A_{gcn}=20db or $\beta=10^{Agcn/20}= 10$

Choose $\omega_{c2} =\omega_{gcn} /10$

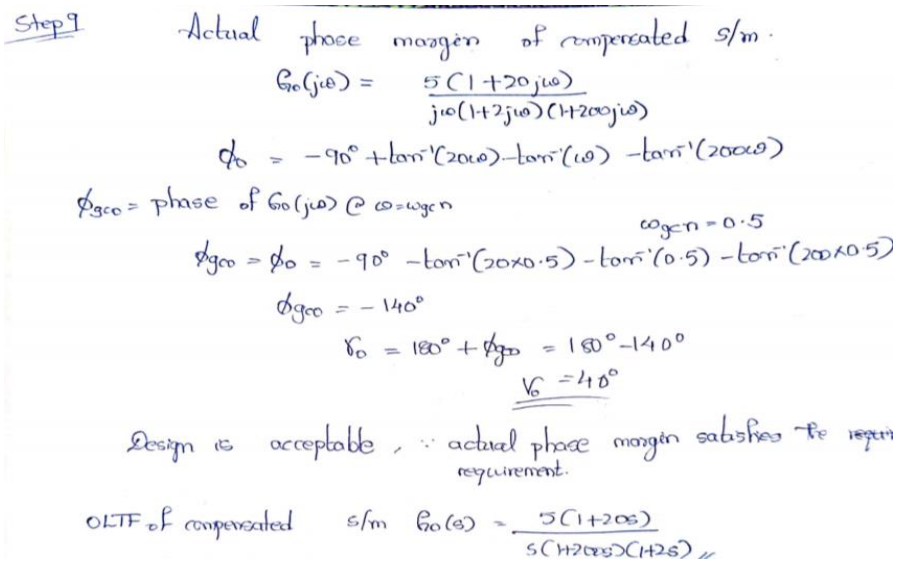
therefore T= $10/\omega_{gcn}$ = 20 sec

Step 6: The transfer function of the lag compensator $G_c(s) = \frac{1+20s}{1+200s}$

Step 7: Determine the open loop transfer function of the compensated system after connecting $G_c(S)$ in cascade with G(S) as given below

$G_o(s) = G(s)*G_c(s) = \frac{5}{s(1+2s)}\frac{1+20s}{1+200s}$

Step 8: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε or Ωn value till the specifications met.



Effects & draw backs of Lag compensator:

- Lag compensator allows high gain at low frequencies i.e it is a low pass filter. Hence improve steady state performance.
- In lag compensator the attenuation characteristics is used for compensation. Thus phase lag characteristics is of no use.
- the gain cross over frequency is reduced.
- Phase margin increases.
- Thus bandwidth is reduced.
- Reduced bandwidth results in slow response i.e high rise time & settling time.
- Thus transient response lasts for longer time.
- System become more sensitive to parameter variations.
- Lag compensator acts similar to PI controller.

Lag-lead Compensator:

- The Lag-lead Compensator is an electrical network which produces a sinusoidal output having the phase lag at low frequencies and phase lead at high frequencies when a sinusoidal input is applied.
- Both transient & steady state response improves.

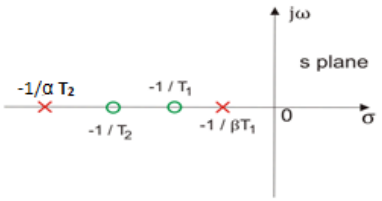
Transfer Function of lag compensator:

$$G_c(s) = \frac{(s+\frac{1}{T_1})(s+\frac{1}{T_2})}{(s+\frac{1}{\beta T_1})(s+\frac{1}{\alpha T_2})}$$ **α<1,β>1, T_1,T_2>0 & T_1>T_2**

two poles at s= -1/β T_1, -1/α T_2

two zeros at s= -1/ T_1, -1/ T_2

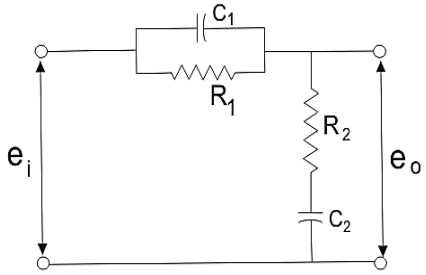
Pole-Zero plot:

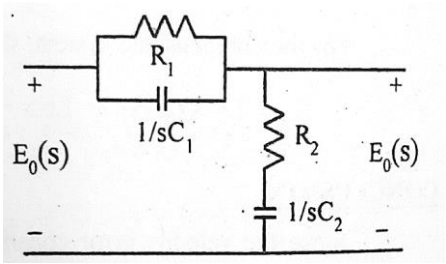


1/β T_1< 1/ T_1< 1/ T_2 < 1/α T_2

At low frequencies acts as lag compensator & at high frequencies acts as lead compensator

Electrical network for Lead compensator:





$$Z_1 = \frac{R_1 * 1/sC_1}{R_1 + 1/sC_1} = \frac{R_1}{R_1C_1s + 1}$$

$$Z_2 = R_2 + 1/sC_2 = \frac{R_2C_2s + 1}{C_2s}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1+Z_2} = \frac{\frac{R_2C_2s+1}{C_2s}}{\frac{R_1}{R_1C_1s+1}+\frac{R_2C_2s+1}{C_2s}} = \frac{(R_1C_1s+1)(R_2C_2s+1)}{R_1C_2s+(R_2C_2s+1)(R_1C_1s+1)}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_1C_1(s+\frac{1}{R_1C_1})R_2C_2(s+\frac{1}{R_2C_2})}{R_1C_1R_2C_2s^2+(R_1C_2+R_1C_1+R_2C_2)s+1}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_1C_1(s+\frac{1}{R_1C_1})R_2C_2(s+\frac{1}{R_2C_2})}{R_1C_1R_2C_2(s^2+(\frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_1})s+\frac{1}{R_1C_1R_2C_2})}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(s+\frac{1}{R_1C_1})(s+\frac{1}{R_2C_2})}{(s^2+(\frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_1})s+\frac{1}{R_1C_1R_2C_2})}$$

$$\boldsymbol{G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)} = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s^2+s\left(\frac{1}{\beta T_1}+\frac{1}{\alpha T_2}\right)+\frac{1}{\alpha\beta T_1T_2}\right)}}$$

By comparing numerator $T_1 = R_1 C_1$ & $T_2 = R_2 C_2$

By comparing constant of denominator $\alpha\beta = 1$ and

$$\boldsymbol{\frac{1}{\beta T_1}+\frac{1}{\alpha T_2}} = \frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_1C_1}$$

So generally we design lag-lead compensator with **$\alpha\beta = 1$**

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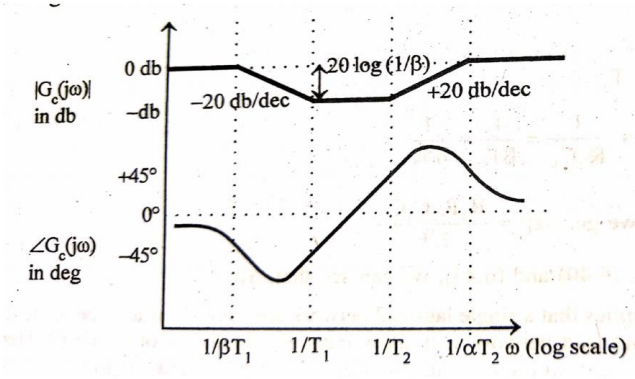
Bode plot & Frequency response:

$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)} = \alpha\beta\ \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} = \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)}$$



Design procedure for lag-lead compensator

STEP1: Determine the open loop gain K of the uncompensated system to satisfy the specified error requirement.

STEP 2: Draw the bode plot of uncompensated system KG(s) and Determine the phase margin of uncompensated system.

STEP3: let γ_s = specified phase margin

Determine the new phase margin using $\gamma_n=\gamma_s+\epsilon$

Where Є-margin of safety as gain cross over frequency may shift due to compensation=5 to 15°. Initially Choose Є=5^0

STEP 4: Determine new phase required at gain cross over frequency ω_{gcn} using

$\Phi_n=\gamma_n-180$ ($\gamma_n=180+\Phi_n$)

Determine new frequency ω_n the frequency at which the phase is Φ_n from bode plot. Choose ω_{gcn} gain crossover frequency of lag compensator some what greater than ω_n.

STEP 5: Determine the parameter β of the compensator. The value of β is given by the magnitude of G(jω) at new gain cross over frequency ω_{gcn}. Find the db gain of (A_{gcn}) at new gain cross over frequency ω_{gcn} from bode plot.

$A_{gcn}=20\log\beta$ or $\beta=10^{Agcn/20}$

Choose higher corner frequency of lag $\omega_{c2}=1/T_1=\omega_{gcn}/10$ (or) $\omega_{gcn}/2$

Therefore $T_1=10/\omega_{gcn}$

With β and T_1 determine the transfer function of lag compensator $=G_{c1}(s)=\frac{(1+sT_1)}{(1+s\beta T_1)}$

STEP 6: calculate $\alpha=1/\beta$

From the bode plot determine the frequency ω_m at which the magnitude is **$20\log\sqrt{\alpha}$** db.

Calculate T_2 using the relation $\omega_m=1/(T_2\sqrt{\alpha})$ → $T_2=1/(\omega_m\sqrt{\alpha})$

The transfer function of the lead compensator $G_{c2}(s)=\frac{1+sT_2}{1+s\alpha T_2}$

STEP 7: Determine the open loop transfer function of the compensated system as given below

$$G_o(s)=KG(s)*\frac{\beta(1+sT1)}{(1+s\beta T1)}\frac{\alpha(1+sT2)}{(1+s\alpha T2)}$$

Step 6: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε value till the specifications met.

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Pb: For a unity feedback system with open loop transfer function is $\mathbf{G(s)} = \frac{K}{s(s+1)(s+2)}$. Design a lag-lead compensator such that the closed loop system will satisfy the following specifications. (i) static velocity error constant should be less than or equal to 10 (ii) PM =50° (iii) GM >=10db

Sol:

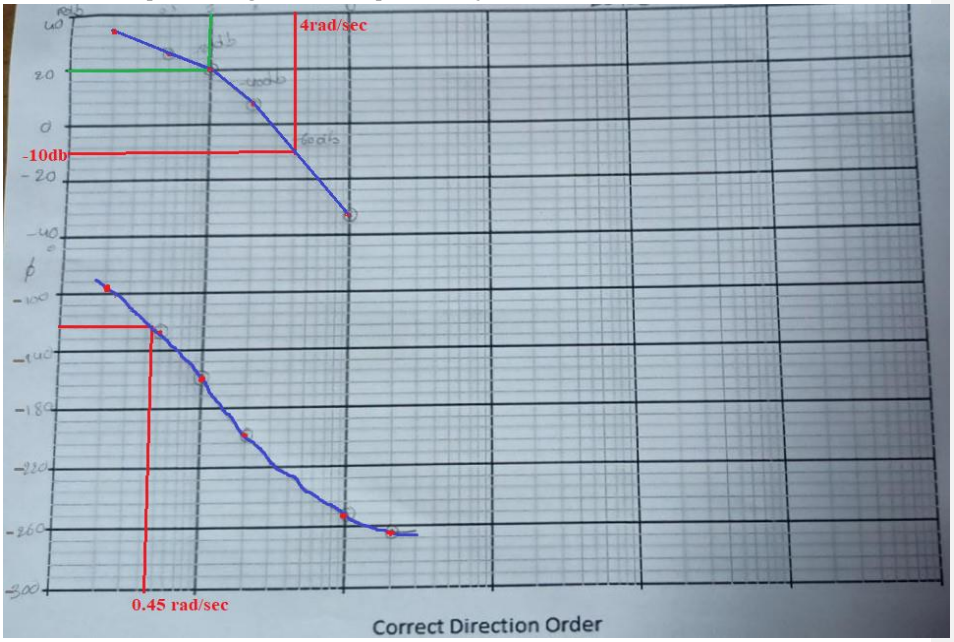
STEP1: Determine the open loop gain K of the uncompensated system to satisfy the specified error requirement.

Transfer function of uncompensated system $G(\mathbf{S}) = \frac{K}{s(s+1)(s+2)}$

$K_v = 10 = \lim_{s\to 0} sG(s) = K/2$

K = 20

STEP 2: Draw the bode plot of uncompensated system $\frac{20}{s(s+1)(s+2)} = \frac{10}{s(1+s)(1+0.5s)}$ and Determine the phase margin of uncompensated system.



STEP3: let γ_s = specified phase margin = 50°

Choose $\epsilon=5^0$

Determine the new phase margin using $\gamma_n=\gamma_s+\epsilon = 55°$

STEP 4: Determine new phase required at gain cross over frequency ω_n using

$$\Phi_n = 55 - 180 = -125° \quad (\gamma_n = 180 + \Phi_n)$$

Determine new frequency ω_n the frequency at which the phase is Φ_n from bode plot.

ω_n = 0.45 rad/sec

Choose ω_{gcn} = 1 rad/sec (some what greater than $\omega_{n.}$)

STEP 5: Find the db gain of (A_{gcn}) at new gain cross over frequency ω_{gcn} from bode plot.

$$A_{gcn} = 20\log\beta = 20 \text{ db} \qquad \text{or} \quad \beta = 10^{A_{gcn}/20} = 10$$

Choose higher corner frequency of lag $\omega_{c2} = 1/T_1 = \omega_{gcn}/10$

Therefore $T_1 = 10/\omega_{gcn} = 10$ sec

With β and T_1 determine the transfer function of lag compensator $= G_{c1}(s) = \frac{1+sT_1}{1+s\beta T_1} = \frac{(1+10s)}{(1+100s)}$

STEP 6: calculate $\alpha = 1/\beta = 0.1$

From the bode plot determine the frequency ω_m at which the magnitude is $20\log\sqrt{\alpha} = -10$ db.

ω_m = 4 rad/sec

Calculate T_2 using the relation $\omega_m = 1/(T_2\sqrt{\alpha}) \rightarrow T_2 = 1/(\omega_m\sqrt{\alpha}) = 0.79$ sec

The transfer function of the lead compensator $G_{c2}(s) = \frac{1+sT_2}{1+s\alpha T_2} = \frac{1+0.79s}{1+0.079s}$

STEP 7: Determine the open loop transfer function of the compensated system as given below

$$G_o(s) = KG(s)^* \frac{(1+sT2)}{(1+s\alpha T2)} \frac{(1+sT1)}{(1+s\beta T1)} = \frac{10(1+10s)(1+0.79s)}{s(1+s)(1+0.5s)(1+100s)(1+0.079s)}$$

Step 6: Draw Bode plot for $G_o(S)$ and verify whether it satisfies the given specifications. If it is not satisfactory then repeat steps by changing ε value till the specifications met.

Effects of Lag Lead Compensator

- The lead characteristics of the cascaded compensator provide improvement in the overall bandwidth.
- So a lag lead network provides a **quick response**
- The low-frequency gain of the system is increased with this network
- Also, due to the use of the lag network, the steady-state performance of the system gets improved.
- Gm & PM increased. So relative stability is improved.

Questions on Compensation:

Theory:

What is compensation. Explain about different types of compensation networks.

Derive the transfer function of lag(/lead/lag-lead) network from electrical circuit and draw its pole zero plot and frequency response plots.

Explain the procedure for design of lag (/lead / lag-lead) compensator.

Distinguish between lag (/lead) and lag-lead compensation.

Problems:

Design lag/lead/lag-lead compensator for a given system and specifications.

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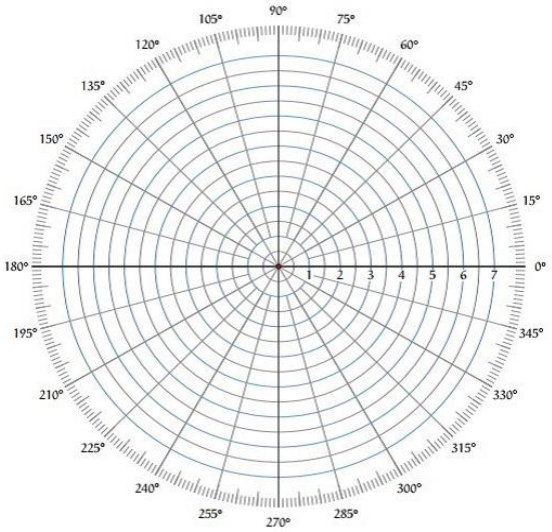
POLAR PLOTS

In the previous classes, we have discussed the Bode plots. There, we have two separate plots, one for magnitude and other for phase as a function of frequency. Let us now discuss about polar plots. Polar plot is also frequency response like bode plot. Polar plot is a plot which is drawn between magnitude and phase. Here, the magnitudes are represented by normal values only.

The polar form of $G(j\omega)H(j\omega)$ is

$$\mathbf{G(j\omega)H(j\omega)=|G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)}$$

The **Polar plot** is a plot, which can be drawn between the magnitude versus the phase angle of $G(j\omega)H(j\omega)$ by varying ω from 0 to ∞. We can draw the polar plot either on polar graph or normal graph. The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circles and radial lines. The **concentric circles** represent the magnitudes and the **radial lines** represent phase angles. The angles in anti-clock wise direction are treated as positive values. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0 in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

Each point on the polar plot is a phasor of magnitude M and phase angle 'φ'. The distance from the origin is proportional to magnitude $G(j\omega)$ i.e $|G(j\omega)|$ and the angle represents the phase of $G(j\omega)$ i.e $\angle G(j\omega)$.

The polar plot representation shows the plot of magnitude versus phase angle on polar coordinates with variation in ω from 0 to ∞. It is used for stability analysis.

In polar plot, a sketch between the magnitude and phase angle of the transfer function G(jω) is formed for different values of ω.

Suppose M represents the magnitude and φ denotes the phase angle, then for the transfer function of a system it is given as:

$$M = | G(j\omega) H(j\omega) |$$

$$\phi = \angle G(j\omega) H(j\omega)$$

So, with the variation in ω from 0 to ∞, the values of M and φ can be determined.

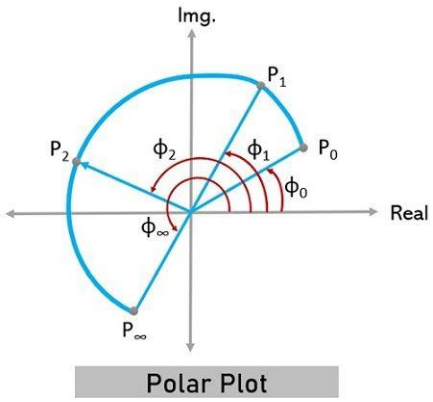
As we have already discussed in the beginning that polar plot is magnitude versus phase angle graph plotted for various values of ω.

So, to construct a polar plot, the different values of magnitude and phase angle is tabulated and further, the sketch is formed. The table is given below:

Frequency	Magnitude	Phase Angle
0	M0	ϕ0
ω1	M1	ϕ1
ω2	M2	ϕ2
|	|	|
|	|	|
∞	M∞	ϕ∞

From the above table, for ω = ω1, M = M1 and φ = φ1 a point in the polar coordinate system is decided that represents M1∠φ1, hence, the point on the plot corresponds to the tip of the phasor of magnitude M1 plotted at an angle φ1.

So, by using the tabulated data, the polar plot can be formed. Thus, in this way, the magnitude vs phase angle plot can be constructed for various values of frequency.



Polar Plot

It is to be noted here that conversion of magnitude into dB or logarithm values is not necessary. Also, the anticlockwise direction represents positive phase angles, while the clockwise direction shows the negative phase angles.

The figure represents the polar plot for ω between 0 to ∞:

Thus, from the above discussion, we can conclude that polar plot is started from a point specifying magnitude and angle for ω = 0 and is terminated at a point specifying magnitude and angle for ω = ∞.

Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

- Substitute, s=jω in the open loop transfer function.
- Write the expressions for magnitude and the phase of $G(j\omega)H(j\omega)$.
- Find the starting magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting ω=0. So, the polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting ω=∞. So, the polar plot ends with this magnitude and the phase angle.
- determine phase cross over frequency. There are two methods to determine this.

  M1:

  make the $\angle G(j\omega)H(j\omega)$ equal to 180° and find the value of ω which is nothing but phase cross over frequency.

  M2:

  make the imaginary term of $G(j\omega)H(j\omega)$ equal to zero and find the value of ω which is nothing but phase cross over frequency.

  For drawing polar plot more clearly, find the magnitude and phase of $G(j\omega)H(j\omega)$ by considering the other value(s) of ω around ω_{pc}(or around corner frequencies of open loop transfer function).

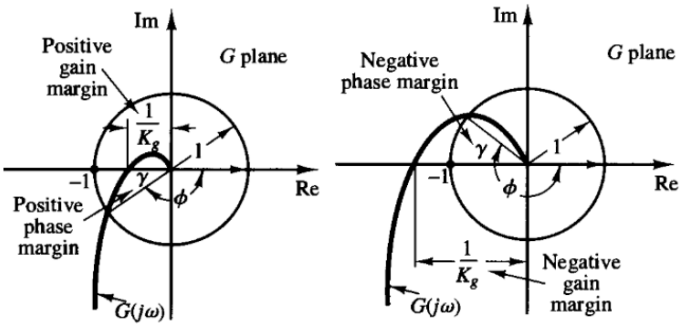
DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT

The **gain margin** is defined as the inverse of the magnitude of G(jω) at phase crossover frequency. The **phase crossover frequency** is the frequency at which the phase of G(jω) is 180°.

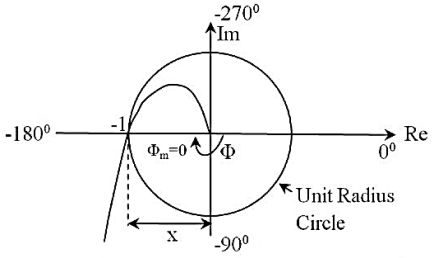
Let the polar plot cut the 180° axis at point **B** and the magnitude circle passing through the point B be G_B. Now the Gain margin, $K_g = 1/G_B$. If the point B lies within unity circle, then the Gain margin is positive otherwise negative. (If the polar plot is drawn in ordinary graph sheet using rectangular coordinates then the point B is the cutting point of G(jω) locus with negative real axis and $K_g = 1/|G_B|$ where G_B is the magnitude corresponding to point B).

The **phase margin** is defined as, phase margin, $\gamma = 180^0 + \phi_{gc}$ where ϕ_{gc} is the phase angle of G(jω) at gain crossover frequency. The **gain crossover frequency** is the frequency at which the magnitude of G(jω) is unity.

Let the polar plot cut the unity circle at point A as shown in fig 3.23 and 3.24. Now the phase margin, γ is given by ∠AOP, i.e. if ∠AOP is below −180° axis then the phase margin is positive and if it is above −180° axis then the phase margin is negative.



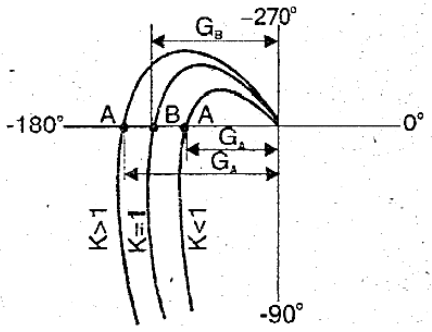
(a) Stable System (b) Unstable System



Marginally stable system: GM=0 dB, PM=0⁰

GAIN ADJUSTMENT USING POLAR PLOT

To Determine K for Specified GM



Draw G(jω) locus with K =1. Let it cut the −180° axis at point B corresponding to a gain of G_B. Let the specified gain margin be x db. For this gain margin, the G(jω) locus will cut −180° at point A whose magnitude is G_A.

$$\text{Now, } 20\log\frac{1}{G_A} = x \Rightarrow \log\frac{1}{G_A} = \frac{x}{20} \Rightarrow \frac{1}{G_A} = 10^{x/20}$$

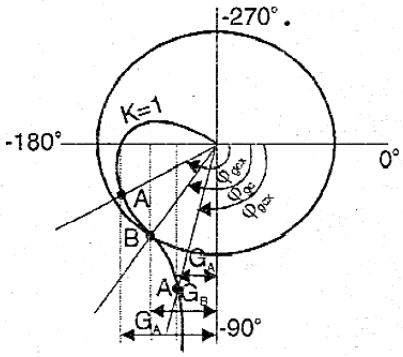
$$\therefore G_A = \frac{1}{10^{x/20}}$$

Now the value of K is given by, $K = \frac{G_A}{G_B}$.

If, K >1, then the system gain should be increased.

If, K < 1, then the system gain should be reduced.

To Determine K for Specified PM



Draw G(jω) locus with K = 1. Let it cut the unity circle at point B. (The gain at point B is G_B and equal to unity). Let the specified phase margin be x°

For a phase margin of x°, let ϕ_{gcx} be the phase angle of G(jω) at gain crossover frequency.

$$\therefore x° = 180° + \phi_{gcx} \Rightarrow \phi_{gcx} = x° - 180°$$

In the polar plot, the radial line corresponding to ϕ_{gcx} will cut the locus of G(jω) with K = 1 at point A and the magnitude corresponding to that point be G_A

$$\text{Now, } K = \frac{G_B}{G_A} = \frac{1}{G_A} \qquad (\because G_B = 1)$$

Example1:**The open loop transfer function of a closed loop control system is G(s)H(s)=5/s(s+1)(2s+1). Draw the polar plot for this system and determine GM, PM & stability.**

Sol:

$$G(s) = \frac{5}{s(s+1)(2s+1)}$$

$$G(j\omega) = \frac{5}{j\omega(j\omega+1)(j2\omega+1)}$$

The magnitude of the open loop transfer function is

$$M = |G(j\omega)| = \frac{5}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \ldots (2)$$

Gain cross over frequency: at ω_{pc} phase = 180°

$\Phi = -180^\circ = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

$-90^\circ = -\tan^{-1}(\omega) - \tan^{-1}(2\omega) = -\tan^{-1}\frac{(\omega+2\omega)}{1-2\omega^2}$

$\text{Tan } 90^\circ = \infty = \frac{(\omega+2\omega)}{1-2\omega^2} \rightarrow 1 - 2\omega^2 = 0 \rightarrow \omega^2 = 1/2 = 0.5 \rightarrow \omega_{pc} = 0.707$ rad/sec

$\omega_{c1} = 0.5$ rad/sec

$\omega_{c2} = 1$ rad/sec

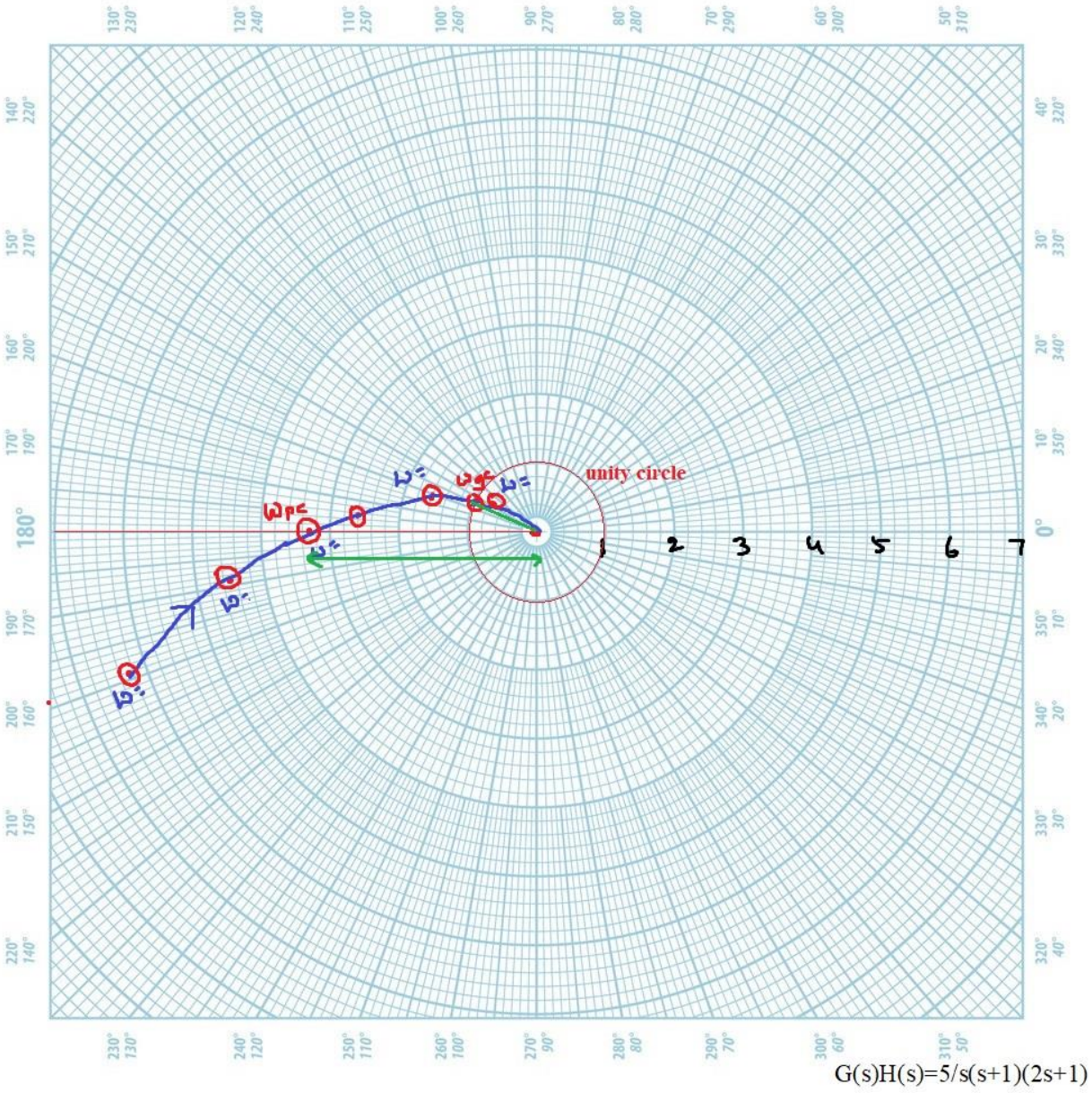
Frequency (rad/sec)	0	0.5	0.6	0.707	0.8	1	1.2	∞
Magnitude	∞	6.3	4.5	3.33	2.6	1.6		0
Phase angle(degrees)	-90°	-162	-171	-180	-186	-198		-270

Gain Margin = $1/|G(j\omega)| = 1/3.15 = 0.317$

Gain Margin (in db) = -10 db

Phase Margin = $180^\circ + \phi_{gc} = 180^\circ - 204^\circ = -24^\circ$

So Unstable system.



Example2:**The open loop transfer function of a closed loop control system is G(s)H(s)=1/s^2(s+1)(2s+1). Draw the polar plot for this system and determine GM, PM & stability.**

Sol:

$$G(s) = \frac{1}{s^2(s+1)(2s+1)}$$

$$G(j\omega) = \frac{1}{j^2\omega^2(j\omega+1)(j2\omega+1)}$$

The magnitude of the open loop transfer function is

$$M = |G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = -180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \ldots (2)$$

Gain cross over frequency: at ω_{pc} phase = 180°

$\Phi = -180^\circ = -180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

$0^\circ = -\tan^{-1}(\omega) - \tan^{-1}(2\omega) \rightarrow \quad -\tan^{-1}(\omega) = \tan^{-1}(2\omega) \rightarrow \omega = 0$

If ω is positive real number phase cross over frequency exist .i.e. intersection point with –ve real axis exist. Otherwise there is no intersection point.

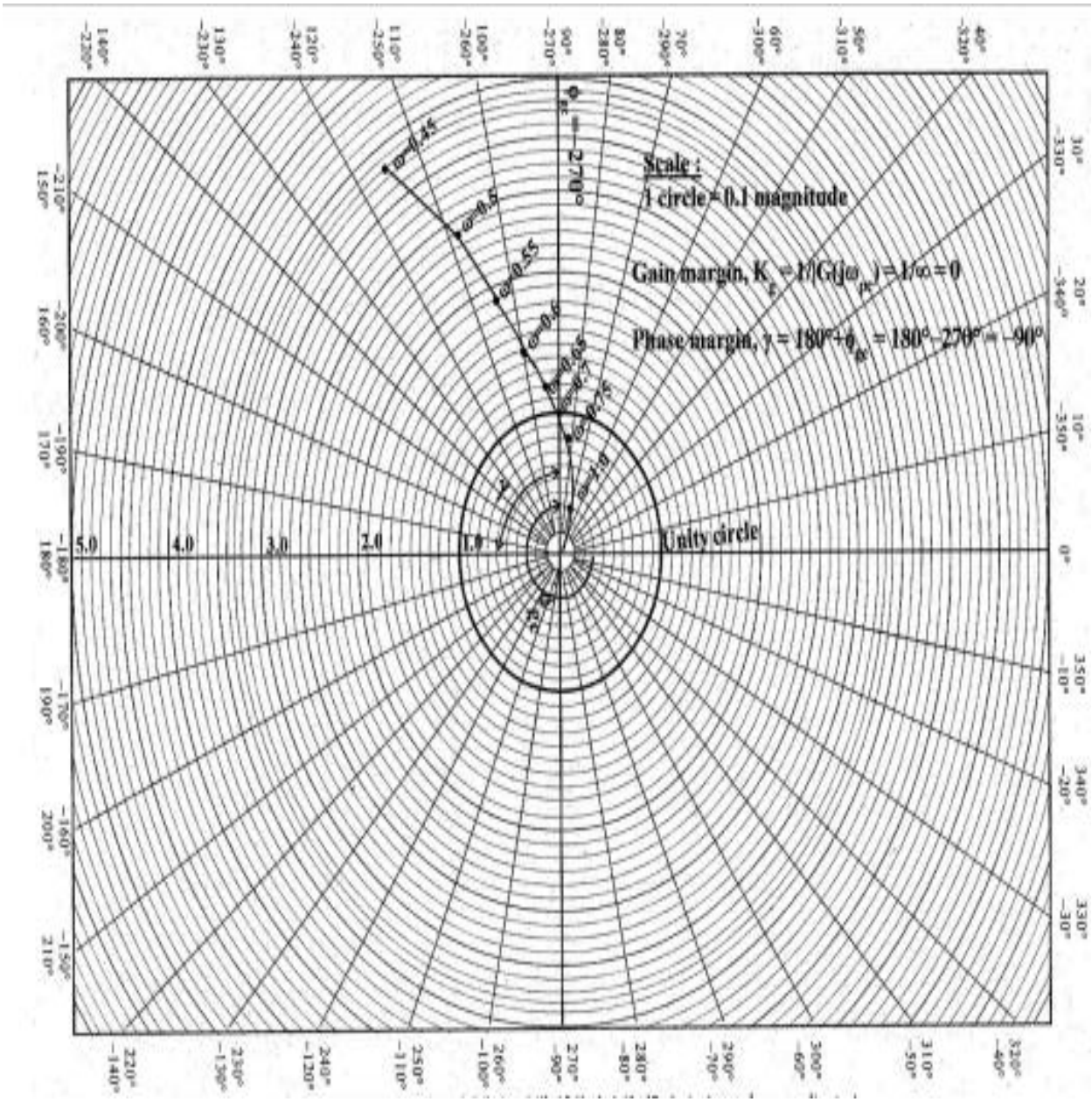
ω_{c1}= 0.5 rad/sec

ω_{c2} =1 rad/sec

Frequency (rad/sec)	0	0.45	0.5	0.6	0.7	1	∞
Magnitude	∞	3.3	2.5	1.5	0.97	0.3	0
Phase angle(degrees)	-180°	-246	-251	-261	-269	-288	-270

Gain Margin = $1/|G(j\omega)| = 1/\infty = 0$

Phase Margin = $180^\circ + \phi_{gc} = 180^\circ - 270^\circ = -90^\circ$


Scale :
1 circle = 0.1 magnitude
Gain margin, $K_g = 1/|G(j\omega_{pc})| = 1/\infty = 0$
Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 270° = -90°$
$\phi_{gc} = -270°$
Unity circle
5.0
4.0
3.0
2.0
1.0

Example3:**The open loop transfer function of a closed loop control system is G(s)H(s)=1/s(1+s)^2. Draw the polar plot for this system and determine GM, PM & stability.**

Sol:

$$G(s) = \frac{1}{s(1+s)^2}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2}$$

The magnitude of the open loop transfer function is

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}^2} = \frac{1}{\omega(1+\omega^2)} = \frac{1}{(\omega+\omega^3)} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = -90^\circ - 2\tan^{-1}(\omega) \ldots (2)$$

Gain cross over frequency: at ω_{pc} phase = 180°

Φ = -180° = $-90^\circ - 2\tan^{-1}(\omega)$

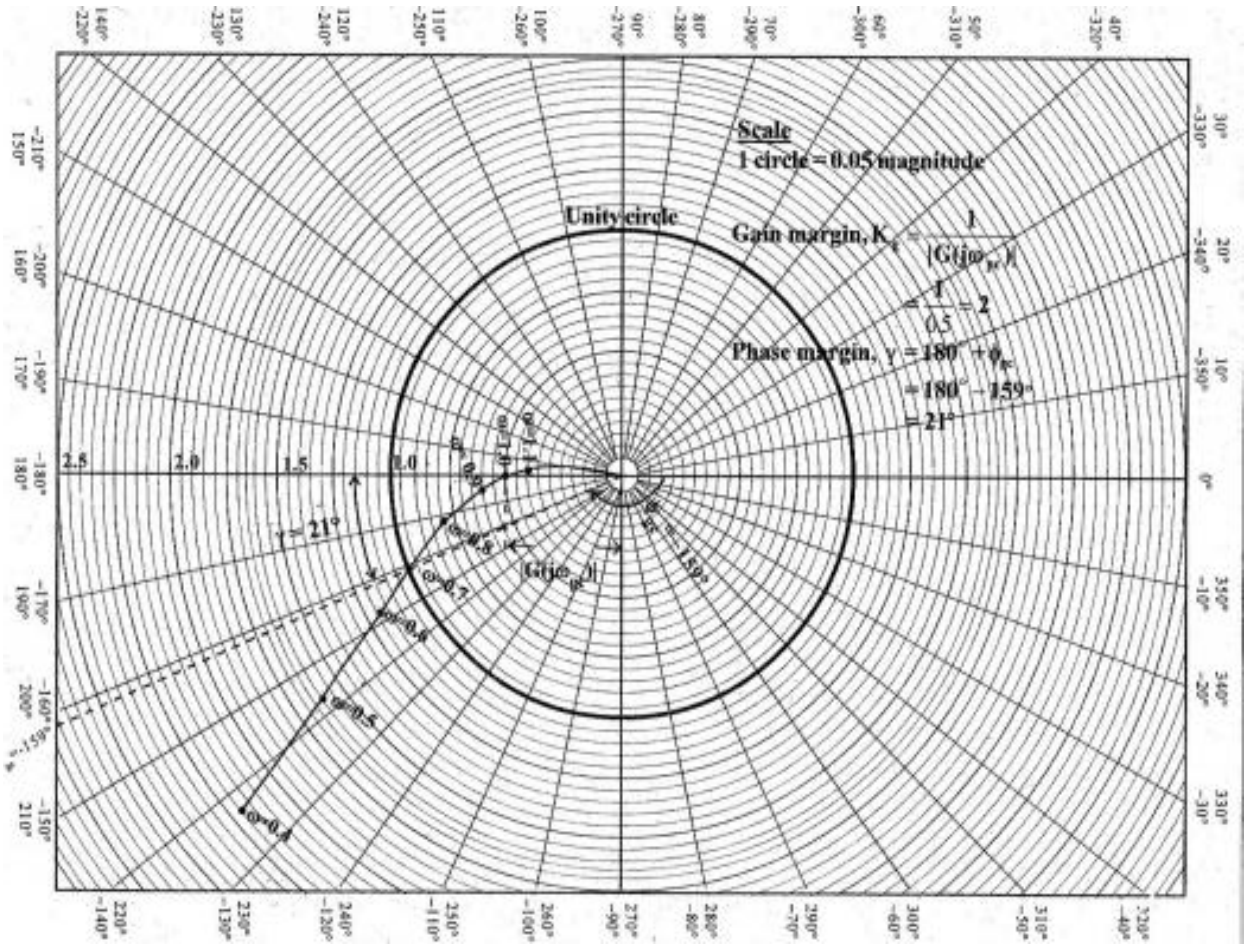
-90° = $-2\tan^{-1}(\omega)$ → $\tan^{-1}(\omega) = 45^\circ$ → ω=1 rad/sec

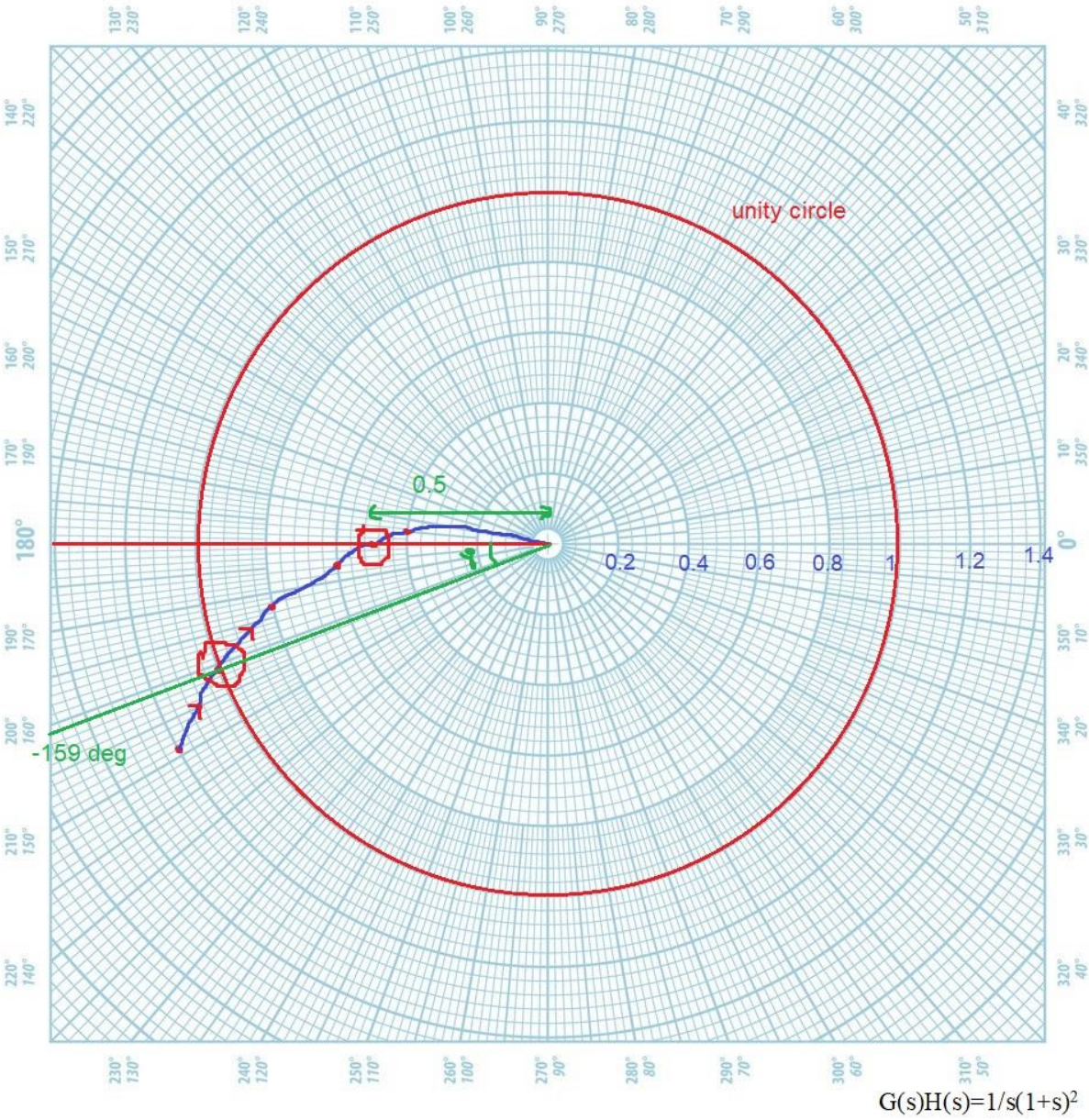
ω_{c1}= 1 rad/sec

Frequency (rad/sec)	0.6	0.7	0.8	0.9	1	1.1	∞
Magnitude	1.2	1	0.8	0.6	0.5	0.4	0
Phase angle(degrees)	-151	-159	-167	-174	-180	-185	-270

Gain Margin = 1/|G(jω)| = 1/0.5 = 2

Phase Margin = 180°+ϕ_{gc} =180°-159°= 21°


Scale
1 circle = 0.05 magnitude
Unity circle
Gain margin, $K_g = \frac{1}{|G(j\omega_{pc})|}$
$= \frac{1}{0.5} = 2$
Phase margin, $\gamma = 180^\circ + \phi_{gc}$
$= 180^\circ - 159^\circ$
$= 21^\circ$
$\gamma = 21^\circ$
$|G(j\omega_{gc})|$
$\phi = -159^\circ$
$\omega = 1.1$
$\omega = 0.9$
$\omega = 0.8$
$\omega = 0.7$
$\omega = 0.6$
$\omega = 0.5$
$\omega = 0.4$
2.5
2.0
1.5
1.0



Example5: The open loop transfer function of a closed loop control system is G(s) $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. Draw the polar plot for this system and determine K value for which(i)GM =18 db, (ii)PM = 60°.

Sol:

$$G(s) = \frac{1}{s(1+0.2s)(1+0.05s)}$$

$$G(j\omega) = \frac{1}{s(1+j0.2\omega)(1+j0.05\omega)}$$

The magnitude of the open loop transfer function is

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+0.04\omega^2}\,\sqrt{1+0.0025\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = -90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega) \quad \ldots(2)$$

Gain cross over frequency: at ω_{pc} phase = -180°

Φ = -180° = $-90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$

-90° = $-\tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$ → -90° = $-\tan^{-1}\left(\frac{0.2\omega+0.05\omega}{1-0.01\omega^2}\right)$

$\tan(90) = \frac{0.2\omega+0.05\omega}{1-0.01\omega^2} = \infty$ → $1 - 0.01\omega^2 = 0$ → $1 = 0.01\omega^2$ → $\omega^2 = 100$

ω_{pc}= 10 rad/sec

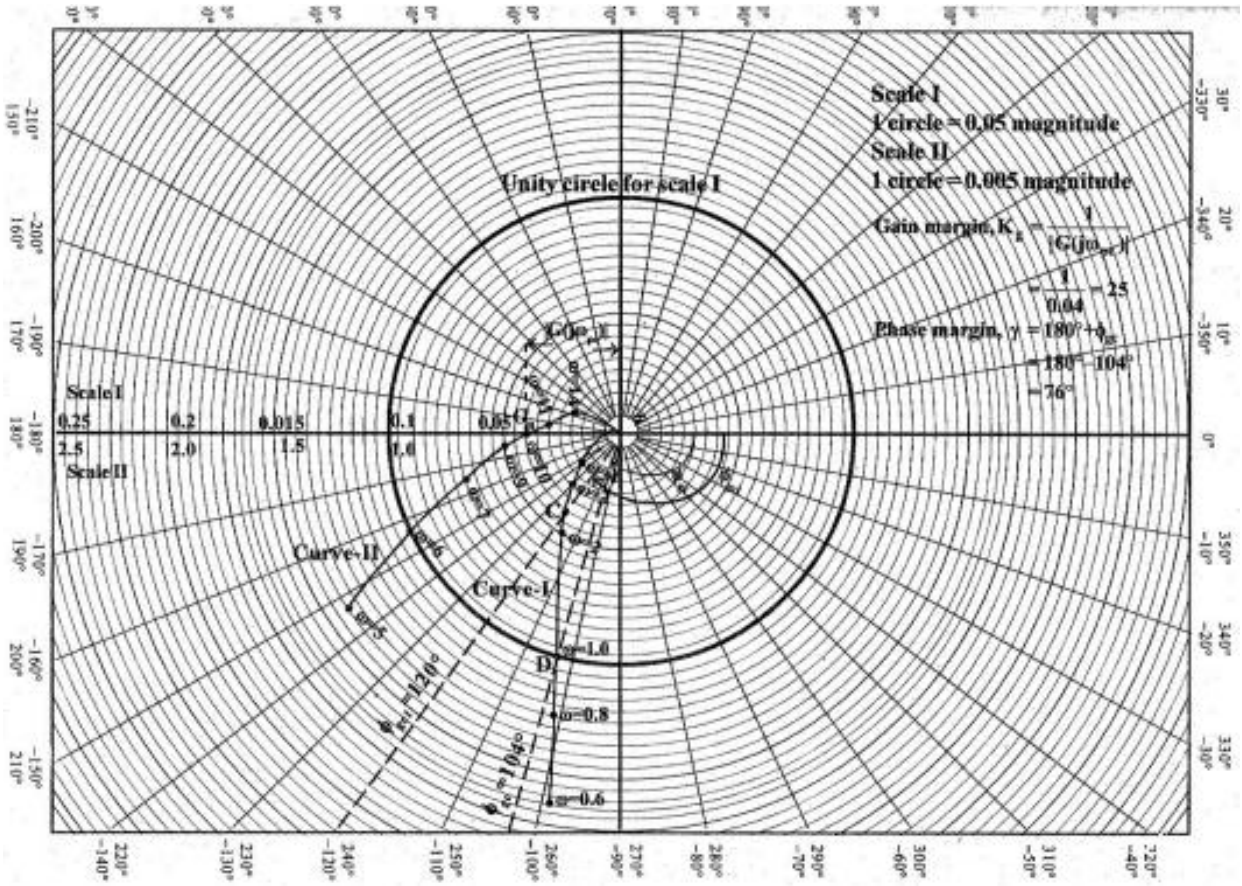
ω_{c1}= 5 rad/sec

ω_{pc}= 20 rad/sec

Frequency (rad/sec)	0.8	1	5	7	10	14	∞
Magnitude	1.23	1	0.14	0.07	0.04	0.02	0
Phase angle(degrees)	-101	-104	-149	-164	-180	-195	-270

ω rad/sec	0.6	0.8	1	2	3	4
|G(jω)|	1.65	1.23	1.0	0.5	0.3	0.2
∠G(jω) deg	−98	−101	−104	−117.5	−129.4	−140

ω rad/sec	5	6	7	9	10	11	14
|G(jω)|	0.14	0.1	0.07	0.05	0.04	0.03	0.02
∠G(jω) deg	−149	−157	−164	−176	−180	−184	−195



From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.04 = 25$.

Gain margin in db = 20 log 25 = 28 db.

Phase margin, $\gamma = 76°$.

Case (i)

With K = 1, let G(jω) cut the −180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$. The gain margin of 28 db with K = 1 has to be reduced to 18 db and so K has to be increased to a value greater than one.

Let G_A be the gain at −180° for a gain margin of 18 db.

$$\text{Now, } 20 \log \frac{1}{G_A} = 18 \quad \Rightarrow \quad \log \frac{1}{G_A} = \frac{18}{20} \quad \Rightarrow \quad \frac{1}{G_A} = 10^{18/20}$$

$$\therefore G_A = \frac{1}{10^{18/20}} = 0.125$$

$$\text{The value of K is given by, } K = \frac{G_A}{G_B} = \frac{0.125}{0.04} = 3.125$$

Case (ii)

With K = 1, the phase margin is 76°. This has to be reduced to 60°. Hence gain has to be increased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 60°

$$\therefore 60° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 60° - 180° = -120°$$

In the polar plot the −120° line cut the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 0.425$ and $G_D = 1$.

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{0.425} = 2.353$$

RESULT

(a) When K = 1, Gain margin, K_g = 25
Gain margin in db = 28db

(b) When K = 1, Phase margin, γ = 76°

(c) For a gain margin of 18 db, K = 3.125

(d) For a phase margin of 60°, K = 2.353

Example4: The open loop transfer function of a closed loop control system is G(s) $G(s) = \frac{K}{s(1+0.5s)(1+4s)}$. Draw the polar plot for this system and determine K value for which(i)GM =20 db, (ii)PM = 30°.

$$G(s) = \frac{K}{s(1+0.5s)(1+4s)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j0.5\omega)(1+j4\omega)}$$

The magnitude of the open loop transfer function is

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is 1.63

$$\Phi = \angle G(j\omega) = -90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(4\omega) \ldots (2)$$

Gain cross over frequency: at ω_{pc} phase = 180°

Φ = -180° = $-90^\circ - \tan^{-1}(0.5\omega) - \tan^{-1}(4\omega)$

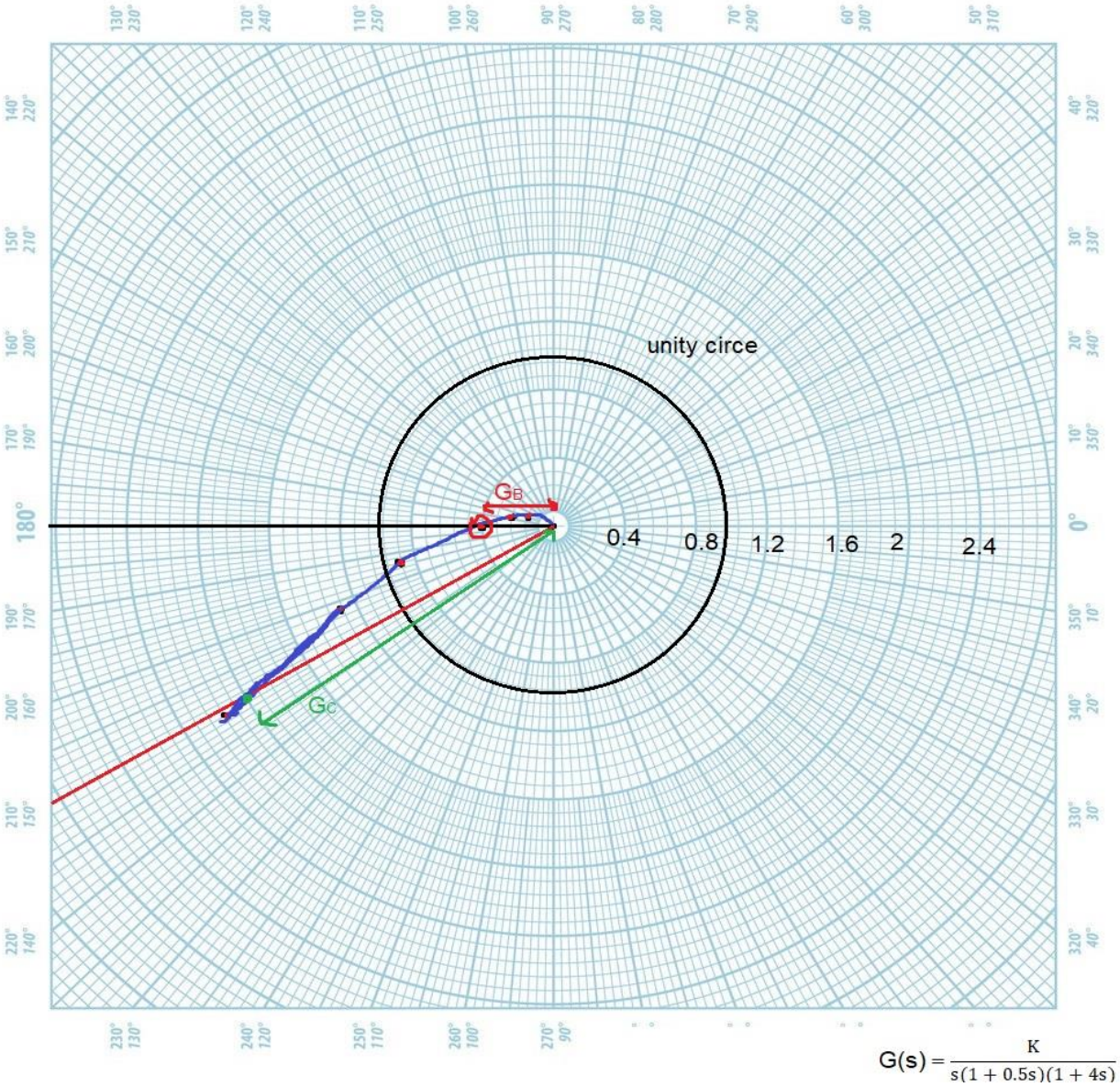
-90° = $-\tan^{-1}(0.5\omega) - \tan^{-1}(4\omega)$ → $-90 = -\tan^{-1}\left(\frac{0.5\omega+4\omega}{1-2\omega^2}\right)$

$\tan 90 = \infty = \frac{0.5\omega+4\omega}{1-2\omega^2}$ → $1-2\omega^2 = 0$ → $\omega^2 = ½$ → $\omega_{pc} = 0.707$ rad/sec

ω_{c1}= 0.25 rad/sec ω_{c1}= 2 rad/sec

ω_{pc} =0.707 rad/sec

Frequency (rad/sec)	0.3	0.4	0.5	0.707	1	1.2	∞
Magnitude	2.11	1.3	0.87	0.4	0.22	0.15	0
Phase angle(degrees)	-149	-159	-167	-180	-193	-199	-270



Case (i):

Desired Gain margin = 20 db → = 20 log (1/ G_A)

Desired magnitude M = G_A =10 $^{-(GM/20)}$ = 0.1

With K=1 M at ωpc = G_B = 0.44

K = G_A/ G_B =0.1/0.44 = 0.227

Case (ii):

Desired Phase margin = 30° = 180 + ϕ_{gc}

ϕ_{gc} = 30 – 180 = -150°

-150 line cuts the polar plot at point C.

Let magnitude at point C = G_C = 2.04

K = 1/ G_C =1/2 = 0.5

Note : *A point in complex plane can be represented by rectangular coordinates or by polar coordinates. Consider a point, z = a+jb in complex plane.*

Now, $|z| = \sqrt{a^2 + b^2}$ *and* $\angle z = \tan^{-1} b/a$.

If the point lies in first or fourth quadrant then the argument as calculated by $\tan^{-1} b/a$ *will be the correct values. But if it lies either in second or third quadrant then a correction should be made in the calculated values of argument, because the calculator will always give the values of* $\tan^{-1} b/a$ *either from 0 to + 90° or from 0 to–90°. The corrections to be made while converting from rectangular to polar coordinates is shown below.*

A point in Ist quadrant, $a + jb = \sqrt{a^2 + b^2}\ \angle \tan^{-1} b/a$

A point in IInd quadrant, $-a + jb = \sqrt{a^2 + b^2}\ \angle(\pi - \tan^{-1} b/a)$

A point in IIIrd quadrant, $-a - jb = \sqrt{a^2 + b^2}\ \angle(\pi + \tan^{-1} b/a)$

A point in IVth quadrant, $a - jb = \sqrt{a^2 + b^2}\ \angle -\tan^{-1} b/a)$

NYQUIST STABILITY CRITERION

Nyquist stability criterion (or **Nyquist criteria**) is a graphical technique used in control systems for determining the stability of a dynamic system.

Before going to discuss about nyquist stability we need to know the following basic concepts so that it is easy to understand **Nyquist stability criterion**.

- Poles and zeros of F(S) = 1+G(S) H(S)
- Concept of enclosed and encirclement
- Single valued function and Analytical Function.
- Mapping theorem or principle of argument
- Nyquist stability criterion

1. Poles and zeros of F(S) = 1+G(S) H(S):

Consider open loop transfer function of a system

$$G(s) = \frac{P(s)}{Q(s)}$$

The open loop poles of the system are roots of Q(S) = 0.

Consider the unity feedback closed loop transfer function of the same system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)} \quad \text{since H(s)=1}$$

Closed loop poles are the roots of the equation 1+G(s)=0.

Consider a function F(s) = 1+G(s)H(s)= 1+G(s).

$$F(s) = 1 + G(s) = 1 + \frac{P(s)}{Q(S)}$$

$$F(s) = \frac{P(s) + Q(S)}{Q(S)}$$

∴Poles of F(s) are roots of Q(s) =0

∴ Poles of F(s) are same as open loop poles of the system.

& Zeros of F(s) are roots of 1+G(s)=0.

∴ Zeros of F(s) are same as closed loop poles of the system.

Ex: Consider $G(s) = \frac{1}{S(S+2)}$

Open loop poles of the system are S=0,-2.

Closed loop transfer function is C(s)/R(s)= G(s)/(1+G(S)) = 1/ S^2+2S+1

Closed loop poles of the system are roots of S^2+2S+1=0

i.e S=-1,-1.

Consider $F(S) = 1 + G(S) = 1 + \frac{1}{s(s+2)} = \frac{s^2+2s+1}{s(s+2)}$

∴ The poles of F(s) are S=0,-2 i.e Open loop poles of the system.

& The zeros of F(S) are roots of eqn S^2+2S+1=0. i.e S=-1,-1. i.e Closed loop poles of the system.

Key Concepts:

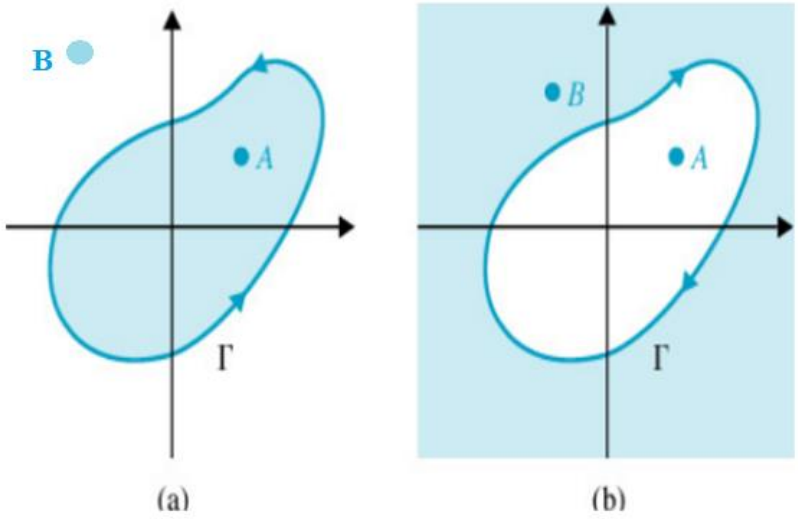
Poles of F(s) are same as open loop poles of the system.

Zeros of F(s) are same as closed loop poles of the system.

2. Concept of enclosed and encirclement:

Enclosed:

A point or region is said to be enclosed by a closed path if it lies to the right of the path when the path is traversed in any prescribed direction.

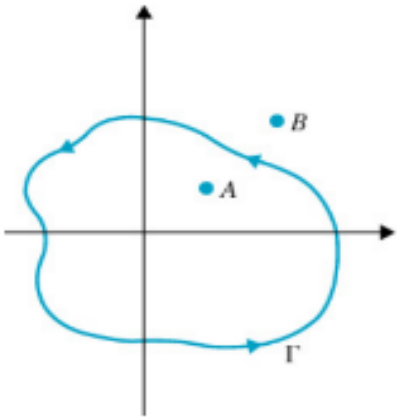


(a) (b)

In figure (a) point A is not enclosed by the path, while point B is enclosed.
In figure (b) point A is enclosed by the path, while point B is not enclosed.
(**Blue colour shaded portion is not enclosed).**

Encirclement:

A point in a complex plane is said to be encircled by a closed path (contour) if it is found inside the path.



Point A is encircled by the closed path (Γ) while point B is NOT encircled by the closed path.

Figure 1

We can say that the region inside the path is encircled in clockwise or in counter clockwise direction depending on direction of travel and the region outside the path is not encircled.

The total number of times a closed path encircles a point is called as number of encirclements.

Let say N is number of encirclements, by definition N is **positive for encirclement in anti-clockwise direction** and is **negative for clockwise direction.**

In figure 1 point A is encircled once in anti clock wise direction. So N=1 for A.

Point B is not encircled. So N=0 for B.

For complex figures shown below it is difficult to identify N.

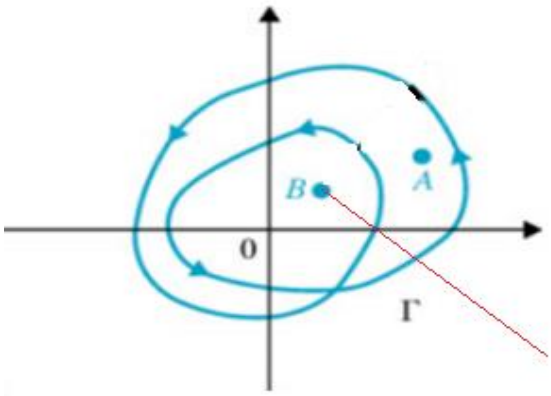
Procedure to find N:

- Draw a line from the point to outside closed path or contour in any direction.
- Identify the intersection points of closed path and this line.
- At intersection point mark the direction of travel with arrow marks.
- Cancel opposite direction arrows.
- Number of encirclements N is equal to no. of remaining arrows(+ for anti clock wise and – for clock wise direction).

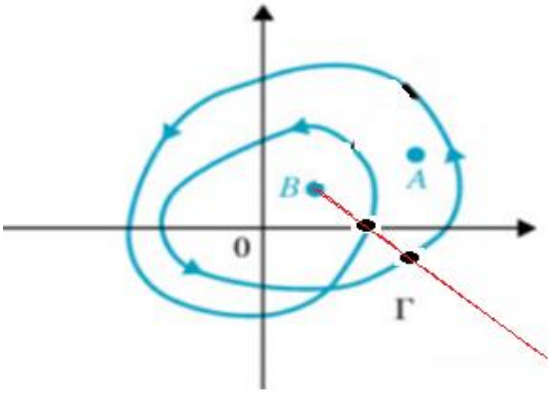
Ex: Find number of encirclements of the points B,A,O by the closed path Γ.

Point B: Above given procedure is followed as shown in below figures.

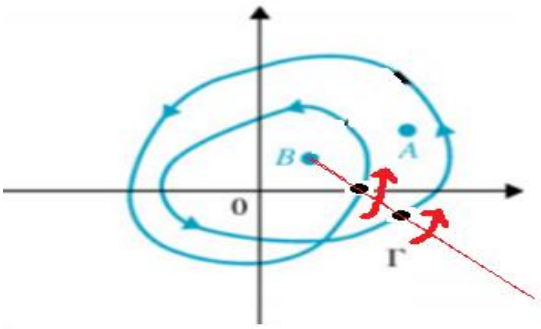
1.Draw a line from the point to outside closed path r in any direction.



2.Identify the intersection points of closed path and this line.



3.At intersection point mark the direction of travel with arrow marks.



Two anti clock wise arrows. So **N=2 for point B**.
Similarly **for point A N=1** and for point o N=2(refer figure 2 and figure 3 respectively).

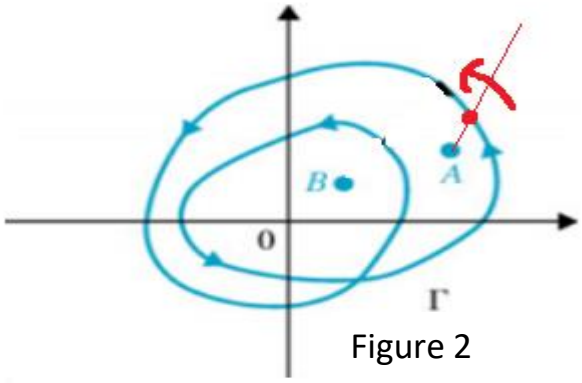


Figure 2

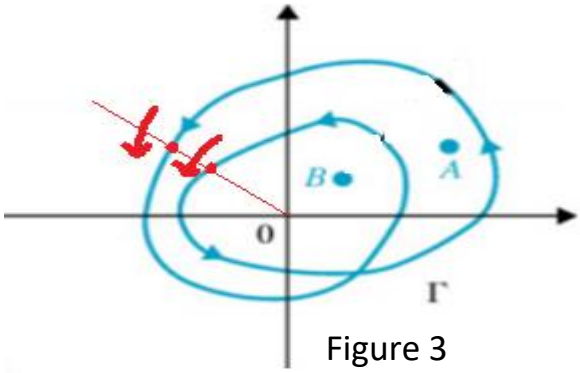
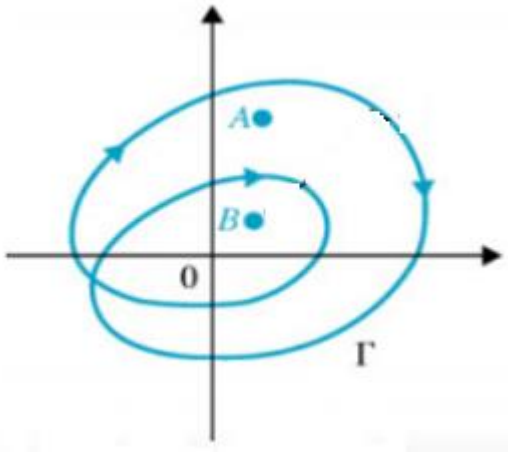


Figure 3

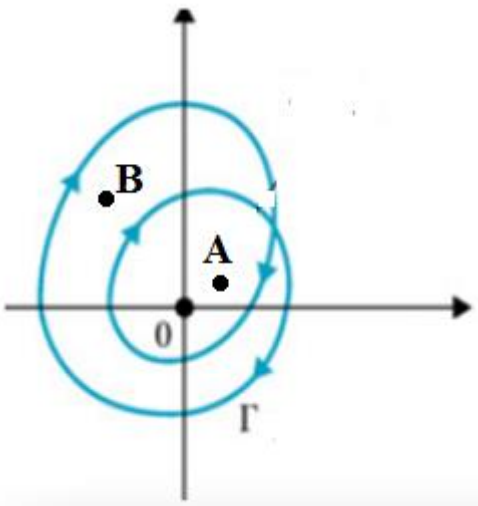
Determine number of encirclements for the points in below figures.



For Point A N=-1(clock wise direction)

For Point B N=-2(clock wise direction)

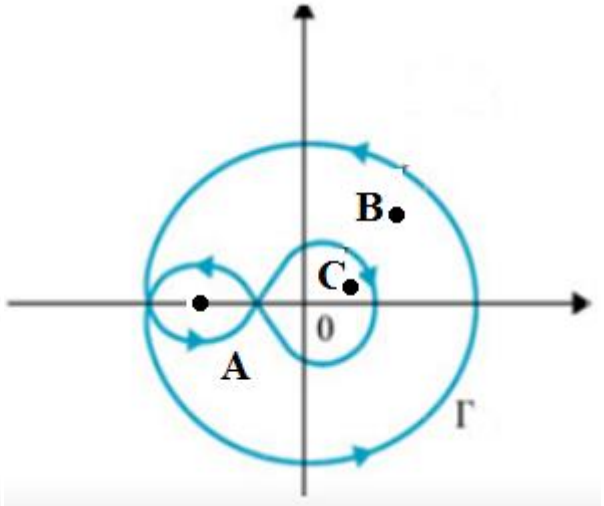
For Point O N=-2(clock wise direction)



For Point A N=-2(clock wise direction)

For Point B N=-1(clock wise direction)

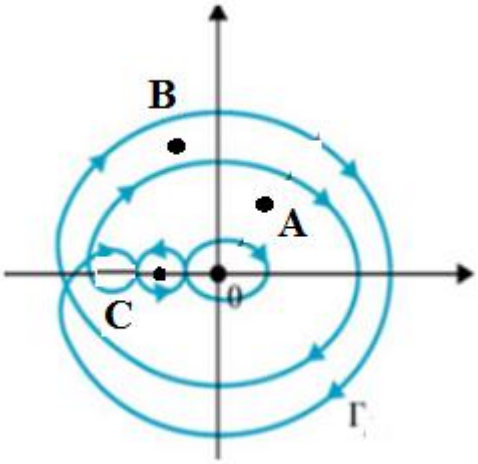
For Point O N=-2(clock wise direction)



For Point A N=2(Anti clock wise direction)

For Point B N=1(Anti clock wise direction)

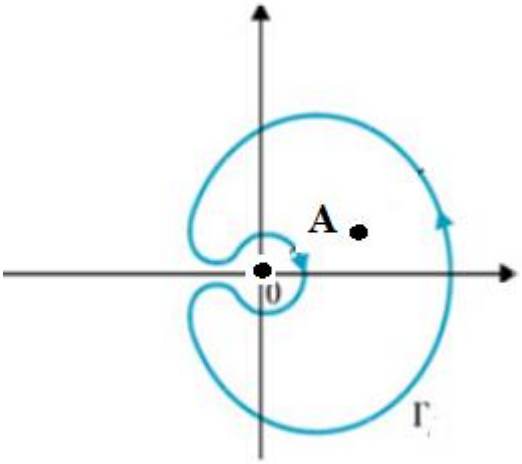
For Point C& O N=0(one anti clockwise and one in clock wise direction. So net is zero)



For Point A N=-2(clock wise direction)

For Point B N=-1(clock wise direction)

For Point C N=-1(net one clock wise direction) and For Point O N=-3(clock wise direction)



For Point A N=1(Anti clock wise direction)

For Point O N=0.

3. Single Valued Function and Analytical Function:

Single valued function: Single valued function F(s) is a function which has only one value for each value of S.

Ex: 1. **F(s) = s+2**

For s= 2 F(s)=4

For s= 2+j2 F(s)=4+j2
For s= -j3 F(s)=2-j3
For each value of 's' the function results only one value. So this is single valued function.

2. F(s)=√s

For s= 4 F(s)=√4=±2
For s= -4 F(s)=±j2
For each value of s the function results more than one value. So this is not a single valued function.

Analytical Function: A function is said to be analytic at a point if for that value of point the function is finite.

Singular points: The points at which the value of that function or its derivatives is not finite (∞) are known as singular points.

Ex: F(s)= 10/S(S+1)(S+2)

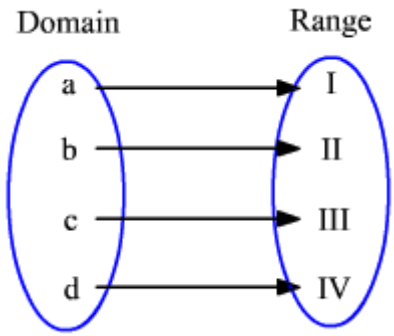
This function is ∞ at S=0,-1,-2.
At all remaining points the function value is finite and it is analytical.

So these points are known as singular points for this function.

Generally functions are analytical functions except at some singular points.
Generally poles of a function are singular points for that function.

4. Mapping of a function:

General meaning of Mapping is a relation such that each element of a given set (the domain) is associated with an element of another set (the range).

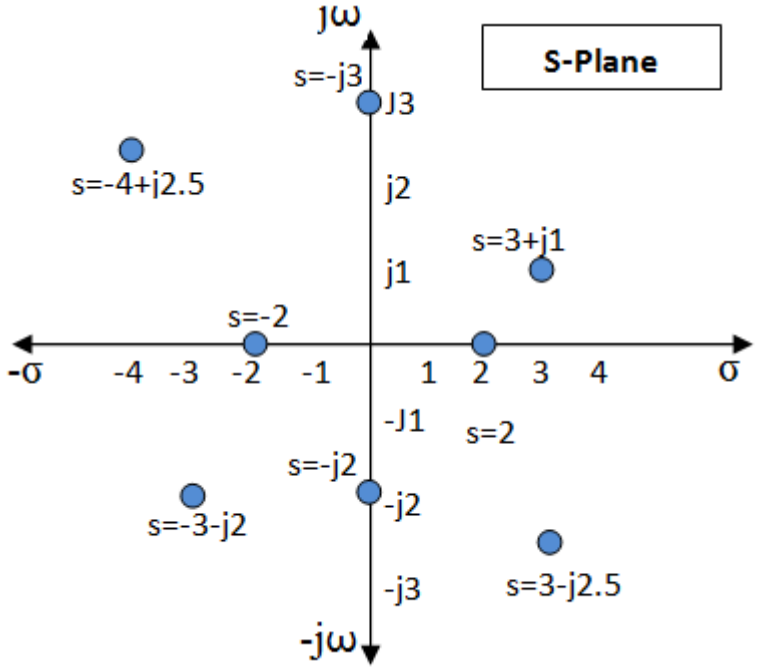


Mapping of a function from one plane to other plane is explained below using an example.

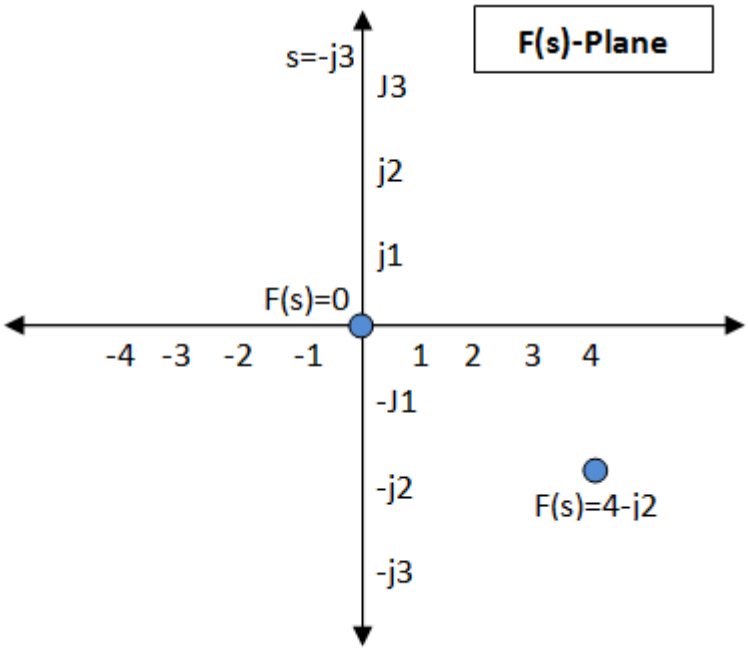
Let us consider a single valued function.

F(s) = s+4

Here we are mapping the function F(s) from S-plane to F(s)-plane. Each value of 's' represents a point in S-plane as shown below.



For each value of 's', F(s) has only one value since it is single valued function.
For example for s=-4 F(s)=0; &
for s=-j2 F(s)=4-j2.
The each value of F(s) can be represented by a point in F(s) plane.

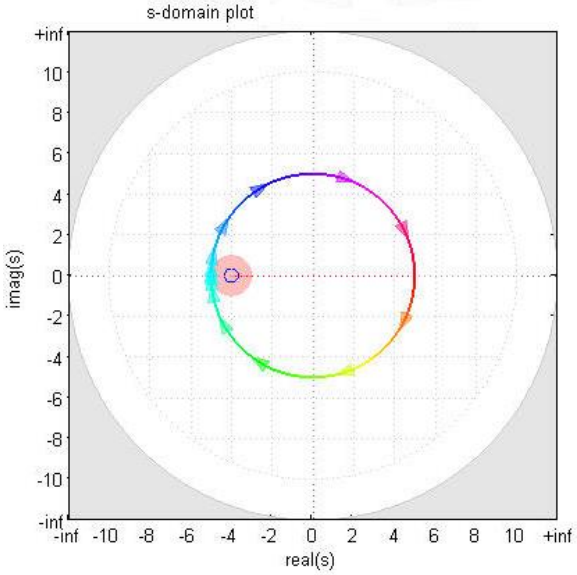


Select an arbitrary closed path(locus) in S-plane such that it does not go through singular points. Closed path is connecting some points in a particular direction in a plane.

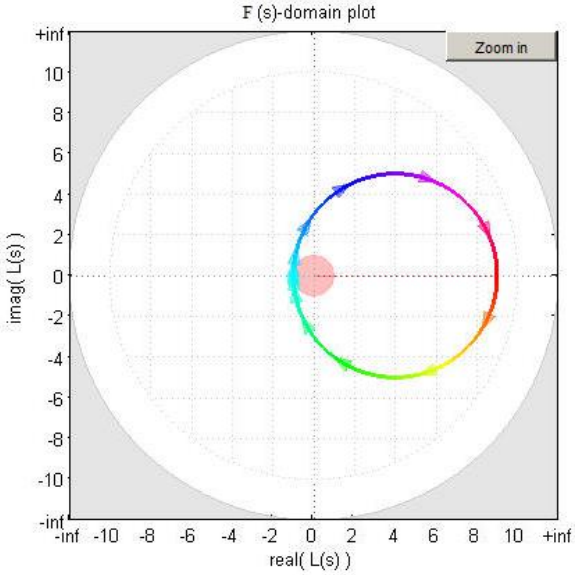
If we calculate F(S) value for these points of 'S' and represent in F(s) plane it is another closed path(mapped locus) in F(s) plane. For every point in closed path of S-plane there is a point in closed path of F(s) plane.

See below figures. First one is the chosen closed path in s-plane which is not passing through singular points of F(s)(poles of F(s)). The direction is clockwise direction. Second figure is the mapped locus in F(s) plane.

Since F(s) = s+4 mapped locus is just shifted by 4 units towards right and the direction of mapped locus is same as locus.



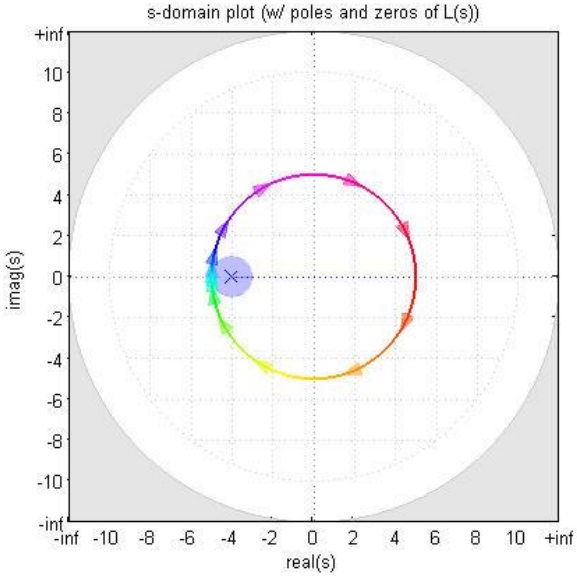
Locus without poles & zero of F(s)

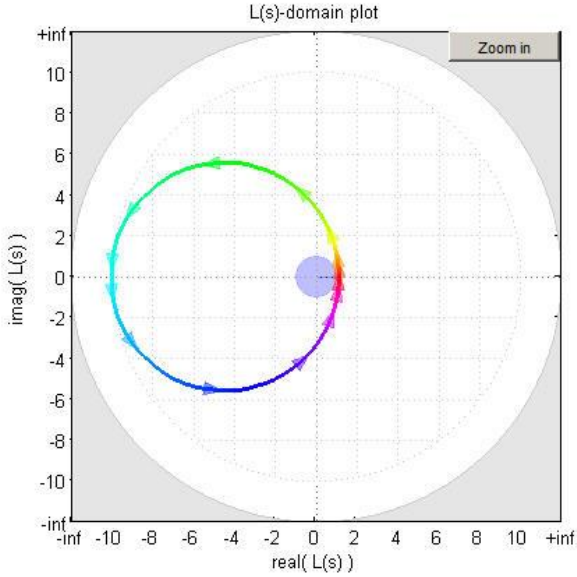


Mapped locus

Now consider another function F(s)=10/(s+4).

The arbitrary closed path Chosen and its mapped locus are shown in below figure.





The mapped locus is shifted by 4 units toward left and is opposite direction to the locus.

This is known as mapping. F(s) is known mapping function and closed path obtained in F(s) plane is known as mapped locus.

Mapping Theorem or The principle of argument:

This theorem gives relation between the poles and zeros enclosed by a closed path in S-plane and number of encirclements of mapped locus in F(s)- plane.

Suppose an arbitrary closed path is chosen in s-plane in clock wise direction so that it does not go through any of the poles or zeros of F(s); the corresponding mapped locus (Γ) in F(s)-plane will encircle the origin as many times as the difference between the number of zeros and poles of F(s) that are enclosed by the s-plane locus.

Mathematically it is represented by

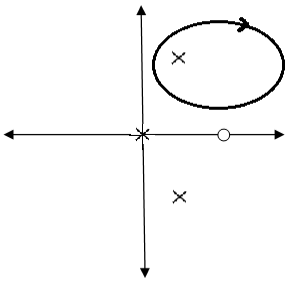
$$N=P-Z$$

Where N = number of encirclements of origin in anti clock wise direction made by the mapped locus(Γ) in F(s) plane

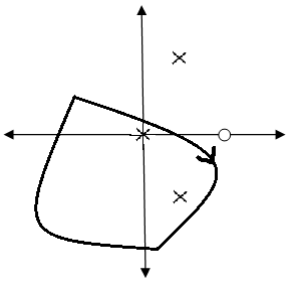
P= number of poles of F(s) enclosed by closed path in S-plane

Z= number of zeros of F(s) enclosed by closed path in S-plane

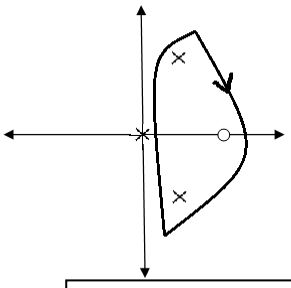
Examples: The following figures shows different arbitrarily chosen closed paths in s-plane for same system which has 3 poles and one zero. So number of encirclements of origin in F(s) is also different which depends upon no. of poles and zeros encircled by closed path in s-lane.



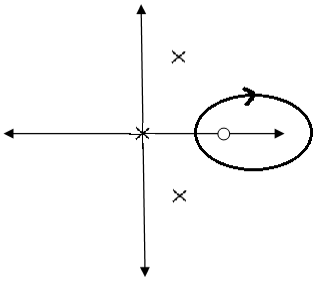
N=1-0=1 (mapped locus encircles origin of F(s) once in anti clock wise)



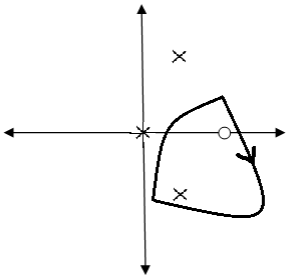
N=2-0=2 (mapped locus encircles origin of F(s) twice in anti clock wise)



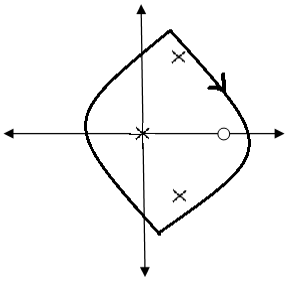
N=2-1=1 (mapped locus encircles origin of F(s) once in anti clock wise)



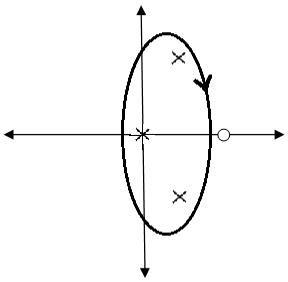
N=0-1=-1 (mapped locus encircles origin of F(s) once in clock wise)



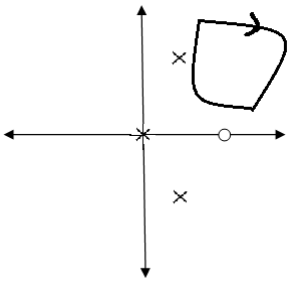
N=1-1=0 (mapped locus does not encircles origin of F(s)) .



N=3-1=2 (mapped locus encircles origin of F(s) twice in anti clock wise)



N=3-0=3 (mapped locus encircles origin of F(s) thrice in anti clock wise)



N=0-0=0 (mapped locus does not encircles origin of F(s)).

N=P-Z If we know any two we can find third one

After reading through the material above, the question arises "So what?". What we have done here is introduce a technique that gives us information about the number encirclements of origin in a plane from poles and zeros enclosed in a closed contour in S-plane. Nyquist used this information to determine stability of closed loop system.
Let us see how it is possible.

Condition for stability:

If all the closed loop poles i.e roots of 1+G(s)H(s)=0 lies in left half of S-plane the system is stable.
If one closed loop pole or root lies in right half of S-plane the system is Unstable.

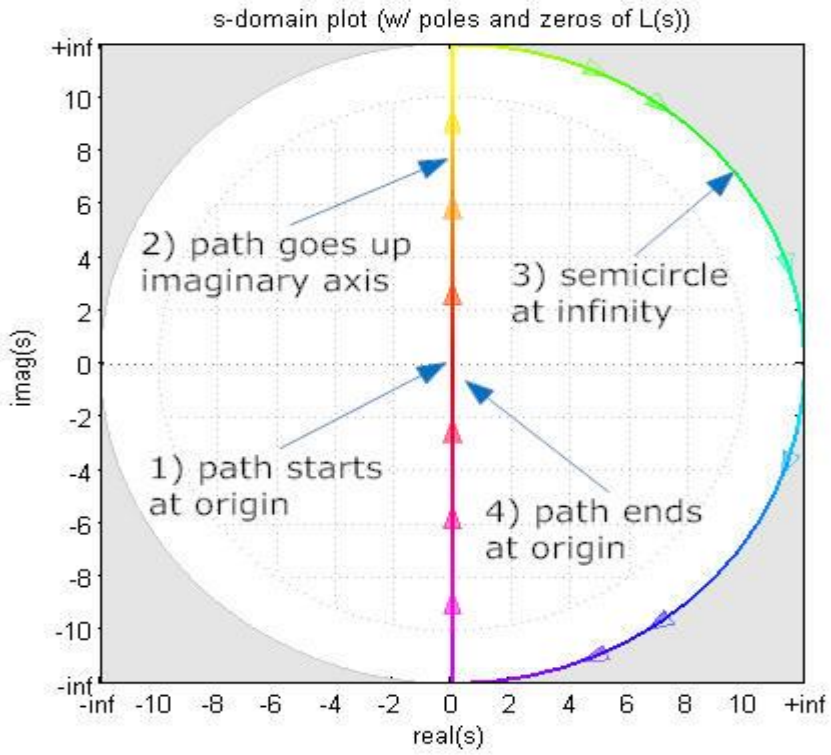
So if no pole in right side of s-plane, system is stable otherwise unstable.

Nyquist Stability Criterion:

The principle of argument can be applied to determine the stability of closed loop system by choosing the s-plane closed path such that it encircles the entire right half of s-plane in clockwise direction and mapping function F(s)=1+G(s)H(s). This path is known as nyquist path.

Nyquist path: Nyquist path is a closed path in S-plane which encircles the entire right half of S-plane and do not pass through the poles of F(s).

Ex:



Let us recall that

Poles of F(s) are same as open loop poles of the system.

Zeros of F(s) are same as closed loop poles of the system.

- For a given open loop transfer function (G(s)H(s)) we know poles of F(s). So we know how many poles(P) are encircled by nyquist path.

- Then we perform a mapping (as explained on the previous section) of the nyquist path with the function "1+F(s)".

- And we count the encirclements (N) of the origin in the "1+F(s)" plane in the anti clockwise direction.

Now we know that

$$N=P-Z$$

Since we know N & P values we can find Z value.

But Z = no. of zeros of F(s) encircled by nyquist path in S-plane.
= no. of closed loop poles encircled by nyquist path in S-plane.
= no. of closed loop poles in right half of S-plane.

So we are able to find the no. of closed loop poles in right half of S-plane if mapped locus of 1+F(s) is drawn and no. of encirclements of origin is counted.

For the system to be stable there should be no poles in right half of S-plane.i.e.

Z=0 for stable system
Or
N=P for stable system

Before continuing we make one small change. Instead of mapping from "s" to "1+F(s)" and counting encirclements of the origin, we map from "s" to "F(s)" and count encirclements of the point -1+j0 in the complex plane. This is because the origin in "1+F(s)" corresponds to the "-1+j0" point in "F(s)" (if F(s)=-1, then 1+F(s)=0).

The reason behind this change is that it is very easy to draw mapped locus for F(s) i.e G(s)H(s). It is completely from polar plot. This mapped locus of F(s) is known as Nyquist Plot.

So finally Nyquist Stability Criterion states that

If number of **encirclements in anticlockwise direction of the point -1+j0** by the Nyquist plot in F(s)-plane is equal to **number of right half** of s-plane **poles** of G(s)H(s) then the system is stable.

Mathematically it is represented as

$$N=P$$

Where N= **no. of encirclements in anticlockwise direction of the point -1+j0** by the Nyquist plot

P= No. of open loops encircled by nyquist path (or)
= **No. of open loop poles in right half of S-plane**

NYQUIST PLOTS

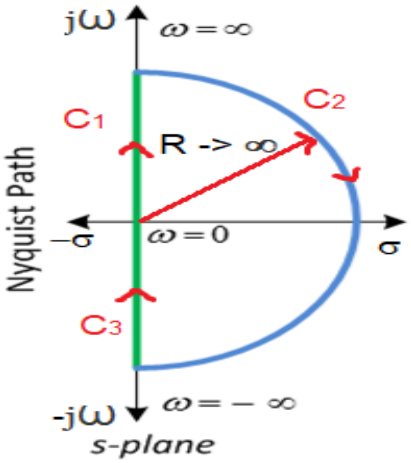
Problems:

1. Draw nyquist plot for a unity feed back system whose open loop transfer function is $G(s) = \frac{1}{(s+2)(s+3)}$. And determine the stability of the closed loop system.

Sol: $G(s) = \frac{1}{(s+2)(s+3)}$

open loop poles s= -2, -3 (no pole at origin)

Nyquist Path: The nyquist path consists 3 sections as shown in figure. The starting and end points of these sections are given below.



Section	Starting	end
C1	ω=0	ω=∞
C2	ω=∞	ω=-∞
C3	ω=-∞	ω=0

Mapping of section C1:

Mapping of C1 is polar plot of G(jω).

$$G(j\omega) = \frac{1}{(j\omega + 2)(j\omega + 3)}$$

$$M = |G(j\omega)| = \frac{1}{\sqrt{4+\omega^2}\sqrt{9+4\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

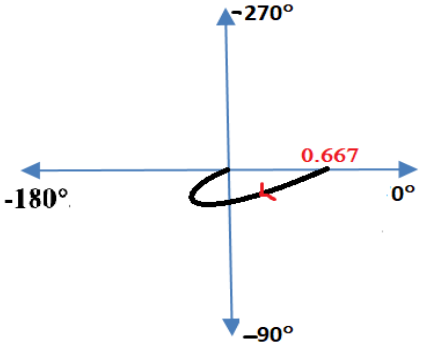
$$\Phi = \angle G(j\omega) = -\tan^{-1}(\omega/2) - \tan^{-1}(\omega/3) \ldots (2)$$

Starting point: at **ω = 0** **M = 1/6 = 0.166** **Φ = 0°**

End point: at **ω = ∞** **M = 0** **Φ = -180°.**

net angle of rotation = -180 -0 = -180°(180° clockwise)

Starting point is 0.166∠0°. from this point plot starts and rotate 180° in clockwise direction and ends at origin(end point magnitude=0).



So it is as shown in figure.

Intersection point with -180°: at ω_{pc} , phase = 180°

$\Phi = -180° = -\tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)$

$-180° = -\tan^{-1}\left(\frac{\omega/2+\omega/3}{1-\frac{\omega^2}{6}}\right) \rightarrow \tan 180 = \frac{\omega/2+\omega/3}{1-\frac{\omega^2}{6}} = 0 \rightarrow \omega = 0$

ω is not a positive real number so phase cross over frequency does not exist i.e. there is no intersection point.

Mapping of section C2:

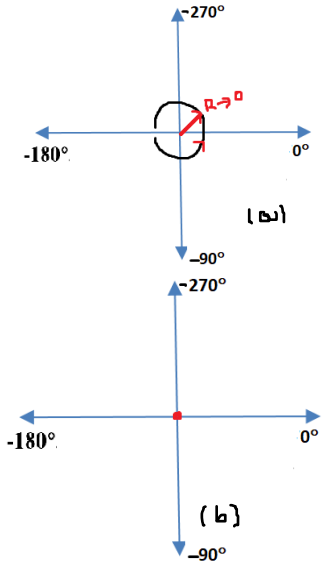
Mapping of C2 is G(jω) as ω varies from ∞ to -∞.

Starting point: at **ω = ∞** **M = 0** **Φ = -180°**

End point: at **ω = -∞** **M = 0** **Φ = 180°.**

net angle of rotation = 180 –(-180) = 360°(360° anti-clockwise)

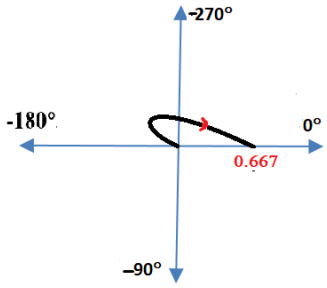
Starting point is 0∠-180°. from this point plot starts and rotate 360° in anti-clockwise direction and ends at 0∠180°.Through out this plot magnitude is zero.So it is as shown in figure(fig a).



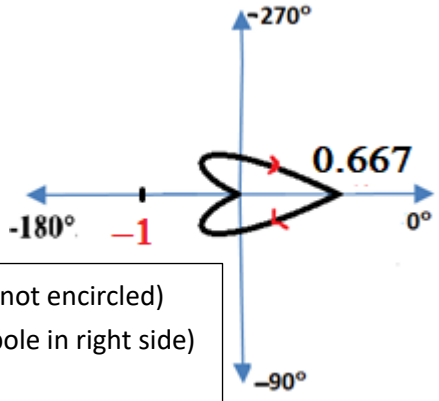
since magnitude is zero we can consider this mapping as a point at origin (fig b).

Mapping of section C3:

Mapping of C3 is mirror image of polar plot of G(s) about real axis .



Nyquist Plot:



N = 0 (-1 not encircled)

P=0 (no pole in right side)

N = P

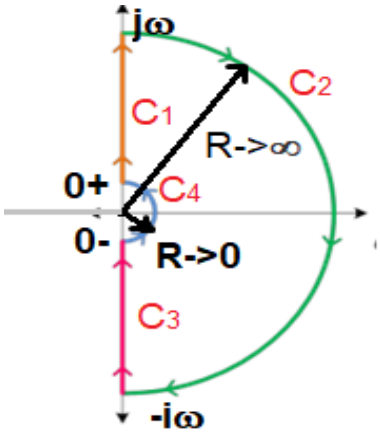
∴ System is Stable

2. Draw nyquist plot for a unity feed back system whose open loop transfer function is $G(s) = \frac{1}{s(s+1)(s+2)}$. And determine the stability of the closed loop system.

Sol: $G(s) = \frac{1}{s(s+1)(s+2)}$

open loop poles s= 0,-1, -2 (pole at origin)

Nyquist Path: The nyquist path consists 4 sections as shown in figure. The starting and end points of these sections are given below.



Section	Starting	end
C1	ω=0+	ω=∞
C2	ω=∞	ω=-∞
C3	ω=-∞	ω=0-
C4	ω=0-	ω=0+

Mapping of section C1:

Mapping of C1 is polar plot of G(jω).

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(j\omega+2)}$$

$$M = |G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = -\tan^{-1}(\omega/0) - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega) \ldots (2)$$

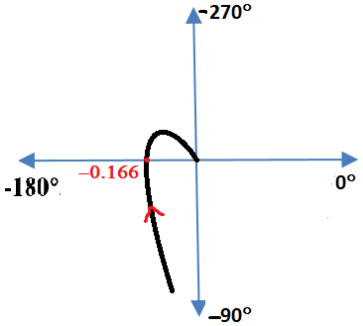
Starting point: at $\omega = 0+$ $M = \infty$ $\Phi = -90°$

End point: at $\omega = \infty$ $M = 0$ $\Phi = -270°$.

net angle of rotation = -270 –(-90) = -180°(180° clockwise)

Starting point is ∞∠-90°. from this point plot starts and rotate 180° in clockwise direction and ends at origin(end point magnitude=0).

So it is as shown in figure.



Intersection point with -180°: at ω_{pc} , phase = 180°

$\Phi = -180° = -90° - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega)$

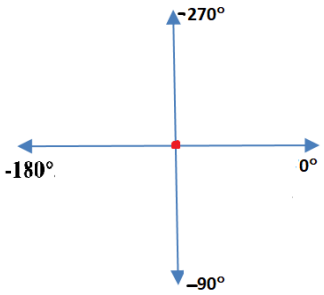
$-90° = -\tan^{-1}\left(\frac{\omega + 0.5\omega}{1-0.5\omega^2}\right) \rightarrow \tan 90 = \frac{\omega + 0.5\omega}{1-0.5\omega^2} = \infty$

$1 - 0.5\omega^2 = 0 \rightarrow \omega^2 = 2 \rightarrow \omega = 1.414$

Substitute ω= 1.414 in magnitude eqn(eqn 1).

$$M = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} = 0.166$$

Therefore intersection point is -0.166.



Mapping of section C2:

Mapping of C2 is G(jω) as ω varies from ∞ to -∞.

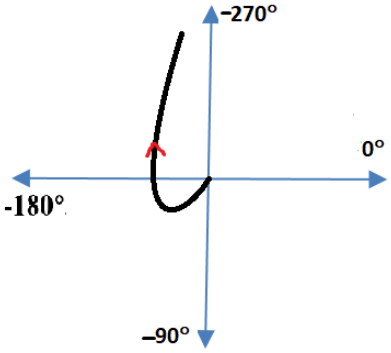
Starting point: at **ω = ∞ M = 0 Φ = -270°**

End point: at **ω = -∞ M = 0 Φ = 270°.**

net angle of rotation = 270 –(-270) = 540°(540° anti-clockwise)

Through out this plot magnitude is zero ,only angle is there.

since magnitude is zero we can consider this mapping as a point at origin.



Mapping of section C3:

Mapping of C3 is mirror image of polar plot of G(s) about real axis .

Mapping of section C4:

Mapping of C4 is G(s) as ω varies from 0^- to 0^+.

Starting point: at **ω = 0- M = ∞ Φ = 90°**

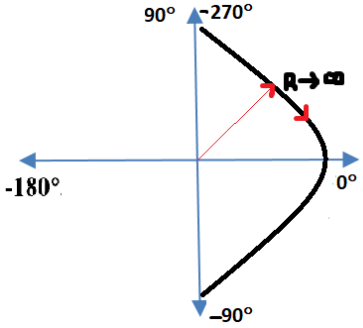
End point: at **ω = 0+ M = ∞ Φ = -90°.**

[$\Phi = \angle G(j\omega) = -\tan^{-1}(\omega/0) - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega)$

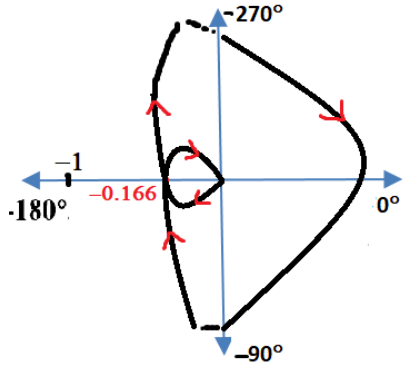
$\Phi = +90°$]

net angle rotation = -90-90=-180°(180° clockwise)

magnitude is ∞ through out this section with 180° angle rotation.



Nyquist Plot:



N = 0 (-1 not encircled)

P=0 (no pole in right side)

N = P

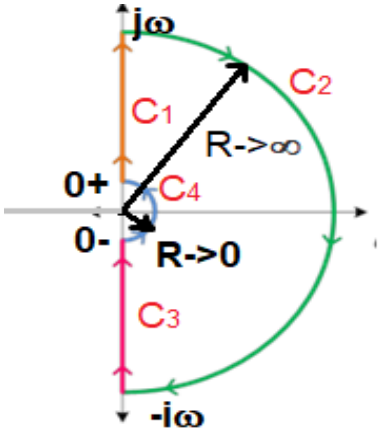
∴System is Stable

3. Draw nyquist plot for a unity feed back system whose open loop transfer function is $G(s) = \frac{K(1+4s)}{s^2(s+1)(1+2s)}$. Determine the range of K for which the closed loop system is stable.

Sol: $G(s) = \frac{(1+4s)}{s^2(s+1)(1+2s)}$

open loop poles s= 0,0,-1, -0.5 (pole at origin)

Nyquist Path: The nyquist path consists 4 sections as shown in figure. The starting and end points of these sections are given below.



Section	Starting	end
C1	ω=0+	ω=∞
C2	ω=∞	ω=-∞
C3	ω=-∞	ω=0-
C4	ω=0-	ω=0+

Mapping of section C1:

Mapping of C1 is polar plot of G(jω).

$$G(j\omega) = \frac{1 + j\omega 4}{(j\omega)^2(j\omega + 1)(j2\omega + 1)}$$

$$M = |G(j\omega)| = \frac{\sqrt{1 + 16\omega^2}}{\omega^2\sqrt{1 + \omega^2}\,\sqrt{1 + 4\omega^2}} \ldots (1)$$

The phase angle of the open loop transfer function is

$$\Phi = \angle G(j\omega) = \tan^{-1}(4\omega)\ -180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \ldots (2)$$

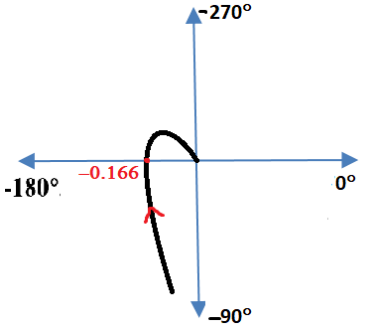
Starting point: at $\omega = 0+$ $M = \infty$ $\Phi = -180°$

End point: at $\omega = \infty$ $M = 0$ $\Phi = -270°$.

net angle of rotation = -270 –(-180) = -90°(-90° clockwise)

Starting point is ∞∠-180°. from this point plot starts and rotate 90° in clockwise direction and ends at origin(end point magnitude=0).

So it is as shown in figure.



Intersection point with -180°: at ω_{pc} , phase = 180°

$\Phi = -180° = \tan^{-1}(4\omega)\ -180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

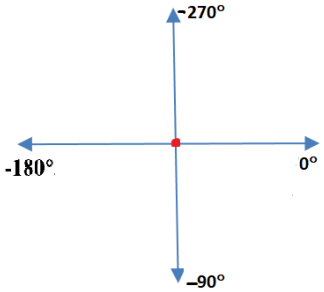
$-\tan^{-1}(4\omega) = -\tan^{-1}(\frac{\omega + 2\omega}{1-2\omega^2}) \rightarrow 4\omega = \frac{3\omega}{1-2\omega^2}$

$1 - 2\omega^2 = 0.75 \rightarrow \omega^2 = 0.125 \rightarrow \omega = 0.35$ rad/sec

Substitute ω= 0.35 in magnitude eqn(eqn 1).

$$M = \frac{K\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}} = 10.66K$$

Therefore intersection point is -10.66K.



Mapping of section C2:

Mapping of C2 is G(jω) as ω varies from ∞ to -∞.

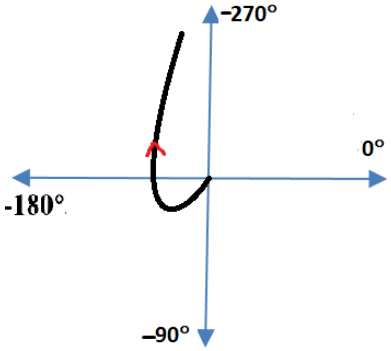
Starting point: at **ω = ∞ M = 0 Φ = -270°**

End point: at **ω = -∞ M = 0 Φ = 270°.**

net angle of rotation = 270 –(-270) = 540°(540° anti-clockwise)

Through out this plot magnitude is zero ,only angle is there.

since magnitude is zero we can consider this mapping as a point at origin.



Mapping of section C3:

Mapping of C3 is mirror image of polar plot of G(s) about real axis .

Mapping of section C4:

Mapping of C4 is G(s) as ω varies from 0^- to 0^+.

Starting point: at **ω = 0- M = ∞ Φ = 180°**

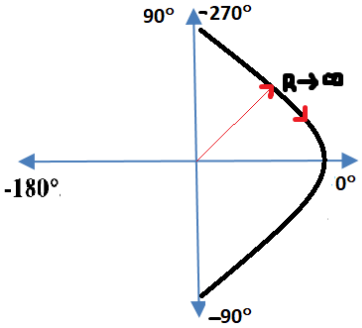
End point: at **ω = 0+ M = ∞ Φ = -180°.**

[at **ω = 0-** $\Phi = \angle G(j\omega) = -\tan^{-1}(4\omega) + 180 - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

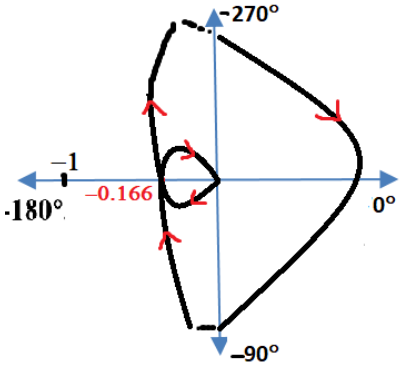
$\Phi = +180°$]

net angle rotation = -180-180=-360°(360° clockwise)

magnitude is ∞ through out this section with 180° angle rotation.



Nyquist Plot:



N = -2 (2 times clockwise)
P=0 (no pole in right side)
N ≠ P
∴ System is Unstable

ANNAMACHARYA UNIVERSITY, RAJAMPET

(ESTD UNDER AP PRIVATE UNIVERSITIES (ESTABLISHMENT AND REGULATION) ACT, 2016
RAJAMPET, Annamayya District, AP, INDIA

Department of Electrical & Electronics Engineering

III B.Tech I Semester

Linear Control Systems (24AEEE51T)
(Electrical and Electronics Engineering)

Unit-5 (State Space Analysis of Continuous Systems)

Syllabus:

Concept of state, state variables and state model, state models – differential equations & Transfer function models, problems, Transfer function from state model, solving the linear time invariant state equations – State transition Matrix and it's properties. System response through state space models. The concepts of controllability and observability, problems.

State Space Model Or State Variable Representation

We learnt from the first unit that there are two types of mathematical models in control system viz.

- Transfer function or input-output model
- State space model or state variable representation

In the first four units of control systems the first model i.e transfer function model is used to design and analysis of control systems. In this unit we are going to learn about state space model.

State space model is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function model has many drawbacks.

Just recall the definition of transfer function.

Transfer function of linear time invariant system is defined as the ratio of Laplace transform of the output to the Laplace transform of input under zero initial conditions.

From the definition itself we can identify many drawbacks as given below.

Drawbacks of Transfer Function Model:

- Transfer function approach can be applied only to linear systems.
- Transfer function approach can be applied only to time invariant systems.
- Transfer function is defined under zero initial conditions.
- It does not give any idea about the internal state of the system.
- It is difficult to apply for multiple input multiple output systems.
- It is comparatively difficult to perform transfer function analysis on computers.

To overcome the above drawbacks, there is a need for another mathematical model i.e. state variable model. State variable analysis can be performed on any type systems and it is very easy to perform state variable analysis on computers. The most interesting feature of state space analysis is that the state variable we choose for describing the system need not be physical quantities related to the system. Variables that are not related to the physical quantities associated with the system can be also selected as the state variables. Even variables that are immeasurable or unobservable can be selected as state variables.

Advantages of state variable analysis:

- It can be applied to both linear &non linear system.
- It can be applied to both time varying and time invariant systems.
- It can be applied to non zero initial conditions.
- It can be applied to multiple input multiple output (MIMO) systems.
- Its gives idea about the internal state of the system.
- It is easy to perform state space analysis on computers.

Basic Definitions:

State of a dynamic system:

The state of a system is the minimum set of variables (known as state variables) whose knowledge at time t= t_0, along with the knowledge of the inputs for time t≥ t_0 completely describes the behaviour of a system for any time t >t_0 .

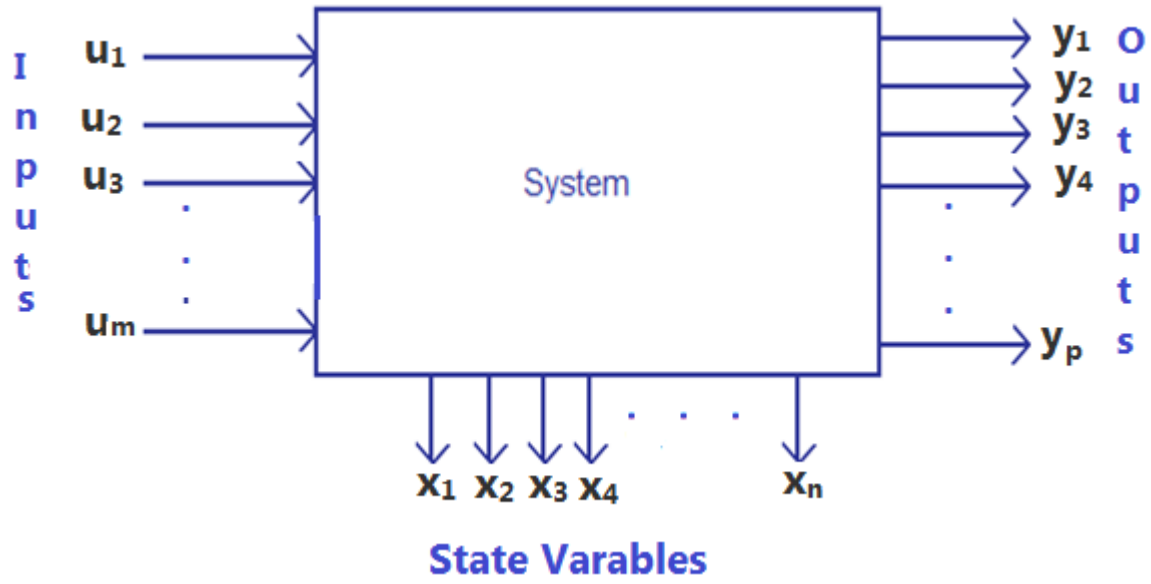
State variables: **State variables** are a set of variables which fully describes a dynamic system at a given instant of time.

State variable representation:

State space model is representing a system with first order differential equations in terms of state variables.

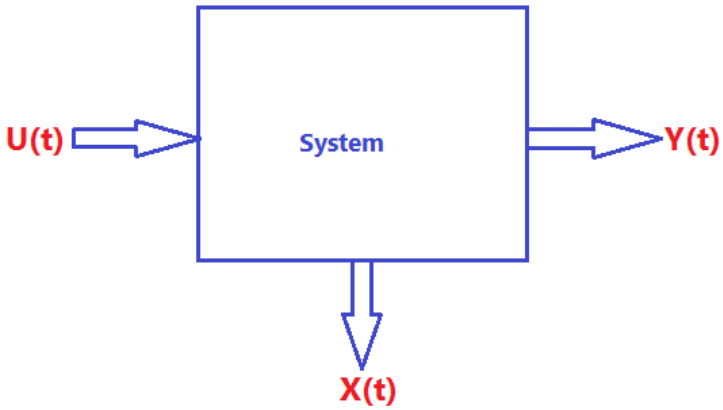
Consider a general n^{th} order system with 'm' inputs and 'p' outputs. It will have 'n' state variables.

Let u_1, u_2, u_3.... u_m are the inputs,
y_1, y_2, y_3.... $y_p$are the outputs and
x_1, x_2, x_3....x_n are the state variables.



(Single arrow represents single variable and double arrow represents a vector.)

In vector form it can be represented as shown below.



Where U(t)= Input Vector= $\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ (A vector of size mx1)

Y(t)=Output Vector= $\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$ (A vector of size px1)

X(t)=State Vector= $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ (A vector of size nx1)

The state space model is representing the system with two set of equations as given below.

- **State equations**

A set of first order differential equations in terms of state variables.

i.e

$$\frac{dx_1}{dt} = f_1(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

$$\frac{dx_2}{dt} = f_2(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

.

.

.

$$\frac{dx_n}{dt} = f_n(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

In short the above equations can be represented as

$$\frac{dX}{dt} = f(X,U,t)$$

- **Output equations**

A system *output* is defined to be any system variable of interest. A set of algebraic equations which gives output in terms of state variables. i.e.

$$y_1 = g_1(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

$$y_2 = g_2(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

$$y_3 = g_3(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

.

.

.

$$y_p = g_p(x_1,x_2,x_3....x_n\ ,u_1,u_2,u_3....u_m,t)$$

In short the above equations can be represented as

$$Y(t)=g(X,U,t)$$

The 't' in above equations represents time..
The above equations represents a general system that can be a non linear or time varying also.

State variable representation of Linear Time Invariant (LTI) Systems:

$$\frac{dX}{dt} = f(X,U,t)$$
$$Y(t)=g(X,U,t)$$

The above equations represents a general system. For time invariant system 't' does not exist in equations and for linear system f & g are linear functions.
So the equations becomes

$$\frac{dX}{dt} = f(X,U) \quad ....(1)$$
$$Y(t)=g(X,U) \quad ....(2)$$

State equations:

If 'f' is linear function equation (1) can be written as

$$\dot{x_1} = \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots . + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots . + b_{1m}u_m$$

$$\dot{x_2} = \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_{21} + \cdots . + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots . + b_{2m}u_m$$

.
.
.

$$\dot{x_n} = \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_{21} + \cdots . + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots . + b_{nm}u_m$$

The above equations may be written compactly in a matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & & b_{2m} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

which may be summarized as:

$$\dot{X} = \mathbf{AX} + \mathbf{BU}$$

where

State vector **X** is a column vector of length *n*,

Input vector **U** is a column vector of length *m*,

State Matrix **A** is an $n \times n$ square matrix of the constant coefficients *aij*, and

Input Matrix or Control matrix **B** is an $n \times m$ matrix of the coefficients *bij*

Output Equations

$$Y(t)=g(X,U) \quad ....(2)$$

If 'g' is linear function equation (2) can be written as

$$y_1 = c_{11}x_1 + c_{12}x_{21} + \cdots . + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots . + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_{21} + \cdots . + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots . + d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_{21} + \cdots . + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots . + d_{pm}u_m$$

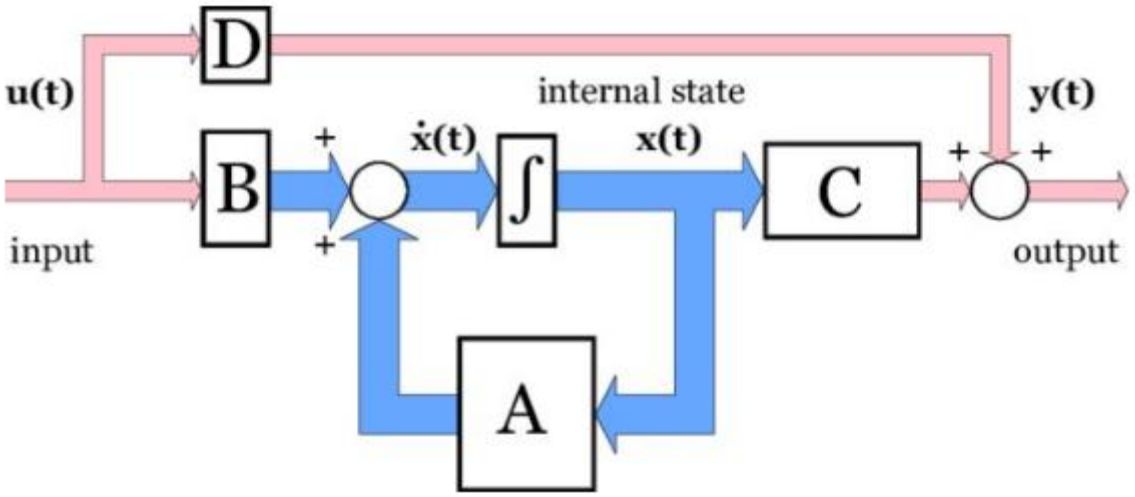
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \dots & d_{1m} \\ d_{21} & & d_{2m} \\ \vdots & & \vdots \\ d_{p1} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

which may be summarized as:

$$Y = \mathbf{CX} + \mathbf{DU}$$

Where
Output vector **Y** is a column vector of length *p*,
Input vector **U** is a column vector of length *m*,
Output Matrix or Observation matrix **C** is an $p \times n$ matrix of the coefficients *cij and* Direct coupling Matrix **D** is an $p \times m$ matrix of the constant coefficients *dij*.

The block diagram of state space model is known as **state diagram** and is as shown below.



Key Concept:

An LTI System is represented in state space model as

$$\dot{X} = AX + BU \quad ...(1)$$
$$Y = CX + DU \quad ...(2)$$

Eqn(1) **State equations** : A set of 'n' first order differential equations by solving which state variables 'x' can be determined.
Eqn(2) **Output equations** : A set of 'p' algebraic equations from which outputs 'y' can be determined.

Obtaining State Space Model

Obtaining State Space Model means determining A,B,C & D matrices for given system. State space model for a given system is not unique. For different state variables chosen we will get different model for a same system.

The state variables are classified as

- Physical variables
- Phase variables
- Canonical variables

State Variable representation using physical variables:

Physical variables are those variables that can be measured like voltage, current, speed etc. When physical system (Circuit) is given we will use these variables to obtain state model.

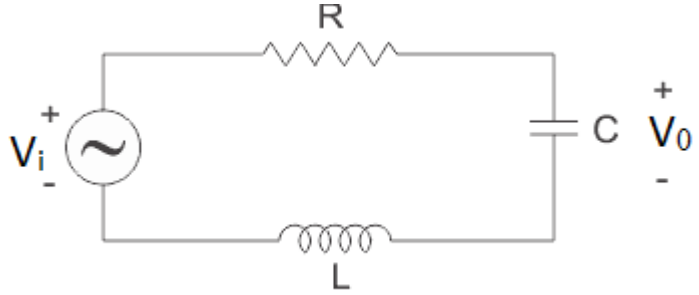
Number of state variables is equal to independent energy storing elements in the system like inductors & capacitors in electrical systems and Mass & Spring in mechanical systems.

For each independent energy storing element, one variable will be considered as state variable.

For electrical systems Current through Inductors and Voltage across capacitors will be considered as state variables.

Solved Examples:

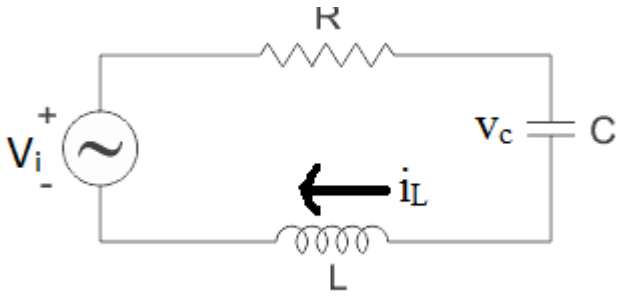
Exp 1: Obtain state space model for the series RLC circuit shown below.



Solution: There are two energy storing elements in the system.

Number of state variables n=2

Let current through inductor is i_L and voltage across capacitor is v_c as shown in below figure.



So state variables are $x_1 = i_L$ and $x_2 = v_c$

One input u= V_i and one output y= V_0

we need two first order differential equations in terms of state variables.

State Equations:

Writing the loop equation

$$V_i = Ri_L + L\frac{di_L}{dt} + \frac{1}{C}\int i_L dt$$

We need only first order differential equation for state space so need to eliminate integral term. We know that

$$V_c = \frac{1}{C}\int i_L dt$$

$$\therefore\ V_i = Ri_L + L\frac{di_L}{dt} + v_c$$

Rearrange the terms

$$L\frac{di_L}{dt} = -Ri_L - v_c + V_i$$

$$\frac{di_L}{dt} = -\frac{R}{L} i_L - \frac{1}{L} v_c + \frac{1}{L} V_i$$

$$\therefore \quad \dot{i}_L = -\frac{R}{L} i_L - \frac{1}{L} v_c + \frac{1}{L} V_i \qquad \ldots . (1)$$

The current through capacitor is given by

$$i_L = C \frac{dv_c}{dt}$$

$$\frac{dv_c}{dt} = \frac{1}{C} i_L$$

$$\dot{v}_c = \frac{1}{C} i_L \qquad \text{....(2)}$$

Equation 1 & 2 can be written in matrix form as shown below.

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} V_i$$

Which is state equations of the system.

Output Equations:

Output $V_0 = v_c$

The output equation of the system is

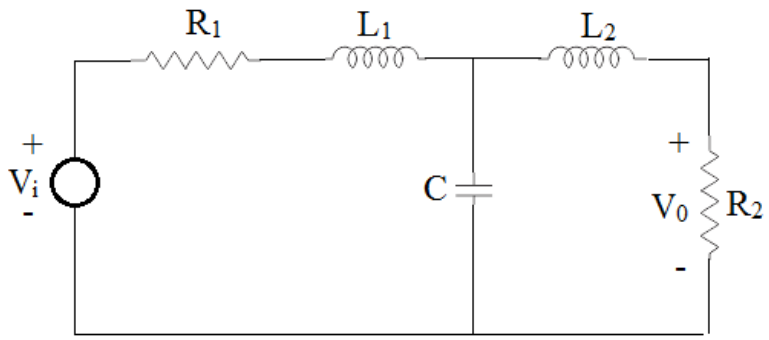
$$V_0 = [0 \quad 1] \begin{bmatrix} i_L \\ v_c \end{bmatrix} + [0] V_i$$

Note:

For each inductor write a loop equation for the loop in which that inductor exist. If capacitor is also exist in that loop, instead of $\frac{1}{C}\int i_L dt$ write v_c.

For each capacitor write a KCL equation at the node to which that capacitor connected. If inductor is also connected to that node, instead of $\frac{1}{L}\int v_c dt$ write i_L.

Exp 2: Obtain state space model for the circuit shown below.

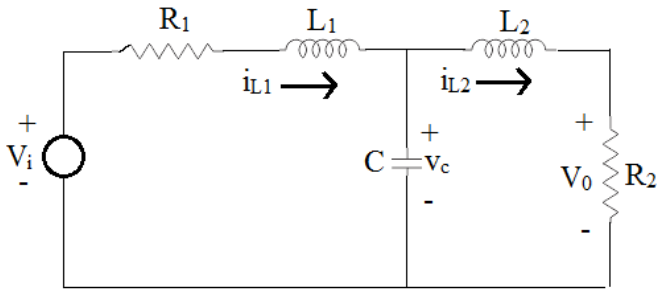


Solution:

There are three energy storing elements in the system.

So Number of state variables n=3

Let current through L_1 is i_{L1}, current through L_2 is i_{L2}, and voltage across capacitor is v_c as shown in below figure.



So state variables are $x_{1=}\ i_{L1}$, $x_{2=}\ i_{L2}$ and $x_3= v_c$

The system has one input $u= V_i$ and one output $y= V_0$

we need three first order differential equations in terms of state variables.

State Equations:

L1 is present in loop1 so write loop equation for loop1.

$$V_i = R_1 i_{L1} + L_1 \frac{di_{L1}}{dt} + \frac{1}{C}\int i_{L1} dt$$

Remember the loop has capacitor o replace integral term with v_c.

$$V_i = R_1 i_{L1} + L_1 \frac{di_{L1}}{dt} + v_c$$

(since Voltage drop across inductor and capacitor are in same polarity.)

$$\Longrightarrow \qquad L1 \frac{di_{L1}}{dt} = -R_1 i_{L1} - v_c + V_i$$

$$\Longrightarrow \qquad \frac{di_{L1}}{dt} = -\frac{R_1}{L_1} i_{L1} - \frac{1}{L_1} v_c + \frac{1}{L_1} V_i$$

$$\dot{i_{L1}} = -\frac{R_1}{L_1} i_{L1} - \frac{1}{L_1} v_c + \frac{1}{L_1} V_i \qquad ...(1)$$

Similarly L_2 is present in loop2 so write loop equation for loop2.

$$0 = R_2 i_{L2} + L_2 \frac{di_{L2}}{dt} + \frac{1}{C}\int i_{L2} dt$$

Remember the loop has capacitor so replace integral term with V_c.

$$0 = R_2 i_{L2} + L_2 \frac{di_{L2}}{dt} - v_c$$

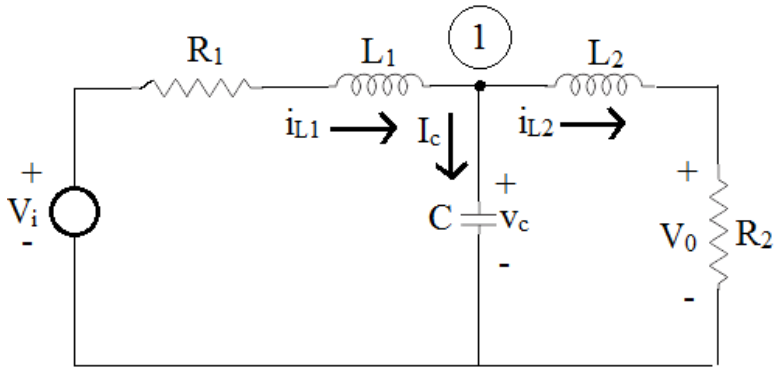
(since Voltage drop across inductor and capacitor are having opposite polarity.)

$$\Longrightarrow \qquad L_2 \frac{di_{L2}}{dt} = -R_2 i_{L2} + v_c$$

$$\Longrightarrow \quad \frac{di_{L2}}{dt} = -\frac{R_2}{L_2} i_{L2} + \frac{1}{L_2} v_c$$

$$\dot{i_{L2}} = -\frac{R_2}{L_2} i_{L2} + \frac{1}{L_2} v_c \quad \text{...(2)}$$

For capacitor we need to write KCL at node 1 shown below.



$$i_{L1} = i_{L2} + I_c$$

$$I_c = i_{L1} - i_{L2}$$

But current through capacitor $I_c = C\frac{dv_c}{dt}$

$$\Longrightarrow \quad i_{L1} - i_{L2} = C\frac{dv_c}{dt}$$

$$\Longrightarrow \quad \frac{1}{C} i_{L1} - \frac{1}{C} i_{L2} = \frac{dv_c}{dt}$$

$$\dot{v_c} = \frac{1}{C} i_{L1} - \frac{1}{C} i_{L2} \quad \text{...(3)}$$

Equations 1-3 can be written in matrix form as shown below and is the state equation of the system.

$$\begin{bmatrix} \dot{i_{L1}} \\ \dot{i_{L2}} \\ \dot{v_c} \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} V_i$$

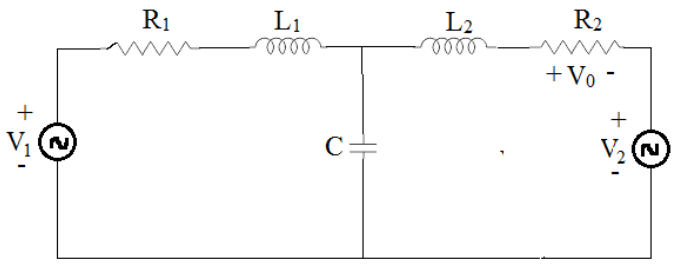
Output Equations:

The output $V_0=i_{L2}\ R_2$

So the output equation in matrix form is

$$V_0 = [0 \quad i_{L2} \quad 0]\begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + [0]V_i$$

Exp 3: Obtain state space model for the circuit shown below.

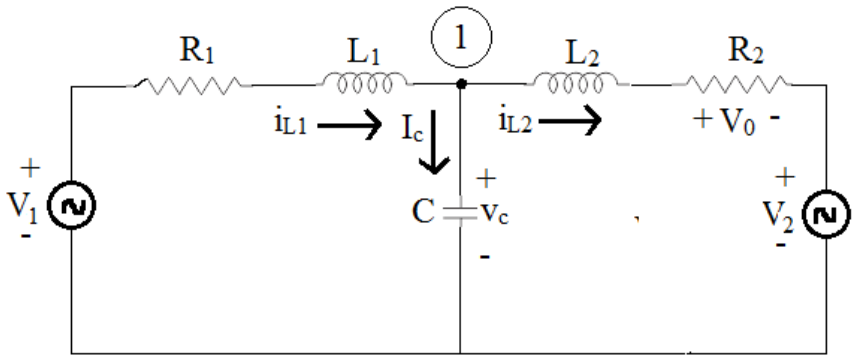


Solution:

There are three energy storing elements in the system.

So Number of state variables n=3

Let current through L_1 is i_{L1}, current through L_2 is i_{L2}, and voltage across capacitor is v_c as shown in below figure.



So state variables are $x_{1=}\ i_{L1}$, $x_{2=}\ i_{L2}$ and $x_3=\ v_c$

The system has two inputs $u_1=\ V_1$ & $u_2=\ V_2$ and one output $y=\ V_0$

we need three first order differential equations in terms of state variables.

State Equations:

L1 is present in loop1 so write loop equation for loop1.

$$V_1 = R_1 i_{L1} + L_1 \frac{di_{L1}}{dt} + \frac{1}{C}\int i_{L1} dt$$

Remember the loop has capacitor o replace integral term with v_c.

$$V_1 = R_1 i_{L1} + L_1 \frac{di_{L1}}{dt} + v_c$$

(since Voltage drop across inductor and capacitor are in same polarity.)

$$\Longrightarrow \quad L1 \frac{di_{L1}}{dt} = -R_1 i_{L1} - v_c + V_1$$

$$\Longrightarrow \quad \frac{di_{L1}}{dt} = -\frac{R_1}{L_1} i_{L1} - \frac{1}{L_1} v_c + \frac{1}{L_1} V_1$$

$$\dot{i_{L1}} = -\frac{R_1}{L_1} i_{L1} - \frac{1}{L_1} v_c + \frac{1}{L_1} V_1 \qquad ...(1)$$

Similarly L_2 is present in loop2 so write loop equation for loop2.

$$-V_2 = R_2 i_{L2} + L_2 \frac{di_{L2}}{dt} + \frac{1}{C}\int i_{L2} dt$$

Remember the loop has capacitor so replace integral term with V_c.

$$-V_2 = R_2 i_{L2} + L_2 \frac{di_{L2}}{dt} - v_c$$

(since Voltage drop across inductor and capacitor are having opposite polarity.)

$$\Longrightarrow \quad L_2 \frac{di_{L2}}{dt} = -R_2 i_{L2} + v_c - V_2$$

$$\Longrightarrow \quad \frac{di_{L2}}{dt} = -\frac{R_2}{L_2} i_{L2} + \frac{1}{L_2} v_c - \frac{1}{L_2} V_2$$

$$\dot{i_{L2}} = -\frac{R_2}{L_2} i_{L2} + \frac{1}{L_2} v_c - \frac{1}{L_2} V_2 \quad ...(2)$$

For capacitor we need to write KCL at node 1.

$$i_{L1} = i_{L2} + I_c$$

$$I_c = i_{L1} - i_{L2}$$

But current through capacitor $I_c = C\frac{dv_c}{dt}$

$$\Longrightarrow \quad i_{L1} - i_{L2} = C\frac{dv_c}{dt}$$

$$\Longrightarrow \quad \frac{1}{C} i_{L1} - \frac{1}{C} i_{L2} = \frac{dv_c}{dt}$$

$$\dot{v_c} = \frac{1}{C} i_{L1} - \frac{1}{C} i_{L2} \quad ...(3)$$

Equations 1-3 can be written in matrix form as shown below and is the state equation of the system.

$$\begin{bmatrix} \dot{i_{L1}} \\ \dot{i_{L2}} \\ \dot{v_c} \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + \begin{bmatrix} 1/L_1 & 0 \\ 0 & -1/L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

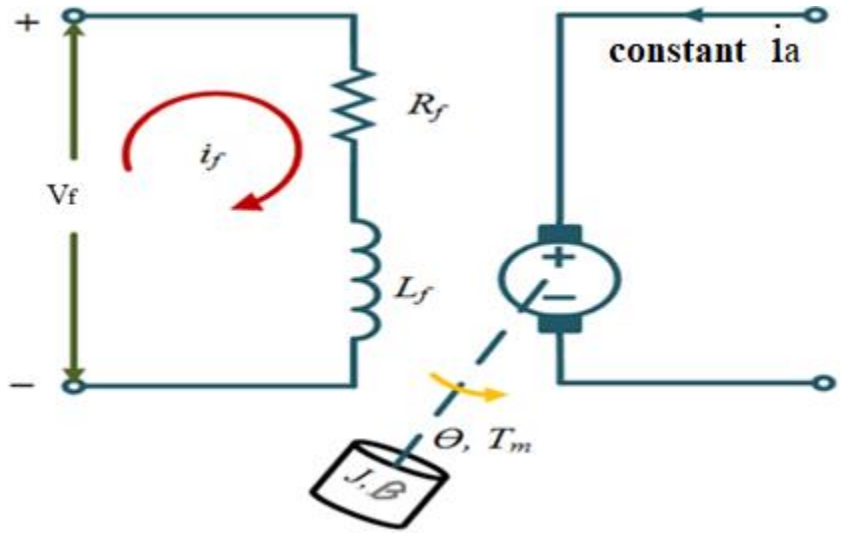
Output Equations:

The output $V_0 = i_{L2} R_2$

So the output equation in matrix form is

$$V_0 = [0 \quad i_{L2} \quad 0] \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + [0] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

State variable Model of DC field controlled servo motor:



R_f= Field Resistance
L_f = Field Inductance
I_f = Field Current
V_f = Field Voltage
I_a = Armature Current
ω = Angular Velocity
Θ= Angular displacement
J = Moment of Inertia
B= friction coefficient of rotor
Tm = Torque Produced

$$V_f = R_f\, i_f + L_f\, (di_f/dt) \ldots(1)$$

$$T_m = K_m K_f\, i_f \ldots.(2)$$

$$T_m = J\ (d^2\theta/dt^2) + B\ (d\theta/dt)\ \ldots(3)$$

$$V_f = R_f\, i_f + L_f\, (di_f/dt) \ldots(1)$$

$$K_m K_f\, i_f = J\ (d^2\theta/dt^2) + B\ (d\theta/dt)$$

$$K\, i_f = J\ (d^2\theta/dt^2) + B\ (d\theta/dt)\ \ldots.(2a)$$

State variable model:

One Input = vf

One Output = ω

Two energy storing elements. (i) Inductor (ii) Mass

There will be two State variables for the system. One is current flowing through inductor. And the other angular velocity of armature.

X1= if

X2=ω

eqn (1) becomes

$$\mathbf{V_f = R_f\ i_f + L_f\ (di_f/dt)\ \ldots(1)}$$

$$L_f \frac{di_f}{dt} = -R_f i_f + V_f$$

$$\frac{di_f}{dt} = -\frac{R_f}{L_f} i_f + \frac{1}{L_f} V_f \ \ldots\ldots(1a)$$

eqn (2) becomes

$$\mathbf{K\ i_f = J\ (d\omega/dt) + B\ (\omega)\ \ldots.(2)}$$

$$J \frac{d\omega}{dt} = K i_f - B\omega$$

$$\frac{d\omega}{dt} = \frac{K}{J} i_f - \frac{B}{J} \omega \ldots.(2a)$$

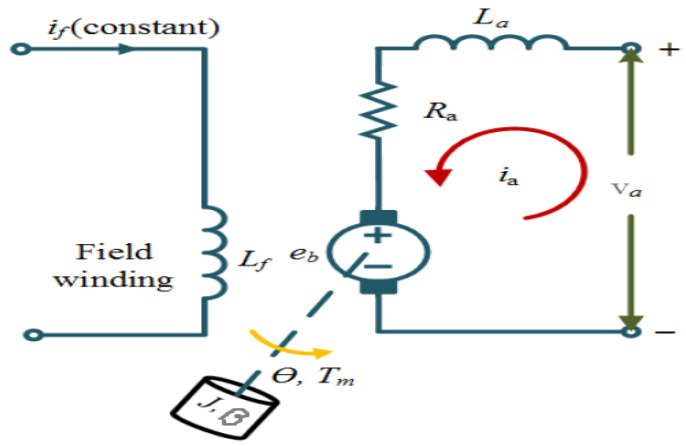
Therefore state eqn. of the system is

$$\begin{bmatrix} \dot{i_f} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 \\ \frac{K}{J} & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} i_f \\ \omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \end{bmatrix} V_f$$

The output eqn. of the system is

$$y = [0 \quad 1] \begin{bmatrix} i_f \\ \omega \end{bmatrix}$$

State variable Model of DC Armature controlled servo motor:



R_f= Field Resistance
L_f = Field Inductance
I_f = Field Current
R_a= Armature Resistance
L_a = Armature Inductance
V_a = Armature Voltage
I_a = Armature Current
eb= back emf
ω = Angular Velocity
Θ= Angular displacement
J = Moment of Inertia
B= friction coefficient of rotor
Tm = Torque Produced

$$V_a = R_a\ i_a + L_a\ (di_a/dt) + e_b \ldots(1)$$

$$T_m = K_T\ i_a \ldots.(2)$$

$$T_m = J\ (d^2\theta/dt^2) + B\ (d\theta/dt) \ldots(2A)$$

$$e_b = K_b\ \omega \ldots(3)$$

$$K\ i_a = J\ (d^2\theta/dt^2) + B\ (d\theta/dt) \ldots(2B)$$

State variable model:

One Input = va

One Output = ω

Two energy storing elements. (i) Inductor (ii) Mass

There will be two State variables for the system. One is current flowing through inductor. And the other angular velocity of armature.

X1= if

X2=ω

eqn (1) becomes

$$V_a = R_a\ i_a + L_a\ (di_a/dt) + e_b$$

$$L_a \frac{di_a}{dt} = -R_a i_a - e_b + V_a$$

$$L_a \frac{di_a}{dt} = -R_a i_a - K_b \omega + V_a (\text{since } e_b = K_b \omega)$$

$$\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_b}{L_a} \omega + \frac{1}{L_a} V_a \ \ldots\ldots (1a)$$

eqn (2B) becomes

$$\mathbf{K\ i_a = J\ (d\omega/dt) + B\ (\omega)\ \ldots.(2)}$$

$$J \frac{d\omega}{dt} = K i_a - B\omega$$

$$\frac{d\omega}{dt} = \frac{K}{J} i_a - \frac{B}{J} \omega \ldots.(2c)$$

Therefore state eqn. of the system is

$$\begin{bmatrix} \ddot{i_a} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} \\ \frac{K}{J} & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} i_f \\ \omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \end{bmatrix} V_f$$

The output eqn. of the system is

$$y = [0 \quad 1] \begin{bmatrix} i_a \\ \omega \end{bmatrix}$$

Obtaining State Space Model

State Variable representation using Phase variables:

Phase variables are those variables which are a system variable and its n-1 derivatives for an nth order system. Generally the system variable is output of the system. When differential equation or transfer function of the system is given these variables are preferred to obtain state model.

Here two cases will be considered based on the terms present in the numerator of the transfer function.

- Transfer function with constant term in numerator (T/F without zeros)
- Transfer function with 'S' terms in numerator (T/F with zeros)

Transfer function with constant term in numerator (T/F without zeros):

Consider the following n^{th} order system without zeros.

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0}$$

Upon cross multiplication

$$(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0)Y(s) = b_0 U(s)$$

By applying inverse Laplace transform on both sides

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u \qquad \text{....(1)}$$

There will be 'n' state variables for this system. We chose output and it's n-1 derivatives as state variables.

So let

$$x_1 = y \quad \text{.....(2)}$$

$$x_2 = \frac{dy}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \qquad \text{.....(3)}$$

$$x_3 = \frac{d^2y}{dt^2} = \frac{dx_2}{dt} = \dot{x}_2 \qquad \text{....(4)}$$

.
.
.

$$x_{n-1} = \frac{d^{n-2}y}{dt^{n-2}} = \frac{dx_{n-2}}{dt} = \dot{x}_{n-2} \quad \text{.....(5)}$$

$$x_n = \frac{d^{n-1}y}{dt^{n-1}} = \frac{dx_{n-1}}{dt} = \dot{x}_{n-1} \quad \text{....(6)}$$

By substituting eqns 2-6 in eqn(1)

$$\dot{x}_n + a_{n-1}x_n + \cdots + a_1x_2 + a_0x_1 = b_0u$$

$$\therefore \quad \dot{x}_n = -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u \quad \text{...(7)}$$

Equations 3-7 can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-2} = x_{n-1}$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u$$

The above equations are state equations for the given transfer function. In matrix form they can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

The output $y=x_1$ (eqn(2)) in matrix form

$$Y = [1 \quad 0 \quad \dots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here D=[0].

Note: Transfer function-

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = [1 \quad 0 \quad \dots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If you observe carefully we can directly write A,B,C,D matrices from the given transfer function.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

→ first row - all are zeros except 2nd element and is 1

→ Second row - all are zeros exept 3 rd element and is 1

⋮

→ n-1 th row - all are zeros except n th element and is 1

→ n th row - -ve of coefficeints of powers of 's' terms in denominator in reverse order

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

→ All elements are zeros except last element and the last one is b_0 which is numerator of transfer function

$$C = [1 \quad 0 \quad \dots \quad 0 \quad 0]$$

↓

All elements are zeros except first one and the first elemnet is 1

And D = [0]

Example

Find the state space model for the system having transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Solution: The highest power of 's' in denominator is 2.
Therefore n=2 i.e. there are two state variables.

Upon cross multiplication of transfer function

$$(s^2+s+10)Y(s) = U(s)$$

By applying inverse Laplace on both side

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 10y = 1u \quad \ldots(1)$$

Let

$$x_1 = y \ldots..(2)$$

$$x_2 = \frac{dy}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \qquad \ldots..(3)$$

By substituting eqns 2-3 in eqn(1)

$$\dot{x}_2 + x_2 + 10x_1 = b_0u$$

$$\therefore \qquad \dot{x}_2 = -10x_1 - x_2 + b_0u \quad \ldots(4)$$

∴ State equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10x_1 - x_2 + b_0u$$

In matrix form

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

And the output equations is $y=x_1$

$$\therefore \qquad Y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

EXP 2: Obtain state space model for given transfer function.

$$G(s) = \frac{10}{s^3 + 11s^2 + 6s + 11}$$

Solution:

$$G(s) = \frac{10}{s^3 + 11s^2 + 6s + 11}$$

Highest power of 's' in denominator =3 = n = no. of state variables
Constant term in denominator = a_0 = 11
Coefficient of 's' term in denominator = a_1 = 6
Coefficient of 's^2' term in denominator = a_1 = 11

Constant term in numerator = b_0 = 10
So the state space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -11 & -6 & -11\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

Transfer function with 'S' terms in numerator (T/F with zeros):

Consider the following n th order system with zeros.(The highest power of numerator can be equal to highest power of 's' in denominator.)

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_2 s^2 + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0}(b_n s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0)$$

The above can be considered as a product of two transfer functions as shown.

$$\frac{Y(s)}{U(s)} = \left(\frac{X(s)}{U(s)}\right)\left(\frac{Y(s)}{X(s)}\right)$$

Where

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0} \quad \ldots(1)$$

$$\frac{Y(s)}{X(s)} = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0 \quad \ldots(2)$$

Consider the transfer function

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0} \qquad \ldots(1)$$

We already discussed how to obtain state model for this type of transfer functions in previous section.
By assuming state variable as given (since the output for this T/F is x)

$$x_1 = x \quad .....(3)$$

$$x_2 = \frac{dx}{dt} = \frac{dx_1}{dt} = \dot{x}_1 \qquad .....(4)$$

$$x_3 = \frac{d^2x}{dt^2} = \frac{dx_2}{dt} = \dot{x}_2 \qquad ....(5)$$

$$\vdots$$

$$x_{n-1} = \frac{d^{n-2}x}{dt^{n-2}} = \frac{dx_{n-2}}{dt} = \dot{x}_{n-2} \quad .....(6)$$

$$x_n = \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-1}}{dt} = \dot{x}_{n-1} \qquad ....(7)$$

The state equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-2} = x_{n-1}$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n + a_{n-1}x_n + \cdots + a_1 x_2 + a_0 x_1 = u$$

The above equations are state equations for the given transfer function. In matrix form they can be written as (since numerator is '1'.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u]$$

Now consider the other transfer function

$\frac{Y(s)}{X(s)} = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0$ ...(2)

Upon cross multiplication

$$Y(s) = (b_n s^n + b_{n-1}s^{n-1} + \cdots + b_2 s^2 + b_1 s + b_0)X(s)$$

By applying inverse Laplace transform on both sides

$$b_n \frac{d^n x}{dt^n} + b_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x = y$$

Substituting derivatives terms with state variables using equations 3-7 it becomes

$$y = b_n \dot{x}_n + b_{n-1}x_n + \cdots + b_1 x_2 + b_0 x_1$$

But we know

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + u$$

Substitute this in y equation

$$y = b_n(-a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + u) + b_{n-1}x_n + \cdots + b_1 x_2 + b_0 x_1$$

By rearranging the terms

$$y = (b_0 - b_n a_0)x_1 + (b_1 - b_n a_1)x_2 + \cdots + (b_{n-1} - b_n a_{n-1})x_n + b_n u$$

Therefore the output equation in matrix form is $[b_n]u$

$$Y = [b_0 - b_n a_0 \quad b_1 - b_n a_1 \quad \dots \quad b_{n-2} - b_n a_{n-2} \quad b_{n-1} - b_n a_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [b_n]\, u$$

If 'b_n=0' (i.e. highest power of 's' in numerator is less than highest power of 's' in denominator) the output equation is

$$Y = [b_0 \quad b_1 \quad \dots \quad b_{n-2} \quad b_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

The state model for transfer function with zeros is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u]$$

$$Y = [b_0 - b_n a_0 \quad b_1 - b_n a_1 \quad \dots \quad b_{n-2} - b_n a_{n-2} \quad b_{n-1} - b_n a_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + [b_n]\, u$$

So no change in A matrix if the T/F has zeros. But following changes will be there in B, C&D. If $\mathbf{b_n \neq 0}$

$$\mathbf{C} = [b_0 - b_n a_0 \quad b_1 - b_n a_1 \quad \dots \quad b_{n-2} - b_n a_{n-2} \quad b_{n-1} - b_n a_{n-1}]$$

And **D=[b_n]**

If $\mathbf{b_n=0}$

$$\mathbf{C} = [b_0 \quad b_1 \quad \dots \quad b_{n-2} \quad b_{n-1}]$$

↓

Coefficients of powers of 'S' in numerator in reverse order

And **D=[0]**

Solved Examples:

EXP 1: Obtain state space model for given transfer function.

$$G(s) = \frac{s+1}{s^2+5s+6}$$

Solution:

$$G(s) = \frac{1*s+1}{s^2+5s+6}$$

Highest power of 's' in denominator =2 = n = no. of state variables
Constant term in denominator = $a_0 = 6$
Coefficient of 's' term in denominator = $a_1 = 5$
Constant term in numerator = $b_0 = 1$
Coefficient of 's' term in denominator = $b_1 = 1$
$b_n=b_2=0$
So the state space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$Y=[\,b_0 \quad b_1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+[0]u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$Y=[\,1 \quad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

EXP 2: Obtain state space model for given transfer function.

$$G(s) = \frac{s^2+3s+3}{s^2+2s+1}$$

Solution:

$$G(s) = \frac{s^2+3s+3}{s^2+2s+1}$$

Highest power of 's' in denominator =2 = n = no. of state variables
Constant term in denominator = $a_0 = 1$
Coefficient of 's' term in denominator = $a_1 = 2$

Constant term in numerator = $b_0 = 3$
Coefficient of 's' term in denominator = $b_1 = 3$
Coefficient of 's' term in denominator = $b_2 = 1$

So the state space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$Y=[\ b_0-b_2\ a_0 \qquad b_1-b_2\ a_1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+[b_2]u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$Y=[\ 2 \qquad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+[1]u$$

Obtaining State Space Model using Canonical Variables

State Variable representation using canonical variables:

Canonical variables are those variables which makes the state matrix diagonal. When transfer function of the system is given these variables are preferred to obtain state model. The state equations are decoupled in canonical form so obtaining the solution is very easy.

Here two cases will be considered based on the roots of the denominator (Poles) of the transfer function.

- T/F with Real distinct poles
- T/F with Multiple poles

Transfer function with distinct poles(non-repeated poles) :

Consider the following n^{th} order system (Here n=m) with distinct poles.

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_2 s^2 + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0}$$

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_2 s^2 + b_1 s + b_0}{(s+p_1)(s+p_2)(s+p_3)\ldots.(s+p_n)}$$

Where $-p_1, -p_2, \ldots\ldots p_n$ are poles of T/F.

The partial fraction expansion of the T/F can be written as

$$\frac{Y(s)}{U(s)} = b_n + \frac{C_1}{s+p_1} + \frac{C_2}{s+p_2} + \cdots + \frac{C_n}{s+p_n}$$

(since n=m we need to divide numerator with denominator before obtaining partial fraction.)

The canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \ddots & & \\ & & & -p_{n-1} & \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_n u$$

Explanation:

$$A = \begin{bmatrix} -p_1 & 0 & \cdots & 0 \\ 0 & -p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -p_{n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ -p_n \end{matrix}$$

→ Diagonal matrix with digonal elements equal to poles of T/F

$$B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

→ Unity column vector i.e. all elements are 1.

$$C = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} & c_n \end{bmatrix}$$

↓

Elements are the residues of the partial fraction expansion of T/F

D =[b_n]

↓

Coefficient of 's^n' in numerator

Ex 1: Obtain state space model for given transfer function.

$$\boldsymbol{G(s) = \frac{10}{s^3 + 11s^2 + 6s + 11}}$$

Solution: The highest power of 's' in denominator is 3.
Therefore n=3 i.e. there are three state variables.
The poles of the T/F are s=-1,-2,-3(roots of the denominator)

The partial faction expansion of the transfer function is

$$G(s) = \frac{10}{(s+1)(s+2)(s+3)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)} + \frac{C_3}{(s+3)}$$

Where C1,C2 & C3 are calculated by

$$C_1 = \frac{10}{(s+2)(s+3)}\Big/_{s=-1} = \frac{10}{(1)(2)} = 5$$

$$C_2 = \frac{10}{(s+1)(s+3)}\Big/_{s=-2} = \frac{10}{(-1)(1)} = -10$$

$$C_3 = \frac{10}{(s+1)(s+2)}\Big/_{s=-3} = \frac{10}{(-2)(-1)} = 5$$

$$G(s) = \frac{5}{(s+1)} + \frac{-10}{(s+2)} + \frac{5}{(s+3)}$$

Therefore the state model is given by

Poles of T/F

All '1's

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

Residues of the factors

$$y = [C_1 \quad C_2 \quad C_3]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

$$y = [5 \quad -10 \quad 5]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

Ex 2: Obtain state space model for given transfer function.

$$G(s) = \frac{s+1}{s^2+5s+6}$$

Solution:

$$G(s) = \frac{s+1}{s^2+5s+6}$$

$$G(s) = \frac{s+1}{(s+2)(s+3)}$$

The poles of the T/F are **S= -2,-3**.
The partial fraction expansion is given by

$$G(s) = \frac{C_1}{(s+2)} + \frac{C_1}{(s+3)}$$

Where

$$C_1 = \frac{s+1}{(s+3)}\Big/_{s=-2} = \frac{-1}{1} = -1$$

$$C_2 = \frac{s+1}{(s+2)}\Big/_{s=-3} = \frac{-2}{-1} = 2$$

$$G(s) = \frac{-1}{(s+2)} + \frac{2}{(s+3)}$$

Therefore the state model is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$Y = \begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Ex 3: Obtain state space model for given transfer function.

$$G(s) = \frac{s^2+3s+3}{s^2+6s+5}$$

Solution:

$$G(s) = \frac{s^2+3s+3}{s^2+6s+5}$$

The poles of the T/ F are S=-1,-5
Since numerator power is equal to denominator power before obtaining the partial fraction we need divide numerator with denominator up to one term.

$$s^2 + 6s + 5) \; s^2 + 3s + 3 \; (1$$
$$\underline{-s^2 \mp 6s \mp 5}$$
$$-3s - 2$$

$$G(s) = \frac{s^2 + 3s + 3}{s^2 + 6s + 5} = 1 + \frac{-3s - 2}{s^2 + 6s + 5}$$

$$\frac{-3s - 2}{(s + 1)(s + 5)} = \frac{C_1}{(s + 1)} + \frac{C_1}{(s + 5)}$$

Where

$$C_1 = \left.\frac{-3s - 2}{(s + 5)}\right/_{s = -1} = \frac{1}{4}$$

$$C_2 = \left.\frac{-3s - 2}{(s + 1)}\right/_{s = -5} = \frac{13}{-4} = -\frac{13}{4}$$

$$G(s) = 1 + \frac{{}^{1}/_{4}}{(s + 1)} + \frac{{}^{-13}/_{4}}{(s + 5)}$$

Therefore the state model is given by (D=[1]since b_n=1)

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & -5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 1\end{bmatrix}u$$

$$Y = [\,1/4 \quad -13/4\,]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + [1]u$$

Transfer function with repeated poles(Jordan canonical form):

To deal with multiple roots, Diagonal Canonical Form must be modified to Jordan Canonical form. Consider, the T/F of 7th order

$$\frac{Y(s)}{U(s)} = \frac{b_7 s^7 + b_6 s^6 + b_5 s^5 + \cdots + b_2 s^2 + b_1 s + b_0}{(s + p_1)(s + p_2)(s + p_3) \ldots . (s + p_7)}$$

Where $-p_1, -p_2, \ldots\ldots p_n$ are poles of T/F.
Let assume $p_1 = p_2 = p_3 = p$ and p_4 & p_5 are distinct and $p_6 = p_7 = q$

The partial fraction expansion of the T/F can be written as

$$\frac{Y(s)}{U(s)} = b_7 + \frac{C_1}{(s+p)} + \frac{C_2}{(s+p)^2} + \frac{C_3}{(s+p)^3} + \frac{C_4}{s+p_4} + \frac{C_5}{s+p_5} + \frac{C_6}{s+q} + \frac{C_7}{(s+q)^2}$$

The state model is

Jordan block

'0's and '1's positions exchanged

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\end{bmatrix} = \left[\begin{array}{ccc|cccc} -p & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -p & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -p & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -p_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -p_5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -q & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -q \end{array}\right] \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 1\\ 1\\ 0\\ 1\end{bmatrix} u$$

In Jordan blocks upper diagonal elements becomes '1' and corresponding B matrix elements becomes '0'.

Jordan block

residue of highest power term of repeated poles

exist if n=m otherwise '0'

$$y = [\,C_3 \quad C_2 \quad C_1 \quad C_4 \quad C_5 \quad C_7 \quad C_6\,] \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\end{bmatrix} + [b_7\,]u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\end{bmatrix} = \begin{bmatrix} -p & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -p & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -p & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -p_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -p_5 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -q & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -q \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 1\\ 1\\ 0\\ 1\end{bmatrix} u$$

$$y = [\,C_3 \quad C_2 \quad C_1 \quad C_4 \quad C_5 \quad C_7 \quad C_6\,] \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\end{bmatrix} + [b_7\,]u$$

EXP 1: Obtain state space model for given transfer function.

$$\boldsymbol{G(s) = \frac{3}{s(s^2 + 2s + 1)}}$$

Solution:

$$\boldsymbol{G(s) = \frac{3}{s(s^2 + 2s + 1)} = \frac{3}{s(s+1)^2}}$$

The poles are s=0,-1,-1('-1' Repeated twice)

So the partial fraction expansion is

$$\boldsymbol{G(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{(s+1)^2}}$$

Where

$$C_1 = \frac{3}{(s+1)^2}\Big/_{s=0} = 3$$

$$C_3 = \frac{3}{s}\Big/_{s=-1} = -3$$

$$C_2 = \frac{d(\frac{3}{s})}{ds}\Big/_{s=-1} = -\frac{3}{s^2}\Big/_{s=-1} = -3$$

$$\boldsymbol{G(s) = \frac{3}{s} + \frac{-3}{s+1} + \frac{-3}{(s+1)^2}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [3 \quad -3 \quad -3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

:

Obtaining Transfer function from State Space Model

From previous classes we learnt how to obtain state space model. In this module let us discuss how to obtain transfer function from state space model.

We know that an LTI System is represented in state space model as

$$\dot{X} = AX + BU \qquad Y = CX + DU$$

Applying Laplace transformation on both sides of these eqns

$$sX(s) = AX(s) + BU(s) \qquad ...(1)$$

$$Y(s) = CX(s) + D\,U(s) \qquad ...(2)$$

Eqn(1) becomes

$$s\,X(s) - AX(s) = BU(s)$$

$$(sI-A)\,X(s) = X(0) + BU(s)$$

(since S is scalar and A is matrix. Here I is identity matrix with same size that of A)

$$X(s) = (sI-A)^{-1}\,B\,U(s)$$

Substitute this in eqn(2)

$$Y(s) = C\,(sI-A)^{-1}\,B\,U(s) + DU(s)$$

$$Y(s) = [C\,(sI-A)^{-1}\,B + D]\,U(s)$$

The transfer function of the system is

$$\frac{Y(s)}{U(s)} = [C(sI - A)^{-1}B + D]$$

Solved examples:

EX 1: Obtain transfer function for the given system.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$\mathbf{Y}=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Sol:

From the given state space model

$A=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}$,$B=\begin{bmatrix}1\\ 0\end{bmatrix}$,C=[0 1] and D=[0]

Transfer function

$$\frac{\mathbf{Y(s)}}{\mathbf{U(s)}}=[\boldsymbol{C(sI-A)^{-1}B+D}]$$

$$(sI-A)=s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}s+1 & 1\\ -1 & s\end{bmatrix}$$

$$(sI-A)^{-1}=\begin{bmatrix}s+1 & 1\\ -1 & s\end{bmatrix}^{-1}=\frac{1}{s(s+1)-(-1*1)}\begin{bmatrix}s & -1\\ 1 & s+1\end{bmatrix}$$

$$(sI-A)^{-1}==\frac{1}{s^2+s+1}\begin{bmatrix}s & -1\\ 1 & s+1\end{bmatrix}$$

$$(sI-A)^{-1}B=\frac{1}{s^2+s+1}\begin{bmatrix}s & -1\\ 1 & s+1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$(sI-A)^{-1}B=\frac{1}{s^2+s+1}\begin{bmatrix}s\\ 1\end{bmatrix}$$

$$C(sI-A)^{-1}B=\begin{bmatrix}0 & 1\end{bmatrix}\frac{1}{s^2+s+1}\begin{bmatrix}s\\ 1\end{bmatrix}=\frac{1}{s^2+s+1}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s\\ 1\end{bmatrix}$$

$$\boldsymbol{C(sI-A)^{-1}B=\frac{1}{s^2+s+1}}$$

Therefore Transfer function is

$$\boldsymbol{\frac{Y(s)}{U(s)}=[C(sI-A)^{-1}B+D]=\frac{1}{s^2+s+1}}$$

Ex 2: Obtain transfer function for the given system.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}3 & -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+[0]u$$

Sol: Given

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix},\ B=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix},\ C=\begin{bmatrix}3 & -3 & -3\end{bmatrix}\ and\ D=[0]$$

$$sI-A=s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix}-\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}s & 0 & 0\\ 0 & s+1 & -1\\ 0 & 0 & s+1\end{bmatrix}$$

$$(sI-A)^{-1}=\begin{bmatrix}s & 0 & 0\\ 0 & s+1 & -1\\ 0 & 0 & s+1\end{bmatrix}^{-1}$$

$$(sI-A)^{-1}=\frac{1}{s(s+1)^2}\begin{bmatrix}(s+1)^2 & 0 & 0\\ 0 & s(s+1) & s\\ 0 & 0 & s(s+1)\end{bmatrix}$$

$$=\frac{1}{s(s^2+2s+1)}\begin{bmatrix}(s+1)^2 & 0 & 0\\ 0 & s(s+1) & s\\ 0 & 0 & s(s+1)\end{bmatrix}$$

$$(sI-A)^{-1}B=\frac{1}{s(s^2+2s+1)}\begin{bmatrix}(s+1)^2 & 0 & 0\\ 0 & s(s+1) & s\\ 0 & 0 & s(s+1)\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$(sI-A)^{-1}B=\frac{1}{s(s^2+2s+1)}\begin{bmatrix}(s+1)^2\\ s\\ s(s+1)\end{bmatrix}$$

$$C(sI-A)^{-1}B=\frac{1}{s(s^2+2s+1)}[3\quad -3\quad -3]\begin{bmatrix}(s+1)^2\\ s\\ s(s+1)\end{bmatrix}$$

$$=\frac{3((s+1)^2)-3s-3s(s+1)}{s(s^2+2s+1)}=\frac{3}{s(s^2+2s+1)}$$

Therefore Transfer function is

$$\frac{\mathbf{Y(s)}}{\mathbf{U(s)}}=[\boldsymbol{C(sI-A)^{-1}B+D}]=\frac{\mathbf{3}}{\boldsymbol{s(s^2+2s+1)}}$$

EX 3: Obtain transfer function for the given system.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}u$$

$$\mathbf{Y=[\ 1/4\quad -13/4]}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\mathbf{+[1]u}$$

Sol:

From the given state space model

A $=\begin{bmatrix}-1 & 0\\ 0 & -5\end{bmatrix}$,B $=\begin{bmatrix}1\\1\end{bmatrix}$,C=[1/4 -13/4] and D=[1]

Transfer function

$$\frac{\mathbf{Y(s)}}{\mathbf{U(s)}} = [\boldsymbol{C(sI - A)^{-1}B + D}]$$

$$(sI - A) = s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 0 & -5\end{bmatrix} = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 0 & -5\end{bmatrix} = \begin{bmatrix}s+1 & 0\\ 0 & s+5\end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix}s+1 & 0\\ 0 & s+5\end{bmatrix}^{-1} = \frac{1}{(s+5)(s+1)}\begin{bmatrix}s+5 & 0\\ 0 & s+1\end{bmatrix}$$

$$(sI - A)^{-1} == \frac{1}{s^2+6s+5}\begin{bmatrix}s+5 & 0\\ 0 & s+1\end{bmatrix}$$

$$(sI - A)^{-1}B = \frac{1}{s^2+6s+5}\begin{bmatrix}s+5 & 0\\ 0 & s+1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$(sI - A)^{-1}B = \frac{1}{s^2+6s+5}\begin{bmatrix}s+5\\ s+1\end{bmatrix}$$

$$C(sI - A)^{-1}B = [1/4 \quad -13/4]\frac{1}{s^2+s+1}\begin{bmatrix}s+5\\ s+1\end{bmatrix}$$

$$= \frac{1}{s^2+6s+5}[1/4 \quad -13/4]\begin{bmatrix}s+5\\ s+1\end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{(s+5-13(s+1))}{4(s^2+6s+5)} = \frac{-3s-2}{s^2+6s+5}$$

Therefore Transfer function is

$$\frac{Y(s)}{U(s)} = [C(sI - A)^{-1}B + D] = \frac{-3s-2}{s^2+6s+5} + 1 = \frac{-3s-2+s^2+6s+5}{s^2+6s+5}$$

$$\frac{\mathbf{Y(s)}}{\mathbf{U(s)}} = \frac{\boldsymbol{s^2+3s+3}}{\boldsymbol{s^2+6s+5}}$$

Solution of State Space Model & STM

State variable representation of a system is

$\dot{X} = AX + BU; \qquad Y = CX + DU$

We need to find X & Y for given U & X(0).

Homogeneous Solution:

When U =0

$\dot{X} = AX$...(1)

$Y = CX$...(2)

Apply laplace transform to eqn 1

sX(s) -X(0) = A X(s)

sX(s) – A X(s) = X(0)

(sI-A) X(s) = X(0)

$X(s) = (sI-A)^{-1} X(0) = \varphi(s) X(0)$ ....(3)

Where $\varphi(s) = [sI-A]^{-1}$ is known as resolvant Matrix.

$X(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} X(0) = \varphi(t) X(0)$

$X(t) = \varphi(t) X(0)$

$\varphi(t) = \mathcal{L}^{-1}\{\varphi(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At}$

$X(t) = \varphi(t) X(0) = e^{At} X(0)$.....(4)

$Y(t) = C X(t) + D U(t)$

The initial state X(0) is transformed to other state through φ(t). So $\varphi(t) = e^{At}$ is known as state transition Matrix(STM).

Non-homogeneous Solution:

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Apply laplace transform to eqn 1

$sX(s) - X(0) = A\,X(s) + BU(s)$

$sX(s) - A\,X(s) = X(0) + BU(s)$

$(sI-A)\,X(s) = X(0) + BU(s)$

$X(s) = (sI-A)^{-1}\,[X(0) + BU(s)] = \varphi(s)\,[X(0) + BU(s)]$

$$\mathbf{X(s) = \varphi(s)\,[X(0) + BU(s)] \quad ....(3)}$$

$X(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\,[X(0) + BU(s)]\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\,X(0)\} + \mathcal{L}^{-1}\{(sI-A)^{-1}\,BU(s)\}$

$$\boldsymbol{X(t) = \varphi(t)X(0) + \int_0^t \emptyset(t-\tau)BU(\tau)d\tau) = e^{At}\ X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau} \quad \mathbf{(4)}$$

State Transition Matrix:

It is the matrix which transform the state from initial state(X(0)) to any other state X(t).

$$X(t) = \varphi(t)X(0)$$

This φ(t) is known as state transition Matrix(STM).Also known as fundamental matrix, system matrix, exponential matrix.

$\varphi(t) = \mathcal{L}^{-1}\{\varphi(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At}$

φ(s) = resolvant Matrix

Properties of STM:

1. $\Phi(0) = e^{0} = I$
2. $\Phi^{-1}(t) = e^{-At} = \Phi(-t)$
3. $\Phi(t)^{k} = \Phi(kt)$
4. $\Phi(t_2)\Phi(t_1) = e^{At2}e^{At1} = \Phi(t_2 + t_1)$
5. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$
6. $x(0) = \Phi(-t)x(t)$

Computation of STM:

Two methods to compute STM.

(i) Using resolvant matrix φ(s)
(ii) Using cayley-Hamilton theorem

Using resolvant matrix φ(s):

$\varphi(t) = e^{At} = \mathcal{L}^{-1}\{\varphi(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

Problems:

1. Find STM for the given State model.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Sol:

$$A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}, C = \begin{bmatrix}1 & 0\end{bmatrix}, D = 0$$

$$sI - A = \begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}$$

$$\phi(s) = (sI-A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}[\phi(s)] = \mathcal{L}^{-1}\left\{\begin{bmatrix}\frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2}\\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2}\end{bmatrix}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1}+\frac{-1}{s+2}\right\} = 2e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{-1}{s+2}\right\} = e^{-t} - e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{-2}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+1}+\frac{2}{s+2}\right\} = -2e^{-t} + 2e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{2}{s+2}\right\} = -e^{-t} + 2e^{-2t}$$

Therefore $\phi(t) = \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$

2. Find STM for the given State matrix.

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Sol:

$$sI - A = \begin{bmatrix} s+3 & -1 & 0 \\ 0 & s+3 & -1 \\ 0 & 0 & s+3 \end{bmatrix}$$

$$\phi(s) = [sI - A]^{-1} = \frac{1}{(s+3)^3}\begin{bmatrix} (s+3)^2 & s+3 & 1 \\ 0 & (s+3)^2 & s+3 \\ 0 & 0 & (s+3)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)^2} & \frac{1}{(s+3)^3} \\ 0 & \frac{1}{s+3} & \frac{1}{(s+3)^2} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}[\phi(s)] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)^2} & \frac{1}{(s+3)^3} \\ 0 & \frac{1}{s+3} & \frac{1}{(s+3)^2} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] = te^{-3t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+3)^3}\right] = t^2e^{-3t}/2$$

$$\phi(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & t^2e^{-3t}/2 \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

3. Find STM for the given State matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D = 0$$

$$sI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s+4 & -3 \\ 1 & 1 & s+2 \end{bmatrix}$$

$$\phi(s) = (sI - A)^{-1} = \frac{1}{s^3 + 6s^2 + 11s + 3} \begin{bmatrix} s^2 + 6s + 11 & s+2 & 3 \\ -3 & s^2 + 2 & 3s \\ s+4 & -s-1 & s^2 + 4s \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}[\phi(s)]$$

Cayley_Hamilton Theorem:

Every matrix satisfies its own characteristic equation.

Procedure:

1. Find eigen values of the given matrix A. eigen values are roots of characteristic eqn. $|\lambda I-A|=0$. Let eigen values are $\lambda_1, \lambda_2, \ldots \lambda_n$.
2. If all eigen values are distinct write the following 'n' eqns.
   $\alpha_0+ \alpha_1\lambda_i + \alpha_2\lambda^2_i +\ldots+ \alpha_{n-1}\lambda^{n-1}_i = e^{\lambda it}$ for I =1….n (1)
3. Substitute λ values in above eqns and solve for $\lambda_1, \lambda_2, \ldots \lambda_n$.
4. Find $\varphi(t) = e^{At} = \alpha_0 I+ \alpha_1 A + \alpha_2 A^2 +\ldots+ \alpha_{n-1} A^{n-1}$ …(2)
5. If eigen values are repeated differentiate corresponding eqns in eqn(1) with respect to λ_i and then substitute λ_i to get other eqn. and proceed.

Problems:

1.Find STM for the given State model.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Sol:

$$A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}, C = \begin{bmatrix}1 & 0\end{bmatrix}, D = 0$$

$$\lambda I - A = \begin{bmatrix}\lambda & -1\\ 2 & \lambda+3\end{bmatrix}$$

$$roots\ of\ |\lambda I - A| = 0\ are\ \lambda_1 = -1, \lambda_1 = -2.$$

$\alpha_0+ \alpha_1\lambda_i + \alpha_2\lambda^2_i +\ldots+ \alpha_{n-1}\lambda^{n-1}_i = e^{\lambda it}$ for I =1….n

here n=2

$\alpha_0+ \alpha_1\lambda_1 = e^{\lambda 1t}$ → $\alpha_0 - \alpha_1 = e^{-t}$ …(1)

$\alpha_0+ \alpha_1\lambda_2 = e^{\lambda 2t}$ → $\alpha_0 - 2\alpha_1 = e^{-2t}$ …(2)

(1)-(2) gives

$\alpha_1 = e^{-t} - e^{-2t}$ & $\alpha_0 = 2e^{-t} - e^{-2t}$

$$e^{At} = \alpha_0 I + \alpha_1 A = \alpha_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \alpha_1\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}\alpha_0 & 0\\ 0 & \alpha_0\end{bmatrix} + \begin{bmatrix}0 & \alpha_1\\ -2\alpha_1 & -3\alpha_1\end{bmatrix} = \begin{bmatrix}\alpha_0 & \alpha_1\\ -2\alpha_1 & \alpha_0 - 3\alpha_1\end{bmatrix}$$

Therefore $\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$

2.Find STM for the given State model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Sol:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$
$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix}$$

$$roots\ of\ |\lambda I - A| = 0\ are\ \lambda_1 = -1, \lambda_2 = -1 (repeated\ eigen\ values).$$

$\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda^2_i + \ldots + \alpha_{n-1}\lambda^{n-1}_i = e^{\lambda i t}$ for I =1….n

here n=2

$\alpha_0 + \alpha_1\lambda_1 = e^{\lambda 1 t}$ → $\alpha_0 - \alpha_1 = e^{-t}$ …(1)

since $\lambda_2 = \lambda_1$ differentiate above eqn w.r.t λ_1

$\frac{d}{d\lambda_1}(\alpha_0 + \alpha_1\lambda_1) = \frac{d}{d\lambda_1}(e^{\lambda 1 t})$ → $\alpha_1 = t\, e^{\lambda 1 t}$ → $\alpha_1 = t\, e^{-t}$ …(2)

Therefore $\alpha_0 = e^{-t} + t\, e^{-t}$

$e^{At} = \alpha_0 I + \alpha_1 A = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 \\ -\alpha_1 & -2\alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 2\alpha_1 \end{bmatrix}$

Therefore $\phi(t) = \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & -e^{-t} - te^{-t} \end{bmatrix}$

4. $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} find\ STM.$

Sol:

$$\lambda I - A = \begin{bmatrix} \lambda + 3 & -1 & 0 \\ 0 & \lambda + 3 & -1 \\ 0 & 0 & \lambda + 3 \end{bmatrix}$$

$$roots\ of\ |\lambda I - A| = 0\ are\ \lambda_1 = -3, \lambda_2 = -3, \lambda_3 = -3. (repeated\ roots)$$

$\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda^2_i + \ldots + \alpha_{n-1}\lambda^{n-1}_i = e^{\lambda i t}$ for I =1….n

here n=3

$\alpha_0+ \alpha_1\lambda_1 +\alpha_2\lambda_1^2 = e^{\lambda 1t}$ → $\alpha_0 - 3\alpha_1 +9\alpha_2 = e^{-3t}$ ...(1)

since $\lambda_2 = \lambda_1$ differentiate above eqn w.r.t λ_1

$\frac{d}{d\lambda_1}(\alpha_0+ \alpha_1\lambda_1 +\alpha_2\lambda_1^2) = \frac{d}{d\lambda_1}(e^{\lambda 1t})$ → $\alpha_1+ 2\alpha_2 \lambda_1= t\, e^{\lambda 1t}$ → $\alpha_1- 6\alpha_2 = t\, e^{-3t}$ ...(2)

since $\lambda_3 = \lambda_2$ differentiate above eqn w.r.t λ_1

$\frac{d}{d\lambda_1}(\alpha_1+ 2\alpha_2 \lambda_1)= \frac{d}{d\lambda_1}(t\, e^{\lambda 1t})$ → $2\alpha_2 = t^2 e^{\lambda 1t}$ → $\alpha_2 = (t^2/2)e^{-3t}$ ...(3)

Therefore $\alpha_1= 3t^2 e^{-3t} +t e^{-3t}$ $\alpha_0 = e^{-3t} +18t^2 e^{-3t} +3t e^{-3t} -9 t^2 e^{-3t} = e^{-3t} + 3t e^{-3t} +9t^2 e^{-3t}$

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 = \alpha_0 \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \alpha_1 \begin{bmatrix}-3 & 1 & 0\\ 0 & -3 & 1\\ 0 & 0 & -3\end{bmatrix} + \alpha_2 \begin{bmatrix}9 & -6 & 1\\ 0 & 9 & -6\\ 0 & 0 & 9\end{bmatrix}$$

$$= \begin{bmatrix}\alpha_0 & 0 & 0\\ 0 & \alpha_0 & 0\\ 0 & 0 & \alpha_0\end{bmatrix} + \begin{bmatrix}-3\alpha_1 & \alpha_1 & 0\\ 0 & -3\alpha_1 & \alpha_1\\ 0 & 0 & -3\alpha_1\end{bmatrix} + \begin{bmatrix}9\alpha_2 & -6\alpha_2 & \alpha_2\\ 0 & 9\alpha_2 & -6\alpha_2\\ 0 & 0 & 9\alpha_2\end{bmatrix}$$

$$= \begin{bmatrix}\alpha_0 - 3\alpha_1 + 9\alpha_2 & \alpha_1 - 6\alpha_2 & \alpha_2\\ 0 & \alpha_0 - 3\alpha_1 + 9\alpha_2 & \alpha_1 - 6\alpha_2\\ 0 & 0 & \alpha_0 - 3\alpha_1 + 9\alpha_2\end{bmatrix}$$

Therefore $\phi(t) = \begin{bmatrix}e^{-3t} & te^{-3t} & t^2e^{-3t}/2\\ 0 & e^{-3t} & te^{-3t}\\ 0 & 0 & e^{-3t}\end{bmatrix}$

Solution:

$X(s) = (sI-A)^{-1} X(0) = \varphi(s) X(0)$ ....(3)

$X(t) = \varphi(t) X(0) = e^{At} X(0)$

Non-homogeneous solution:

$X(s) = \varphi(s) [X(0) + BU(s)]$

$X(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{(sI-A)^{-1}[X(0) + BU(s)]\} = \mathcal{L}^{-1}\{(sI-A)^{-1} X(0)\} + \mathcal{L}^{-1}\{(sI-A)^{-1} BU(s)\}$

$$\boldsymbol{X(t) = \varphi(t)X(0) + \int_0^t \emptyset(t-\tau)BU(\tau)d\,\tau) = e^{At}\ X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\,\tau}$$ **(4)**

Y(t) = C X(t) + D U(t)

Problems:

1. Solve state eqns for unit step input of a system given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$\mathrm{X}^T(0) = [1 \quad 0]$$

2. find (i) STM (ii) Transfer function (iii)the output for unit step input of a system under zero initial conditions given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$Y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

3. find the outputfor unit step input of a system given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 5\end{bmatrix}u$$

$$Y = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \& \ \mathrm{x}_1(0) = 1, \mathrm{x}_2(0) = 1$$

4. find the response of a system given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$y = [1 \quad -1]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \& \ \mathrm{X}(0) = \begin{bmatrix}1\\ 1\end{bmatrix}$$

Controllability & Obseravability

Controllability and observability represent two major concepts of modern control system theory. These concepts were introduced by R. Kalman in 1960.

Controllability: In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.

Observability: In order to see what is going on inside the system under observation, the system must be observable

Controllability

State variable representation of a system is

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

A system represented by $\dot{X} = AX + BU$ & $Y = CX + DU$ is said to be controllable if it is possible to transfer the system state from any initial state x(t_0) to any desired state x(t) in finite time 't' by a control input U(t). Otherwise the system is state uncontrollable.

A system is completely controllable if all the state variables are controllable.

Controllability is a property of the linkage between the input and the state & thus involves the matrices A and B.

Kalman's Method of testing controllability:

For an nth order linear time invariant system, controllable can be determined from the below statement.

The system is state controllable if and only if the rank of state controllability matrix Q_C is equal to 'n'.

Where Q_C = [B AB A^2B ..... $A^{n-1}B$]

Ex: find whether the given system is controllable or not.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \end{bmatrix} u$$

Sol:

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}, B=\begin{bmatrix}0\\0\\1\end{bmatrix}, AB=\begin{bmatrix}0\\1\\-6\end{bmatrix}, A^2B=\begin{bmatrix}1\\-6\\25\end{bmatrix}$$

$$Q_c=\begin{bmatrix}B & AB & A^2B\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -6\\ 1 & -6 & 25\end{bmatrix}$$

The determinant of Q_c =1 ≠0 , so rank of Q_c = 3 =n hence this system is controllable.

2. Consider a system represented by the two state equations

$\dot{x}_1=-2x_1+u$, $\dot{x}_2=-3x_2+d\,x_1$

The output of the system is $y=x_2$. Determine the condition of controllability.

$$\dot{X}=\begin{bmatrix}-2 & 0\\ d & -3\end{bmatrix}X+\begin{bmatrix}1\\0\end{bmatrix}u \quad , \quad y=\begin{bmatrix}0 & 1\end{bmatrix}x+\begin{bmatrix}0\end{bmatrix}u$$

$$B=\begin{bmatrix}1\\0\end{bmatrix} \; and \; AB=\begin{bmatrix}-2 & 0\\ d & -3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-2\\d\end{bmatrix}$$

$$Q_c=\begin{bmatrix}1 & -2\\ 0 & d\end{bmatrix}$$

The determinant of Qc is equal to d, which is nonzero only when d is nonzero.

Observability

A system represented by $\dot{X}=AX+BU$ & $Y=CX+DU$ is said to be observable if each state x(t) could be determined from measurement of the output with a known input over a finite interval of time.

A system is completely observable if all the state variables are observable.

Observability is a property of linkage between the output & the state. Thus involves the matrices A and C.

Kalman's Method of testing Observability:

For an nth order linear time invariant system, Observability can be determined from the below statement.

The system is observable if and only if the rank of state observable matrix Q_0 is equal to 'n'.

Where

$$Q_0^T = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

Ex: find whether the given system is observable or not.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \end{bmatrix} u$$

$$sol \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad , \quad CA^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$Q_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of Q_o =1 ≠0 , so rank of Q_o = 3 =n hence this system is observable.

$$\mathbf{A}_d = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}, \quad \mathbf{C}_d = [1 \quad 2]$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}$$

so that $\text{rank}\mathcal{O} = 1 < 2$, and the system is unobservable.

Example 5.3: Given the linear continuous-time system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \\ -6 & 7 & 8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & -1 \\ 2 & -3 \\ 4 & -5 \end{bmatrix} \mathbf{u}$$

The controllability matrix for this third-order system is given by

$$\mathcal{C} = \left[\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B}\right] = \left[\begin{array}{ccccc} 0 & -1 & \vdots & -6 & 7 \\ 2 & -3 & \vdots & 12 & -10 \\ 4 & -5 & \vdots & 46 & -55 \end{array} \vdots \; \mathbf{A}^2\mathbf{B}\right]$$

Since the first three columns are linearly independent we can conclude that $\text{rank}\mathcal{C} = 3$. Hence there is no need to compute $\mathbf{A}^2\mathbf{B}$ since it is well known from linear algebra that the row rank of the given matrix is equal to its column rank. Thus, $\text{rank}\mathcal{C} = 3 = n$ implies that the system under consideration is controllable.

Gilberts method to find controllability & observability:

This method is useful when the canonical state variable representation is given.

Case 1: When the eigen values of the system matrix are distinct.

Consider a system with state model represented by the equations in the canonical form as follows,

$$\dot{Z} = \hat{A}Z + \hat{B}U$$

$$Y = \hat{C}Z + DU$$

The system is completely state controllable if the matrix $\hat{B}$ does not have any row with all zeros.

The system is completely Observable if the matrix $\hat{C}$ does not have any column with all zeros.

Case 2: Eigen values of the system matrix are repeated (Jordan Canonical form)

In this situation it is not possible to diagonalize the system matrix and it can be converted to Jordan canonical form.

The system will be completely state controllable if elements of any row of B that correspond to the last row of each Jordan block are not all zero and the rows corresponding to other state variables must not have all zeros.

The system will be completely state observable if elements of any column of C that correspond to the first row of each Jordan block are not all zero and the columns corresponding to other state variables must not have all zeros.

Gilbert's test is only applicable if matrix A is in **Jordan canonical form** or **Diagonal canonical form**.

Let us understand with the help of an example of Jordan canonical form.

$$A = \begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Here matrix A is in Jordan canonical form. The Jordan block JB_1 and JB_2 are

$$A = \begin{bmatrix} \overbrace{-\lambda_1}^{JB_1} & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & \underbrace{-\lambda_2}_{JB_2} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$JB_1 = \begin{bmatrix} -\lambda_1 & 1 \\ 0 & -\lambda_1 \end{bmatrix}, JB_2 = [-\lambda_2]$$

To check controllability, we see the last row of Jordan blocks.

The last row of Jordan block JB_1 corresponds to the second row of matrix A. The last row of Jordan block JB_2 corresponds to the third row of matrix A.

Hence according to Gilbert's test, we check the corresponding second and third rows of matrix B.

If these second and third rows of matrix B are non-zero then the system is controllable.

Let us understand with the help of an example of Diagonal canonical form.

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Now we treat it as Jordan canonical form with each Jordan block of order 1 × 1.

The Jordan block JB_1, JB_2 and JB_3 are

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \text{ (JB}_1\text{, JB}_2\text{, JB}_3\text{)} \quad , B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$JB_1 = [-\lambda_1] \ , JB_2 = [-\lambda_2] \ , JB_3 = [-\lambda_3]$$

To check controllability, we see the last row of Jordan blocks.

The last row of Jordan block JB_1 corresponds to the **first row** of matrix A. The last row of Jordan block JB_2 corresponds to the **second row** of matrix A. The last row of Jordan block JB_3 corresponds to the **third row** of matrix A.

Hence according to Gilbert's test, we check the corresponding first, second, and third rows of matrix B.

If these first, second, and third rows of matrix B are non-zero then the system is **controllable**.

Gilbert's Test to check observability

It states that

A system is said to be observable if columns of C corresponding to the first row of each Jordan Block is non-zero.

A system is said to be observable if all columns of C is non-zero in case matrix A is in diagonal canonical form.

Gilbert's test is only applicable if matrix A is in **Jordan canonical form** or **Diagonal canonical form**.

Let us understand with the help of an example of Jordan canonical form.

$$A = \begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{bmatrix} \ , C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Here matrix A is in Jordan canonical form. The Jordan block JB_1 and JB_2 are

$$A = \begin{bmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

JB_1 JB_2

$$JB_1 = \begin{bmatrix} -\lambda_1 & 1 \\ 0 & -\lambda_1 \end{bmatrix}, JB_2 = [-\lambda_2]$$

To check observability, we see the first row of Jordan blocks.

The first row of Jordan block JB_1 corresponds to the **first row** of matrix A. The first row of Jordan block JB_2 corresponds to the **third row** of matrix A.

Hence according to Gilbert's test, we check the corresponding first and third columns of matrix C.

If these first and third columns of matrix C are non-zero then the system is **unobservable**.

Let us understand with the help of an example of Diagonal canonical form.

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Now we treat it as Jordan canonical form with each Jordan block of order 1 × 1.

The Jordan block JB_1, JB_2 and JB_3 are

$$A = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

JB_1 JB_2 JB_3

$$JB_1 = [-\lambda_1], JB_2 = [-\lambda_2], JB_3 = [-\lambda_3]$$

To check observability, we see the first row of Jordan blocks.

The first row of Jordan block JB_1 corresponds to the **first row** of matrix A. The first row of Jordan block JB_2 corresponds to the **second row** of matrix A. The first row of Jordan block JB_3 corresponds to the **third row** of matrix A.

Hence according to Gilbert's test, we check the corresponding first, second, and third columns of matrix C.

If these first, second, and third columns of matrix C are non-zero then the system is **observable**.

Example: Consider the system given by

$$\dot{X} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \quad and \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} X$$

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad and \quad AB = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

det Q_c=0 and rank(Q_c)=1≠2. Thus, the system is not controllable

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad and \quad CA = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

det Q_o=0 and rank(Q_o)=1≠2. Thus, the system is not observable.

Problems: